Shtuka cohomology and special values of Goss L-functions

Mornev, M.

CHAPTER 7

The motive of a Drinfeld module

We recall the notion of a Drinfeld module [7], its motive as introduced by Anderson [1], and a construction of Drinfeld [8] associating a shtuka to a Drinfeld module. More precisely: let C over $\mathbb{F}_q$ be a smooth projective curve and $\infty \in C$ a closed point. Denote $A = \Gamma(C \backslash \{\infty\}, \mathcal{O}_C)$. If E is a Drinfeld A-module over an $\mathbb{F}_q$-algebra B then its motive M is a left $A \otimes B\{\tau\}$-module which is a locally free $A \otimes B$-module. Drinfeld's construction yields a canonical shtuka $\mathcal{E}_{-1} \rightrightarrows \mathcal{E}_0$ on $C \times B$ which restricts to M on $\operatorname{Spec} A \otimes B$.

This construction provides a compactification of M in the direction of the coefficient curve C. In the subsequent chapters we will combine it with a compactification along a base curve. Both compactifications play an essential role in the proof of the class number formula.

1. Forms of the additive group

In this section we work over a fixed $\mathbb{F}_q$-algebra B. To simplify the exposition *we assume that B is reduced.* The theory which we attempt to present here can be developed without this assumption. However it then becomes more subtle. In applications we will only need the case of reduced B.

We equip B with a τ-ring structure given by the q-th power map. The Frobenius τ and its powers are in a natural way $\mathbb{F}_q$-linear endomorphisms of the additive group scheme $\mathbb{G}_a$ over B.

Lemma 1.1. *Let* $\operatorname{End}(\mathbb{G}_a)$ *be the ring of* $\mathbb{F}_q$*-linear endomorphisms of* $\mathbb{G}_a$. *The natural map* $B\{\tau\} \to \operatorname{End}(\mathbb{G}_a)$ *is an isomorphism.* □

Lemma 1.2. $B\{\tau\}^\times = B^\times$.

Proof. Let $\varphi \in B\{\tau\}^\times$. If K is a B-algebra which is a field then the image of φ in $K\{\tau\}$ must be an element of $K^\times$. Therefore the constant coefficient of φ is a unit and all other coefficients are nilpotent. Since B is assumed to be reduced we conclude that $\varphi \in B^\times$. □

Recall that an $\mathbb{F}_q$-vector space scheme E is an abelian group scheme equipped with a compatible $\mathbb{F}_q$-multiplication.

Definition 1.3. We say that an $\mathbb{F}_q$-vector space scheme E is *a form of* $\mathbb{G}_a$ if it is Zariski-locally isomorphic to $\mathbb{G}_a$.

Our main object of study is the motive of E which we now introduce.

Definition 1.4. Let E be a form of $\mathbb{G}_a$. The *motive of E* is the abelian group $M = \operatorname{Hom}(E, \mathbb{G}_a)$ of $\mathbb{F}_q$-linear group scheme morphisms from E to $\mathbb{G}_a$. The endomorphism ring of $\mathbb{G}_a$ acts on M by composition making it into a left $B\{\tau\}$-module.

Lemma 1.5. *The formation of the motive of E commutes with arbitrary base change.*

Proof. For a scheme Y over $X = \operatorname{Spec} B$ set $\mathcal{M}(Y) = \operatorname{Hom}(E_Y, \mathbb{G}_{a,Y})$ where E_Y denotes the pullback of E to Y. Zarski descent for morphisms of schemes implies that $\mathcal{M}$ is a sheaf on the big Zariski site of X. The abelian group $\mathcal{M}(Y)$ carries a natural action of $\Gamma(Y, \mathcal{O}_Y)$ on the left. Together these actions make $\mathcal{M}$ into a sheaf of $\mathcal{O}_X$-modules. The formation of $\mathcal{M}$ is functorial in E.

If $E = \mathbb{G}_a$ then $\mathcal{M}$ is the quasi-coherent sheaf defined by the left B-module $B\{\tau\}$. Since E is Zariski-locally isomorphic to $\mathbb{G}_a$ we conclude that $\mathcal{M}$ is quasi-coherent. Therefore the natural map $S \otimes_B \mathcal{M}(X) \to \mathcal{M}(\operatorname{Spec} S)$ is an isomorphism for every B-algebra S. $\square$

Proposition 1.6. *Let E be a form of $\mathbb{G}_a$ and let M be its motive. There exists a unique invertible B-submodule $M^0 \subset M$ such that the natural homomorphism of left $B\{\tau\}$-modules*

$$B\{\tau\} \otimes_B M^0 \to M, \quad \varphi \otimes m \mapsto \varphi \cdot m$$

is an isomorphism.

Definition 1.7. Let E be a form of $\mathbb{G}_a$ and let M be its motive. We define the *degree filtration* M^* on M in the following way. For $n \geqslant 0$ we let $M^n = B\{\tau\}^n \cdot M^0$ where $B\{\tau\}^n \subset B\{\tau\}$ is the submodule of τ-polynomials of degree at most n. For $n < 0$ we set $M^n = 0$.

