Shtuka cohomology and special values of Goss L-functions

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CHAPTER 5

Regulators

Let F be a local field and let K be a finite product of local fields. As usual, the conventions and the notation of Section 3.5 apply to F and K. In particular F and K are supposed to contain $\mathbb{F}_q$. We denote $\mathcal{O}_F \subset F$ and $\mathcal{O}_K \subset K$ the rings of integers, $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_F$ and $\mathfrak{m}_K$ the Jacobson radical of $\mathcal{O}_K$. We omit the subscript F for the ideal $\mathfrak{m} \subset \mathcal{O}_F$ to improve the legibility.

We mainly work with $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtukas. In agreement with the conventions of Section 3.11 the τ-structure on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is given by the endomorphism which acts as the identity on $\mathcal{O}_F$ and as the q-Frobenius on $\mathcal{O}_K$.

The aim of this chapter is to construct for a certain class of shtukas $\mathcal{M}$ on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ a natural quasi-isomorphism $\rho\colon \mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\nabla\mathcal{M})$ called the *regulator*. We do it for *elliptic shtukas*, a natural class of shtukas generalizing the models of Drinfeld modules in the sense of Chapter 9. The key definitions and results of this chapter are as follows:

- Theorem 4.2 gives a sufficient condition for the cohomology modules of a shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ to be finitely generated free $\mathcal{O}_F$-modules. It applies in particular to elliptic shtukas.
- Definition 6.2 introduces elliptic shtukas.
- Definition 14.1 introduces the regulator for elliptic shtukas.
- The existence and unicity of the regulator is affirmed by Theorem 14.4.

It is easy to characterize the regulator as a natural transformation of functors on the category of elliptic shtukas (see Definition 14.1). However its construction is a bit involved.

The content of this chapter is new save for the preliminary Sections 1 and 2. Even in the case of Section 1 we are not aware of a reference in the literature for the key Lemma 1.1. Our search for a shtuka-theoretic regulator was motivated by the article [17] of V. Lafforgue.

1. Topological preliminaries

In this section we give a topological criterion for an $\mathcal{O}_F$-module to be finitely generated. We will use it to prove that cohomology modules of certain shtukas are finitely generated.

Lemma 1.1. *Let M be a compact Hausdorff $\mathcal{O}_F$-module. The following are equivalent:*

1. *$M/\mathfrak{m}$ is finite as a set.*
2. *M is finitely generated as an $\mathcal{O}_F$-module without topology.*

Proof. (1) $\Rightarrow$ (2). Let z be a uniformizer of $\mathcal{O}_F$. The submodule $\mathfrak{m}M \subset M$ is the image of M under multiplication by z so it is closed, compact and Hausdorff. Furthermore multiplication by z defines a surjective map $M/\mathfrak{m} \to (\mathfrak{m}M)/\mathfrak{m}$ so that $(\mathfrak{m}M)/\mathfrak{m}$ is finite. By induction we conclude that the submodules $\mathfrak{m}^n M \subset M$ are closed and of finite index, hence open.

Let us show that the open submodules $\mathfrak{m}^n M$ form a fundamental system of neighbourhoods of zero. Proposition 2.4.1 and Lemma 3.4.3 imply that M admits a fundamental system of open $\mathcal{O}_F$-submodules. If $U \subset M$ is an open submodule then M/U is finite so there exists an $n > 0$ such that z^n acts by zero on M/U. Hence $\mathfrak{m}^n M \subset U$.

Now M is a compact $\mathcal{O}_F$-module so it is complete as a topological $\mathbb{F}_q$-vector space. As the submodules $\mathfrak{m}^n M$ form a fundamental system we conclude that $M = \lim_{n>0} M/\mathfrak{m}^n$.

Let r be the dimension of $M/\mathfrak{m}$ as an $\mathcal{O}_F/\mathfrak{m}$-vector space. For every $n > 0$ let H_n be the set of surjective $\mathcal{O}_F$-linear maps from $\mathcal{O}_F^{\oplus r}$ to $M/\mathfrak{m}^n$. The sets H_n form a projective system H_* in a natural way. Every point of the limit of H_* defines a continuous morphism from $\mathcal{O}_F^{\oplus r}$ to M. Such a morphism is surjective since it has dense image by construction and its domain $\mathcal{O}_F^{\oplus r}$ is compact.

By definition the set H_1 is nonempty so Nakayama's lemma implies that every H_n is nonempty. Since all H_n are finite and nonempty we conclude that the projective system H_* has a nonempty limit. □

It is worth mentioning that Lemma 1.1 works for any nonarchimedean local field F and more generally for any local noetherian ring $\mathcal{O}_F$ with finite residue field. Indeed one can show that a (locally) compact Hausdorff $\mathcal{O}_F$-module M admits a fundamental system of open submodules and the rest of the argument applies essentially as is.

2. Algebraic preliminaries

Let us review some elementary algebraic properties of the ring $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$.

Lemma 2.1. *An ideal $I \subset \mathcal{O}_K$ is open if and only if it is a free $\mathcal{O}_K$-module of rank* 1.

Proof. By definition $\mathcal{O}_K$ is a finite product of complete discrete valuation rings. The ideal I is open if and only if it projects to a nonzero ideal in every factor. Whence the result. □

Let $I \subset \mathcal{O}_K$ be an open ideal. We will often use natural homomorphisms

$$f_I \colon \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \otimes \mathcal{O}_K/I,$$
$$g_I \colon \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to F \otimes \mathcal{O}_K/I.$$

The homomorphism f_I is the completion of the natural map $\mathcal{O}_F \otimes_{\mathrm{c}} \mathcal{O}_K \to \mathcal{O}_F \otimes_{\mathrm{c}} \mathcal{O}_K/I$. We use the fact that $\mathcal{O}_K/I$ is finite to identify $\mathcal{O}_F \mathbin{\widehat{\otimes}} (\mathcal{O}_K/I)$ with $\mathcal{O}_F \otimes \mathcal{O}_K/I$. The homomorphism g_I is the composition of f_I with the natural map $\mathcal{O}_F \otimes \mathcal{O}_K/I \to F \otimes \mathcal{O}_K/I$. By construction g_I factors over $F \otimes_{\mathcal{O}_F} (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K)$. The notation f_I and g_I will not be used. The same constructions apply to a nonzero ideal $J \subset \mathcal{O}_F$.

By definition $\mathcal{O}_F = k[[z]]$ for a finite field extension k of $\mathbb{F}_q$ and a uniformizer $z \in \mathcal{O}_F$. In a similar way

$$\mathcal{O}_K = \prod_{i=1}^{d} k_i[[\zeta_i]]$$

where k_i are finite field extensions of $\mathbb{F}_q$ and ζ_i are uniformizers of the factors of $\mathcal{O}_K$. As a consequence

$$\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K = \prod_{i=1}^{d} (k \otimes k_i)[[z, \zeta_i]]$$

is a finite product of power series rings in two variables. With this observation in mind the following lemmas become obvious.

Lemma 2.2. *Let $I \subset \mathcal{O}_K$ be an open ideal.*

(1) *The ideal $I \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) = \mathcal{O}_F \mathbin{\widehat{\otimes}} I$ is a free $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module of rank 1.*
(2) *The sequence $0 \to I \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \otimes \mathcal{O}_K/I \to 0$ is exact.* □

Lemma 2.3. *Let $J \subset \mathcal{O}_F$ be an open ideal.*

(1) *The ideal $J \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) = J \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is a free module of rank 1.*
(2) *The sequence $0 \to J \cdot (\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F/J \otimes \mathcal{O}_K \to 0$ is exact.* □

The next lemma will only be used in the proof of Proposition 10.3.

Lemma 2.4. *If $I \subset \mathcal{O}_K$ and $J \subset \mathcal{O}_F$ are open ideals then the natural sequence*

$$0 \to \frac{\mathcal{O}_F \mathbin{\widehat{\otimes}} I}{J \mathbin{\widehat{\otimes}} I} \oplus \frac{J \mathbin{\widehat{\otimes}} \mathcal{O}_K}{J \mathbin{\widehat{\otimes}} I} \to \frac{\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K}{J \mathbin{\widehat{\otimes}} I} \to \frac{\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K}{J \mathbin{\widehat{\otimes}} \mathcal{O}_K + \mathcal{O}_F \mathbin{\widehat{\otimes}} I} \to 0$$

is exact.

Proof. Follows since $J \mathbin{\widehat{\otimes}} I = J \mathbin{\widehat{\otimes}} \mathcal{O}_K \cap \mathcal{O}_F \mathbin{\widehat{\otimes}} I$. □

3. A lemma on nilpotent shtukas

Fix a finite $\mathbb{F}_q$-algebra S which is a local artinian ring. Let $\mathfrak{m} \subset S$ be the maximal ideal. In this section we work with the ring $S \otimes \mathcal{O}_K$. We equip it with the τ-ring structure given by the endomorphism which acts as the identity on S and as the q-Frobenius on $\mathcal{O}_K$.

The sole result of this section is a lemma on shtukas over $S \otimes \mathcal{O}_K$. We place it in a separate section since it is used in several independent parts of our theory.

Lemma 3.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes \mathcal{O}_K$. If $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent then:*

(1) *$\mathcal{M}(S \otimes K)$ is nilpotent,*
(2) *$\mathrm{H}^0(\mathcal{M}) = 0$,*
(3) *$\mathrm{H}^1(\mathcal{M})$ is a free S-module of finite rank.*

Proof. (1) The ring $S \otimes K$ is noetherian and complete with respect to the ideal $\mathfrak{m} \otimes K$. As the ideal $\mathfrak{m}$ is nilpotent the result follows from Proposition 1.9.4.

(2) Since $\mathcal{M}$ is locally free the natural map $\mathrm{H}^0(\mathcal{M}) \to \mathrm{H}^0(S \otimes K, \mathcal{M})$ injective. However $\mathcal{M}(S \otimes K)$ is nilpotent by (1) so $\mathrm{H}^0(S \otimes K, \mathcal{M}) = 0$ by Propostion 1.9.3.

(3) First let us prove that $\mathrm{H}^1(\mathcal{M})$ is a flat S-module. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

The cohomology complex $\mathrm{R}\Gamma(\mathcal{M})$ is represented by the associated complex

$$\Gamma_{\mathrm{a}}(\mathcal{M}) = [M_0 \xrightarrow{i-j} M_1].$$

It is a comlex of flat S-modules since $\mathcal{M}$ is locally free by assumption. As a consequence

$$\Gamma_{\mathrm{a}}(\mathcal{M}) \otimes^{\mathbf{L}}_S S/\mathfrak{m} = \Gamma_{\mathrm{a}}(\mathcal{M}) \otimes_S S/\mathfrak{m}.$$

However $\Gamma_{\mathrm{a}}(\mathcal{M}) \otimes_S S/\mathfrak{m}$ is the complex representing $\mathrm{R}\Gamma(S/\mathfrak{m} \otimes \mathcal{O}_K, \mathcal{M})$. Applying the argument (2) above to the shtuka $\mathcal{M}(S/\mathfrak{m} \otimes \mathcal{O}_K)$ we deduce that $\mathrm{R}\Gamma(S/\mathfrak{m} \otimes \mathcal{O}_K, \mathcal{M})$ is concentrated in degree 1. As a consequence $\mathrm{H}^1(\mathcal{M}) \otimes^{\mathbf{L}}_S S/\mathfrak{m}$ is concentrated in degree 0. In other words $\mathrm{Tor}_n(\mathrm{H}^1(\mathcal{M}), S/\mathfrak{m}) = 0$ for $n > 0$. Therefore $\mathrm{H}^1(\mathcal{M})$ is a flat S-module [051K].

