Shtuka cohomology and special values of Goss L-functions

Mornev, M.

Author: Mornev, M.
Title: Shtuka cohomology and special values of Goss L-functions
Issue Date: 2018-02-13

CHAPTER 6

Trace formula

Let X be a smooth projective curve over $\mathbb{F}_q$. As in Chapter 4 we fix an open dense affine subscheme $\operatorname{Spec} R \subset X$. Its complement consists finitely many closed points $x_1, \ldots, x_n \in X$. We denote K the product of the local fields of X at $x_1, \ldots, x_n$. We use the notation and the terminology of Section 3.5 in regard to K. In particular $\mathcal{O}_K \subset K$ stands for the product of the rings of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ denotes the Jacobson radical.

Fix a local field F containing $\mathbb{F}_q$. Let $\mathcal{O}_F \subset F$ be the ring of integers and $\mathfrak{m}_F \subset \mathcal{O}_F$ the maximal ideal. In this chapter we mainly work with the scheme $\mathcal{O}_F \times X$. We equip it with a τ-structure given by the endomorphism which acts as the identity on $\mathcal{O}_F$ and as the absolute q-Frobenius on X.

Let $\mathfrak{f} \subset \mathfrak{m}_K$ be an open ideal. We say that a shtuka $\mathcal{M}$ on $\mathcal{O}_F \times X$ is *elliptic of conductor* $\mathfrak{f}$ if it has the following properties:

(1) $\mathcal{M}$ is locally free,
(2) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent,
(3) $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is an elliptic shtuka of conductor $\mathfrak{f}$ in the sense of Definition 5.6.2.

Using the theory of Chapter 5 we will construct for every elliptic shtuka $\mathcal{M}$ a natural quasi-isomorphism

$$\rho_{\mathcal{M}} \colon \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(\nabla\mathcal{M}),$$

the *regulator* of $\mathcal{M}$. We will also define a numerical invariant $L(\mathcal{M}) \in \mathcal{O}_F^\times$. This invariant is given by an infinite product of local factors

$$\prod_{\mathfrak{m}} \frac{1}{L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m}))}$$

where $\mathfrak{m}$ runs over the maximal ideals of R.

The main result of this chapter is Theorem 10.4 which states that

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}})$$

under a certain cohomological condition on $\mathcal{M}$. Here $\zeta_{\mathcal{M}}$ is the ζ-isomorphism of $\mathcal{M}$ in the sense of Definition 1.10.2. We call this expression the trace formula for regulators of elliptic shtukas. Theorem 10.4 is based on Anderson's trace formula [2] in the form given to it by Böckle-Pink [3] and V. Lafforgue [17]. The statement of Theorem 10.4 has its roots in the article [17] of V. Lafforgue as well.

Besides the τ-scheme $\mathcal{O}_F \times X$ we will use more general τ-schemes $S \times X$ and $S \otimes B$ where S is a local noetherian ring and B an $\mathbb{F}_q$-algebra. In all the cases the τ-structure is given by the endomorphism which acts as the identity on S and as the absolute q-Frobenius on the other factor.

1. A lemma on cohomology of locally free sheaves

In this section we prove an auxillary statement on cohomology of locally free sheaves over $S \times X$ where S is a local noetherian ring. We denote $\mathfrak{m}_S \subset S$ the maximal ideal.

Lemma 1.1. *Let $I \subset \mathcal{O}_K$ be an open ideal which is different from $\mathcal{O}_K$. Denote $\mathcal{I} \subset \mathcal{O}_{S\times X}$ the unique invertible ideal sheaf such that $\mathcal{I}(\operatorname{Spec} S \otimes R) = S \otimes R$ and $\mathcal{I}(\operatorname{Spec} S \otimes \mathcal{O}_K) = S \otimes I$. If $\mathcal{E}$ is a locally free sheaf on $S \times X$ then for all $n \gg 0$ the following is true:*

(1) $\mathrm{H}^0(\mathcal{I}^n\mathcal{E}) = 0$,
(2) $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$ *is a free S-module of finite rank.*

Proof. The base change theorem for coherent cohomology [07VK] states that the S-module complex $\mathrm{R}\Gamma(\mathcal{I}^n\mathcal{E})$ is perfect and

$$\mathrm{R}\Gamma(\mathcal{I}^n\mathcal{E}) \otimes^{\mathbf{L}}_S S/\mathfrak{m}_S = \mathrm{R}\Gamma(S/\mathfrak{m}_S \times X, \mathcal{I}^n\mathcal{E}).$$

In particular $\mathrm{R}\Gamma(\mathcal{I}^n\mathcal{E}) \otimes^{\mathbf{L}}_S S/\mathfrak{m}_S$ is concentrated in degrees $[0,1]$. By [068V] it follows that $\mathrm{R}\Gamma(S \times X, \mathcal{I}^n\mathcal{E})$ has Tor amplitude $[0,1]$. Now [0658] shows that $\mathrm{R}\Gamma(\mathcal{I}^n\mathcal{E})$ is represented by a two-term complex

$$\big[P_0 \xrightarrow{d} P_1\big]$$

of finitely generated free S-modules.

Let Ω be the dualizing sheaf of $S \times X$ over S. Consider the locally free sheaf

$$\mathcal{F}(n) = \mathcal{H}\mathrm{om}(\mathcal{I}^n\mathcal{E}, \Omega)$$

on $S \times X$. Since the open ideal $I \subset \mathcal{O}_K$ is different from $\mathcal{O}_K$ the dual of the invertible sheaf $\mathcal{I}$ is ample. So the cohomology module $\mathrm{H}^1(\mathcal{F}(n))$ vanishes for all $n \gg 0$. Furthermore Grothendieck-Serre duality shows that

$$\mathrm{RHom}_S(\mathrm{R}\Gamma(\mathcal{I}^n\mathcal{E}), S) = \mathrm{R}\Gamma(\mathcal{F}(n))[1].$$

However we know that $\mathrm{R}\Gamma(\mathcal{I}^n\mathcal{E})$ is represented by a two term complex

$$\big[P_0 \xrightarrow{d} P_1\big]$$

with P_0, P_1 finitely generated free S-modules. So from vanishing of $\mathrm{H}^1(\mathcal{F}(n))$ we conclude that the S-linear dual of the map d is surjective. This dual is therefore split, and d itself is a split injection. Whence $\mathrm{H}^0(\mathcal{I}^n\mathcal{E}) = 0$ and $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$ is a finitely generated free S-module. □

2. Euler products in the artinian case

In this section we work with a finite $\mathbb{F}_q$-algebra S which is a local artinian ring. We denote $\mathfrak{m}_S \subset S$ the maximal ideal.

Lemma 2.1. *Let k be a finite field extension of $\mathbb{F}_q$. Let $\mathcal{M}$ be a locally free shtuka on $S \otimes k$ given by a diagram*

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

If $\mathcal{M}(S/\mathfrak{m}_S \otimes k)$ is nilpotent then the following holds:

(1) *M_0 is a free S-module of finite rank,*
(2) *$i\colon M_0 \to M_1$ is an isomorphism,*
(3) *$i^{-1}j\colon M_0 \to M_0$ is an S-linear nilpotent endomorphism.*

Proof. (1) is clear. The ring $S \otimes k$ is noetherian and complete with respect to the nilpotent ideal $\mathfrak{m}_S \otimes R$ so Proposition 1.9.4 implies that $\mathcal{M}$ is nilpotent. (2) and (3) follow by definition of a nilpotent shtuka. $\square$

Definition 2.2. Let k be a finite field extension of $\mathbb{F}_q$. Let $\mathcal{M}$ be a locally free shtuka on $S \otimes k$ given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

Assuming that $\mathcal{M}(S/\mathfrak{m}_S \otimes k)$ is nilpotent we define

$$L(\mathcal{M}) = \det{}_S(1 - i^{-1}j \mid M_0) \in S^\times.$$

Lemma 2.3. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes R$. If $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent then $\mathcal{M}$ itself is nilpotent.*

Proof. The ring $S \otimes R$ is noetherian and complete with respect to the nilpotent ideal $\mathfrak{m}_S \otimes R$. Whence Proposition 1.9.4 implies that $\mathcal{M}$ is nilpotent. $\square$

Lemma 2.4. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes R$. If $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent then for almost all maximal ideals $\mathfrak{m} \subset R$ we have $L(\mathcal{M}(S \otimes R/\mathfrak{m})) = 1$.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

Let $\mathfrak{m} \subset R$ be a maximal ideal. Lemma 8.1.3 in the book of Böckle-Pink [3] shows that

$$\det{}_S(1 - i^{-1}j \mid M_0/\mathfrak{m}) = \det{}_{S \otimes R/\mathfrak{m}}\big(1 - (i^{-1}j)^d \mid M_0/\mathfrak{m}\big)$$

where d is the degree of the finite field $R/\mathfrak{m}$ over $\mathbb{F}_q$. However $i^{-1}j\colon M_0 \to M_0$ is a nilpotent endomorphism by Lemma 2.3. Hence $(i^{-1}j)^d = 0$ for $d \gg 0$ and we get the result. $\square$

Definition 2.5. Let $\mathcal{M}$ be a locally free shtuka on $S \times X$. Assuming that $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent we define

$$L(\mathcal{M}) = \prod_{\mathfrak{m}} \frac{1}{L(\mathcal{M}(S \otimes R/\mathfrak{m}))} \in S^\times$$

where $\mathfrak{m} \subset R$ ranges over the maximal ideals. This product is well-defined by Lemma 2.4. Note that the closed points in the complement of $\operatorname{Spec} R$ in X are not taken into account.

Lemma 2.6. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ such that $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent. If S is a field then $L(\mathcal{M}) = 1$.*

Proof. If V is a vector space over a field and N a nilpotent endomorphism of V then $\det(1 - N \mid V) = 1$. Hence for every maximal ideal $\mathfrak{m} \subset R$ the invariant $L(\mathcal{M}(S \otimes R/\mathfrak{m}))$ is equal to 1. □

3. Anderson's trace formula

We continue working with a finite $\mathbb{F}_q$-algebra S which is a local artinian ring. As before $\mathfrak{m}_S \subset S$ stands for the maximal ideal.