Proof of Proposition 1.6. First let us prove unicity. If $M^0, N^0 \subset M$ are invertible B-submodules such that the natural maps $B\{\tau\} \otimes_B M^0 \to M$ and $B\{\tau\} \otimes_B N^0 \to M$ are isomorphisms then we get an induced isomorphism $B\{\tau\} \otimes_B M^0 \cong B\{\tau\} \otimes_B N^0$ of left $B\{\tau\}$-modules. Now Lemma 1.2 implies that this isomorphism comes from a unique B-linear isomorphism $M^0 \cong N^0$ which is compatible with the inclusions $M^0 \hookrightarrow M$ and $N^0 \hookrightarrow M$. As a consequence the submodules M^0 and N^0 of M coincide.

Next let us prove the existence. If $E = \mathbb{G}_a$ then we can take for M^0 the submodule $B \cdot \tau^0 \subset B\{\tau\} = M$. For an affine open subscheme $\operatorname{Spec} S \subset \operatorname{Spec} B$ let E_S be the pullback of E to $\operatorname{Spec} S$ and let M_S be the motive of E_S. If E_S is isomorphic to $\mathbb{G}_{a,S}$ then by the remark above we have an invertible S-submodule $M_S^0 \subset M_S$ satisfying the condition of the proposition. Now the natural map $S \otimes_B M \to M_S$ is an isomorphism by Lemma 1.5 so the unicity part of the proposition implies that M_S^0 glue to an invertible B-submodule $M^0 \subset M$. The natural map $B\{\tau\} \otimes_B M^0 \to M$ is an isomorphism since it is so after the pullback to every affine open subscheme $\operatorname{Spec} S \subset \operatorname{Spec} B$ such that $E_S \cong \mathbb{G}_{a,S}$. $\square$

Without the assumption that B is reduced the existence part of Proposition 1.6 still holds. However the submodule $M^0 \subset M$ is not unique anymore. The group $\operatorname{Aut} E$ acts transitively on the set of all such submodules with stabilizers isomorphic to $B^\times$.

Lemma 1.8. *The degree filtration on M is stable under base change to an arbitrary* reduced *B-algebra.*

Proof. Let S be a reduced B-algebra. By Proposition 1.6 it is enough to show that the formation of M^0 commutes with the base change. Let S be a B-algebra and let M_S be the motive of E over S. The natural map $S\{\tau\} \otimes_B M^0 \to S \otimes_B M$ is an isomorphism by definition of M^0. Lemma 1.5 shows that the natural map $S \otimes_B M \to M_S$ is an isomorphism. In particular $S \otimes_B M^0$ is in a natural way an S-submodule of M_S. Now if S is reduced then Propostion 1.6 implies that the image of $S \otimes_B M^0$ in M_S is $(M_S)^0$. □

Proposition 1.9. *If E is a form of $\mathbb{G}_a$ with motive M then for every B-algebra S the map*

$$E(S) \to \operatorname{Hom}_B(M^0, S), \quad e \mapsto \big(m \mapsto m(e)\big)$$

is an $\mathbb{F}_q$-linear isomorphism.

Proof. Let E_0 be the $\mathbb{F}_q$-vector space scheme defined by the functor of points $E_0(S) = \operatorname{Hom}_B(M^0, S)$. The natural map above defines a morphism of $\mathbb{F}_q$-vector space schemes $E \to E_0$. By Lemma 1.8 the formation of M^0 commutes with localization of B. So it is enough to prove that $E \to E_0$ is an isomorphism after a localization of B. However if $E = \mathbb{G}_a$ then the map $E \to E_0$ is clearly an isomorphism. □

Let E be a form of $\mathbb{G}_a$ and M its motive. We denote $M^{\geqslant 1} = B\{\tau\}^{\geqslant 1} \otimes_B M^0$ where $B\{\tau\}^{\geqslant 1} \subset B\{\tau\}$ is the ideal of τ-polynomials which have constant coefficient 0. In other words $M^{\geqslant 1}$ consists of those morphisms which induce the zero map from the Lie algebra of E to the Lie algebra of $\mathbb{G}_a$.

Proposition 1.10. *Let E be a form of $\mathbb{G}_a$ with motive M. The adjoint $\tau^* M \to M$ of the multiplication map $\tau\colon M \to M$ is injective with image $M^{\geqslant 1}$.* □

Proposition 1.11. *If E is a form of $\mathbb{G}_a$ with motive M then for every B-algebra S the map*

$$\operatorname{Lie}_E(S) \to \operatorname{Hom}_B(M/M^{\geqslant 1}, S), \quad \varepsilon \mapsto \big(m \mapsto dm(\varepsilon)\big)$$

is an isomorphism of S-modules. □

Here dm denotes the map from Lie_E to $\operatorname{Lie}_{\mathbb{G}_a}$ induced by $m\colon E \to \mathbb{G}_a$.