Next we prove that $\mathrm{H}^1(\mathcal{M})$ is finitely generated. The $\mathcal{O}_K$-modules M_0 and M_1 are finitely generated by assumption. Hence they carry a natural topology given by the powers of the ideal $\mathfrak{m}_K$. We would like to prove that the map $(i-j)\colon M_0 \to M_1$ is open. Since M_1 is a compact $\mathcal{O}_K$-module it then follows that $M_1/(i-j)M_0 = \mathrm{H}^1(\mathcal{M})$ is a finite set.

Consider the locally compact K-vector spaces $V_0 = M_0 \otimes_{\mathcal{O}_K} K$ and $V_1 = M_1 \otimes_{\mathcal{O}_K} K$. By (1) the shtuka $\mathcal{M}(S \otimes K)$ is nilpotent whence $i\colon V_0 \to V_1$ is an isomorphism and the endomorphism $i^{-1}j$ of V_0 is nilpotent. The isomorphism $i^{-1}\colon V_1 \to V_0$ is continuous by K-linearity. The map $j\colon V_0 \to V_1$ is continuous since it is a Frobenius-linear morphism of finite-dimensional K-vector spaces. As a consequence $i^{-1}j\colon V_0 \to V_0$ is continuous. Since it is nilpotent we conclude that the endomorphism $(1 - i^{-1}j)^{-1}$ is continuous. Therefore $1 - i^{-1}j$ is open. However

$$i - j = i \circ (1 - i^{-1}j).$$

So $(i - j)\colon V_0 \to V_1$ is a composition of open maps. We conclude that $(i - j)\colon M_0 \to M_1$ is open. $\square$

4. Finiteness of cohomology

In this section we prove that under certain natural conditions the cohomology groups of shtukas over $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ are finitely generated free $\mathcal{O}_F$-modules.

Lemma 4.1. *Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent then the $\mathcal{O}_F$-modules $\mathrm{H}^0(\mathcal{M})$ and $\mathrm{H}^1(\mathcal{M})$ have the following properties:*

(1) $\mathrm{H}^0(\mathcal{M})$ *is uniquely divisible,*
(2) $\mathrm{H}^1(\mathcal{M})$ *is torsion free,*
(3) $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}$ *is finite as a set.*

Proof. Let $z \in \mathcal{O}_F$ be a uniformizer. Consider the short exact sequence

$$0 \to \mathcal{M} \xrightarrow{z} \mathcal{M} \to \mathcal{M}/z \to 0.$$

By Lemma 2.3 $\mathcal{M}/z$ is the restriction of $\mathcal{M}$ to $\mathcal{O}_F/\mathfrak{m} \otimes \mathcal{O}_K$. Since $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent Lemma 3.1 implies that $\mathrm{H}^0(\mathcal{M}/z) = 0$ and $\mathrm{H}^1(\mathcal{M}/z)$ is finite as a set. Taking the long exact cohomology sequence we deduce the result. $\square$

The first main result of this chapter is the following:

Theorem 4.2. *Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent then the following holds:*

(1) $\mathrm{H}^0(\mathcal{M}) = 0$.
(2) $\mathrm{H}^1(\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

The $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-modules M_0, M_1 come equipped with a canonical topology given by powers of the ideal

$$I = \mathfrak{m} \mathbin{\widehat{\otimes}} \mathcal{O}_K + \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathfrak{m}_K.$$

The map i is continuous in this topology since it is $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-linear. The partial Frobenius $\tau\colon \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ maps I to itself. As j is τ-linear it follows that j is continuous. Hence $i - j$ is continuous.

Consider the complex

$$\big[M_0 \xrightarrow{i-j} M_1\big].$$

The ring $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is compact with respect to the I-adic topology. Therefore the finitely generated $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-modules M_0, M_1 are compact Hausdorff. It follows that the image of M_0 in M_1 is closed and the quotient topology on $\mathrm{H}^1(\mathcal{M})$ is compact Hausdorff. So is the subspace topology on $\mathrm{H}^0(\mathcal{M})$.

The natural map $\mathcal{O}_F \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is continuous and the differential $i - j$ in the complex above is $\mathcal{O}_F$-linear. As a consequence $\mathrm{H}^0(\mathcal{M})$ and $\mathrm{H}^1(\mathcal{M})$ are topological $\mathcal{O}_F$-modules. By Lemma 4.1 the modules $\mathrm{H}^0(\mathcal{M})$ and $\mathrm{H}^1(\mathcal{M})$ are torsion free, $\mathrm{H}^0(\mathcal{M})/\mathfrak{m} = 0$ and $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}$ is finite as a set. Now $\mathrm{H}^0(\mathcal{M})$ and $\mathrm{H}^1(\mathcal{M})$ are compact Hausdorff so Lemma 1.1 shows that they are finitely generated free. As $\mathrm{H}^0(\mathcal{M})$ is uniquely divisible it must be zero. □

5. Artinian regulators

Let us fix a finite $\mathbb{F}_q$-algebra S which is a local artinian ring. We denote $\mathfrak{m} \subset S$ the maximal ideal. In this section we work over the ring $S \otimes \mathcal{O}_K$. We equip it with the τ-ring structure given by the endomorphism which acts as the identity on S and as the q-Frobenius on $\mathcal{O}_K$.

We study locally free shtukas on $S \otimes \mathcal{O}_K$ which restrict to nilpotent shtukas on $S/\mathfrak{m} \otimes K$. Under certain conditions we will define a regulator map for such shtukas, the *artinian regulator*.

Lemma 5.1. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes \mathcal{O}_K$ given by a diagram*

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

If $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent then the map $i\colon M_0 \otimes_{\mathcal{O}_K} K \to M_1 \otimes_{\mathcal{O}_K} K$ is an isomorphism.

Proof. By Lemma 3.1 the shtuka $\mathcal{M}(S \otimes K)$ is nilpotent. The i-map of such a shtuka is an isomorphism by definition. □

Definition 5.2. Let a locally free shtuka $\mathcal{M}$ on $S \otimes \mathcal{O}_K$ be given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Suppose that $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent. We say that the artinian regulator is defined for $\mathcal{M}$ if the endomorphism $i^{-1}j$ of the S-module $M_0 \otimes_{\mathcal{O}_K} K$ preserves the submodule M_0. In this case we define a map of S-module complexes $\rho_{\mathcal{M}}\colon \Gamma_{\mathrm{a}}(\mathcal{M}) \to \Gamma_{\mathrm{a}}(\nabla\mathcal{M})$ by the diagram

$$\begin{array}{ccc} [M_0 & \xrightarrow{i-j} & M_1] \\ {\scriptstyle 1-i^{-1}j}\downarrow & & \downarrow{\scriptstyle 1} \\ [M_0 & \xrightarrow{i} & M_1]. \end{array}$$

We call $\rho_{\mathcal{M}}$ the *artinian regulator* of $\mathcal{M}$.

In a moment we will give a sufficient condition for the regulator to be defined. Before that let us study its properties.

Lemma 5.3. *The regulator of Definition 5.2 has the following properties.*

(1) $\rho_{\mathcal{M}}$ *is natural in* $\mathcal{M}$.
(2) $\rho_{\mathcal{M}}$ *is an isomorphism.*

Proof. (1) Clear. (2) Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

The shtuka $\mathcal{M}(S \otimes K)$ is nilpotent by Lemma 3.1 whence the endomorphism $i^{-1}j$ of $M_0 \otimes_{\mathcal{O}_K} K$ is nilpotent. As a consequence the endomorphism $1 - i^{-1}j$ is in fact an automorphism. □

Definition 5.4. Let $I \subset \mathcal{O}_K$ be an ideal. Let a quasi-coherent shtuka $\mathcal{M}$ on $S \otimes \mathcal{O}_K$ be given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

We define

$$I\mathcal{M} = \Big[IM_0 \underset{j}{\overset{i}{\rightrightarrows}} IM_1 \Big].$$

By assumption $\mathcal{O}_K$ is a finite product of complete discrete valuation rings. Recall that an ideal $I \subset \mathcal{O}_K$ is open if and only if it projects to a nonzero ideal in every factor of $\mathcal{O}_K$. Such an ideal is necessarily a free $\mathcal{O}_K$-module of rank 1.

Lemma 5.5. *Let* $\mathcal{M}$ *be a locally free shtuka on* $S \otimes \mathcal{O}_K$ *such that* $\mathcal{M}(S/\mathfrak{m} \otimes K)$ *is nilpotent. Let* $I \subset \mathcal{O}_K$ *be an open ideal.*

(1) $I\mathcal{M}$ *is a locally free shtuka which restricts to a nilpotent shtuka on* $S/\mathfrak{m} \otimes K$.
(2) *If the regulator is defined for* $\mathcal{M}$ *then it is defined for* $I\mathcal{M}$.

Proof. (1) Clear. (2) Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

We know that $j(M_0) \subset i(M_0)$. As a consequence $j(IM_0) \subset i(I^q M_0)$ which implies that IM_0 is invariant under $i^{-1}j$. □

Lemma 5.6. *Let* $I \subset \mathcal{O}_K$ *be an open ideal and let* $\mathcal{M}$ *be a locally free shtuka on* $S \otimes \mathcal{O}_K$. *If* $\mathcal{M}(S/\mathfrak{m} \otimes K)$ *is nilpotent then the short exact sequence of shtukas*

$$0 \to I\mathcal{M} \to \mathcal{M} \to \mathcal{M}/I \to 0$$

induces a long exact sequence of cohomology

$$0 \to \mathrm{H}^0(\mathcal{M}/I) \xrightarrow{\delta} \mathrm{H}^1(I\mathcal{M}) \longrightarrow \mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/I) \to 0. \tag{5.1}$$

The exact sequence (5.1) will play an important role in our theory. It will mainly appear as the sequence (10.1) for elliptic shtukas on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$.

Proof of Lemma 5.6. By Lemma 5.5 $I\mathcal{M}$ is a locally free shtuka on $S \otimes \mathcal{O}_K$ whose restriction to $S/\mathfrak{m} \otimes K$ is nilpotent. Hence $\mathrm{H}^0(I\mathcal{M})$ and $\mathrm{H}^0(\mathcal{M})$ vanish by Lemma 3.1. $\square$

If a shtuka $\mathcal{M}$ on $S \otimes \mathcal{O}_K$ is linear then $\mathrm{R}\Gamma(\mathcal{M})$ is represented by a complex with an $\mathcal{O}_K$-linear differential. As a consequence $\mathrm{R}\Gamma(\mathcal{M})$ carries a natural action of $\mathcal{O}_K$.

Lemma 5.7. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes \mathcal{O}_K$ such that $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent. Let $I \subset \mathcal{O}_K$ be an open ideal. If $\mathcal{M}$ is linear then the following are equivalent:*

(1) $I \cdot \mathrm{H}^1(\mathcal{M}) = 0$.
(2) *The map* $\mathrm{H}^0(\mathcal{M}/I) \xrightarrow{\delta} \mathrm{H}^1(I\mathcal{M})$ *in* (5.1) *is an isomorphism.*
(3) *The map* $\mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/I)$ *in* (5.1) *is an isomorphism.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \overset{i}{\underset{0}{\rightrightarrows}} M_1.$$

By definition $\mathrm{H}^1(\mathcal{M}) = M_1/i(M_0)$. Hence the following are equivalent:

(1) $I \cdot \mathrm{H}^1(\mathcal{M}) = 0$.
(1′) $IM_1 \subset i(M_0)$.
(1″) The natural map $\mathrm{H}^1(I\mathcal{M}) \to \mathrm{H}^1(\mathcal{M})$ is zero.