Our trace formula for the shtuka-theoretic regulator is based on the following narrow variant of Anderson's trace formula:

Theorem 3.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{E} \overset{1}{\underset{j}{\rightrightarrows}} \mathcal{E}.$$

Suppose that

(1) $\mathrm{H}^0(\mathcal{E}) = 0$ *and* $\mathrm{H}^1(\mathcal{E})$ *is a free S-module of finite rank,*
(2) $\mathcal{M}$ *is nilpotent,*
(3) $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ *is linear.*

Then $L(\mathcal{M}) = \det_S(1 - j \mid \mathrm{H}^1(\mathcal{E}))$.

We will deduce it from Proposition 3.2 in the article [17] of V. Lafforgue. However our setting differs slightly from Lafforgue's. In [17, Section 3] the coefficient ring (denoted A) is a power series ring while we work with a local artinian ring S. The next lemma helps to bridge this gap. In the following $\mathcal{I} \subset \mathcal{O}_{S\times X}$ stands for the unique invertible ideal sheaf such that

$$\mathcal{I}(\operatorname{Spec} S \otimes R) = S \otimes R,$$
$$\mathcal{I}(\operatorname{Spec} S \otimes \mathcal{O}_K) = S \otimes \mathfrak{m}_K.$$

Lemma 3.2. *Let $\mathcal{E}$ be a locally free sheaf on $S \times X$. If $\mathrm{H}^0(\mathcal{E}) = 0$ and $\mathrm{H}^1(\mathcal{E})$ is a free S-module of finite rank then for every $n \geqslant 0$ the following holds:*

(1) $\mathrm{H}^0(\mathcal{I}^n\mathcal{E}) = 0$ *and* $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$ *is a free S-module of finite rank.*
(2) *The natural map* $\mathrm{H}^1(\mathcal{I}^n\mathcal{E}) \to \mathrm{H}^1(\mathcal{E})$ *is a split surjection.*

Proof. Consider the short exact sequence of coherent sheaves $0 \to \mathcal{I}^n\mathcal{E} \to \mathcal{E} \to \mathcal{E}/\mathcal{I}^n\mathcal{E} \to 0$. Since $\mathrm{H}^0(\mathcal{E}) = 0$ it follows that $\mathrm{H}^0(\mathcal{I}^n\mathcal{E}) = 0$. Furthermore $\mathcal{E}/\mathcal{I}^n\mathcal{E}$ is supported at a closed affine subscheme of $S \times X$. We thus have a short exact sequence

$$0 \to \mathrm{H}^0(\mathcal{E}/\mathcal{I}^n\mathcal{E}) \to \mathrm{H}^1(\mathcal{I}^n\mathcal{E}) \to \mathrm{H}^1(\mathcal{E}) \to 0.$$

Now $S \otimes \mathcal{O}_K/\mathfrak{m}_K^n$ is finite flat over S. Since $\mathcal{E}/\mathcal{I}^n\mathcal{E}$ is a locally free sheaf on $\mathrm{Spec}(S \otimes \mathcal{O}_K/\mathfrak{m}_K^n)$ it follows that $\mathrm{H}^0(\mathcal{E}/\mathcal{I}^n\mathcal{E})$ is a free S-module of finite rank. Hence so is $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$. The surjection $\mathrm{H}^1(\mathcal{I}^n\mathcal{E}) \to \mathrm{H}^1(\mathcal{E})$ is split since $\mathrm{H}^1(\mathcal{E})$ is free by assumption. □

We also need a lemma on shtukas:

Lemma 3.3. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{E} \underset{j}{\overset{1}{\rightrightarrows}} \mathcal{E}.$$

Suppose that

(1) $\mathrm{H}^0(\mathcal{E}) = 0$ *and* $\mathrm{H}^1(\mathcal{E})$ *is a free S-module of finite rank,*
(2) $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ *is linear.*

Then for every $n \geqslant 0$ the following holds:

(1) $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$ *is a free S-module*
(2) $\det_S(1 - j \mid \mathrm{H}^1(\mathcal{I}^n\mathcal{E})) = \det_S(1 - j \mid \mathrm{H}^1(\mathcal{E}))$.

Proof. Part (1) is immediate from Lemma 3.2. The same lemma implies that the short exact sequence of coherent sheaves

$$0 \to \mathcal{I}^{n+1}\mathcal{E} \to \mathcal{I}^n\mathcal{E} \to (\mathcal{I}^n\mathcal{E})/\mathcal{I} \to 0$$

induces a short exact sequence of cohomology modules

$$0 \to \mathrm{H}^0(\mathcal{I}^n\mathcal{E}/\mathcal{I}) \to \mathrm{H}^1(\mathcal{I}^{n+1}\mathcal{E}) \to \mathrm{H}^1(\mathcal{I}^n\mathcal{E}) \to 0$$

with $\mathrm{H}^0(\mathcal{I}^n\mathcal{E}/\mathcal{I})$ a free S-module of finite rank. As a consequence

$$\det_S\big(1 - j \mid \mathrm{H}^1(\mathcal{I}^{n+1}\mathcal{E}\big) = \det_S\big(1 - j \mid \mathrm{H}^0(\mathcal{I}^n\mathcal{E}/\mathcal{I})\big) \cdot \det_S\big(1 - j \mid \mathrm{H}^1(\mathcal{I}^n\mathcal{E})\big).$$

By assumption $j(\mathcal{E}) \subset \mathcal{I}\mathcal{E}$. Since $\tau(\mathcal{I}) \subset \mathcal{I}^q$ and j is τ-linear we conclude that $j(\mathcal{I}^n\mathcal{E}) \subset \mathcal{I}^{qn+1}\mathcal{E}$. Hence j is zero on the quotient $\mathcal{I}^n\mathcal{E}/\mathcal{I}$ and we get the result. □

Proof of Theorem 3.1. Let Ω be the canonical sheaf of $S \times X$ over S. We define

$$\mathcal{V} = \mathrm{H}^0(S \otimes R, \mathcal{H}\mathrm{om}(\mathcal{E}, \Omega)),$$
$$\mathcal{V}_t = \mathrm{H}^0(\mathcal{H}\mathrm{om}(\mathcal{I}^t\mathcal{E}, \Omega)).$$

Grothendieck-Serre duality identifies $\mathcal{V}_t$ with the (-1)-st cohomology module of the complex

$$\mathrm{RHom}_S(\mathrm{R}\Gamma(\mathcal{I}^t\mathcal{E}), S).$$

However Lemma 3.2 shows that $\mathrm{H}^0(\mathcal{I}^t\mathcal{E}) = 0$ and $\mathrm{H}^1(\mathcal{I}^t\mathcal{E})$ is a free S-module of finite rank. Hence Grothendieck-Serre duality gives a natural isomorphism

$$\mathrm{Hom}_S(\mathrm{H}^1(\mathcal{I}^t\mathcal{E}), S) \cong \mathcal{V}_t.$$

In particular every $\mathcal{V}_t$ is a free S-module of finite rank. By Lemma 3.2 the natural maps $\mathrm{H}^1(\mathcal{I}^{t+1}\mathcal{E}) \to \mathrm{H}^1(\mathcal{I}^t\mathcal{E})$ are split surjections whence the inclusions $\mathcal{V}_t \subset \mathcal{V}_{t+1}$ are split.

The endomorphism j of $\mathcal{E}$ induces for every $n \geqslant 0$ an endomorphism of $\mathrm{H}^1(\mathcal{I}^n\mathcal{E})$. Grothendieck-Serre duality identifies it with the Cartier-linear endomorphisms $\kappa_{\mathcal{V}}\colon \mathcal{V}_t \to \mathcal{V}_t$ which are used in Section 3 of [17].

As we explained above the S-modules $\mathcal{V}_t$ are free of finite rank and the inclusions $\mathcal{V}_t \subset \mathcal{V}_{t+1}$ are split. So the argument of [17, Proposition 3.2] applies with only one minor change. Namely, one employs Lemma 1.1 to ensure that the auxillary locally free sheaves $\mathcal{F}$ on $S \times X$ constructed in the course of the proof have the property that $\mathrm{H}^0(\mathcal{F}) = 0$ and $\mathrm{H}^1(\mathcal{F})$ is a free S-module of finite rank. The rest of the argument works without change. It shows that for $t \gg 0$ we have an equality of power series in $S[[T]]$:

$$\det\nolimits_S(1 - T\kappa_{\mathcal{V}} \mid \mathcal{V}_t) = \prod_{\mathfrak{m}} \det\nolimits_S\big(1 - Tj \mid \mathcal{E}(S \otimes R/\mathfrak{m})\big)^{-1}$$

where $\mathfrak{m} \subset R$ runs over the maximal ideals.

Now recall the endomorphism $j\colon \mathcal{E} \to \tau_*\mathcal{E}$ is assumed to be nilpotent. Furthermore the maximal ideal of S is nilpotent too. As a consequence Lemma 8.1.4 of [3] implies that only finitely many factors in the product on the right hand side are different from 1. Thus we can evaluate the equality above at $T = 1$ and conclude that

$$\det\nolimits_S(1 - \kappa_{\mathcal{V}} \mid \mathcal{V}_t) = \prod_{\mathfrak{m}} \det\nolimits_S\big(1 - j \mid \mathcal{E}(S \otimes R/\mathfrak{m})\big)^{-1} = L(\mathcal{M}).$$

Therefore $\det_S(1-j \mid \mathrm{H}^1(\mathcal{I}^t\mathcal{E})) = L(\mathcal{M})$. Lemma 3.3 implies that this equality holds already for $t = 0$. ☐

4. Artinian regulators

We keep the assumption that S is a finite $\mathbb{F}_q$-algebra which is a local artinian ring.