2. Coefficient rings

Let A be an $\mathbb{F}_q$-algebra of finite type which is a Dedekind domain. To such an algebra A one can functorially associate a smooth connected projective curve C over $\mathbb{F}_q$ together with an open embedding $\operatorname{Spec} A \subset C$. We call C the compactification of $\operatorname{Spec} A$. The closed points of C correspond in one-to-one manner to the maximal discrete valuation subrings in the fraction field $\operatorname{Frac} A$.

Definition 2.1. We say that A is a *coefficient ring* if the complement of $\operatorname{Spec} A$ in C consists of a single point. This point is called the point of A at infinity.

A typical example of a coefficient ring is $\mathbb{F}_q[t]$. Every coefficient ring can be constructed in the following way. Let C be a smooth projective connected curve over $\mathbb{F}_q$. Pick a closed point $\infty \in C$ and set $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$. In this case C is the compactification of $\operatorname{Spec} A = C - \{\infty\}$. Hence A is a coefficient ring.

Recall that an element $a \in A$ is called constant if it is algebraic over $\mathbb{F}_q$. Since we do not assume A to be geometrically irreducible there may be constant elements not in $\mathbb{F}_q \subset A$.

Lemma 2.2. *Let A be a coefficient ring. If $a \in A$ is not constant then the natural map $\mathbb{F}_q[a] \to A$ is finite flat.*

Proof. If the map $\mathbb{F}_q[a] \to A$ is not injective then a satisfies a polynomial equation with coefficients in $\mathbb{F}_q$, a contradiction. Hence $\mathbb{F}_q[a] \to A$ is an injection. Since A has no $\mathbb{F}_q[a]$-torsion it folows that A is flat over $\mathbb{F}_q[a]$. The only nontrivial claim is that it is finite.

Let C be the compactification of $\operatorname{Spec} A$ and let ∞ be the point in the complement of $\operatorname{Spec} A$ in C. The inclusion $\mathbb{F}_q[a] \subset A$ induces a morphism $C \to \mathbb{P}^1_{\mathbb{F}_q}$. This morphism is automatically proper. The only point of C which does not map to $\operatorname{Spec} \mathbb{F}_q[a] \subset \mathbb{P}^1_{\mathbb{F}_q}$ is ∞. Hence the preimage of $\operatorname{Spec} \mathbb{F}_q[a]$ in C is $\operatorname{Spec} A$. We conclude that the map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{F}_q[a]$ is proper. As a consequence it is finite [01WN]. □

Attached to A one has its local field at infinity F. It is the completion of the function field of C at the point ∞. One has a canonical inclusion $A \hookrightarrow F$.

Definition 2.3. Let A be a coefficient ring. We define the degree map

$$\deg\colon A - \{0\} \to \mathbb{Z}$$

in the following way. Let F be the local field of A at infinity and let $\nu\colon F^\times \to \mathbb{Z}$ be its normalized valuation. We set $\deg(a) = -\nu(a)$.

Observe that $\deg(a) = 0$ if and only if a is a nonzero constant.

Lemma 2.4. *Let A be a coefficient ring, F the local field at infinity, k the residue field of F. If $a \in A$ is not constant then*

$$f \cdot \deg(a) = d$$

where $f = [k : \mathbb{F}_q]$, $d = [A : \mathbb{F}_q[a]]$.

Proof. Let F_0 be the local field of $\mathbb{F}_q[a]$ at infinity. By construction a^{-1} is a uniformizer of F_0. Hence $\deg(a)$ equals the ramification index e of F over F_0. Moreover f coincides with the inertia index of F over F_0. Since $ef = [F : F_0]$ we only need to prove that $[F : F_0] = d$. Both A and $\mathbb{F}_q[a]$ have a single point at infinity. Thus

$$F_0 \otimes_{\mathbb{F}_q[a]} A = F.$$

Since the inclusion $\mathbb{F}_q[a] \subset A$ is finite flat it follows that $[F : F_0] = d$. □

3. Action of coefficient rings

We keep working over the fixed $\mathbb{F}_q$-algebra B. As before we suppose that B is reduced. Throughout this section we fix an $\mathbb{F}_q$-vector space scheme E over B which is a form of $\mathbb{G}_a$. We denote M its motive.

We assume that E is equipped with an action of a fixed coefficient ring A. In other words we are given a homomorphism $\varphi\colon A \to \mathrm{End}(E)$. The ring A acts on $M = \mathrm{Hom}(E, \mathbb{G}_a)$ on the right. As A is commutative we can view it as a left action. Thus M acquires a structure of a left $A \otimes B\{\tau\}$-module. In this section we study how the $A \otimes B$-module structure on M interacts with the degree filtration.