By Lemma 5.6 either of the conditions (2) or (3) is equivalent to (1″). $\square$

Lemma 5.8. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes \mathcal{O}_K$ such that $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent. Suppose that the regulator is defined for $\mathcal{M}$. Let $I \subset \mathcal{O}_K$ be an open ideal. The following are equivalent:*

(1) $I \cdot \mathrm{H}^1(\nabla\mathcal{M}) = 0$.
(2) *The map* $\mathrm{H}^0(\mathcal{M}/I) \xrightarrow{\delta} \mathrm{H}^1(I\mathcal{M})$ *in* (5.1) *is an isomorphism.*
(3) *The map* $\mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/I)$ *in* (5.1) *is an isomorphism.*

Proof. Lemma 5.5 tells that the regulator is defined for the shtuka $I\mathcal{M}$ as well. By Lemma 5.3 regulators are natural. We thus get a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathrm{a}}(I\mathcal{M}) \arrow[r] \arrow[d, "\rho_{I\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}/I) \arrow[r] \arrow[d, "\rho_{\mathcal{M}/I}"] & 0 \\
0 \arrow[r] & \Gamma_{\mathrm{a}}(\nabla I\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}/I) \arrow[r] & 0
\end{tikzcd}$$

where $\rho_{\mathcal{M}/I}$ is induced by $\rho_{\mathcal{M}}$. The regulators $\rho_{\mathcal{M}}$ and $\rho_{I\mathcal{M}}$ are isomorphisms by Lemma 5.3. As a consequence $\rho_{\mathcal{M}/I}$ is an isomorphism. Taking H^1 of the diagram above we conclude that the following are equivalent:

(3) The reduction map $\mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/I)$ is an isomorphism.

(3′) The reduction map $\mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M}/I)$ is an isomorphism.

According to Lemma 5.7 the condition (1) is equivalent to (3′). Hence the condition (1) is equivalent to (3). The conditions (2) and (3) are equivalent by Lemma 5.6. □

Proposition 5.9. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes \mathcal{O}_K$ such that $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent. Let $I \subset \mathcal{O}_K$ be an open ideal. Assume that*

(1) $I \cdot \mathrm{H}^1(\nabla\mathcal{M}) = 0$,

(2) *$\mathcal{M}/I$ is linear.*

Then the following holds:

(1) *The regulator is defined for $\mathcal{M}$.*

(2) *The map $\mathrm{H}^0(\mathcal{M}/I) \xrightarrow{\delta} \mathrm{H}^1(I\mathcal{M})$ in (5.1) is an isomorphism.*

(3) *The map $\mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/I)$ in (5.1) is an isomorphism.*

Proof. Suppse that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Now $j(M_0) \subset IM_1$ by assumption (2) and $IM_1 \subset i(M_0)$ by assumption (1). Hence M_0 is preserved by $i^{-1}j$. We conclude that the regulator is defined for $\mathcal{M}$. In view of this fact the results (2) and (3) follow from the assumption (1) by Lemma 5.8. □

Proposition 5.10. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes \mathcal{O}_K$ such that $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent. Let $I \subset \mathcal{O}_K$ be an open ideal. Assume that $\mathcal{M}/I$ is linear. If the regulator is defined for $\mathcal{M}$ then the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\text{red.}"] \arrow[d, "\rho_{\mathcal{M}}"'] & \mathrm{H}^1(\mathcal{M}/I) \arrow[d, "1"] \\
\mathrm{H}^1(\nabla\mathcal{M}) \arrow[r, "\text{red.}"] & \mathrm{H}^1(\nabla\mathcal{M}/I).
\end{tikzcd}$$

is commutative.

Proof. The regulator is defined for $I\mathcal{M}$ by Lemma 5.5. Since the regulators are natural by Lemma 5.3 we obtain a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathrm{a}}(I\mathcal{M}) \arrow[r] \arrow[d, "\rho_{I\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}) \arrow[r] \arrow[d, "\rho_{\mathcal{M}}"'] & \Gamma_{\mathrm{a}}(\mathcal{M}/I) \arrow[r] \arrow[d, "\rho_{\mathcal{M}/I}"] & 0 \\
0 \arrow[r] & \Gamma_{\mathrm{a}}(\nabla I\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}) \arrow[r] & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}/I) \arrow[r] & 0
\end{tikzcd}$$

where $\rho_{\mathcal{M}/I}$ is the morphism induced by $\rho_{\mathcal{M}}$. The regulator $\rho_{\mathcal{M}}$ is given by the identity map in degree 1. So the same is true for $\rho_{\mathcal{M}/I}$. Taking H^1 of the diagram above we get the result. □

Proposition 5.11. *Let $\mathcal{M}$ be a locally free shtuka on $S \otimes \mathcal{O}_K$ such that $\mathcal{M}(S/\mathfrak{m} \otimes K)$ is nilpotent. Let $I \subset \mathcal{O}_K$ be an open ideal. Assume that*

(1) $I \cdot \mathrm{H}^1(\nabla\mathcal{M}) = 0$,
(2) $\mathcal{M}/I^2$ *is linear.*

Then the regulator is defined for $I\mathcal{M}$ and the square

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{M}/I) & \xrightarrow{\delta} & \mathrm{H}^1(I\mathcal{M}) \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \rho_{I\mathcal{M}}} \\ \mathrm{H}^0(\nabla\mathcal{M}/I) & \xrightarrow{\delta} & \mathrm{H}^1(\nabla I\mathcal{M}) \end{array}$$

is commutative. Here the maps δ are the boundary homomorphisms of the long exact sequence (5.1).

Proof. The regulator is defined for $\mathcal{M}$ by Proposition 5.9. Hence it is defined for $I\mathcal{M}$ by Lemma 5.5. We then have a diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_{\mathrm{a}}(I\mathcal{M}) & \longrightarrow & \Gamma_{\mathrm{a}}(\mathcal{M}) & \longrightarrow & \Gamma_{\mathrm{a}}(\mathcal{M}/I) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_{I\mathcal{M}}} & & \downarrow{\scriptstyle \rho_{\mathcal{M}}} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & \Gamma_{\mathrm{a}}(\nabla I\mathcal{M}) & \longrightarrow & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}) & \longrightarrow & \Gamma_{\mathrm{a}}(\nabla\mathcal{M}/I) & \longrightarrow & 0 \end{array} \tag{5.2}$$

with short exact rows. We will show that the right square commutes. The result then follows since (5.2) induces a morphism of long exact cohomology sequences.

The right square of (5.2) commutes in degree 1 since $\rho_{\mathcal{M}}$ is given by the identity map in degree 1. We thus need to show commutativity in degree 0. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

To show commutativity of (5.2) is is enough to prove that $i^{-1}j(M_0) \subset IM_0$. By assumption (2) we have $j(M_0) \subset I^2M_1$. Assumption (1) implies that $I^2M_1 \subset i(IM_0)$. Hence $i^{-1}j(M_0) \subset IM_0$. □

6. Elliptic shtukas

Starting from this section we work over the ring $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$.

Definition 6.1. Throughout the rest of the chapter we fix an open ideal $\mathfrak{f} \subset \mathcal{O}_K$. We call $\mathfrak{f}$ the *conductor*.

Definition 6.2. Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$. We say that $\mathcal{M}$ is an *elliptic shtuka* of conductor $\mathfrak{f}$ if the following holds:

(E0) $\mathcal{M}$ is locally free.
(E1) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent.
(E2) $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent.

(E3) $\mathfrak{m} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{f} \cdot \mathrm{H}^1(\nabla\mathcal{M})$.
(E4) $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{f})$ is linear.

As the conductor $\mathfrak{f}$ is fixed throughout the chapter, in the following we speak simply of elliptic shtukas instead of elliptic shtukas of conductor $\mathfrak{f}$.

Example. Let $\mathcal{O}_F = \mathbb{F}_q[[z]]$, $\mathcal{O}_K = \mathbb{F}_q[[\zeta]]$. We have $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K = \mathbb{F}_q[[z,\zeta]]$. The endomorphism τ of this ring preserves $\mathbb{F}_q[[z]]$ and sends ζ to ζ^q. Our conductor $\mathfrak{f} \subset \mathbb{F}_q[[\zeta]]$ will be the ideal generated by ζ.

Consider the shtuka

$$\mathcal{M} = \Big[\ \mathbb{F}_q[[z,\zeta]] \overset{\zeta - z}{\underset{z\zeta\cdot\tau}{\rightrightarrows}} \mathbb{F}_q[[z,\zeta]]\ \Big].$$

In fact $\mathcal{M}$ is (a part of) a model of the Carlitz module. We claim that $\mathcal{M}$ is an elliptic shtuka of conductor $\mathfrak{f}$. Indeed $\mathcal{M}$ is locally free by construction. Furthermore

$$\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K) = \Big[\ \mathbb{F}_q(\!(\zeta)\!) \overset{\zeta}{\underset{0}{\rightrightarrows}} \mathbb{F}_q(\!(\zeta)\!)\ \Big]$$
$$\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K) = \Big[\ \mathbb{F}_q(\!(z)\!) \overset{-z}{\underset{0}{\rightrightarrows}} \mathbb{F}_q(\!(z)\!)\ \Big].$$

Hence the restrictions of $\mathcal{M}$ to $\mathcal{O}_F/\mathfrak{m}\otimes K$ and $F\otimes\mathcal{O}_K/\mathfrak{m}_K$ are nilpotent. The cohomology of $\nabla\mathcal{M}$ is easy to compute:

$$\mathrm{H}^1(\nabla\mathcal{M}) = \mathbb{F}_q[[z,\zeta]]/(\zeta - z) = \mathbb{F}_q[[\zeta]].$$

The element $z \in \mathbb{F}_q[[z]]$ acts on $\mathrm{H}^1(\nabla\mathcal{M})$ by multiplication by ζ. So $\mathfrak{m} \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{f} \cdot \mathrm{H}^1(\nabla\mathcal{M})$. Finally the linearity condition holds since

$$\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{f}) = \Big[\ \mathbb{F}_q[[z]] \overset{-z}{\underset{0}{\rightrightarrows}} \mathbb{F}_q[[z]]\ \Big].$$

Proposition 6.3. *If a shtuka $\mathcal{M}$ is elliptic then so is $\nabla\mathcal{M}$.*

Proof. Indeed the functor ∇ commutes with arbitrary restrictions and preserves nilpotence so that $\nabla\mathcal{M}$ satisfies the conditions (E1) and (E2) of Definition 6.2. The condition (E3) is tautologically satisfied and (E4) follows since the shtuka $\nabla\mathcal{M}$ is already linear. □

Theorem 6.4. *If $\mathcal{M}$ is an elliptic shtuka then the following holds:*

(1) $\mathrm{H}^0(\mathcal{M}) = 0$
(2) $\mathrm{H}^1(\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*

Proof. Indeed $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent by (E1) so the result follows from Theorem 4.2. □

7. Twists and quotients

Definition 7.1. Let $I \subset \mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ be a τ-invariant ideal. Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

We define

$$I\mathcal{M} = \Big[IM_0 \overset{i}{\underset{j}{\rightrightarrows}} IM_1 \Big].$$

We call $I\mathcal{M}$ the *twist* of $\mathcal{M}$ by I. The shtuka $I\mathcal{M}$ comes equipped with a natural embedding $I\mathcal{M} \hookrightarrow \mathcal{M}$. We denote the quotient $\mathcal{M}/I$.

Warning. Given invariant ideals $I, J \subset \mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ we denote $I\mathcal{M}/J$ the quotient $(I\mathcal{M})/(JI\mathcal{M})$. In other words we first do the twist by I and then take the quotient by J.