Lemma 4.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$. If $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent then the following holds:*

(1) *The natural map $\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \mathcal{M})$ is a quasi-isomorphism.*
(2) *The complex $\mathrm{R}\Gamma(\mathcal{M})$ is concentrated in degree 1.*
(3) *$\mathrm{H}^1(\mathcal{M})$ is a free S-module of finite rank.*

Proof. (1) Consider the Čech complex

$$\mathrm{R}\check{\Gamma}(\mathcal{M}) = \Big[\mathrm{R}\Gamma(S \otimes R, \mathcal{M}) \oplus \mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S \otimes K, \mathcal{M})\Big].$$

According to Theorem 4.4.11 there is a natural quasi-isomorphism $\mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\check{\Gamma}(\mathcal{M})$. The composition of this quasi-isomorphism with the natural map $\mathrm{R}\check{\Gamma}(\mathcal{M}) \to \mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \mathcal{M})$ is the pullback map $\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \mathcal{M})$. So to prove that this pullback map is a quasi-isomorphism it is enough to demonstrate that

$$\mathrm{R}\Gamma(S \otimes R, \mathcal{M}) = 0, \quad \mathrm{R}\Gamma(S \otimes K, \mathcal{M}) = 0.$$

Now the ring $S \otimes R$ is noetherian and complete with respect to the ideal $\mathfrak{m}_S \otimes R$. As the shtuka $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent Proposition 1.9.4 implies that $\mathrm{R}\Gamma(S \otimes R, \mathcal{M}) = 0$. The shtuka $\mathcal{M}(S/\mathfrak{m}_S \otimes K)$ is nilpotent since nilpotence is preserved under pullbacks. So the same argument shows that $\mathrm{R}\Gamma(S \otimes K, \mathcal{M}) = 0$. Whence the result. In view of (1) the results (2) and (3) follow instantly from Lemma 5.3.1. ☐

Lemma 4.2. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

If $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent then $i\colon \mathcal{M}_0(S \otimes R) \to \mathcal{M}_0(S \otimes R)$ is an isomorphism.

Proof. Indeed $\mathcal{M}(S \otimes R)$ is nilpotent by Lemma 2.3, and the i-arrow of such a shtuka is an isomorphism by definition. ☐

Definition 4.3. Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Suppose that $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent. We say that the artinian regulator is defined for $\mathcal{M}$ if the τ-linear endomorphism $i^{-1}j$ of $\mathcal{M}_0(S \otimes K)$ preserves the submodule $\mathcal{M}_0(S \otimes \mathcal{O}_K)$. In this case we define the *artinian regulator* $\rho_{\mathcal{M}}\colon \mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\nabla\mathcal{M})$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma(S \times X, \mathcal{M}) \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d, "\wr"'] & \mathrm{R}\Gamma(S \times X, \nabla\mathcal{M}) \arrow[d, "\wr"] \\
\mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \mathcal{M}) \arrow[r, "\rho_{\mathcal{M}(S\otimes\mathcal{O}_K)}"] & \mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \nabla\mathcal{M})
\end{tikzcd}$$

Here $\rho_{\mathcal{M}(S\otimes\mathcal{O}_K)}$ is the artinian regulator of the shtuka $\mathcal{M}(S \otimes \mathcal{O}_K)$ as in Definition 5.5.2. It is given by a morphism of complexes

$$\begin{tikzcd}
\big[\mathcal{M}_0(S \otimes \mathcal{O}_K) \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathcal{M}_1(S \otimes \mathcal{O}_K)\big] \arrow[d, "1"] \\
\big[\mathcal{M}_0(S \otimes \mathcal{O}_K) \arrow[r, "i"] & \mathcal{M}_1(S \otimes \mathcal{O}_K)\big]
\end{tikzcd}$$

Lemma 4.4. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by the diagram*

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Suppose that $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent. If the artinian regulator is defined for $\mathcal{M}$ then the following holds:

1. *The τ-linear endomorphism $i^{-1}j$ of $\mathcal{M}_0(S \otimes R)$ extends uniquely to a τ-linear endomorphism $i^{-1}j$ of $\mathcal{M}_0$.*
2. *The endomorphism $i^{-1}j$ of $\mathcal{M}_0$ is nilpotent.*

Here we identify the S-module sheaves $\mathcal{M}_0$ and $\tau_*\mathcal{M}_0$ using the fact that τ is the identity on the underlying topological space of $S \times X$.

Proof of Lemma 4.4. (1) Let $\operatorname{Spec} R' \subset X$ be an open affine open subscheme containing $\operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K$. Shrinking $\operatorname{Spec} R'$ if necessary we can find an element $r' \in R$ which is a uniformizer of $\mathcal{O}_K$. Observe that the following holds:

- $S \otimes \mathcal{O}_K$ is the completion of $S \otimes R'$ at the ideal generated by $1 \otimes r'$.
- The fibre product of $\operatorname{Spec}(S \otimes R)$ and $\operatorname{Spec}(S \otimes \mathcal{O}_K)$ over $S \times X$ is $\operatorname{Spec}(S \otimes K)$.

The endomorphism $i^{-1}j$ of $\mathcal{M}_0(S \otimes K)$ extends to $\mathcal{M}_0(S \otimes \mathcal{O}_K)$ by assumption. Now as $\mathcal{M}_0$ is locally free the Beauville-Laszlo glueing theorem [0BP2] implies that there exists a unique morphism $\tau^*\mathcal{M}_0 \to \mathcal{M}_0$ restricting to the adjoints

$$(i^{-1}j)^a \colon \tau^*\mathcal{M}_0(S \otimes R) \to \mathcal{M}_0(S \otimes R),$$
$$(i^{-1}j)^a \colon \tau^*\mathcal{M}_0(S \otimes \mathcal{O}_K) \to \mathcal{M}_0(S \otimes \mathcal{O}_K).$$

of the endomorphisms $i^{-1}j$ on $S \otimes R$ respectively $S \otimes \mathcal{O}_K$. (2) Indeed $i^{-1}j$ is nilpotent on $S \otimes R$ and so on $S \otimes K$. The natural map $\mathcal{M}_0(S \otimes \mathcal{O}_K) \to \mathcal{M}_0(S \otimes K)$ is injective since $\mathcal{M}_0$ is locally free. As a consequence $i^{-1}j$ is nilpotent on $S \otimes \mathcal{O}_K$. $\square$

Let $\mathcal{M}$ be a shtuka on $S \times X$ given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Theorem 1.5.6 provides $\mathcal{M}$ with a natural distinguished triangle

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1].$$

Lemma 4.5. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by the diagram*

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Suppose that $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent. If the artinian regulator is defined for $\mathcal{M}$ then the diagram

$$\begin{tikzcd} \mathrm{R}\Gamma(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] \arrow[d, "1"] & {[1]} \\ \mathrm{R}\Gamma(\nabla\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] & {[1]} \end{tikzcd} \tag{4.1}$$

is morphism of distinguished triangles. Here $i^{-1}j\colon \mathcal{M}_0 \to \tau_*\mathcal{M}_0$ is the morphism described in Lemma 4.4.

Proof. We will control the distinguished triangles in (4.1) through similar distinguished triangles for Čech cohomology which we constructed in Section 4.4 (see Definition 4.4.7).

According to Definition 4.4.1 the complex $\mathrm{R}\check{\Gamma}(\mathcal{M})$ is the total complex of the double complex

$$\begin{tikzcd} \mathcal{M}_0(S\otimes K) \arrow[rr, "j-i"] & & \mathcal{M}_1(S\otimes K) \\ \mathcal{M}_0(S\otimes R)\oplus\mathcal{M}_0(S\otimes\mathcal{O}_K) \arrow[u, "\text{difference}"] \arrow[rr, "i-j"] & & \mathcal{M}_1(S\otimes R)\oplus\mathcal{M}_1(S\otimes\mathcal{O}_K) \arrow[u, "\text{difference}"'] \end{tikzcd}$$

Similarly $\mathrm{R}\check{\Gamma}(\nabla\mathcal{M})$ is given by the double complex

$$\begin{tikzcd} \mathcal{M}_0(S\otimes K) \arrow[rr, "-i"] & & \mathcal{M}_1(S\otimes K) \\ \mathcal{M}_0(S\otimes R)\oplus\mathcal{M}_0(S\otimes\mathcal{O}_K) \arrow[u, "\text{difference}"] \arrow[rr, "i"] & & \mathcal{M}_1(S\otimes R)\oplus\mathcal{M}_1(S\otimes\mathcal{O}_K) \arrow[u, "\text{difference}"'] \end{tikzcd}$$

Let $\breve{\rho}\colon \mathrm{R}\check{\Gamma}(\mathcal{M}) \to \mathrm{R}\check{\Gamma}(\nabla\mathcal{M})$ be the morphism given by $1-i^{-1}j$ in the column of $\mathcal{M}_0$ and by the identity map in the column of $\mathcal{M}_1$. Consider the diagram

$$\begin{tikzcd} \mathrm{R}\check{\Gamma}(\mathcal{M}) \arrow[r] \arrow[d, "\breve{\rho}"'] & \mathrm{R}\check{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathrm{R}\check{\Gamma}(\mathcal{M}_1) \arrow[r] \arrow[d, "1"] & {[1]} \\ \mathrm{R}\check{\Gamma}(\nabla\mathcal{M}) \arrow[r] & \mathrm{R}\check{\Gamma}(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{R}\check{\Gamma}(\mathcal{M}_1) \arrow[r] & {[1]} \end{tikzcd}$$

where the rows are the distinguished triangles for Čech cohomology of Definition 4.4.7. Using the explicit discription of the distinguished triangle for the associated complex functor Γ_{a} given in Definition 1.5.4 one readily verifies that the diagram above is a morphism of distinguished triangles.

Theorem 4.4.11 shows that the natural morphisms from the Čech cohomology to the usual cohomology define an isomorphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\check{\Gamma}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd}$$

and similarly for $\nabla\mathcal{M}$. We thus get a morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{M}) \arrow[r] \arrow[d, "\rho"'] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] \arrow[d, "1"] & {[1]} \\
\mathrm{R}\Gamma(\nabla\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd}$$

Here we denote ρ the map induced by $\breve{\rho}\colon \mathrm{R}\check{\Gamma}(\mathcal{M}) \to \mathrm{R}\check{\Gamma}(\nabla\mathcal{M})$. However the natural square

$$\begin{tikzcd}
\mathrm{R}\check{\Gamma}(\mathcal{M}) \arrow[r] \arrow[d, "\breve{\rho}"'] & \mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \mathcal{M}) \arrow[d, "\rho_{\mathcal{M}(S\otimes\mathcal{O}_K)}"] \\
\mathrm{R}\check{\Gamma}(\nabla\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(S \otimes \mathcal{O}_K, \nabla\mathcal{M})
\end{tikzcd}$$

is commutative by construction of $\breve{\rho}$. Comparing with the definition of the artinian regulator $\rho_{\mathcal{M}}$ we conclude that $\rho = \rho_{\mathcal{M}}$. □

5. Trace formula for artinian regulators

We continue working with a finite $\mathbb{F}_q$-algebra S which is a local artinian ring. As before $\mathfrak{m}_S \subset S$ denotes the maximal ideal.

Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ such that $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent. Our goal is to compare the artinian regulator $\rho_{\mathcal{M}}$ with the ζ-isomorphism of $\mathcal{M}$. This isomorphism is constructed in the following way (see Definition 1.10.2). Suppose that $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

Assume that the S-modules $\mathrm{H}^n(\mathcal{M})$, $\mathrm{H}^n(\mathcal{M}_0)$ and $\mathrm{H}^n(\mathcal{M}_1)$, $n \geqslant 0$, are finitely generated free. The ζ-isomorphism

$$\zeta_{\mathcal{M}}\colon\ \det_S \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_S \mathrm{R}\Gamma(\nabla\mathcal{M})$$

is the composition of isomorphisms

$$\det_S \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_S \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_S \det_S^{-1} \mathrm{R}\Gamma(\mathcal{M}_1) \xrightarrow{\sim} \det_S \mathrm{R}\Gamma(\nabla\mathcal{M})$$

induced by the natural distinguished triangles

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1],$$
$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1].$$

of Theorem 1.5.6.

Even though the the S-module complexes $\mathrm{R}\Gamma(\mathcal{M})$, $\mathrm{R}\Gamma(\mathcal{M}_0)$ and $\mathrm{R}\Gamma(\mathcal{M}_1)$ are perfect the additional hypothesis on the cohomology of $\mathcal{M}$, $\mathcal{M}_0$ and $\mathcal{M}_1$ is necessary to make this construction work. The reason is that in the theory of Knudsen-Mumford [16, Corollary 2 after Theorem 2] a distinguished triangle

$$A \to B \to C \to [1]$$

of perfect S-module complexes determines a canonical isomorphisms

$$\det_S A \xrightarrow{\sim} \det_S B \otimes_S \det_S^{-1} C$$

only if the cohomology modules of the complexes A, B and C are themselves perfect. A module over the local artinian ring S is perfect if and only if it is free of finite rank while the cohomology modules of a perfect S-module complex can be arbitrary finitely generated S-modules.

Now we are finally ready to state the main result of this section.

Theorem 5.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Assume that

(1) $\mathrm{H}^0(\mathcal{M}_0) = 0$ *and* $\mathrm{H}^1(\mathcal{M}_0)$ *is a free S-module of finite rank,*
(2) $\mathrm{H}^0(\mathcal{M}_1) = 0$ *and* $\mathrm{H}^1(\mathcal{M}_1)$ *is a free S-module of finite rank,*
(3) $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ *is nilpotent,*
(4) *There exists an open ideal* $I \subset \mathfrak{m}_K$ *such that* $I \cdot \mathrm{H}^1(S \otimes \mathcal{O}_K, \nabla\mathcal{M}) = 0$ *and* $\mathcal{M}(S \otimes \mathcal{O}_K/I^2)$ *is linear.*

Then the following holds:

(1) *The artinian regulator* $\rho_{\mathcal{M}}$ *is defined for* $\mathcal{M}$*.*
(2) *The ζ-isomorphism* $\zeta_{\mathcal{M}}$ *is defined for* $\mathcal{M}$*.*
(3) $\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_S(\rho_{\mathcal{M}})$*.*

Proof. (1) According to Proposition 5.5.9 the assumptions (3) and (4) imply that the artinian regulator is defined for $\mathcal{M}(S \otimes \mathcal{O}_K)$. Hence it is defined for $\mathcal{M}$.

(2) The cohomology modules of the complexes $\mathrm{R}\Gamma(\mathcal{M}_0)$ and $\mathrm{R}\Gamma(\mathcal{M}_1)$ are perfect by assumptions (1) and (2). Since $\mathcal{M}$ is locally free and $\mathcal{M}(S/\mathfrak{m}_S \otimes R)$ is nilpotent by assumption (3) Lemma 4.1 demonstrates that the cohomology modules of $\mathrm{R}\Gamma(\mathcal{M})$ are also perfect. Whence the result.

(3) According to Lemma 4.1 the complex $\mathrm{R}\Gamma(\mathcal{M})$ is concentrated in degree 1. So the natural distinguished triangle of Theorem 1.5.6

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1]$$

induces a short exact sequence

$$0 \to \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{H}^1(\mathcal{M}_1) \to 0.$$

Similarly the natural distinguished triangle for $\mathrm{R}\Gamma(\nabla\mathcal{M})$ induces a short exact sequence

$$0 \to \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}_0) \xrightarrow{i} \mathrm{H}^1(\mathcal{M}_1) \to 0.$$

Lemma 4.1 also implies that all the objects in these sequences are free S-modules of finite rank. So the ζ-isomorphism

$$\zeta_{\mathcal{M}} \colon \det_S \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_S \mathrm{R}\Gamma(\nabla\mathcal{M})$$

is the composition of the isomorphisms

$$\det_S \mathrm{R}\Gamma(\mathcal{M}) \cong \det_S \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_S \det_S^{-1} \mathrm{R}\Gamma(\mathcal{M}_1) \cong \det_S \mathrm{R}\Gamma(\nabla\mathcal{M})$$

determined by the short exact sequences above.

The endomorphism $i^{-1}j$ of $\mathcal{M}_0(S \otimes R)$ extends to an endomorphism $i^{-1}j$ of $\mathcal{M}_0$ by Lemma 4.4. Lemma 4.5 shows that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d, "1-i^{-1}j"'] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] & 0
\end{tikzcd}$$

is commutative. As a consequence

$$\zeta_{\mathcal{M}} = \det_S \big(1 - i^{-1}j \bigm| \mathrm{H}^1(\mathcal{M}_0)\big) \cdot \det_S(\rho_{\mathcal{M}}).$$

We thus need to show that

$$\det_S \big(1 - i^{-1}j \bigm| \mathrm{H}^1(\mathcal{M}_0)\big) = L(\mathcal{M}).$$

Consider a locally free shtuka

$$\mathcal{N} = \Big[\mathcal{M}_0 \underset{i^{-1}j}{\overset{1}{\rightrightarrows}} \mathcal{M}_0\Big].$$

This shtuka has the following properties:

(i) $\mathrm{H}^0(\mathcal{M}_0) = 0$ and $\mathrm{H}^1(\mathcal{M}_0)$ is a free S-module of finite rank,
(ii) $\mathcal{N}$ is nilpotent,
(iii) $\mathcal{N}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is linear.

Indeed (i) is just the assumption (1) and (ii) holds by Lemma 4.4. Let us prove that (iii) also holds. We temporarily denote

$$M_0 = \mathcal{M}_0(S \otimes \mathcal{O}_K), \quad M_1 = \mathcal{M}_1(S \otimes \mathcal{O}_K).$$

According to the assumption (4) we have $IM_1 \subset i(M_0)$. Hence $I^2M_1 \subset i(IM_0)$. At the same time the assumption (4) implies that $j(M_0) \subset I^2M_1$. As a consequence $j(M_0) \subset i(IM_0)$. We conclude that the shtuka $\mathcal{N}(S \otimes \mathcal{O}_K/I)$ is linear. However $I \subset \mathfrak{m}_K$ by assumption so we get the property (iii).

We now apply Theorem 3.1 to $\mathcal{N}$ and conclude that

$$\det_S \left(1 - i^{-1}j \mid \mathrm{H}^1(\mathcal{M}_0)\right) = L(\mathcal{N}).$$

To get the result it remains to observe that $L(\mathcal{N}) = L(\mathcal{M})$ by construction. $\square$

6. A lemma on determinants

Our aim for the moment is to prove a technical lemma on determinants of finite-dimensional F-vector spaces. In this section we omit the subscript F of det and $\otimes$ in order to simlify the notation.

In the theory of Knudsen-Mumford [16] a distinguished triangle $P \to Q \to R \to [1]$ of perfect complexes of finite-dimensional F-vector spaces determines a natural isomorphism

$$\det P \xrightarrow{\sim} \det Q \otimes \det^{-1} R.$$

[16, Corollary 2 after Theorem 2]. In particular a short exact sequence $0 \to P \to Q \to R \to 0$ of finite-dimensional F-vector spaces gives rise to such a natural isomorphism.

Lemma 6.1. *Consider a diagram of finite-dimensional F-vector spaces:*

$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "p"] \arrow[d, equal] & B \oplus A_0 \arrow[r, "q"] \arrow[d] & B \oplus A_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & A_0 \arrow[r, "g"] & A_1 \arrow[r] & 0
\end{tikzcd}$$

Assume the following:

1. *The rows are short exact sequences.*
2. *The vertical arrows in the second and third column are natural inclusions respectively projections.*
3. *The diagram is commutative.*

Then the natural square of determinants

$$\begin{tikzcd}
\det A \arrow[r, "\sim"] \arrow[d, equal] & \det(B \oplus A_0) \otimes \det^{-1}(B \oplus A_1) \arrow[d, "\wr"] \\
\det A \arrow[r, "\sim"] & \det A_0 \otimes \det^{-1} A_1
\end{tikzcd}$$

commutes up to the sign $(-1)^{nm}$ *where* $n = \dim A$, $m = \dim B$. *The right vertical arrow in this square is induced by the natural isomorphism* $\det B \otimes \det^{-1} B \xrightarrow{\sim} F$.

Morally this lemma is a special case of [16, Proposition 1 (ii)]. There is, however, a gap between morality and reality which one has to fill with a sound argument.