Lemma 3.1. *Assume that B is noetherian. If M^0 is an $A \otimes B$-submodule of M then M is* not *a finitely generated $A \otimes B$-module.*

Proof. Indeed in this case every $M^n \subset M$ is an $A \otimes B$-submodule. Since $M^n/M^{n-1} \cong B$ we conclude that M contains an infinite increasing chain of $A \otimes B$-submodules. As $A \otimes B$ is noetherian it follows that M can not be a finitely generated $A \otimes B$-module. □

Lemma 3.2. *Let $a \in A$ and let $d \geqslant 0$. The following are equivalent:*

(1) *$M^0 a \subset M^d$ and the induced map $a\colon M^0 \to M^d/M^{d-1}$ is an isomorphism.*
(2) *The same holds after base change to every B-algebra K which is a field.*

Proof. (1) $\Rightarrow$ (2) is a consequence of Lemma 1.8. (2) $\Rightarrow$ (1). Thanks to Lemma 1.8 we may assume that $E = \mathbb{G}_a$. In this case the action of A on E is given by a homomorphism $\varphi\colon A \to B\{\tau\}$. The condition (2) means that for every B-algebra K which is a field the polynomial $\varphi(a)$ has degree d in $K\{\tau\}$. Therefore the coefficient of $\varphi(a)$ at τ^d is a unit while the coefficient at τ^n is nilpotent for every $n > d$. By assumption of this section B is reduced. Hence $\varphi(a)$ is of degree d with top coefficient a unit. □

Lemma 3.3. *Assume that $A = \mathbb{F}_q[t]$. Let $r \geqslant 1$. The following are equivalent:*

(1) *$M^0 t \subset M^r$ and the induced map $t\colon M^0 \to M^r/M^{r-1}$ is an isomorphism of B-modules.*

(2) *The natural map* $A \otimes M^{r-1} \to M$ *is an isomorphism of* $A \otimes B$-*modules.*

(3) M *is a locally free* $A \otimes B$-*module of rank* r.

Proof. Thanks to Lemma 1.8 we may assume that $E = \mathbb{G}_a$. In this case $M = B\{\tau\}$ and the degree filtration on M is the filtration by degree of τ-polynomials. The action of A is given by a homomorphism $\varphi\colon \mathbb{F}_q[t] \to B\{\tau\}$. We split the rest of the proof into several steps.

Step 1. *If* (1) *holds then the natural map* $A \otimes M^{r-1} \to M$ *is surjective.*

By assumption $\varphi(t) \in M^r$. Write $\varphi(t) = \psi + \alpha_r \tau^r$ with ψ of degree less than r. The coefficient α_r is invertible since the induced map $t\colon M^0 \to M^r/M^{r-1}$ is an isomorphism. Therefore

$$\tau^r = \frac{1}{\alpha_r}(\varphi(t) - \psi).$$

Multiplying both sides by τ^n on the left we obtain a relation

$$\tau^{r+n} = \frac{1}{\tau^n(\alpha_r)}(\tau^n\varphi(t) - \tau^n\psi).$$

Induction over n now shows that the image of the natural map $A \otimes M^{r-1} \to M$ is the whole of M.

Step 2. *If* B *is a field then* (1) *implies* (2).

According to Step 1 the natural map $A \otimes M^{r-1} \to M$ is surjective. We need to prove that it is injective. Observe that for every nonzero $\alpha \in B$ and $n, d \geqslant 0$ the τ-polynomial $\alpha\tau^n\varphi(t^d)$ is of degree $rd + n$. Hence if $f \in A \otimes B = B[t]$ is a polynomial of degree d then $f \cdot \tau^n$ is of degree $rd + n$.

Now let $f_0, \ldots, f_{r-1} \in B[t]$. If one of the f_n is nonzero then there exists a unique $n \in \{0, \ldots, r-1\}$ such that $r \cdot \deg f_n + n$ is maximal. From the observation above we deduce that $f_n \cdot \tau^n$ is of degree $r \deg f_n + n$ while for every $m \neq n$ the element $f_m \cdot \tau^m$ is of lesser degree. We conclude that

$$f_0 \cdot 1 + f_1 \cdot \tau + \ldots + f_{r-1} \cdot \tau^{r-1} \neq 0.$$

Step 3. *If* B *is noetherian then* (1) *implies* (2).

According to Step 1 the natural map $A \otimes M^{r-1} \to M$ is surjective. We thus have a short exact sequence

$$0 \to N \to A \otimes M^{r-1} \to M \to 0. \tag{3.1}$$

Since B is noetherian it follows that N is a finitely generated $A \otimes B$-module. By construction M^{r-1} and M are flat B-modules. Therefore (3.1) is a short exact sequence of flat B-modules.

Let K be a B-algebra which is a field. Lemma 1.8 tells that the formation of M commutes with base change to K and that the base change preserves the degree filtration. Therefore Step 3 shows that the second arrow of (3.1) becomes an isomorphism after base change to K. Since (3.1) is a sequence of flat B-modules we conclude that $N \otimes_B K = 0$. As N is a finitely generated $A \otimes B$-module Nakayama's lemma implies that $N = 0$.