We will use the following invariant ideals:

- $\mathfrak{m}^n \widehat{\otimes} \mathcal{O}_K$ for $n \geqslant 0$.
- $\mathcal{O}_F \widehat{\otimes} I$ for $I \subset \mathcal{O}_K$ an open ideal.
- $\mathfrak{m}^n \widehat{\otimes} \mathcal{O}_K + \mathcal{O}_F \widehat{\otimes} I$ for $\mathfrak{m}^n$ and I as above.
- $\mathfrak{m}^n \widehat{\otimes} I$ for $\mathfrak{m}^n$ and I as above.

To simplify the notation we will write $\mathfrak{m}^n\mathcal{M}$ instead of $(\mathfrak{m}^n \widehat{\otimes} \mathcal{O}_K)\mathcal{M}$. The same applies to I, $\mathfrak{m}^n + I$ and the quotients by the ideals of these three types. The twist of $\mathcal{M}$ by the last ideal will be denoted $\mathfrak{m}^n I\mathcal{M}$ and the quotient will be denoted $\mathcal{M}/\mathfrak{m}^n I$.

Lemma 7.2. *Let $n \geqslant 0$ and let $I \subset \mathcal{O}_K$ be an open ideal. If $\mathcal{M}$ is a shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ then*

$$\mathcal{M}/\mathfrak{m}^n = \mathcal{M}(\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K),$$
$$\mathcal{M}/I = \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/I),$$
$$\mathcal{M}/(\mathfrak{m}^n + I) = \mathcal{M}(\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K/I).$$

Proof. By Lemma 2.3 we have $\mathcal{O}_F/\mathfrak{m}^n \otimes_{\mathcal{O}_F} (\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K) = \mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$ so the first formula holds. In a similar way Lemma 2.2 implies the second formula. The last formula follows from the first two. $\square$

Proposition 7.3. *If $\mathcal{M}$ is an elliptic shtuka then so is $\mathfrak{m}\mathcal{M}$.*

Proof. Indeed the shtukas $\mathcal{M}$ and $\mathfrak{m}\mathcal{M}$ are isomorphic. $\square$

Lemma 7.4. *Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. For every open ideal $I \subset \mathcal{O}_K$ the following holds:*

(1) *$\mathcal{M}(F \otimes \mathcal{O}_K/I)$ is nilpotent,*
(2) *$\mathrm{H}^0(\mathcal{M}/I) = 0$.*

Proof. (1) It is enough to assume that K is a single local field. In this case $I = \mathfrak{m}_K^n$ for some $n \geqslant 0$. The ring $F \otimes \mathcal{O}_K/\mathfrak{m}_K^n$ is noetherian and complete with respect to the τ-invariant ideal $F \otimes \mathfrak{m}_K/\mathfrak{m}_K^n$. Since $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent Proposition 1.9.4 implies that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K^n)$ is nilpotent.

(2) Lemma 7.2 shows that $\mathcal{M}/I = \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/I)$. The natural map

$$\mathrm{H}^0(\mathcal{O}_F \otimes \mathcal{O}_K/I, \mathcal{M}) \to \mathrm{H}^0(F \otimes \mathcal{O}_K/I, \mathcal{M})$$

is injective since $\mathcal{M}$ is locally free. However the shtuka $\mathcal{M}(F \otimes \mathcal{O}_K/I)$ is nilpotent by (1). So Proposition 1.9.3 shows that $\mathrm{H}^0(F \otimes \mathcal{O}_K/I, \mathcal{M}) = 0$. □

Lemma 7.5. *Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$. If $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then $(\mathfrak{f}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent.*

Proof. We have a short exact sequence of shtukas

$$0 \to \mathfrak{f}\mathcal{M}/\mathfrak{m}_K\mathfrak{f}\mathcal{M} \to \mathcal{M}/\mathfrak{m}_K\mathfrak{f}\mathcal{M} \to \mathcal{M}/\mathfrak{m}_K\mathcal{M} \to 0.$$

Using Lemma 7.2 we rewrite it as follows:

$$0 \to (\mathfrak{f}\mathcal{M})(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{m}_K\mathfrak{f}) \to \mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to 0.$$

Localizing at a uniformizer of $\mathcal{O}_F$ we get a short exact sequence

$$0 \to (\mathfrak{f}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K\mathfrak{f}) \to \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K) \to 0.$$

The third shtuka is nilpotent by assumption while the second shtuka is nilpotent by Lemma 7.4. Hence the first shtuka is nilpotent. □

Proposition 7.6. *Let $\mathcal{M}$ be a shtuka on $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$. For every $n \geqslant 0$ the shtuka $(\mathfrak{f}^n\mathcal{M})/\mathfrak{f}^n$ is linear.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

We need to prove that $j(\mathfrak{f}^n M_0) \subset \mathfrak{f}^{2n} M_1$. The endomorphism τ of $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ sends $\mathfrak{f}$ to $\mathfrak{f}^q$. Since j is τ-linear it follows that

$$j(\mathfrak{f}^n M_0) \subset \mathfrak{f}^{nq} M_1.$$

The result follows since $q > 1$. □

Proposition 7.7. *If $\mathcal{M}$ is an elliptic shtuka then so is $\mathfrak{f}\mathcal{M}$.*

Proof. Let us verify the conditions of Definition 6.2 for $\mathfrak{f}\mathcal{M}$.

(E0) Lemma 2.2 shows that $\mathcal{O}_F \widehat{\otimes} \mathfrak{f}$ is a free $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$-module of rank 1. Hence $\mathfrak{f}\mathcal{M}$ is a locally free shtuka.

(E1) The shtukas $\mathcal{M}$ and $\mathfrak{f}\mathcal{M}$ coincide on $(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K) \otimes_{\mathcal{O}_K} K$ so that $(\mathfrak{f}\mathcal{M})(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent.

(E2) Follows by Lemma 7.5.

(E3) Consider the short exact sequence of shtukas

$$0 \to \nabla\mathfrak{f}\mathcal{M} \to \nabla\mathcal{M} \to \nabla\mathcal{M}/\mathfrak{f} \to 0.$$

The module $\mathrm{H}^0(\nabla\mathcal{M}/\mathfrak{f})$ vanishes by Lemma 7.4. Taking the cohomology sequence we conclude that the natural map $\mathrm{H}^1(\nabla\mathfrak{f}\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ is injective. This map is $\mathcal{O}_F \otimes \mathcal{O}_K$-linear by construction. Now the image of $\mathrm{H}^1(\nabla\mathfrak{f}\mathcal{M})$ in $\mathrm{H}^1(\nabla\mathcal{M})$ is $\mathfrak{f}\cdot\mathrm{H}^1(\nabla\mathcal{M})$ by definition. Therefore

$$\mathfrak{m}\cdot\mathrm{H}^1(\nabla\mathfrak{f}\mathcal{M}) = \mathfrak{m}\mathfrak{f}\cdot\mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{f}\mathfrak{m}\cdot\mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{f}\mathfrak{f}\cdot\mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{f}\cdot\mathrm{H}^1(\nabla\mathfrak{f}\mathcal{M}).$$

(E4) Indeed the shtuka $(\mathfrak{f}\mathcal{M})/\mathfrak{f}$ is linear according to Proposition 7.6. □

8. Filtration on cohomology

An elliptic shtuka $\mathcal{M}$ carries a natural filtration by elliptic subshtukas $\mathfrak{f}^n\mathcal{M}$. In this section we would like to describe the induced filtration on $\mathrm{H}^1(\mathcal{M})$. If the elliptic shtuka $\mathcal{M}$ is linear then

$$\mathfrak{m}\cdot\mathrm{H}^1(\mathcal{M}) = \mathfrak{f}\cdot\mathrm{H}^1(\mathcal{M}) = \mathrm{H}^1(\mathfrak{f}\mathcal{M})$$

by the condition (E3). As a consequence the filtration on $\mathrm{H}^1(\mathcal{M})$ induced by $\mathfrak{f}^n\mathcal{M}$ is the filtration by powers of $\mathfrak{m}$. Our goal is to prove that the same is true without the linearity assumption.

Lemma 8.1. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$. The shtuka $\mathcal{N} = \mathcal{M}/\mathfrak{m}^n$ has the following properties:*

(1) *$\mathcal{N}$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$-shtuka.*
(2) *$\mathcal{N}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent.*
(3) *$\mathfrak{f}^n \cdot \mathrm{H}^1(\nabla\mathcal{N}) = 0$.*

Proof. By Lemma 7.2 we have $\mathcal{N} = \mathcal{M}(\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K)$. So (1) and (2) follow since $\mathcal{M}$ is an elliptic shtuka.

The natural map $\mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{N})$ is surjective and $\mathcal{O}_F \otimes \mathcal{O}_K$-linear. According to the condition (E3) we have

$$\mathfrak{f}\cdot\mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{m}\cdot\mathrm{H}^1(\nabla\mathcal{M}).$$

As a consequence

$$\mathfrak{f}^n\cdot\mathrm{H}^1(\nabla\mathcal{N}) = \mathfrak{m}^n\cdot\mathrm{H}^1(\nabla\mathcal{N}).$$

However $\mathfrak{m}^n$ acts on this module by zero since $\nabla\mathcal{N}$ is a shtuka on $\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$. We thus get (3). □

Proposition 8.2. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$.*

(1) *The natural sequence*

$$0 \to \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \to 0$$

is exact.

(2) *The image of $\mathrm{H}^1(\mathfrak{m}^n\mathcal{M})$ in $\mathrm{H}^1(\mathcal{M})$ is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$.*

Proof. (1) By Lemma 8.1 the quotient $\mathcal{M}/\mathfrak{m}^n$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$-shtuka whose restriction to $\mathcal{O}_F/\mathfrak{m} \otimes K$ is nilpotent. Hence $\mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n) = 0$ by Lemma 3.1. Taking the cohomology sequence of the short exact sequence $0 \to \mathfrak{m}^n\mathcal{M} \to \mathcal{M} \to \mathcal{M}/\mathfrak{m}^n \to 0$ we get the result. (2) is clear. $\square$

Lemma 8.3. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$. If $\mathcal{M}/\mathfrak{f}^n$ is linear then the shtuka $\mathcal{N} = \mathcal{M}/\mathfrak{m}^n$ has the following properties:*

(1) *The artinian regulator is defined for $\mathcal{N}$ (Definition 5.2).*
(2) *The reduction map $\mathrm{H}^1(\mathcal{N}) \to \mathrm{H}^1(\mathcal{N}/\mathfrak{f}^n)$ is an isomorphism.*

Proof. We claim that the shtuka $\mathcal{N}$ has the following properties:

(i) $\mathcal{N}$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$-module shtuka whose restriction to $\mathcal{O}_F/\mathfrak{m} \otimes K$ is niltpotent.
(ii) $\mathfrak{f}^n \cdot \mathrm{H}^1(\nabla\mathcal{N}) = 0$.
(iii) $\mathcal{N}/\mathfrak{f}^n$ is linear.

Indeed Lemma 8.1 implies (i) and (ii) while (iii) follows since the quotient $\mathcal{M}/\mathfrak{f}^n$ is linear by assumption. We then apply Proposition 5.9 to $\mathcal{N}$ with $S = \mathcal{O}_F/\mathfrak{m}^n$ and $I = \mathfrak{f}^n$ and conclude that $\mathcal{N}$ has the properties (1) and (2). $\square$

Lemma 8.4. *If $\mathcal{M}$ is an elliptic shtuka then the reduction map $\mathrm{H}^1(\mathcal{M}/\mathfrak{f}) \to \mathrm{H}^1(\mathcal{M}/(\mathfrak{m}+\mathfrak{f}))$ is an isomorphism.*

Proof. The short exact sequence of shtukas $0 \to (\mathfrak{m}\mathcal{M})/\mathfrak{f} \to \mathcal{M}/\mathfrak{f} \to \mathcal{M}/(\mathfrak{m} + \mathfrak{f}) \to 0$ induces an exact sequence of cohomology

$$\mathrm{H}^1(\mathfrak{m}\mathcal{M}/\mathfrak{f}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{f}) \to \mathrm{H}^1(\mathcal{M}/(\mathfrak{m}+\mathfrak{f})) \to 0.$$

So to prove the lemma it is enough to demonstrate that the first map in this sequence is zero.