Proof of Lemma 6.1. Let $s\colon A_1 \to A_0$ be a section of the surjection $g\colon A_0 \to A_1$. Pick an element $a_1 \in \det A_1$. It is easy to show that the natural isomorphism $\det A \xrightarrow{\sim} \det A_0 \otimes \det^{-1} A_1$ sends $a \in \det A$ to the element

$$\big(f(a) \wedge s(a_1)\big) \otimes a_1^* \tag{6.1}$$

where $a_1^*\colon \det A_1 \to F$ is the unique linear map such that $a_1^*(a_1) = 1$. This element depends neither on the choice of s nor on the choice of a_1. We would like to obtain a similar explicit formula for the map determined by the short exact sequence

$$0 \to A \xrightarrow{p} B \oplus A_0 \xrightarrow{q} B \oplus A_1 \to 0.$$

The commutativity of the diagram implies that the map q is given by a matrix

$$\begin{pmatrix} 1 & h \\ 0 & g \end{pmatrix}$$

where $h\colon A_0 \to B$ is a certain map. Similarly

$$p = \begin{pmatrix} \delta \\ f \end{pmatrix}$$

where $\delta\colon A \to B$ is a certain map.

Let $s\colon A_1 \to A_0$ be a section of g. A quick computation shows that the map

$$t = \begin{pmatrix} 1 & -hs \\ 0 & s \end{pmatrix} : B \oplus A_1 \to B \oplus A_0$$

is a section of q. Next, fix elements $b \in \det B$ and $a_1 \in \det A_1$. Let $m = \dim B$ and $d_1 = \dim A_1$. By a slight abuse of notation we write $b \wedge a_1 \in \Lambda^{m+d_1}(B \oplus A_1)$ for the element of $\det(B \oplus A_1)$ defined by b, a_1. For every $a \in \det A$ we need to compute

$$p(a) \wedge t(b \wedge a_1).$$

Suppose that

$$a = a^1 \wedge \ldots \wedge a^n, \quad b = b^1 \wedge \ldots \wedge b^m, \quad a_1 = a_1^1 \wedge \ldots \wedge a_1^{d_1}.$$

In this case

$$t(b \wedge a_1) = b_1 \wedge \ldots \wedge b^m \wedge \big(s(a_1^1) - hs(a_1^1)\big) \wedge \ldots \wedge \big(s(a_1^{d_1}) - hs(a_1^{d_1})\big) = b \wedge s(a_1).$$

Furthermore

$$p(a) = \big(\delta(a^1) + f(a^1)\big) \wedge \ldots \wedge \big(\delta(a^n) + f(a^n)\big)$$

so that

$$p(a) \wedge t(b \wedge a_1) = f(a) \wedge b \wedge s(a_1) = (-1)^{nm} b \wedge f(a) \wedge s(a_1).$$

We conclude that the natural isomorphism $\det A \xrightarrow{\sim} \det(B \oplus A_0) \otimes \det^{-1}(B \oplus A_1)$ sends the element $a \in \det A$ to

$$(-1)^{nm} b \otimes \big(f(a) \wedge s(a_1)\big) \otimes b^* \otimes a_1^*$$

where $b^* \colon \det B \to F$ and $a_1^* \colon \det A_1 \to F$ are the unique elements such that $b^*(b) = 1$, $a_1^*(a_1) = 1$. Comparing this formula with (6.1) we get the result. $\square$

7. A functoriality statement for ζ-isomorphisms

Lemma 7.1. *Let $\mathcal{M}$ be a shtuka on $F \times X$. If $\mathcal{M}$ is coherent then the ζ-isomorphism is defined for $\mathcal{M}$.*

Proof. Indeed F is a regular ring so the result is a consequence of Proposition 4.9.1. $\square$

Lemma 7.2. *Let $0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{Q} \to 0$ be a short exact sequence of coherent shtukas on $F \times X$. Suppose that $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{Q}$ are given by diagrams*

$$\mathcal{N} = \Big[\mathcal{N}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{N}_1\Big], \quad \mathcal{M} = \Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big], \quad \mathcal{Q} = \Big[\mathcal{Q}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{Q}_1\Big].$$

Assume the following:

(1) $\mathrm{R\Gamma}(\mathcal{M})$ *and* $\mathrm{R\Gamma}(\nabla\mathcal{M})$ *are concentrated in degree* 1.
(2) $\mathrm{H}^0(\mathcal{M}_*) = 0$ *for* $* \in \{0, 1\}$.
(3) $\mathrm{R\Gamma}(\mathcal{Q}) = 0$ *and* $\mathrm{R\Gamma}(\nabla\mathcal{Q}) = 0$.
(4) $\mathrm{H}^1(\mathcal{Q}_*) = 0$ *for* $* \in \{0, 1\}$.
(5) $\mathcal{Q}$ *is linear.*

Then the following holds:

(1) *The natural map* $\mathrm{R\Gamma}(\mathcal{N}) \to \mathrm{R\Gamma}(\mathcal{M})$ *is a quasi-isomorphism.*
(2) *The natural map* $\mathrm{R\Gamma}(\nabla\mathcal{N}) \to \mathrm{R\Gamma}(\nabla\mathcal{M})$ *is a quasi-isomorphism.*
(3) *The natural square*

$$\begin{tikzcd}
\det_F \mathrm{R\Gamma}(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathcal{N}}"', "\wr"] & \det_F \mathrm{R\Gamma}(\mathcal{M}) \arrow[d, "\wr"', "\zeta_{\mathcal{M}}"] \\
\det_F \mathrm{R\Gamma}(\nabla\mathcal{N}) \arrow[r, "\sim"] & \det_F \mathrm{R\Gamma}(\nabla\mathcal{M})
\end{tikzcd}$$

commutes up to the sign $(-1)^{nm}$ where

$$n = \dim_F \mathrm{H}^1(\nabla\mathcal{M}) + \dim_F \mathrm{H}^1(\mathcal{M}),$$
$$m = \dim_F \mathrm{H}^0(\mathcal{Q}_0).$$

Lemma 7.2 should certainly hold without the conditions (1), (2) and (4). However to establish it in this generality one should prove a stronger version of Lemma 6.1. Unfortunately such a proof appears to be quite messy. We opt not to do it since the present version of Lemma 7.2 is all we need to prove the main result of this text, the class number formula.

Proof of Lemma 7.2. In the following we drop the subscript F of det and $\otimes$ to improve the legibility. The claims (1) and (2) are immediate consequences of the assumption (3). Let us now prove (3). Assumptions (1) and (2) imply that we have a short exact sequence

$$0 \to \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{H}^1(\mathcal{M}_1) \to 0.$$

Assumption (2) also implies that $\mathrm{H}^0(\mathcal{N}_*) = 0$ for $* \in \{0,1\}$. The natural map $\mathrm{R}\Gamma(\mathcal{N}) \to \mathrm{R}\Gamma(\mathcal{M})$ is a quasi-isomorphism due to assumption (3). We thus have a short exact sequence

$$0 \to \mathrm{H}^1(\mathcal{N}) \to \mathrm{H}^1(\mathcal{N}_0) \xrightarrow{i-j} \mathrm{H}^1(\mathcal{N}_1) \to 0.$$

As $\mathrm{H}^0(\mathcal{N}_*) = 0$ the assumption (4) implies that the cohomology sequence of the short exact sequence $0 \to \mathcal{N}_* \to \mathcal{M}_* \to \mathcal{Q}_* \to 0$ has the form

$$0 \to \mathrm{H}^0(\mathcal{Q}_*) \xrightarrow{\delta} \mathrm{H}^1(\mathcal{N}_*) \to \mathrm{H}^1(\mathcal{M}_*) \to 0.$$

Altogether we have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{H}^0(\mathcal{Q}_0) \arrow[r, "i", "\sim"'] \arrow[d, "\delta"'] & \mathrm{H}^0(\mathcal{Q}_1) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{N}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1(\mathcal{N}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{H}^1(\mathcal{N}_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The arrow $\mathrm{H}^0(\mathcal{Q}_0) \to \mathrm{H}^0(\mathcal{Q}_1)$ is labelled i since the shtuka $\mathcal{Q}$ is linear by assumption (5). Let us denote $B = \mathrm{H}^0(\mathcal{Q}_0)$. Taking splittings of the maps δ

we replace the diagram above with an isomorphic diagram

(7.1)
$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{N}) \arrow[r] \arrow[d, "\wr"] & B \oplus \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & B \oplus \mathrm{H}^1(\mathcal{M}_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{M}) \arrow[r] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] & 0
\end{tikzcd}$$

The same argument applied to the shtukas $\nabla\mathcal{N}$, $\nabla\mathcal{M}$, $\nabla\mathcal{Q}$ shows that we have a commutative diagram

(7.2)
$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^1(\nabla\mathcal{N}) \arrow[r] \arrow[d, "\wr"] & B \oplus \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i"] \arrow[d] & B \oplus \mathrm{H}^1(\mathcal{M}_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] & \mathrm{H}^1(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{H}^1(\mathcal{M}_1) \arrow[r] & 0
\end{tikzcd}$$

Observe that the vertical arrows in the second and third column of (7.2) can be chosen to be the same as the corresponding arrows in (7.1). Indeed they only depend on the chosen splittings of the injections $\mathrm{H}^0(\mathcal{Q}_0) \to \mathrm{H}^1(\mathcal{N}_0)$ and $\mathrm{H}^0(\mathcal{Q}_1) \to \mathrm{H}^1(\mathcal{N}_1)$ and on the map $i\colon \mathrm{H}^0(\mathcal{Q}_0) \to \mathrm{H}^0(\mathcal{Q}_1)$. Here we again use the assumption (5) that $\mathcal{Q}$ is linear.

Applying Lemma 6.1 to (7.1) we conclude that the natural square of determinants

(7.3)
$$\begin{tikzcd}
\det \mathrm{H}^1(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \det\big(B \oplus \mathrm{H}^1(\mathcal{M}_0)\big) \otimes \det^{-1}\big(B \oplus \mathrm{H}^1(\mathcal{M}_1)\big) \arrow[d, "\wr"] \\
\det \mathrm{H}^1(\mathcal{M}) \arrow[r, "\sim"] & \det \mathrm{H}^1(\mathcal{M}_0) \otimes \det^{-1} \mathrm{H}^1(\mathcal{M}_1)
\end{tikzcd}$$

commutes up to the sign $(-1)^{n_1 m}$ where $n_1 = \dim \mathrm{H}^1(\mathcal{M})$, $m = \dim B$. Lemma 6.1 applied to (7.2) shows that the natural square of determinants

(7.4)
$$\begin{tikzcd}
\det \mathrm{H}^1(\nabla\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \det\big(B \oplus \mathrm{H}^1(\mathcal{M}_0)\big) \otimes \det^{-1}\big(B \oplus \mathrm{H}^1(\mathcal{M}_1)\big) \arrow[d, "\wr"] \\
\det \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r, "\sim"] & \det \mathrm{H}^1(\mathcal{M}_0) \otimes \det^{-1} \mathrm{H}^1(\mathcal{M}_1)
\end{tikzcd}$$

commutes up to $(-1)^{n_2 m}$ with $n_2 = \dim \mathrm{H}^1(\nabla\mathcal{M})$.