Step 4. (1) *implies* (2). Write

$$\varphi(t) = \alpha_0 + \alpha_1\tau + \ldots + \alpha_r\tau^r.$$

By assumption α_r is a unit. Let B_0 be the $\mathbb{F}_q$-subalgebra of B generated by α_i and α_r^{-1}. Let $E_0 = \mathbb{G}_{a,B_0}$ equipped with the action of $\mathbb{F}_q[t]$ given by φ. As α_r is a unit in B_0 it follows that the assumption (1) holds for E_0. Step 3 now implies that (2) holds for E_0. Lemma 1.8 shows that (2) holds for the base change of E_0 to B. As $E_0 \otimes_{B_0} B = E$ by construction the result follows.

Step 5. (2) *implies* (3). By construction M^{r-1} is a locally free B-module of rank r. Therefore $A \otimes M^{r-1}$ is a locally free $A \otimes B$-module of rank r.

Step 6. (3) *implies* (1). Thanks to Lemma 3.2 we may suppose that B is a field. If $\varphi(t)$ is of degree 0 then M^0 is an $A \otimes B$-submodule of M. Lemma 3.1 then shows that M is not a finitely generated $A \otimes B$-module, a contradiction. Hence $\varphi(t)$ is of positive degree d. Now Step 3 shows that M is locally free of rank d whence $d = r$. The induced map $t\colon M^0 \to M^r/M^{r-1}$ is an isomorphism since the top coefficient of $\varphi(t)$ is not zero. □

We now return to a general coefficient ring A. Let F be the local field of A at infinity and let k be the residue field of F. We denote

$$f = [k : \mathbb{F}_q]$$

the degree of the residue field extension at infinity. Let $\deg\colon A - \{0\} \to \mathbb{Z}$ be the degree map of A as in Definition 2.3. Recall that $\deg(a) = -\nu(a)$ where ν is the normalized valuation of F.

Proposition 3.4. *Let $r \geqslant 1$ and let $a \in A$ be a nonconstant element. The following are equivalent:*

1. *M is a locally free $A \otimes B$-module of rank r.*
2. *$M^0 a \subset M^{fr\deg a}$ and the induced map*

$$M^0 \xrightarrow{a} M^{fr\deg a}/M^{fr\deg a-1}$$

is an isomorphism of B-modules.

Proof. According to Lemma 2.2 the natural map $\mathbb{F}_q[a] \to A$ is finite flat. Hence M is locally free of rank r as an $A \otimes B$-module if and only if it is locally free of rank rd as an $\mathbb{F}_q[a] \otimes B$-module where $d = [A : \mathbb{F}_q[a]]$. Since $d = f\deg a$ by Lemma 2.4 the result follows from Lemma 3.3 applied to $t = a$. □

Assuming that the base ring B is noetherian we next show that the motive M is a finitely generated $A \otimes B$-module if and only if it is locally free. We include this result only for illustrative purposes. It will not be used in the proof of the class number formula.

Lemma 3.5. *Assume that $A = \mathbb{F}_q[t]$ and B is a field. If M is a finitely generated $A \otimes B$-module then it is locally free of rank $\geqslant 1$.*

Proof. If $M^0t \subset M^0$ then Lemma 3.1 shows that M is not finitely generated as an $\mathbb{F}_q[t] \otimes B$-module, t contradiction. Therefore $M^0t \subset M^n$ for some $n \geqslant 1$. Without loss of generality we may assume that $M^0t \not\subset M^{n-1}$. In this case the induced map $t\colon M^0 \to M^n/M^{n-1}$ is nonzero. As B is t field it is an isomorphism. Lemma 3.3 then shows that M is t locally free $A \otimes B$-module of rank $r \geqslant 1$. □

Lemma 3.6. *Assume that $A = \mathbb{F}_q[t]$ and B is a DVR. Let K be the fraction field and k the residue field of B. If M is a finitely generated $A \otimes B$-module then* $\operatorname{rank}_{A\otimes K} M \otimes_B K \geqslant \operatorname{rank}_{A\otimes k} M \otimes_B k$.

Proof. The Picard group of B is trivial so by Lemma 1.6 we may assume that $E = \mathbb{G}_a$. In this case the A-action is given by a homomorphism $\varphi\colon \mathbb{F}_q[t] \to B\{\tau\}$. Let r_K be the degree of $\varphi(t)$ in $K\{\tau\}$ and let r_k be its degree in $k\{\tau\}$. Lemma 3.3 shows that $M \otimes_B K$ is a locally free $A \otimes K$-module of rank r_K while $M \otimes_B k$ is a locally free $A \otimes k$-module of rank r_k. As $r_K \geqslant r_k$ by construction the result follows. □

Proposition 3.7. *If B is noetherian and* $\operatorname{Spec} B$ *is connected then the following are equivalent:*

(1) *M is a finitely generated $A \otimes B$-module.*
(2) *M is a locally free $A \otimes B$-module of rank r for some $r \geqslant 1$.*

Proof. (1) ⇒ (2). If $a \in A$ is a nonconstant element then the map $\mathbb{F}_q[a] \to A$ is finite flat by Lemma 2.2. Hence to deduce (2) it is enough to assume that $A = \mathbb{F}_q[t]$.