The shtukas $\mathcal{M}/\mathfrak{f}$ and $(\mathfrak{m}\mathcal{M})/\mathfrak{f}$ are linear since $\mathcal{M}$ is elliptic. So we can assume without loss of generality that $\mathcal{M}$ is itself linear. The natural map $\mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{f})$ is $\mathcal{O}_F \otimes \mathcal{O}_K$-linear and surjective. Furthermore $\mathfrak{m} \cdot \mathrm{H}^1(\mathcal{M}) = \mathfrak{f} \cdot \mathrm{H}^1(\mathcal{M})$ by definition of $\mathfrak{f}$. As a consequence

$$\mathfrak{m} \cdot \mathrm{H}^1(\mathcal{M}/\mathfrak{f}) = \mathfrak{f} \cdot \mathrm{H}^1(\mathcal{M}/\mathfrak{f}).$$

However $\mathcal{M}/\mathfrak{f}$ is a linear shtuka on $\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{f}$ so $\mathfrak{f}$ acts on $\mathrm{H}^1(\mathcal{M}/\mathfrak{f})$ by zero. Thus $\mathfrak{m} \cdot \mathrm{H}^1(\mathcal{M}/\mathfrak{f}) = 0$ which implies that the natural map $\mathrm{H}^1(\mathfrak{m}\mathcal{M}/\mathfrak{f}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{f})$ is zero. $\square$

Theorem 8.5. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$.*

(1) *The natural sequence*

$$0 \to \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n) \to 0$$

is exact.
(2) *The image of $\mathrm{H}^1(\mathfrak{f}^n\mathcal{M})$ in $\mathrm{H}^1(\mathcal{M})$ is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$.*

So as we claimed at the beginning of this section the filtration on $\mathrm{H}^1(\mathcal{M})$ induced by the subshtukas $\mathfrak{f}^n\mathcal{M}$ is the filtration by powers of $\mathfrak{m}$.

Proof of Theorem 8.5. (1) Lemma 7.4 claims that $\mathrm{H}^0(\mathcal{M}/\mathfrak{f}^n) = 0$. So the natural sequence above is exact. (2) By Proposition 7.7 the shtuka $\mathfrak{f}^n\mathcal{M}$ is elliptic. It is thus enough to treat the case $n = 1$. Consider the natural commutative square

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathcal{M}/\mathfrak{f}) \arrow[d] \\ \mathrm{H}^1(\mathcal{M}/\mathfrak{m}) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}+\mathfrak{f}) \end{tikzcd}$$

We just demonstrated that the top arrow in this square is surjective with kernel $\mathrm{H}^1(\mathfrak{f}\mathcal{M})$. According to Proposition 8.2 the left arrow is surjective with kernel $\mathfrak{m}\mathrm{H}^1(\mathcal{M})$. The right arrow is an isomorphism by Lemma 8.4. Since $\mathcal{M}/\mathfrak{f}$ is linear by (E4) Lemma 8.3 shows that the bottom arrow is an isomorphism. So the result follows. ☐

9. Overview of regulators

Now as our discussion of elliptic shtukas has gained some substance we can give an overview of the regulator theory which will follow. Let $\mathcal{M}$ be an elliptic shtuka. If $\mathcal{M}$ is linear then one tautologically has a natural isomorphism

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\rho} \mathrm{H}^1(\nabla\mathcal{M})$$

induced by the identity of the shtukas $\mathcal{M}$ and $\nabla\mathcal{M}$, the *regulator* of $\mathcal{M}$. We would like to extend it to all elliptic shtukas $\mathcal{M}$. A rough idea is to approximate $\mathcal{M}$ with linear pieces.

Definition. A natural transformation

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\rho} \mathrm{H}^1(\nabla\mathcal{M})$$

of functors on the category of elliptic shtukas is called a *regulator* if for every $\mathcal{M}$ such that $\mathcal{M}/\mathfrak{f}^{2n}$ is linear the diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^n) \end{tikzcd}$$

is commutative.

Note that we demand the quotient $\mathcal{M}/\mathfrak{f}^{2n}$ to be linear, not just $\mathcal{M}/\mathfrak{f}^n$. Even though the square above makes sense if merely $\mathcal{M}/\mathfrak{f}^n$ is linear the regulator will fail to exist if one demands all such squares to commute.

By naturality a regulator ρ will send the submodule $\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \subset \mathrm{H}^1(\mathcal{M})$ to $\mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}) \subset \mathrm{H}^1(\nabla\mathcal{M})$. According to Proposition 7.6 the subquotients $(\mathfrak{f}^{2n}\mathcal{M})/\mathfrak{f}^{2n} = \mathfrak{f}^{2n}\mathcal{M}/\mathfrak{f}^{4n}\mathcal{M}$ are linear for all $n \geqslant 0$. Hence the regulator is unique. It also follows that ρ is an isomorphism. However the existence of

ρ is a different question altogether. Its construction occupies the rest of the chapter.

In Sections 10 and 11 we present some auxillary results on cohomology of elliptic shtukas and their subquotients. These sections are of a technical nature. The level of technicality reaches its high point in Section 12 where we study the tautological regulators. They are the basic building blocks for the regulator on $\mathrm{H}^1(\mathcal{M})$. Already in Section 13 the statements and proofs become much more natural. The construction of the regulator on $\mathrm{H}^1(\mathcal{M})$ in Section 14 is actually quite simple. Besides this construction, the main results of Section 14 are Theorem 14.5 which gives a criterion for an $\mathcal{O}_F$-linear map $f\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ to coincide with the regulator, and Theorem 14.6 which serves as a link with the trace formula of Chapter 6.

10. Cohomology of subquotients

In this section we present several technical propositions which will be used in the construction of the regulator map. First we use Theorem 8.5 to derive two exact sequences of cohomology modules.

Proposition 10.1. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the short exact sequence of shtukas*

$$0 \to \mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n \to \mathcal{M}/\mathfrak{m}^n \to \mathcal{M}/(\mathfrak{m}^n + \mathfrak{f}^n) \to 0$$

induces an exact sequence of cohomology modules

$$(10.1) \quad 0 \to \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{f}^n)) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \xrightarrow{0} \\ \longrightarrow \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{f}^n)) \to 0.$$

The middle map in this sequence is zero and the adjacent maps are isomorphisms.

Proof. According to Lemma 8.1 the shtuka $\mathcal{M}/\mathfrak{m}^n$ is a locally free $\mathcal{O}_F/\mathfrak{m}^n \otimes \mathcal{O}_K$-module shtuka whose restriction to $\mathcal{O}_F/\mathfrak{m} \otimes K$ is nilpotent. So Lemma 3.1 implies that $\mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n) = 0$. Therefore the sequence (10.1) is exact. To prove the result it is enough to show that the middle map in this sequence is zero. This map fits into a natural commutative square

$$\begin{array}{ccc} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) & \dashrightarrow & \mathrm{H}^1(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) & \longrightarrow & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \end{array}$$

By Proposition 8.2 the vertical maps in this square are surjections and the kernel of the right map is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$. However Theorem 8.5 shows that the image of the top map is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$ whence the composition of the top and the right maps is zero. Since the left map is surjective we conclude that the bottom map is zero. ∎

Proposition 10.2. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the short exact sequence of shtukas*

$$0 \to \mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n \to \mathcal{M}/\mathfrak{f}^n \to \mathcal{M}/(\mathfrak{m}^n + \mathfrak{f}^n) \to 0$$

induces an exact sequence of cohomology modules

$$\begin{aligned}(10.2)\quad 0 \to \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{f}^n)) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \xrightarrow{0} \\ \longrightarrow \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{f}^n)) \to 0.\end{aligned}$$

The middle map in this sequence is zero and the adjacent maps are isomorphisms.

Proof. The cohomology module $\mathrm{H}^0(\mathcal{M}/\mathfrak{f}^n)$ vanishes according to Lemma 7.4. Thus the sequence (10.2) is exact. To prove the proposition it is enough to show that the middle map in (10.2) is zero. This map fits into a natural commutative square

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{H}^1(\mathcal{M}) \arrow[d, dashed] \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n). \end{tikzcd}$$

Theorem 8.5 shows that the vertical maps are surjective and that the kernel of the right map is $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$. So the composition of the top and the right maps is zero. As the left map is surjective we conclude that the bottom map is zero. $\square$

Next we study the boundary homomorphisms δ of the sequences (10.1) and (10.2).

Proposition 10.3. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the natural diagram*

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^n + \mathfrak{f}^n)) \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \arrow[d] \\ \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n) \end{tikzcd}$$

is anticommutative. *Here the vertical δ is the boundary homomorphism of* (10.1) *while the horizontal δ is the boundary homomorphism of* (10.2).

Proof. To improve legibility we will write I for $\mathfrak{f}^n$ and J for $\mathfrak{m}^n$. Since the shtuka $\mathcal{M}$ is locally free Lemma 2.4 implies that the natural sequence

$$0 \to I\mathcal{M}/J \oplus J\mathcal{M}/I \to \mathcal{M}/IJ \to \mathcal{M}/(I+J) \to 0$$

is exact. Now consider the natural diagram

$$\begin{tikzcd}
0 \arrow[r] & I\mathcal{M}/J \oplus J\mathcal{M}/I \arrow[r] \arrow[d, "1"] & \mathcal{M}/IJ \arrow[r] \arrow[d, "{(\mathrm{red.}, \mathrm{red.})}"] & \mathcal{M}/(I+J) \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & I\mathcal{M}/J \oplus J\mathcal{M}/I \arrow[r] & \mathcal{M}/I \oplus \mathcal{M}/J \arrow[r] & (\mathcal{M}/I+J)^{\oplus 2} \arrow[r] & 0
\end{tikzcd}$$

Here Δ means the diagonal map. Observe that the lower row is the direct sum of the short exact sequences

$$0 \to J\mathcal{M}/I \to \mathcal{M}/J \to \mathcal{M}/(I+J) \to 0,$$
$$0 \to I\mathcal{M}/J \to \mathcal{M}/I \to \mathcal{M}/(I+J) \to 0$$

which give rise to the cohomology sequences (10.1) and (10.2) respectively. The diagram above is clearly commutative. So it induces a morphism of long exact sequences. A part of it looks like this:

$$\begin{tikzcd}
\mathrm{H}^0(\mathcal{M}/(I+J)) \arrow[r] \arrow[d, "\Delta"] & \mathrm{H}^1(I\mathcal{M}/J) \oplus \mathrm{H}^1(J\mathcal{M}/I) \arrow[r] \arrow[d, "1"] & \mathrm{H}^1(\mathcal{M}/IJ) \arrow[d, "{(\mathrm{red.}, \mathrm{red.})}"] \\
\mathrm{H}^0(\mathcal{M}/(I+J))^{\oplus 2} \arrow[r, "\delta \oplus \delta"] & \mathrm{H}^1(I\mathcal{M}/J) \oplus \mathrm{H}^1(J\mathcal{M}/I) \arrow[r] & \mathrm{H}^1(\mathcal{M}/J) \oplus \mathrm{H}^1(\mathcal{M}/I)
\end{tikzcd}$$

As a consequence the boundary homomorphism

$$\mathrm{H}^0(\mathcal{M}/(I+J)) \to \mathrm{H}^1(I\mathcal{M}/J) \oplus \mathrm{H}^1(J\mathcal{M}/I)$$

in the top row coincides with (δ, δ). Since the composition of the adjacent homomorphisms

$$\mathrm{H}^0(\mathcal{M}/(I+J)) \xrightarrow{(\delta,\delta)} \mathrm{H}^1(I\mathcal{M}/J) \oplus \mathrm{H}^1(J\mathcal{M}/I) \longrightarrow \mathrm{H}^1(\mathcal{M}/IJ)$$

is zero we get the result. □

Proposition 10.4. *Let $n \geqslant 0$. If $\mathcal{M}$ is an elliptic shtuka then the natural maps*

$$\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n),$$
$$\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n)$$

are injective.