Now the composition of the bottom horizontal isomorphisms in (7.3) and (7.4) is the ζ-isomorphism of $\mathcal{M}$ by definition while the composition of the top horizontal isomorphisms is the ζ-isomorphism of $\mathcal{N}$ by construction of the diagrams (7.1) and (7.2). As the right vertical isomorphisms in (7.3) and (7.4) are the same we get the result. $\square$

8. Elliptic shtukas

Starting from this section we focus on shtukas over $\mathcal{O}_F \times X$.

Definition 8.1. Throughout the rest of the chapter we fix an open ideal $\mathfrak{f} \subset \mathcal{O}_K$. We call $\mathfrak{f}$ the *conductor*.

Definition 8.2. Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \times X$. We say that $\mathcal{M}$ is an *elliptic shtuka of conductor* $\mathfrak{f}$ if the following holds:

(1) $\mathcal{M}$ is locally free.
(2) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent.
(3) $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is an elliptic shtuka of conductor $\mathfrak{f}$ in the sense of Definition 5.6.2.

As the conductor $\mathfrak{f}$ is fixed, in the following we speak simply of elliptic shtukas instead of elliptic shtukas of conductor $\mathfrak{f}$.

Lemma 8.3. *If a shtuka $\mathcal{M}$ is elliptic then $\nabla\mathcal{M}$ is elliptic.*

Proof. Follows immediately from Proposition 5.6.3. $\square$

Lemma 8.4. *Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \times X$. If $\mathcal{M}$ is elliptic then the following holds:*

(1) *The natural map* $\mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})$ *is a quasi-isomorphism.*
(2) $\mathrm{R\Gamma}(\mathcal{M})$ *is concentrated in degree* 1.
(3) $\mathrm{H}^1(\mathcal{M})$ *is a free* $\mathcal{O}_F$*-module of finite rank.*

Proof. Indeed $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent so Theorem 4.7.11 shows that the natural map $\mathrm{R\Gamma}(\mathcal{M}) \to \mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})$ is a quasi-isomorphism. Now $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is an elliptic shtuka of conductor $\mathfrak{f}$ by definition. So Theorem 5.6.4 shows that the complex $\mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})$ is concentrated in degree 1 and $\mathrm{H}^1(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})$ is a free $\mathcal{O}_F$-module of finite rank. $\square$

Definition 8.5. Let $\mathcal{M}$ be an elliptic shtuka on $\mathcal{O}_F \times X$. We define the *regulator* $\rho_{\mathcal{M}}$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{R\Gamma}(\mathcal{M}) \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d, "\wr"'] & \mathrm{R\Gamma}(\nabla\mathcal{M}) \arrow[d, "\wr"] \\
\mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \arrow[r, "\rho"] & \mathrm{R\Gamma}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \nabla\mathcal{M})
\end{tikzcd}$$

where ρ is the regulator of the elliptic shtuka $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ constructed in Theorem 5.14.4. The vertical maps are the natural quasi-isomorphisms of Lemma 8.4.

Lemma 8.6. *The regulator has the following properties:*

(1) *It is natural.*
(2) *It is a quasi-isomorphism.*

Proof. Follows immediately from Theorem 5.14.4. □

Lemma 8.7. *If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \times X$ then the ζ-isomorphism is defined for $\mathcal{M}$.*

Proof. Follows from Proposition 4.9.1 since $\mathcal{O}_F$ is regular and $\mathcal{M}$ is coherent. □

Our goal is to compare the determinant of the regulator with the ζ-isomorphism. To do it we first study how the regulator and the ζ-isomorphism of an elliptic shtuka behave under restriction to $\mathcal{O}_F/\mathfrak{m}_F^n \times X$.

Definition 8.8. Let $J \subset \mathcal{O}_F$ be a nonzero ideal.

(1) If $\mathcal{E}$ is a sheaf of modules on $\mathcal{O}_F \times X$ then $\mathcal{E}/J$ denotes its restriction to the closed subscheme $\mathcal{O}_F/J \times X$.
(2) If $\mathcal{M}$ is a shtuka on $\mathcal{O}_F \times X$ then $\mathcal{M}/J$ denotes its restriction to the closed subscheme $\mathcal{O}_F/J \times X$.

Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \times X$. In Section 4.8 we introduced a natural morphism

$$\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/J \to \mathrm{R}\Gamma(\mathcal{M}/J)$$

which is induced by the pullback map $\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}/J)$ (see Definition 4.8.2).

Lemma 8.9. *Let $J \subset \mathcal{O}_F$ be a nonzero ideal. If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \times X$ then the natural map $\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/J \to \mathrm{R}\Gamma(\mathcal{M}/J)$ is a quasi-isomorphism.*

Proof. It is an immediate consequence of Proposition 4.8.4. □

Lemma 8.10. *Let $J = \mathfrak{m}_F^d \subset \mathcal{O}_F$ be a nonzero ideal. Let $\mathcal{M}$ be an elliptic shtuka. If $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{f}^{2d})$ is linear then the following holds:*

(1) *The artinian regulator $\rho_{\mathcal{M}/J}$ is defined for $\mathcal{M}/J$.*
(2) *The natural diagram*

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/J \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d, "\wr"'] & \mathrm{R}\Gamma(\nabla\mathcal{M}) \otimes^{\mathbf{L}}_{\mathcal{O}_F} \mathcal{O}_F/J \arrow[d, "\wr"] \\
\mathrm{R}\Gamma(\mathcal{M}/J) \arrow[r, "\rho_{\mathcal{M}/J}"] & \mathrm{R}\Gamma(\nabla\mathcal{M}/J)
\end{tikzcd}$$

is commutative.

Proof. (1) As $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{f}^{2d})$ is linear Theorem 5.14.6 shows that the artinian regulator is defined for $\mathcal{M}(\mathcal{O}_F/J \otimes \mathcal{O}_K)$. According to Definition 4.3 the artinian regulator then makes sense for the shtuka $\mathcal{M}/J$ as well. (2) By Lemma 8.4 the complexes $\mathrm{R}\Gamma(\mathcal{M})$ and $\mathrm{R}\Gamma(\nabla\mathcal{M})$ are quasi-isomorphic to free $\mathcal{O}_F$-modules of finite rank placed in degree 1. Similarly Lemma 4.1 shows that the complexes $\mathrm{R}\Gamma(\mathcal{M}/J)$ and $\mathrm{R}\Gamma(\nabla\mathcal{M}/J)$ are quasi-isomorphic to free $\mathcal{O}_F/J$ modules of finite rank placed in degree 1. So we need to prove that the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/J \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d, "\mathrm{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/J \arrow[d, "\wr"', "\mathrm{red.}"] \\
\mathrm{H}^1(\mathcal{M}/J) \arrow[r, "\rho_{\mathcal{M}/J}"] & \mathrm{H}^1(\nabla\mathcal{M}/J)
\end{tikzcd}$$

is commutative.

Let $\mathcal{N}$ be the restriction of $\mathcal{M}$ to $\mathcal{O}_F/J \otimes \mathcal{O}_K$. Lemma 4.1 shows that the natural reduction maps in the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}/J) \arrow[r, "\rho_{\mathcal{M}/J}"] \arrow[d, "\mathrm{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}/J) \arrow[d, "\wr"', "\mathrm{red.}"] \\
\mathrm{H}^1(\mathcal{N}) \arrow[r, "\rho_{\mathcal{N}}"] & \mathrm{H}^1(\nabla\mathcal{N})
\end{tikzcd}$$

are isomorphisms. Here $\rho_{\mathcal{N}}$ is the artinian regulator of $\mathcal{N}$. The diagram commutes by definition of $\rho_{\mathcal{M}/J}$. Similarly the natural pullback maps in the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho_{\mathcal{M}}"] \arrow[d, "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\wr"'] \\
\mathrm{H}^1(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \arrow[r, "\rho"] & \mathrm{H}^1(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \nabla\mathcal{M})
\end{tikzcd}$$

are isomorphisms by Lemma 8.4. Here ρ is the regulator of the restriction of $\mathcal{M}$ to $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. The diagram commutes by the definition of $\rho_{\mathcal{M}}$. However Theorem 5.14.6 shows that the natural diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})/J \arrow[r, "\rho"] \arrow[d, "\mathrm{red.}"', "\wr"] & \mathrm{H}^1(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K, \nabla\mathcal{M})/J \arrow[d, "\wr"', "\mathrm{red.}"] \\
\mathrm{H}^1(\mathcal{N}) \arrow[r, "\rho_{\mathcal{N}}"] & \mathrm{H}^1(\nabla\mathcal{N})
\end{tikzcd}$$

is commutative. The vertical arrows are isomorphisms by Proposition 5.8.2. We thus get the result. □

Lemma 8.11. *Let $J \subset \mathcal{O}_F$ be a nonzero ideal. Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \times X$ given by a diagram $\mathcal{M}_0 \rightrightarrows \mathcal{M}_1$. Assume that*

(1) *$\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent,*

(2) $\mathrm{H}^0(\mathcal{M}_0) = 0$ *and* $\mathrm{H}^1(\mathcal{M}_0)$ *is a free* $\mathcal{O}_F$*-module of finite rank,*
(3) $\mathrm{H}^0(\mathcal{M}_1) = 0$ *and* $\mathrm{H}^1(\mathcal{M}_1)$ *is a free* $\mathcal{O}_F$*-module of finite rank.*

Then the following holds:

(1) *The* ζ*-isomorphism is defined for* $\mathcal{M}/J$*.*
(2) *The natural diagram*

$$\begin{tikzcd}
\big(\det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M})\big)/J \arrow[r, "\zeta_{\mathcal{M}}"] \arrow[d] & \big(\det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M})\big)/J \arrow[d] \\
\det_{\mathcal{O}_F/J} \mathrm{R}\Gamma(\mathcal{M}/J) \arrow[r, "\zeta_{\mathcal{M}/J}"] & \det_{\mathcal{O}_F/J} \mathrm{R}\Gamma(\nabla\mathcal{M}/J)
\end{tikzcd}$$

is commutative.