Let $r\colon \operatorname{Spec} B \to \mathbb{Z}_{\geqslant 1}$ be the function which sends a prime $\mathfrak{p} \subset B$ to the rank of $M \otimes_B \operatorname{Frac} B/\mathfrak{p}$ as an $A \otimes \operatorname{Frac} B/\mathfrak{p}$-module. We will show that r is lower semi-continuous. Let $\mathfrak{p} \subset \mathfrak{q}$ be primes of B such that $\mathfrak{p} \neq \mathfrak{q}$. According to [054F] there exists a discrete valuation ring V and a morphism $B \to V$ such that the generic point of $\operatorname{Spec} V$ maps to $\mathfrak{p}$ and the closed point maps to $\mathfrak{q}$. Applying Lemma 3.6 to the base change of E to V we deduce that $r(\mathfrak{p}) \geqslant r(\mathfrak{q})$. Hence r is lower semi-continuous. However $A \otimes B$ is noetherian and M is a finitely generated $A \otimes B$-module. The function r is therefore also upper semicontinous. We conclude that it is in fact constant. Let us denote this constant r.

Let K be a B-algebra which is a field. Lemma 3.3 shows that $(M^0 \otimes_B K)t \subset (M^r \otimes_B K)$ and the induced map $M^0 \otimes_B K \to (M^r/M^{r-1}) \otimes_B K$ is an isomorphism. Hence Lemma 3.2 shows that the same holds already on the level of B. Applying Lemma 3.3 again we conclude that M is a locally free $\mathbb{F}_q[t] \otimes B$-module of rank $r \geqslant 1$. □

4. Drinfeld modules

We keep working over a fixed reduced $\mathbb{F}_q$-algebra B. Let A be a coefficient ring as in Definition 2.1. Let F be the local field of A at infinity and let k be the residue field of F. We denote

$$f = [k : \mathbb{F}_q]$$

the degree of the residue field extension at infinity. Let $\deg\colon A - \{0\} \to \mathbb{Z}$ be the degree map of A. According to Definition 2.3 $\deg(a) = -\nu(a)$ where ν is the normalized valuation of F.

Definition 4.1. A *Drinfeld A-module* of rank $r \geqslant 1$ over B is an $\mathbb{F}_q$-vector space scheme E over B equipped with an action of A and satisfying the following conditions:

(1) E is a form of $\mathbb{G}_a$.
(2) The motive $M = \operatorname{Hom}(E, \mathbb{G}_a)$ is a locally free $A \otimes B$-module of rank r.

Proposition 3.4 implies that condition (2) is equivalent to

(2′) There exists a nonconstant element $a \in A$ such that $M^0 a \subset M^{fr \deg a}$ and the induced map $a\colon M^0 \to M^{fr \deg a}/M^{fr \deg a - 1}$ is an isomorphism.

Using this fact it is easy to show that our definition is equivalent with Drinfeld's original definition [7]. As in [7] the rank of our Drinfeld modules is constant on $\operatorname{Spec} B$.

If B is noetherian and E satisfies (1) then M is a locally free $A \otimes B$-module if and only if it is finitely generated (see Proposition 3.7).

Proposition 4.2. *Let E be a Drinfeld A-module of rank r and let M be its motive. The degree filtration M^* has the following properties.*

(1) *M^* is exhaustive.*
(2) *M^n is a locally free B-module of rank* $\max(0, n+1)$.
(3) *For every $n \geqslant 0$ and every nonzero $a \in A$ we have $M^n a \subset M^{n+fr \deg a}$ and the induced map*

$$M^n/M^{n-1} \to M^{n+fr \deg a}/M^{n+fr \deg a - 1}$$

is an isomorphism.

Proof. (1) and (2) are immediate from the definition of the degree filtration. Let us prove (3). If $a \in A$ is constant then a is a unit of A and $\deg a = 0$ so the condition is vacuous. Assume that a is not constant. Proposition 3.4 shows that (3) holds for $n = 0$. By construction the natural map $B\{\tau\}^n \otimes_B M^0 \to M^n$, $\varphi \otimes m \mapsto \varphi m$ is an isomorphism. Using this fact and the fact that A acts on M on the right we deduce that (3) holds for every $n \geqslant 0$. ☐

A detailed study of motives of Drinfeld modules over arbitrary, not necessarily reduced base rings can be found in the preprint [12] of Urs Hartl.

5. Drinfeld's construction

In this section we recall Drinfeld's construction [8] of a shtuka attached to a Drinfeld module. No originality is claimed. All nontrivial results are Drinfeld's.

Fix a coefficient ring A. Let C be the projective compactification of $\operatorname{Spec} A$ and $\infty \in C$ the closed point in the complemenet of $\operatorname{Spec} A$. Let F be the local field of C at ∞. As in the previous sections f denotes the degree of the residue field of F over $\mathbb{F}_q$. Let B be an $\mathbb{F}_q$-algebra. We do not assume B to be reduced.