Proof. Consider the short exact sequence $0 \to \mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n \to \mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n \to \mathcal{M}/\mathfrak{f}^n \to 0$. By Lemma 7.4 the module $\mathrm{H}^0(\mathcal{M}/\mathfrak{f}^n)$ vanishes. Hence the natural map $\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n)$ is injective. In a similar way Lemma 3.1 implies that the map $\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n)$ is injective. □

11. Natural isomorphisms on cohomology

Let $\mathcal{M}$ be an elliptic shtuka. According to Theorem 8.5 the images of $\mathrm{H}^1(\mathfrak{m}\mathcal{M})$ and $\mathrm{H}^1(\mathfrak{f}\mathcal{M})$ inside $\mathrm{H}^1(\mathcal{M})$ are equal to $\mathfrak{m}\mathrm{H}^1(\mathcal{M})$.

Definition 11.1. We define the natural isomorphism

$$\gamma_{\mathcal{M}} \colon \mathrm{H}^1(\mathfrak{m}\mathcal{M}) \to \mathrm{H}^1(\mathfrak{f}\mathcal{M})$$

as the unique map which makes the triangle

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}\mathcal{M}) \arrow[rr, dashed, "\gamma_{\mathcal{M}}", "\sim"'] \arrow[dr] & & \mathrm{H}^1(\mathfrak{f}\mathcal{M}) \arrow[dl] \\
& \mathrm{H}^1(\mathcal{M}) &
\end{tikzcd}$$

commutative.

To simplify the expressions we denote $\gamma^n_{\mathcal{M}}$ the composition

$$\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \xrightarrow{\gamma_{\mathfrak{m}^{n-1}\mathcal{M}}} \mathrm{H}^1(\mathfrak{f}\mathfrak{m}^{n-1}\mathcal{M}) \xrightarrow{\gamma_{\mathfrak{f}\mathfrak{m}^{n-2}\mathcal{M}}} \ldots \xrightarrow{\gamma_{\mathfrak{f}^{n-1}\mathcal{M}}} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}).$$

We write γ^n instead of $\gamma^n_{\mathcal{M}}$ if the corresponding shtuka $\mathcal{M}$ is clear from the context.

We should note that the map $\gamma^n_{\mathcal{M}}$ will only be used in Sections 12 and 13.

Lemma 11.2. *If $n, k \geqslant 0$ are integers then the composition*

$$\mathrm{H}^1(\mathfrak{m}^{n+k}\mathcal{M}) \xrightarrow{\gamma^n_{\mathfrak{m}^k\mathcal{M}}} \mathrm{H}^1(\mathfrak{f}^n\mathfrak{m}^k\mathcal{M}) \xrightarrow{\gamma^k_{\mathfrak{f}^n\mathcal{M}}} \mathrm{H}^1(\mathfrak{f}^{n+k}\mathcal{M})$$

is equal to $\gamma^{n+k}_{\mathcal{M}}$. □

Our goal is to relate γ^n to the maps which appear in the natural sequences (10.1) and (10.2).

Lemma 11.3. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ there exists a unique natural isomorphism $\overline{\gamma}_n \colon \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \xrightarrow{\sim} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n)$ such that the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n", "\sim"'] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[d, "\text{red.}"] \\
\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \arrow[r, "\overline{\gamma}_n", "\sim"'] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n)
\end{tikzcd}$$

is commutative.

The map $\overline{\gamma}_n$ will only be used in Section 12. The same remark applies to the related map $\overline{\varepsilon}_n$ which we introduce below.

Proof of Lemma 11.3. By Theorem 8.5 the left reduction map is surjective with kernel $\mathfrak{m}^n\mathrm{H}^1(\mathfrak{m}^n\mathcal{M})$. Similarly Proposition 8.2 shows that the right reduction map is surjective with kernel $\mathfrak{m}^n\mathrm{H}^1(\mathfrak{f}^n\mathcal{M})$. Since γ^n is an isomorphism the result follows. □

Proposition 11.4. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ the maps δ of* (10.1), (10.2) *are isomorphisms and the triangle*

$$\begin{tikzcd} & \mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n + \mathfrak{f}^n) \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \arrow[rr, "\overline{\gamma}_n"] & & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \end{tikzcd}$$

is anticommutative.

Proof. The maps δ are isomorphisms by Propositions 10.1 and 10.2. Next, consider the diagram

$$\begin{tikzcd} & \mathrm{H}^0(\mathcal{M}/\mathfrak{m}^n + \mathfrak{f}^n) \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \arrow[rr, "\overline{\gamma}_n"] \arrow[dr] & & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \arrow[dl] \\ & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n) & \end{tikzcd}$$

The square in this diagram commutes by Proposition 10.3. The bottom diagonal maps are injective by Proposition 10.4. Hence it is enough to prove that the bottom triangle commutes. By definition of $\overline{\gamma}_n$ we need to show the commutativity of the triangle

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[rr, "\gamma^n"] \arrow[dr] & & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[dl] \\ & \mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n\mathfrak{f}^n) & \end{tikzcd}$$

However the two diagonal maps factor over the natural maps to $\mathrm{H}^1(\mathcal{M})$. Hence the commutativity follows from the definition of γ^n. □

Lemma 11.5. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ there exists a unique natural isomorphism $\overline{\varepsilon}_n \colon \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \xrightarrow{\sim} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^n)$ such that the square*

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[r, equal] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \arrow[r, "\overline{\varepsilon}_n", "\sim"'] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^n) \end{tikzcd}$$

is commutative.

Proof. Indeed the left reduction map is surjective with kernel $\mathfrak{m}^n\mathrm{H}^1(\mathfrak{f}^n\mathcal{M})$ by Proposition 8.2 while the right reduction map is surjective with the same kernel by Theorem 8.5. □

Lemma 11.6. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ the square*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n"] \arrow[d, "\mathrm{red.}"'] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[d, "\mathrm{red.}"] \\
\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}/\mathfrak{f}^n) \arrow[r, "\overline{\varepsilon}_n \circ \overline{\gamma}_n"] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^n)
\end{tikzcd}$$

is commutative.

Proof. Follows instantly from the definitions of $\overline{\gamma}_n$ and $\overline{\varepsilon}_n$. □

Proposition 11.7. *Let $\mathcal{M}$ be an elliptic shtuka and let $n \geqslant 0$.*

(1) *The reduction maps*

$$\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \xrightarrow{\mathrm{red.}} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{f}^n),$$
$$\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^n) \xrightarrow{\mathrm{red.}} \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{f}^n)$$

are isomorphisms.

(2) *The diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \arrow[rr, "\overline{\varepsilon}_n"] \arrow[dr, "\mathrm{red.}"'] & & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^n) \arrow[dl, "\mathrm{red.}"] \\
& \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{f}^n) &
\end{tikzcd}$$

is commutative.

Proof. (1) follows from Propositions 10.1 and 10.2. (2) Indeed the outer square in the diagram

$$\begin{tikzcd}
& \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[dl, "\mathrm{red.}"'] \arrow[dr, "\mathrm{red.}"] & \\
\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n) \arrow[rr, "\overline{\varepsilon}_n"] \arrow[dr, "\mathrm{red.}"'] & & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^n) \arrow[dl, "\mathrm{red.}"] \\
& \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{m}^n + \mathfrak{f}^n) &
\end{tikzcd}$$

commutes by definition of $\overline{\varepsilon}_n$. The top left reduction map is surjective by Proposition 8.2 So the result follows. □

12. Tautological regulators

Lemma 12.1. *Let $d \geqslant 1$. If $\mathcal{M}$ is an elliptic shtuka then the reduction map $\mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^d)$ induces an isomorphism $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \to \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^d)$.*

Proof. Indeed Theorem 8.5 states that the reduction map is surjective with kernel $\mathfrak{m}^d\mathrm{H}^1(\mathcal{M})$. □

Let $\mathcal{M}$ be an elliptic shtuka and let $d \geqslant 1$. If $\mathcal{M}/\mathfrak{f}^d$ is linear then the shtukas $\mathcal{M}/\mathfrak{f}^d$ and $\nabla\mathcal{M}/\mathfrak{f}^d$ coincide tautologically. We thus get a natural isomorphism

$$\mathrm{H}^1(\mathcal{M}/\mathfrak{f}^d) \xrightarrow{1} \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^d).$$

Using this map we will now define a natural isomorphism $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d$.

Definition 12.2. Let $d \geqslant 1$ and let $\mathcal{M}$ be an elliptic shtuka such that $\mathcal{M}/\mathfrak{f}^d$ is linear. We define the map $\overline{\rho}_d$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "\wr"', "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "\wr"', "\text{red.}"] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{f}^d) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^d).
\end{tikzcd}$$

Here the vertical maps are the isomorphisms of Lemma 12.1. We call $\overline{\rho}_d$ the *tautological regulator of order d.* By construction $\overline{\rho}_d$ is a natural isomorphism.

For the duration of this section and Section 13 we fix a uniformizer $z \in \mathcal{O}_F$. As it will be shown in Section 13 our results do not depend on the choice of z. However this choice simplifies the exposition. With the uniformizer z fixed we have for every elliptic shtuka $\mathcal{M}$ a natural isomorphism $\varpi\colon \mathcal{M} \to \mathfrak{m}\mathcal{M}$.

Definition 12.3. Let $\mathcal{M}$ be an elliptic shtuka. We define a natural isomorphism

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{s} \mathrm{H}^1(\mathfrak{f}\mathcal{M})$$

as the composition

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\varpi} \mathrm{H}^1(\mathfrak{m}\mathcal{M}) \xrightarrow{\gamma} \mathrm{H}^1(\mathfrak{f}\mathcal{M}).$$

Here γ is the natural isomorphism of Definition 11.1. We call s the *sliding isomorphism.*

Apart from this section the sliding isomorphism s will only be used in Section 13.

Lemma 12.4. *Let $\mathcal{M}$ be an elliptic shtuka. For every $n \geqslant 0$ the natural diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[d, "\varpi^n"'] \arrow[dr, "s^n"] & \\
\mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n"] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M})
\end{tikzcd}$$

is commutative.

Proof. This diagram commutes for $n = 0$. Assuming that it commutes for some n we prove that it does so for $n+1$. Consider the natural diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\varpi^n"] \arrow[dr, "\varpi^{n+1}"'] & \mathrm{H}^1(\mathfrak{m}^n\mathcal{M}) \arrow[r, "\gamma^n_{\mathcal{M}}"] \arrow[d, "\varpi"] & \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[d, "\varpi"] \arrow[dr, "s"] & \\
& \mathrm{H}^1(\mathfrak{m}^{n+1}\mathcal{M}) \arrow[r, "\gamma^n_{\mathfrak{m}\mathcal{M}}"] & \mathrm{H}^1(\mathfrak{m}\mathfrak{f}^n\mathcal{M}) \arrow[r, "\gamma_{\mathfrak{f}^n\mathcal{M}}"] & \mathrm{H}^1(\mathfrak{f}^{n+1}\mathcal{M})
\end{tikzcd}$$

The left triangle in this diagram commutes by definition of ϖ, the square commutes by naturality of γ and the right triangle commutes by definition of s. By assumption $s^n = \gamma^n(\mathcal{M}) \circ \varpi^n$. However

$$\gamma_{\mathfrak{f}^n\mathcal{M}} \circ \gamma^n_{\mathfrak{m}\mathcal{M}} = \gamma^{n+1}_{\mathcal{M}}$$

by definition of γ so the result follows. □

Recall that the natural isomorphism $\overline{\rho}_d$ is defined only under assumption that $\mathcal{M}/\mathfrak{f}^d$ is linear. We would like to extend it to all elliptic shtukas. To that end we will prove that the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^d", "\wr"'] \\
\mathrm{H}^1(\mathfrak{f}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{f}^d\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

commutes provided the shtuka $\mathcal{M}/\mathfrak{f}^{2d}$ is linear. The proof is a bit technical. We split it into a chain of auxillary lemmas. In the following the integer d and the elliptic shtuka $\mathcal{M}$ will be fixed. To improve the legibility we will write I in place of $\mathfrak{f}^d$.