Proof. (1) The base change theorem [07VK] implies that $\mathrm{H}^0(\mathcal{M}_*/J) = 0$ and $\mathrm{H}^1(\mathcal{M}_*/J)$ is a free $\mathcal{O}_F/J$-module of finite rank for $* \in \{0,1\}$. As $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent Lemma 4.1 implies that $\mathrm{H}^0(\mathcal{M}/J) = 0$ and $\mathrm{H}^1(\mathcal{M}/J)$ is a free S-module of finite rank. Hence the ζ-isomorphism is defined for $\mathcal{M}/J$. (2) Follows immediately from Proposition 4.9.2. ☐

As in the case of elliptic shtukas on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ we will need a twisting construction for elliptic shtukas on $\mathcal{O}_F \times X$.

Definition 8.12. Let $\mathcal{E}$ be a quasi-coherent sheaf on $\mathcal{O}_F \times X$ and let $n \geqslant 0$. We define

$$\mathfrak{f}^n\mathcal{E} = \mathcal{I}^n \cdot \mathcal{E}$$

where $\mathcal{I} \subset \mathcal{O}_{\mathcal{O}_F \times X}$ is the unique ideal sheaf such that

$$\mathcal{I}(\operatorname{Spec} \mathcal{O}_F \otimes R) = \mathcal{O}_F \otimes R,$$
$$\mathcal{I}(\operatorname{Spec} \mathcal{O}_F \otimes \mathcal{O}_K) = \mathcal{O}_F \otimes \mathfrak{f}.$$

Lemma 8.13. *If* $\mathcal{E}$ *is a locally free sheaf on* $\mathcal{O}_F \times X$ *then* $\mathfrak{f}\mathcal{E}$ *is locally free.*

Proof. Indeed the ideal sheaf $\mathcal{I}$ above is locally free. ☐

Definition 8.14. Let $\mathcal{M}$ be a quasi-coherent shtuka on $\mathcal{O}_F \times X$ given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

Let $n \geqslant 0$. We define the shtuka $\mathfrak{f}^n\mathcal{M}$ by the diagram

$$\mathfrak{f}^n\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathfrak{f}^n\mathcal{M}_1$$

where $\mathfrak{f}^n\mathcal{M}_0$ and $\mathfrak{f}^n\mathcal{M}_1$ are as in Definition 8.12. We call $\mathfrak{f}^n\mathcal{M}$ the *n-th twist* of $\mathcal{M}$. By construction we have a natural embedding $\mathfrak{f}^n\mathcal{M} \hookrightarrow \mathcal{M}$.

Lemma 8.15. *Let* $\mathcal{M}$ *be a shtuka over* $\mathcal{O}_F \times X$*. If* $\mathcal{M}$ *is elliptic then* $\mathfrak{f}\mathcal{M}$ *is elliptic.*

Proof. Lemma 8.13 implies that $\mathfrak{f}\mathcal{M}$ is locally free. Furthermore $\mathcal{M}(\mathcal{O}_F \otimes R) = (\mathfrak{f}\mathcal{M})(\mathcal{O}_F \otimes R)$ so that $(\mathfrak{f}\mathcal{M})(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent. Finally $(\mathfrak{f}\mathcal{M})(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$ is the twist of $\mathcal{M}(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$ by $\mathfrak{f}$ in the sense of Definition 5.7.1. Hence Proposition 5.7.3 shows that $(\mathfrak{f}\mathcal{M})(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K)$ is an elliptic shtuka in the sense of Definition 5.6.2. □

Lemma 8.16. *Let $\mathcal{M}$ be an elliptic shtuka and let $\mathcal{N} = \mathfrak{f}\mathcal{M}$.*

(1) $F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}/\mathcal{N}) = 0$.
(2) *The natural map $F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M})$ is a quasi-isomorphism.*

Proof. (1) Let $\mathcal{Q} = \mathcal{M}/\mathcal{N}$ and let $\mathcal{Q}_F$ be the pullback of $\mathcal{Q}$ to $F \times X$. By construction $\mathcal{Q}_F$ is the restriction of the locally free shtuka $\mathcal{M}_F$ to the closed affine subscheme $\mathrm{Spec}(F \otimes \mathcal{O}_K/\mathfrak{f})$ of $F \times X$. In particular the underlying sheaves of $\mathcal{Q}$ are coherent and flat over $\mathcal{O}_F$. So Proposition 4.8.4 shows that $F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{Q}) = \mathrm{R}\Gamma(\mathcal{Q}_F)$. Since $\mathcal{Q}_F$ is nilpotent and supported on a closed affine subscheme Proposition 1.9.3 demonstrates that $\mathrm{R}\Gamma(\mathcal{Q}_F) = 0$. (2) follows from (1). □

Lemma 8.17. *If $\mathcal{M}$ is an elliptic shtuka and $\mathcal{N} = \mathfrak{f}\mathcal{M}$ then the natural square*

$$\begin{tikzcd} F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\rho_{\mathcal{N}}"', "\wr"] & F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \arrow[d, "\wr"', "\rho_{\mathcal{M}}"] \\ F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{N}) \arrow[r, "\sim"] & F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \end{tikzcd}$$

is commutative.

Proof. The horizontal arrows are quasi-isomorphisms by Lemma 8.16. The square itself is commutative since the regulators ρ are natural. □

Lemma 8.18. *Let $\mathcal{M}$ be an elliptic shtuka and let $\mathcal{N} = \mathfrak{f}\mathcal{M}$. If the underlying sheaves of $\mathcal{M}$ have cohomology concentrated in degree 1 then the natural square*

$$\begin{tikzcd} F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathcal{N}}"', "\wr"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \arrow[d, "\wr"', "\zeta_{\mathcal{M}}"] \\ F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{N}) \arrow[r, "\sim"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \end{tikzcd}$$

is commutative.

Proof. Let $\mathcal{Q} = \mathcal{M}/\mathcal{N}$. Let $\mathcal{M}_F$, $\mathcal{N}_F$, $\mathcal{Q}_F$ denote the pullbacks of the respective shtukas to $F \times X$. By construction we have a short exact sequence $0 \to \mathcal{N}_F \to \mathcal{M}_F \to \mathcal{Q}_F \to 0$. We would like to apply Lemma 7.2 to this short exact sequence. To do it we should verify that the following conditions are met:

(1) $\mathrm{R}\Gamma(\mathcal{M}_F)$ and $\mathrm{R}\Gamma(\nabla\mathcal{M}_F)$ are concentrated in degree 1.

(2) The underlying sheaves of $\mathcal{M}_F$ have cohomology concentrated in degree 1.
(3) $\mathrm{R}\Gamma(\mathcal{Q}) = 0$ and $\mathrm{R}\Gamma(\nabla\mathcal{Q}) = 0$.
(4) The underlying sheaves of $\mathcal{Q}_F$ have cohomology concentrated in degree 0.
(5) $\mathcal{Q}_F$ is linear.

(1) follows by Lemma 8.4 and (2) holds by assumption. Lemma 8.16 implies (3). By construction $\mathcal{Q}_F$ is supported at a closed affine subscheme $\operatorname{Spec} F \otimes \mathcal{O}_K/\mathfrak{f}$ of $F \times X$. Thus the condition (4) is satisfied. Finally the condition (5) follows since $\mathcal{M}$ is elliptic. Hence Lemma 7.2 demonstrates that the natural square

$$\begin{tikzcd} \det_F \mathrm{R}\Gamma(\mathcal{N}_F) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathcal{N}_F}"', "\wr"] & \det_F \mathrm{R}\Gamma(\mathcal{M}_F) \arrow[d, "\zeta_{\mathcal{M}_F}", "\wr"'] \\ \det_F \mathrm{R}\Gamma(\nabla\mathcal{N}_F) \arrow[r, "\sim"] & \det_F \mathrm{R}\Gamma(\nabla\mathcal{M}_F) \end{tikzcd}$$

commutes up to the sign $(-1)^{nm}$ where

$$n = \dim_F \mathrm{H}^1(\mathcal{M}_F) + \dim_F \mathrm{H}^1(\nabla\mathcal{M}_F).$$

However $\dim_F \mathrm{H}^1(\mathcal{M}_F) = \dim_F \mathrm{H}^1(\nabla\mathcal{M}_F)$ since the shtuka $\mathcal{M}$ is elliptic and so admits a regulator isomorphism $\rho\colon \mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})$. Thus the square above is in fact commutative. Proposition 4.9.2 now implies the result. □

9. Euler products for $\mathcal{O}_F$

In this section we will define the Euler products $L(\mathcal{M})$ for shtukas $\mathcal{M}$ over $\mathcal{O}_F \times X$.

Lemma 9.1. *Let k be a finite field extension of $\mathbb{F}_q$. Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \otimes k$ given by a diagram*

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes k)$ is nilpotent then the following holds:

(1) *M_0 is a free $\mathcal{O}_F$-module of finite rank,*
(2) *$i\colon M_0 \to M_1$ is an isomorphism,*
(3) *$(1 - i^{-1}j)\colon M_0 \to M_0$ is an $\mathcal{O}_F$-linear isomorphism.*

Proof. Indeed the ring $\mathcal{O}_F \otimes k$ is noetherian and complete with respect to a τ-invariant ideal $\mathfrak{m}_F \otimes k$. So Proposition 1.9.4 implies that

(i) $\mathrm{R}\Gamma(\nabla\mathcal{M}) = 0$.
(ii) $\mathrm{R}\Gamma(\mathcal{M}) = 0$.