Fix an ample line bundle on C which corresponds to the divisor ∞. Let $\mathcal{O}(1)$ be the pullback of this bundle to $C \times B$ and $\mathcal{O}$ the structure sheaf of $C \times B$. As $\mathcal{O}(1)$ is defined by a divisor, it comes equipped with an inclusion $\mathcal{O} \subset \mathcal{O}(1)$. So every locally free $\mathcal{O}$-module $\mathcal{E}$ sits in a natural system of inclusions

$$\ldots \subset \mathcal{E}(-1) \subset \mathcal{E} \subset \mathcal{E}(1) \subset \mathcal{E}(2) \subset \ldots$$

Definition 5.1. A locally free $\mathcal{O}$-module $\mathcal{E}$ is called *generic* if for every $n \in \mathbb{Z}$ either $H^0(C \times B, \mathcal{E}(n)) = 0$ or $H^1(C \times B, \mathcal{E}(n)) = 0$.

To a locally free $\mathcal{O}$-module $\mathcal{E}$ we associate the $A \otimes B$-module

$$M = \mathrm{H}^0(\operatorname{Spec} A \otimes B, \mathcal{E})$$

equipped with the filtration M^* by B-submodules $M^n = H^0(C \times B, \mathcal{E}(n)) \subset M$.

Proposition 5.2. *Let $r \geqslant 1$. The construction $\mathcal{E} \mapsto M$ defines an equivalence between the following categories:*

(1) *The category of locally free generic $\mathcal{O}$-modules $\mathcal{E}$ of rank r whose Euler characteristic is constant on* $\operatorname{Spec} B$.
(2) *The category of $A \otimes B$-modules M equipped with an increasing filtration M^* which has the following properties:*
   (a) *M^* is exhaustive.*
   (b) *There exists a $\chi \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$ the B-module M^n is locally free of rank* $\max(0, \chi + frn)$.
   (c) *For each nonzero $a \in A$ we have $aM^n \subset M^{n+\deg a}$ and the induced map*

$$M^n/M^{n-1} \to M^{n+\deg a}/M^{n+\deg a-1}$$

   is an inclusion of a direct summand.

The constant χ in the condition (b) agrees with the Euler characteristic of $\mathcal{E}$.

Proof of Proposition 5.2. See [8, Proposition 1]. Note that in [8] Drinfeld fixes the Euler characteristic χ of the category of generic locally free $\mathcal{O}$-modules. However the proof works with varying χ as well. □

Now consider a diagram

$$\ldots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots$$

of inclusions of locally free $\mathcal{O}$-modules of rank r. Assume that the quotients $\mathcal{E}_{n+1}/\mathcal{E}_n$ are supported at the complement of $\operatorname{Spec} A \otimes B$. To such a diagram we associate the $A \otimes B$-module $M = \mathrm{H}^0(\operatorname{Spec} A \otimes B, \mathcal{E}_0)$ equipped with the filtration M^* by B-submodules $M^n = \mathrm{H}^0(C \times B, \mathcal{E}_n)$.

Proposition 5.3. *Let $r \geqslant 1$. The construction $\mathcal{E}_* \mapsto M$ defines an equivalence between the following categories:*

(1) *The category of diagrams $\mathcal{E}_*$ of inclusions of locally free $\mathcal{O}$-modules*

$$\ldots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots$$

of rank r satisfying the following conditions:
   (a) $\mathcal{E}_n(1) = \mathcal{E}_{n+fr}$.
   (b) $\mathrm{H}^0(C \times B, \mathcal{E}_n/\mathcal{E}_{n-1})$ *is an invertible B-module for all n.*
   (c) *There exists $\chi \in \mathbb{Z}$ such that* $\mathrm{H}^0(C \times B, \mathcal{E}_{-\chi-1}) = 0 = \mathrm{H}^1(C \times B, \mathcal{E}_{-\chi-1})$.

(2) *The category of $A \otimes B$-modules M equipped with an increasing filtration M^* by B-submodules satisfying the following conditions:*
   (a) *M^* is exhaustive.*
   (b) *M^n is a locally free B-module of rank* $\max(0, \chi + n + 1)$.
   (c) *For all nonzero $a \in A$ we have $aM^n \subset M^{n+fr\deg a}$ and the induced map*

$$M^n/M^{n-1} \to M^{n+fr\deg a}/M^{n+fr\deg a-1}$$

   is an inclusion of a direct summand.

Proof. See [8, Corollary 1]. □

Before we state the main result of this section let us introduce an auxillary notion.

Definition 5.4. Let R be a τ-ring and let M be an R-module shtuka given by a diagram

$$\Big[M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1\Big].$$

We say that M is *co-nilpotent* if the adjoint $j^a \colon \tau^* M_0 \to M_1$ of the map $j \colon M_0 \to M_1$ is an isomorphism and for $n \gg 0$ the compsition

$$\tau^{*n}(u) \circ \ldots \circ u, \quad u = (j^a)^{-1} \circ i,$$

is zero.