Lemma 12.5. *The shtuka $(I\mathcal{M})/I$ is linear.*

Proof. Follows instantly from Proposition 7.6. □

Consider the shtuka $I\mathcal{M}/\mathfrak{m}^d$. According to Lemma 8.1 it is a locally free shtuka on $\mathcal{O}_F/\mathfrak{m}^d \otimes \mathcal{O}_K$ whose restriction to $\mathcal{O}_F/\mathfrak{m}^d \otimes K$ is nilpotent. In Section 5 we equipped the shtukas of this kind with a natural isomorphism

$$\rho\colon \mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \to \mathrm{H}^1(\nabla I\mathcal{M}/\mathfrak{m}^d)$$

called the artinian regulator. It is defined only under certain conditions.

Lemma 12.6. *If $\mathcal{M}/I$ is linear then the artinian regulator is defined for $I\mathcal{M}/\mathfrak{m}^d$.*

Proof. Indeed $(I\mathcal{M})/I$ is linear by Lemma 12.5 whence the result follows from Lemma 8.3 applied to the shtuka $I\mathcal{M}$. □

In Section 11 we introduced natural isomorphisms

$$\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/I) \xrightarrow{\overline{\gamma}_d} \mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \xrightarrow{\overline{\varepsilon}_d} \mathrm{H}^1(I\mathcal{M}/I).$$

In the following we drop the indices d for legibility. Our next step is to study how the artinian regulator ρ of the shtuka $I\mathcal{M}/\mathfrak{m}^d$ interacts with $\overline{\gamma}$ and $\overline{\varepsilon}$.

Lemma 12.7. *If $\mathcal{M}/I$ is linear then the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] \arrow[d, "\overline{\varepsilon}"', "\wr"] & \mathrm{H}^1(\nabla I\mathcal{M}/\mathfrak{m}^d) \arrow[d, "\wr"', "\overline{\varepsilon}"] \\
\mathrm{H}^1(I\mathcal{M}/I) \arrow[r, "1"] & \mathrm{H}^1(\nabla I\mathcal{M}/I)
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] \arrow[d, "\mathrm{red.}"', "\wr"] & \mathrm{H}^1(\nabla I\mathcal{M}/\mathfrak{m}^d) \arrow[d, "\wr"', "\mathrm{red.}"] \\
\mathrm{H}^1(I\mathcal{M}/(\mathfrak{m}^d+I)) \arrow[r, "1"] & \mathrm{H}^1(\nabla I\mathcal{M}/(\mathfrak{m}^d+I)) \\
\mathrm{H}^1(I\mathcal{M}/I) \arrow[r, "1"] \arrow[u, "\mathrm{red.}", "\wr"'] & \mathrm{H}^1(\nabla I\mathcal{M}/I) \arrow[u, "\wr", "\mathrm{red.}"']
\end{tikzcd}$$

The bottom square is clearly commutative. Applying Proposition 5.10 to $I\mathcal{M}/\mathfrak{m}^d$ we conclude that the top square is commutative. Now according to Proposition 11.7 the isomorphism $\overline{\varepsilon}$ is the composition of reduction isomorphisms

$$\mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \xrightarrow{\sim} \mathrm{H}^1(I\mathcal{M}/(\mathfrak{m}^d+I)) \xleftarrow{\sim} \mathrm{H}^1(I\mathcal{M}/I).$$

Applying the same Proposition to $\nabla I\mathcal{M}/\mathfrak{m}^d$ we get the result. □

Lemma 12.8. *If $\mathcal{M}/I^2$ is linear then the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/I) \arrow[r, "1"] \arrow[d, "\overline{\gamma}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/I) \arrow[d, "\wr"', "\overline{\gamma}"] \\
\mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] & \mathrm{H}^1(\nabla I\mathcal{M}/\mathfrak{m}^d)
\end{tikzcd}$$

is commutative.

Proof. Consider the boundary homomorphisms

$$\mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+I)) \xrightarrow{\delta} \mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d),$$

$$\mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+I)) \xrightarrow{\delta} \mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/I)$$

of the exact sequences (10.1) and (10.2). Using them we construct a diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/I) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/I) \\
\mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+I)) \arrow[r, "1"] \arrow[u, "\delta", "\wr"'] \arrow[d, "\delta"', "\wr"] & \mathrm{H}^0(\nabla\mathcal{M}/(\mathfrak{m}^d+I)) \arrow[u, "\delta", "\wr"'] \arrow[d, "\delta"', "\wr"] \\
\mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] & \mathrm{H}^1(\nabla I\mathcal{M}/\mathfrak{m}^d)
\end{tikzcd} \tag{12.1}$$

The top square is clearly commutative. We apply Proposition 5.11 to deduce the commutativity of the bottom square. To use it we need to verify that the following conditions hold for the shtuka $\mathcal{N} = \mathcal{M}/\mathfrak{m}^d$:

(1) $I \cdot \mathrm{H}^1(\nabla\mathcal{N}) = 0$.
(2) $\mathcal{N}/I^2$ is linear.

Lemma 8.1 implies (1) while (2) follows since $\mathcal{M}/I^2$ is linear. We conclude that (12.1) is commutative. Now Proposition 11.4 shows that the maps δ are isomorphisms and that the composition

$$\mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/I) \xrightarrow{\delta^{-1}} \mathrm{H}^0(\mathcal{M}/(\mathfrak{m}^d+I)) \xrightarrow{\delta} \mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d)$$

is equal to $-\overline{\gamma}$. The same Proposition applies to $\nabla\mathcal{M}$ as well. So the commutativity of (12.1) implies our result. □

We are finally ready to obtain the key result of this section.

Proposition 12.9. *Let $d \geqslant 1$ and let $\mathcal{M}$ be an elliptic shtuka. If $\mathcal{M}/\mathfrak{f}^{2d}$ is linear then the square*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "\wr"', "s^d"] \\
\mathrm{H}^1(\mathfrak{f}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{f}^d\mathcal{M})/\mathfrak{m}^d.
\end{tikzcd}$$

is commutative.

Proof. We continue to use the notation I for $\mathfrak{f}^d$. By Lemma 12.5 the shtuka $(I\mathcal{M})/I$ is linear so that the square makes sense. We proceed by repeated splitting of this square till the problem is reduced to its core.

Using Lemma 12.4 we split the square as follows:

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "\varpi^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "\wr"', "\varpi^d"] \\ \mathrm{H}^1(\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "\gamma^d"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[d, "\wr"', "\gamma^d"] \\ \mathrm{H}^1(I\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla I\mathcal{M})/\mathfrak{m}^d. \end{tikzcd} \tag{12.2}$$

The top square commutes by functoriality of $\overline{\rho}_d$ so we concentrate on the bottom square. It is necessary to split this square even further.

Recall that $\overline{\rho}_d$ is defined as the composition

$$\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/I) \xrightarrow{1} \mathrm{H}^1(\nabla\mathcal{M}/I) \xleftarrow{\text{red.}} \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d$$

So we can rewrite the bottom square as follows:

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\text{red.}"] \arrow[d, "\gamma^d"', "\wr"] & \mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/I) \arrow[r, "1"] \arrow[d, "(\overline{\varepsilon}\circ\overline{\gamma})_{\mathcal{M}}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/I) \arrow[d, "\wr"', "(\overline{\varepsilon}\circ\overline{\gamma})_{\nabla\mathcal{M}}"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M})/\mathfrak{m}^d \arrow[l, "\text{red.}"'] \arrow[d, "\wr"', "\gamma^d"] \\ \mathrm{H}^1(I\mathcal{M})/\mathfrak{m}^d \arrow[r, "\text{red.}"] & \mathrm{H}^1(I\mathcal{M}/I) \arrow[r, "1"] & \mathrm{H}^1(\nabla I\mathcal{M}/I) & \mathrm{H}^1(\nabla I\mathcal{M})/\mathfrak{m}^d \arrow[l, "\text{red.}"'] \end{tikzcd}$$

Here the notation gets a bit confusing, so let us elaborate on it. The compositions $(\overline{\varepsilon}\circ\overline{\gamma})_{\mathcal{M}}$ and $(\overline{\varepsilon}\circ\overline{\gamma})_{\nabla\mathcal{M}}$ have the same source and target. So one would expect that the middle square comutes tautologically. However the actual situation is more complicated. The maps $(\overline{\varepsilon}\circ\overline{\gamma})_{\mathcal{M}}$ and $(\overline{\varepsilon}\circ\overline{\gamma})_{\nabla\mathcal{M}}$ are defined in terms of data which comes from completely different shtukas $\mathcal{M}$ and $\nabla\mathcal{M}$. We add the subscripts $\mathcal{M}$ and $\nabla\mathcal{M}$ to emphasise this fact.

The left and right squares in the diagram above commute by Lemma 11.6. Let us consider the middle square. We split it in two:

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{m}^d\mathcal{M}/I) \arrow[r, "1"] \arrow[d, "\overline{\gamma}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{m}^d\mathcal{M}/I) \arrow[d, "\wr"', "\overline{\gamma}"] \\ \mathrm{H}^1(I\mathcal{M}/\mathfrak{m}^d) \arrow[r, "\rho"] \arrow[d, "\overline{\varepsilon}"', "\wr"] & \mathrm{H}^1(\nabla I\mathcal{M}/\mathfrak{m}^d) \arrow[d, "\wr"', "\overline{\varepsilon}"] \\ \mathrm{H}^1(I\mathcal{M}/I) \arrow[r, "1"] & \mathrm{H}^1(\nabla I\mathcal{M}/I). \end{tikzcd}$$

Here ρ is the artinian regulator of the shtuka $I\mathcal{M}/\mathfrak{m}^d$ which is defined by Lemma 12.6. The bottom square commutes by Lemma 12.7. Since $\mathcal{M}/I^2$ is linear the top square commutes by Lemma 12.8. So we are done. □

13. Regulators of finite order

Definition 13.1. Let $d \geqslant 1$. An $\mathcal{O}_F/\mathfrak{m}^d$-linear natural transformation

$$\rho_d \colon \mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \longrightarrow \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d$$

of functors on the category of elliptic shtukas is called a *regulator of order d* if the following holds:

(1) If $\mathcal{M}$ is an elliptic shtuka such that $\mathcal{M}/\mathfrak{f}^{2d}$ is linear then ρ_d coincides with the tautological regulator $\overline{\rho}_d$ of Definition 12.2.
(2) For every elliptic shtuka $\mathcal{M}$ the natural diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^d"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^d", "\wr"'] \\
\mathrm{H}^1(\mathfrak{f}^d\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] & \mathrm{H}^1(\nabla\mathfrak{f}^d\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

is commutative.