Now (i) means that $i\colon M_0 \to M_1$ is an isomorphism while (ii) implies that the $\mathcal{O}_F$-linear map

$$(i - j)\colon M_0 \to M_1$$

is an isomorphism. So we get the result. $\square$

Definition 9.2. Let k be a finite field extension of $\mathbb{F}_q$. Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \otimes k$ given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Assuming that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes k)$ is nilpotent we define

$$L(\mathcal{M}) = \det_{\mathcal{O}_F}(1 - i^{-1}j \mid M_0) \in \mathcal{O}_F^\times.$$

Lemma 9.3. *Let $n \geqslant 1$ be an integer. If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \otimes R$ such that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then for almost all maximal ideals $\mathfrak{m} \subset R$ we have*

$$L\big(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})\big) \equiv 1 \pmod{\mathfrak{m}_F^n}$$

Proof. Let $\mathfrak{m} \subset R$ be a maximal ideal. By construction

$$L\big(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})\big) \equiv L\big(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F^n \otimes R/\mathfrak{m})\big) \pmod{\mathfrak{m}_F^n}$$

Applying Lemma 2.4 to $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F^n \otimes R)$ we get the result. $\square$

Definition 9.4. Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \times X$. Assuming that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent we define

$$L(\mathcal{M}) = \prod_{\mathfrak{m}} \frac{1}{L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m}))} \in \mathcal{O}_F^\times$$

where $\mathfrak{m} \subset R$ ranges over the maximal ideals. This product converges by Lemma 9.3. Note that the closed points in the complement of $\operatorname{Spec} R$ in X are not taken into account.

Lemma 9.5. *If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \times X$ such that $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then for every $n \geqslant 1$ we have $L(\mathcal{M}) \equiv L\big(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F^n \times X)\big) \pmod{\mathfrak{m}_F^n}$.* $\square$

Proposition 9.6. *If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \times X$ whose restriction to $\operatorname{Spec}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then $L(\mathcal{M}) \equiv 1 \pmod{\mathfrak{m}_F}$.*

Proof. Indeed $L(\mathcal{M}) \equiv L(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \times X)) \pmod{\mathfrak{m}_F}$. Since $\mathcal{O}_F/\mathfrak{m}_F$ is a field Lemma 2.6 implies that $L(\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \times X)) = 1$. $\square$

10. Trace formula

Lemma 10.1. *If $\mathcal{M}$ is an elliptic shtuka then the invariant $L(\mathcal{M})$ is defined for $\mathcal{M}$.*

Proof. By definition $\mathcal{M}$ is locally free and $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent. $\square$

Lemma 10.2. *Let $d \geqslant 1$. Let $\mathcal{M}$ be an elliptic shtuka of conductor $\mathfrak{f}$ on $\mathcal{O}_F \times X$. Suppose that $\mathcal{M}$ is given by a diagram*

$$\mathcal{M}_0 \rightrightarrows \mathcal{M}_1.$$

Assume that the following holds:

(1) $\mathrm{H}^0(\mathcal{M}_0) = 0$ *and* $\mathrm{H}^1(\mathcal{M}_0)$ *is a free* $\mathcal{O}_F$*-module of finite rank,*
(2) $\mathrm{H}^0(\mathcal{M}_1) = 0$ *and* $\mathrm{H}^1(\mathcal{M}_1)$ *is a free* $\mathcal{O}_F$*-module of finite rank,*
(3) $\mathcal{M}/\mathfrak{f}^{2d}$ *is linear.*

If the conductor $\mathfrak{f}$ *is contained in* $\mathfrak{m}_K$ *then* $\zeta_{\mathcal{M}} \equiv L(\mathcal{M})\cdot\det_{\mathcal{O}_F}(\rho_{\mathcal{M}}) \pmod{\mathfrak{m}_F^d}$.

Proof. Set $S = \mathcal{O}_F/\mathfrak{m}^d$. Let $\mathcal{N} = \mathcal{M}/\mathfrak{m}_F^d$ and let

$$\mathcal{N}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{N}_1$$

be the diagram of $\mathcal{N}$. We claim that the shtuka $\mathcal{N}$ has the following properties:

(a) $\mathrm{H}^0(\mathcal{N}_0) = 0$ and $\mathrm{H}^1(\mathcal{N}_0)$ is a free S-module of finite rank,
(b) $\mathrm{H}^0(\mathcal{N}_1) = 0$ and $\mathrm{H}^1(\mathcal{N}_1)$ is a free S-module of finite rank,
(c) $\mathcal{N}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent,
(d) $\mathcal{N}(S \otimes \mathcal{O}_K/\mathfrak{f}^{2d})$ is linear,
(e) $\mathfrak{f}^d \cdot \mathrm{H}^1(S \otimes \mathcal{O}_K, \nabla\mathcal{N}) = 0$.

The properties (a) and (b) follow from the assumptions (1) and (2) by the base change theorem [07VK]. The property (c) hold since $\mathcal{M}$ is elliptic and (d) is a consequence of the assumption (3). Observe that $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is an elliptic shtuka in the sense of Definition 5.6.2. One thus gets (e) by applying Lemma 5.8.1 to the shtuka $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)/\mathfrak{m}_F^d = \mathcal{N}(S \otimes \mathcal{O}_K)$.

Now we apply Theorem 5.1 to $\mathcal{N}$ with $I = \mathfrak{f}^d$ and conclude that the following is true:

(i) The artinian regulator $\rho_{\mathcal{N}}$ is defined for $\mathcal{N}$.
(ii) The ζ-isomorphism $\zeta_{\mathcal{N}}$ is defined for $\mathcal{N}$.
(iii) $\zeta_{\mathcal{N}} = L(\mathcal{N}) \cdot \det_S(\rho_{\mathcal{N}})$.

The congruence $L(\mathcal{M}) \equiv L(\mathcal{N}) \pmod{\mathfrak{m}_F^d}$ holds by construction. Moreover $\rho_{\mathcal{N}} \equiv \rho_{\mathcal{M}} \pmod{\mathfrak{m}_F^d}$ by Lemma 8.10 and $\zeta_{\mathcal{N}} \equiv \zeta_{\mathcal{M}} \pmod{\mathfrak{m}_F^d}$ by Lemma 8.11. Whence the result. ☐

Lemma 10.3. *Let* $\mathcal{M}$ *be an elliptic shtuka and let* $\mathcal{N} = \mathfrak{f}\mathcal{M}$*. Suppose that the underlying sheaves of* $\mathcal{M}$ *have cohomology concentrated in degree* 1*. If* $\alpha \in \mathcal{O}_F^\times$ *is the unique element such that* $\zeta_{\mathcal{M}} = \alpha\cdot\det_{\mathcal{O}_F}(\rho_{\mathcal{M}})$ *then* $\zeta_{\mathcal{N}} = \alpha\cdot\det_{\mathcal{O}_F}(\rho_{\mathcal{N}})$.

Proof. Lemma 8.17 implies that the natural square

$$\begin{tikzcd}
F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\det_{\mathcal{O}_F}(\rho_{\mathcal{N}})"', "\wr"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \arrow[d, "\wr"', "\det_{\mathcal{O}_F}(\rho_{\mathcal{M}})"] \\
F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{N}) \arrow[r, "\sim"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M})
\end{tikzcd}$$

is commutative. At the same time Lemma 8.18 shows that the natural square

$$\begin{tikzcd} F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\zeta_{\mathcal{N}}"', "\wr"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \arrow[d, "\zeta_{\mathcal{M}}", "\wr"'] \\ F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{N}) \arrow[r, "\sim"] & F \otimes_{\mathcal{O}_F} \det_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \end{tikzcd}$$

is commutative. So we get the result. □

Finally we are ready to prove the trace formula for regulators of elliptic shtukas.

Theorem 10.4. *Let $\mathcal{M}$ be an elliptic shtuka of conductor $\mathfrak{f}$. Suppose that the underlying sheaves of $\mathcal{M}$ have cohomology concentrated in degree* 1. *If the conductor $\mathfrak{f}$ is contained in $\mathfrak{m}_K$ then*

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}}).$$

This theorem should certainly hold without the assumption on on the cohomology of sheaves underlying $\mathcal{M}$. The bottleneck which prevents us from establishing this more natural version of the trace formula is Lemma 6.1. A more general variant of this lemma is needed. The proof of such a lemma appears to be too messy at the moment. Alas, the determinant theory of [16] is not too user-friendly. Nevertheless Theorem 10.4 is still enough to prove the class number formula for Drinfeld modules.

Proof of Theorem 10.4. Let $\alpha \in \mathcal{O}_F^\times$ be the unique element such that $\zeta_{\mathcal{M}} = \alpha \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}})$. We will show that $\alpha \equiv L(\mathcal{M}) \pmod{\mathfrak{m}_F^d}$ for every $d \geqslant 1$.

Let $\mathcal{N} = \mathfrak{f}\mathcal{M}$. Observe that $L(\mathcal{M}) = L(\mathcal{N})$. Indeed the invariant L depends only on the restriction of a shtuka to $\mathcal{O}_F \otimes R$ and $\mathcal{M}(\mathcal{O}_F \otimes R) = \mathcal{N}(\mathcal{O}_F \otimes R)$ by construction. At the same time Lemma 10.3 shows that $\zeta_{\mathcal{N}} = \alpha \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{N}})$. Furthermore the cohomology of sheaves underlying $\mathcal{N}$ is concentrated in degree 1 as well. We are thus free to replace $\mathcal{M}$ by its twists $\mathfrak{f}^n\mathcal{M}$.

Twisting $\mathcal{M}$ we can ensure that $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{f}^{2d})$ is linear. Indeed $(\mathfrak{f}\mathcal{M})(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is the twist of $\mathcal{M}(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ by $\mathfrak{f}$ in the sense of Definition 5.7.1. So Proposition 5.7.6 implies that $(\mathfrak{f}^n\mathcal{M})(\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K/\mathfrak{f}^{2d})$ is linear for all $n \geqslant 2d$.

Suppose that $\mathcal{M}$ is given by a diagram $\mathcal{M}_0 \rightrightarrows \mathcal{M}_1$. Since $\mathfrak{f}$ is different from $\mathcal{O}_K$ Lemma 1.1 implies that $\mathrm{H}^0(\mathfrak{f}^n\mathcal{M}_*) = 0$ and $\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}_*)$ is a free $\mathcal{O}_F$-module of finite rank for all $n \gg 0$ and $* \in \{0,1\}$. So replacing $\mathcal{M}$ by a suitable twist we can ensure that $\mathcal{M}$ meets the conditions of Lemma 10.2. Whence the result. □