Let $\mathcal{O}_F$ be the completed local ring of C at ∞. We denote $\iota \colon \operatorname{Spec} A \otimes B \hookrightarrow C \times B$ the open immersion and $\alpha \colon \operatorname{Spec} \mathcal{O}_F/\mathfrak{m}_F \otimes B \to C \times B$ the closed complement.

Theorem 5.5. *Assume that B is reduced. Let E be a Drinfeld A-module of rank r over B and let M be its motive. There exists a unique subshtuka*

$$\mathcal{E} = \Big[\mathcal{E}_{-1} \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{E}_0\Big] \subset \iota_*\Big[M \underset{\tau}{\overset{1}{\rightrightarrows}} M\Big].$$

such that the following holds:

(1) *$\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are locally free of rank r.*
(2) *For every $n \in \mathbb{Z}$ we have*

$$\mathrm{H}^0(C \times B, \mathcal{E}_0(n)) = M^{nfr},$$
$$\mathrm{H}^0(C \times B, \mathcal{E}_{-1}(n)) = M^{nfr-1}$$

as B-submodules of M.

Moreover the shtuka $\alpha^*\mathcal{E}$ on $\operatorname{Spec} \mathcal{O}_F/\mathfrak{m}_F \otimes B$ is co-nilpotent.

The fact that $\alpha^*\mathcal{E}$ is co-nilpotent is of fundamental importance to our study. It implies that certain shtukas we construct out of $\mathcal{E}$ are nilpotent which in turn allows us to apply the theory of Chapters 5 and 6 to Drinfeld modules.

Proof of Theorem 5.5. Uniqueness follows from (2). For existence note that by Proposition 4.2 the degree filtration M^* satisfies the conditions (2a)–(2c) of Proposition 5.3 with $\chi = 0$. In particular we have a diagram $\mathcal{E}_*$ of locally free rank r subsheaves in ι_*M such that $\mathcal{E}_{-1}$ and $\mathcal{E}_0$ satisfy (2). This already produces a subdiagram

$$\Big[\mathcal{E}_{-1} \xrightarrow{i} \mathcal{E}_0\Big] \subset \iota_*\Big[M \xrightarrow{1} M\Big]$$

where $i\colon \mathcal{E}_{-1} \hookrightarrow \mathcal{E}_0$ is the inclusion.

It follows from condition (1b) that $\mathrm{L}^1\tau^*(\mathcal{E}_n/\mathcal{E}_{n-1}) = 0$. So the sequence

$$0 \to \tau^*\mathcal{E}_{n-1} \to \tau^*\mathcal{E}_n \to \tau^*(\mathcal{E}_n/\mathcal{E}_{n-1}) \to 0$$

is exact. Moreover $\mathrm{H}^0(C \times B, \tau^*(\mathcal{E}_n/\mathcal{E}_{n-1}))$ is an invertible B-module. If we set $\mathcal{F}_n = \tau^*\mathcal{E}_{n-1}$ then $\mathcal{F}_*$ satisfies conditions (1a)–(1c) of Proposition 5.3 with $\chi = 1$. The diagram $\mathcal{F}_*$ corresponds to the $A \otimes B$-module τ^*M equipped with the following filtration. By Propsition 1.10 we can identify τ^*M with $M^{\geqslant 1}$, and we set

$$(\tau^*M)^n = M^{\geqslant 1} \cap M^n.$$

The adjoint $\tau^*M \to M$ of the map $\tau\colon M \to M$ induces a morphism of diagrams $\mathcal{F}_* \to \mathcal{E}_*$. It gives us a morphism $\tau^*\mathcal{E}_{-1} \to \mathcal{E}_0$, restricting the map $\tau^*\iota_*M \to \iota_*M$.

It remains to show that $\alpha^*\mathcal{E}$ is co-nilpotent. Identifying $\mathcal{E}_n/\mathcal{E}_{n-1}$ with M^n/M^{n-1} we see that the map $\tau^*\mathcal{E}_n/\mathcal{E}_{n-1} \to \mathcal{E}_{n+1}/\mathcal{E}_n$ induced by $\tau^*M \to M$ is an isomorphism. Using the natural filtration we deduce that the map $\tau^*\mathcal{E}_{-1}/\mathcal{E}_{-fr-1} \to \mathcal{E}_0/\mathcal{E}_{-fr}$ is an isomorphism. By property (1a) in Proposition

5.3 this map can be identified with $\alpha^*(j^a)\colon \tau^*\alpha^*\mathcal{E}_{-1} \to \alpha^*\mathcal{E}_0$. Property (1a) now implies that the composition

$$\tau^{*fr}(u) \circ \ldots \circ u, \quad u = \alpha^*(j^a)^{-1} \circ \alpha^*(i)$$

is zero. $\square$