The exponent $2d$ in the condition (1) does not look natural. Indeed the tautological regulator is defined even if $\mathcal{M}/\mathfrak{f}^d$ is linear. However with the exponent d the regulators will fail to exist.

It is worth noting that the definition of the regulator for the quotient $\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d$ is actually more complicated than the definition for the module $\mathrm{H}^1(\mathcal{M})$ itself (Definition 14.1). In the latter case one does not need the condition (2).

Proposition 13.2. *Let $d \geqslant 1$. A regulator of order d exists and has the following properties:*

(1) *It is unique.*
(2) *It is an isomorphism.*

Proof. Let $\mathcal{M}$ be an elliptic shtuka. According to Proposition 7.6 the shtuka $(\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{f}^{2d}$ is linear. In particular we have the tautological regulator $\overline{\rho}_d$ for $\mathfrak{f}^{2d}\mathcal{M}$. We define the regulator ρ_d for $\mathcal{M}$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^{2d}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{2d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

Due to condition (1) of Definition 13.1 this diagram should commute for any regulator of order d. We thus get the unicity of ρ_d.

Let us prove that the map ρ_d we just defined is a regulator. It is a natural $\mathcal{O}_F/\mathfrak{m}^d$-linear isomorphism since the maps s^{2d} and $\overline{\rho}_d$ are so. If $\mathcal{M}/\mathfrak{f}^{2d}$ is itself linear then the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^{2d}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{2d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

commutes by Proposition 12.9. Hence the condition (1) of Definition 13.1 is satisfied. To check the condition (2) consider the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^{d}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{f}^{d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^{d}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{f}^{d}\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] \arrow[d, "s^{d}"', "\wr"] & \mathrm{H}^1(\nabla\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^{d}", "\wr"'] \\
\mathrm{H}^1(\mathfrak{f}^{3d}\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{f}^{3d}\mathcal{M})/\mathfrak{m}^d
\end{tikzcd}$$

We need to prove that the top square commutes. The rectangles of height 2 commute by definition of ρ_d. The bottom square commutes by Proposition 12.9. As a consequence the middle square commutes which implies the commutativity of the top square. ☐

Proposition 13.3. *The regulator ρ_d does not depend on the choice of the uniformizer $z \in \mathcal{O}_F$ in the definition of the sliding isomorphism s (Definition 12.3).*

Proof. We will show that ρ_d satisfies the condition (2) of Definition 13.1 with any choice of z. According to Lemma 12.4 the sliding isomorphism s^d is the composition

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\varpi^d} \mathrm{H}^1(\mathfrak{m}^d\mathcal{M}) \xrightarrow{\gamma^d} \mathrm{H}^1(\mathfrak{f}^d\mathcal{M})$$

Here γ^d does not depend on the choice of z. The natural isomorpism $\varpi\colon \mathcal{M} \to \mathfrak{m}\mathcal{M}$ is the unique map whose composition with the natural embedding $\mathfrak{m}\mathcal{M} \hookrightarrow \mathcal{M}$ is the multiplication by z.

Now let $u \in \mathcal{O}_F^\times$. The regulator ρ_d commutes with multiplication by u^d since it is $\mathcal{O}_F/\mathfrak{m}^d$-linear. However $u\varpi$ is the natural isomorphism $\mathcal{M} \to \mathfrak{m}\mathcal{M}$ with the choice of uniformizer uz. We conclude that ρ_d satisfies the condition (2) of Definition 13.1 with the uniformizer uz as well. ☐

To construct the regulator isomorphism on the entire module $\mathrm{H}^1(\mathcal{M})$ we would like to take the limit of regulators ρ_d for $d \to \infty$. To do it we first need to show that these regulators agree.

Proposition 13.4. *If $k \geqslant d \geqslant 1$ are integers then the regulators ρ_d and ρ_k coincide modulo $\mathfrak{m}^d$.*

Proof. Set $n = 2dk$. Let $\mathcal{M}$ be an elliptic shtuka. According to Proposition 7.6 the shtuka $(\mathfrak{f}^n\mathcal{M})/\mathfrak{f}^n$ is linear. The tautological regulators $\overline{\rho}_d$ and $\overline{\rho}_{dk}$ are defined for $\mathfrak{f}^n\mathcal{M}$. Since d and dk divide n the diagrams

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \arrow[r, "\rho_d"] \arrow[d, "s^n", "\wr"'] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d \arrow[d, "s^n", "\wr"'] \\ \mathrm{H}^1(\mathfrak{f}^n\mathcal{M})/\mathfrak{m}^d \arrow[r, "\overline{\rho}_d"] & \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M})/\mathfrak{m}^d \end{tikzcd} \qquad \begin{tikzcd} \mathrm{H}^1(\mathcal{M})/\mathfrak{m}^{dk} \arrow[r, "\rho_{dk}"] \arrow[d, "s^n", "\wr"'] & \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^{dk} \arrow[d, "s^n", "\wr"'] \\ \mathrm{H}^1(\mathfrak{f}^n\mathcal{M})/\mathfrak{m}^{dk} \arrow[r, "\overline{\rho}_{dk}"] & \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M})/\mathfrak{m}^{dk} \end{tikzcd}$$

commute by definition of ρ_d and ρ_{dk}. However $\overline{\rho}_d \equiv \overline{\rho}_{dk} \pmod{\mathfrak{m}^d}$ by construction. We conclude that $\rho_d \equiv \rho_{dk} \pmod{\mathfrak{m}^d}$. Applying the same argument with d and k interchanged we deduce that

$$\rho_k \equiv \rho_{dk} \equiv \rho_d \pmod{\mathfrak{m}^d}. \quad \square$$

14. Regulators

Definition 14.1. An $\mathcal{O}_F$-linear natural transformation

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\ \rho\ } \mathrm{H}^1(\nabla\mathcal{M})$$

of functors on the category of elliptic shtukas is called a *regulator* if for every $\mathcal{M}$ such that $\mathcal{M}/\mathfrak{f}^{2n}$ is linear the diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\ \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^n) \end{tikzcd}$$

is commutative.

Lemma 14.2. *Let $f\colon M \to N$ be a morphism of $\mathcal{O}_F$-modules. If M and N are finitely generated free then the following are equivalent:*

(1) $f = 0$.
(2) *For every $d > 0$ there exists an $n \geqslant 0$ such that $f(\mathfrak{m}^n M) \subset \mathfrak{m}^{n+d} N$.* $\square$

Lemma 14.3. *Let $\mathcal{M}$ be an elliptic shtuka and let $f\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ be an $\mathcal{O}_F$-linear map.*

(1) *For every $n \geqslant 0$ the map f sends the submodule $\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \subset \mathrm{H}^1(\mathcal{M})$ to $\mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}) \subset \mathrm{H}^1(\nabla\mathcal{M})$.*
(2) *If for every $d > 0$ there exists an $n \geqslant 0$ such that the composition*

$$\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \xrightarrow{f} \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^d)$$

is zero then f is itself zero.

Proof. By Theorem 8.5 for every elliptic shtuka $\mathcal{M}$ the natural sequence

$$0 \to \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \longrightarrow \mathrm{H}^1(\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n) \to 0$$

is exact and the first arrow has image $\mathfrak{m}^n\mathrm{H}^1(\mathcal{M})$. So (1) is clear and (2) follows from Lemma 14.2. ☐

Theorem 14.4. *The regulator exists and has the following properties:*

(1) *It is unique.*
(2) *It is an isomorphism.*

Proof. (1) Let ρ and ρ' be two regulators and let $\mathcal{M}$ be an elliptic shtuka. By Proposition 7.6 for every $d \geqslant 1$ the shtuka $(\mathfrak{f}^{2d}\mathcal{M})/\mathfrak{f}^{2d}$ is linear. The difference $\rho - \rho'$ maps $\mathrm{H}^1(\mathfrak{f}^{2d}\mathcal{M})$ to the kernel of the reduction map $\mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^d)$. Hence $\rho = \rho'$ by Lemma 14.3.

(2) Now let us construct the regulator and prove that it is an isomorphism. According to Proposition 13.2 for every $d \geqslant 1$ there exists a unique regulator of order d

$$\rho_d \colon \mathrm{H}^1(\mathcal{M})/\mathfrak{m}^d \to \mathrm{H}^1(\nabla\mathcal{M})/\mathfrak{m}^d.$$

It is a natural $\mathcal{O}_F/\mathfrak{m}^d$-linear isomorphism. The regulators of different orders are compatible by Proposition 13.4 and do not depend on the auxillary choice of a uniformizer $z \in \mathcal{O}_F$ by Proposition 13.3. Now take their limit for $d \to \infty$. Since $\mathrm{H}^1(\mathcal{M})$ and $\mathrm{H}^1(\nabla\mathcal{M})$ are finitely generated $\mathcal{O}_F$-modules we get a natural $\mathcal{O}_F$-linear isomorphism

$$\rho \colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M}).$$

It satisfies the condition of Definition 14.1 since every regulator ρ_d satisfies it up to order d. ☐

Theorem 14.5. *Let $\mathcal{M}$ be an elliptic shtuka and let $f \colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ be an $\mathcal{O}_F$-linear map. Suppose that for every $d > 0$ there exists an $n \geqslant 0$ such that the shtuka $\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^{2d}$ is linear and the diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[r, "f"] \arrow[d] & \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}) \arrow[d] \\
\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^d) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^d)
\end{tikzcd}$$

is commutative. Then f coincides with the regulator ρ of $\mathcal{M}$.

Proof. Indeed the condition implies that the composition

$$\mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \xrightarrow{f-\rho} \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}) \xrightarrow{\text{red.}} \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}/\mathfrak{f}^d)$$

is zero. So the result follows from Lemma 14.3. ☐

The next theorem connects the regulator ρ of this section with the artinian regulator of Section 5. It is essential to the proof of the trace formula in Chapter 6. It can also be used as an alternative definition for the regulator.

Theorem 14.6. *Let $\mathcal{M}$ be an elliptic shtuka of conductor $\mathfrak{f}$ and let $n > 0$. If $\mathcal{M}/\mathfrak{f}^{2n}$ is linear then the following holds:*

(1) *The artinian regulator $\rho_{\mathcal{M}/\mathfrak{m}^n}$ is defined for the shtuka $\mathcal{M}/\mathfrak{m}^n$.*

(2) *The diagram*

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \arrow[r, "\rho_{\mathcal{M}/\mathfrak{m}^n}"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{m}^n)
\end{tikzcd}$$

is commutative.

Proof. (1) follows from Lemma 8.3 since $\mathcal{M}/\mathfrak{f}^n$ is linear. Let us concentrate on (2).

Consider the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{m}^n) \arrow[r, "\rho_{\mathcal{M}/\mathfrak{m}^d}"] \arrow[d, "\text{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{m}^n) \arrow[d, "\text{red.}", "\wr"'] \\
\mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n+\mathfrak{f}^n)) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/(\mathfrak{m}^n+\mathfrak{f}^n))
\end{tikzcd}$$

The bottom square in this diagram commutes by Proposition 5.10. The vertical arrows in the bottom square are isomorphisms by Proposition 10.1. Hence the top square of this diagram commutes if and only if the outer rectangle commutes. Now we can split the outer rectangle in two other squares:

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\text{red.}"'] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\text{red.}"] \\
\mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n) \arrow[r, "1"] \arrow[d, "\text{red.}"', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^n) \arrow[d, "\text{red.}", "\wr"'] \\
\mathrm{H}^1(\mathcal{M}/(\mathfrak{m}^n+\mathfrak{f}^n)) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/(\mathfrak{m}^n+\mathfrak{f}^n))
\end{tikzcd}$$

The bottom square commutes tautologically. As the shtuka $\mathcal{M}/\mathfrak{f}^{2n}$ is linear the top square commutes by definition of ρ. Hence we are done. $\square$