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Shtuka cohomology and special values of Goss L-functions

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CHAPTER 4

Cohomology of shtukas

Fix a locally compact noetherian $\mathbb{F}_q$-algebra S and a smooth projective curve X over $\mathbb{F}_q$. We call S the *coefficient ring* and X the *base curve*. Consider the product $\operatorname{Spec} S \times X$. To simplify the notation we will write $S \times X$ instead of $\operatorname{Spec} S \times X$. We equip $S \times X$ with the τ-scheme structure given by the endomorphism which acts as the identity on S and as the q-power map on X. In this chapter we study the cohomology of locally free shtukas on $S \times X$ and related schemes.

Fix an open dense affine subscheme $\operatorname{Spec} R \subset X$. Its complement consists of finitely many closed points $x_1, \ldots, x_n \in X$. We denote K the product of the local fields of X at $x_1, \ldots, x_n$. We use the notation and the terminology of Section 3.5 in regard to K. In particular $\mathcal{O}_K \subset K$ stands for the product of the rings of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ denotes the Jacobson radical. By construction $\operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K = \{x_1, \ldots, x_n\} \subset X$. The natural topology on $\mathcal{O}_K$ makes it into a compact open $\mathbb{F}_q$-subalgebra of a locally compact $\mathbb{F}_q$-algebra K.

In the first two sections we study the cohomology of shtukas in a local situation. Let $\mathcal{M}$ be a locally free shtuka on $S \check{\otimes} \mathcal{O}_K$. In Section 1 we introduce the germ cohomology complex $\mathrm{R}\Gamma_g(S \check{\otimes} K, \mathcal{M})$. As suggested by the notation it depends only on the restriction of $\mathcal{M}$ to $S \check{\otimes} K$. The germ cohomology is modelled on the germ spaces of Section 2.11. With some degree of caution it can be regarded as compactly supported cohomology for shtukas on $S \check{\otimes} K$ with respect to the compactification given by the ring $S \check{\otimes} \mathcal{O}_K$.

The germ cohomology is related to the usual cohomology via the local germ map

$$\mathrm{R}\Gamma(S \check{\otimes} \mathcal{O}_K, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma_g(S \check{\otimes} K, \mathcal{M})$$

which we construct in Section 2. This map is defined only if $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and is always a quasi-isomorphism. The local germ map will play an important role in Chapters 8 – 12.

Starting from Section 3 we shift the focus to a global situation. Let $\mathcal{M}$ be a locally free shtuka over $S \times X$. In Section 4 we introduce a Čech method which computes $\mathrm{R}\Gamma(S \times X, \mathcal{M})$ in terms of the complexes

$$\mathrm{R}\Gamma(S \otimes R, \mathcal{M}),\ \mathrm{R}\Gamma(S^{\#} \widehat{\otimes} \mathcal{O}_K, \mathcal{M}),\ \mathrm{R}\Gamma(S^{\#} \widehat{\otimes} K, \mathcal{M})$$

This is our main tool to handle the cohomology of shtukas over X.

The compactly supported cohomology functor $\mathrm{R}\Gamma_{\mathrm{c}}(S\otimes R,\mathcal{M})$ is introduced in Section 5. As suggested by the notation $\mathrm{R}\Gamma_{\mathrm{c}}(S\otimes R,\mathcal{M})$ depends only on the restriction of $\mathcal{M}$ to $S\otimes R$. It comes equipped with a natural map $\mathrm{R}\Gamma(S\times X,\mathcal{M})\to\mathrm{R}\Gamma_{\mathrm{c}}(S\otimes R,\mathcal{M})$. We prove that this map is a quasi-isomorphism if $\mathcal{M}(S\otimes\mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. One can interpet the nilpotence condition as saying that $\mathcal{M}$ is an extension by zero of a shtuka on $S\otimes R$. We use $\mathrm{R}\Gamma_{\mathrm{c}}(S\otimes R,\mathcal{M})$ to construct the global germ map

$$\mathrm{R}\Gamma(S\times X,\mathcal{M})\to\mathrm{R}\Gamma_g(S\,\check{\otimes}\,K,\mathcal{M}).$$

Similarly to its local counterpart the global germ map is defined under assumption that $\mathcal{M}(S\otimes\mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. However it is not a quasi-isomorphism in general.

Section 6 is devoted to the proof of Theorem 6.1, a compatibility statement for the local and global germ maps. This statement is vital for the proof of the class number formula.

In Section 7 we present an advanced version of the Čech method for shtukas on $\mathcal{O}_F\times X$ where $\mathcal{O}_F$ is the ring of integers of a local field F. This method enables us to prove the following: if $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F\otimes R)$ is nilpotent then the natural map

$$\mathrm{R}\Gamma(\mathcal{O}_F\times X,\mathcal{M})\to\mathrm{R}\Gamma(\mathcal{O}_F\,\widehat{\otimes}\,\mathcal{O}_K,\mathcal{M})$$

is a quasi-isomorphism (Theorem 7.11). Informally speaking, the cohomology of $\mathcal{M}$ concentrates on $\mathcal{O}_F\,\widehat{\otimes}\,\mathcal{O}_K$. This phenomenon is important to the theory of regulator developed in Chapters 5 and 6.

Finally in Sections 8 and 9 we study how the change of the coefficient ring S reflects on shtuka cohomology and ζ-isomorphisms. The results of these sections will be used in Chapters 6 and 12.

The cohomology functors in this chapter are typically given by mapping fibers

$$\Big[\mathrm{R}\Gamma(Y,\mathcal{M})\to\mathrm{R}\Gamma(Z,\mathcal{M})\Big]$$

where Y and Z are affine schemes (see Definition 3.1 in the Chapter "Notation and conventions"). A word of warning about them is necessary. The complexes $\mathrm{R}\Gamma(Y,\mathcal{M})$ and $\mathrm{R}\Gamma(Z,\mathcal{M})$ are well-defined only as objects in the derived category. As a consequence one gets a problem with functoriality. If $\mathcal{M}\to\mathcal{N}$ is a morphism of shtukas then the induced maps

$$\begin{aligned}\mathrm{R}\Gamma(Y,\mathcal{M})&\to\mathrm{R}\Gamma(Y,\mathcal{N}),\\ \mathrm{R}\Gamma(Z,\mathcal{M})&\to\mathrm{R}\Gamma(Z,\mathcal{N})\end{aligned}\tag{1}$$

do not determine a unique morphism of the mapping fibers. The reason is that this morphism depends on the choice of non-derived representatives for the maps (1).

We solve this problem in the following way. Since the schemes Y and Z are affine Theorem 1.8.1 provides us with canonical non-derived representatives for the complexes $\mathrm{R}\Gamma(Y,\mathcal{M})$ and $\mathrm{R}\Gamma(Z,\mathcal{M})$, namely the associated complexes $\Gamma_{\mathrm{a}}(Y,\mathcal{M})$ and $\Gamma_{\mathrm{a}}(Z,\mathcal{M})$. The mapping fiber construction is functorial on the

level of the non-derived category of complexes. According to the convention of Section 1.8 we identify $\mathrm{R}\Gamma(Y,\mathcal{M})$ with $\Gamma_{\mathrm{a}}(Y,\mathcal{M})$ and $\mathrm{R}\Gamma(Z,\mathcal{M})$ with $\Gamma_{\mathrm{a}}(Z,\mathcal{M})$. We thus regain the functoriality.

1. Germ cohomology

In this section we fix locally compact $\mathbb{F}_q$-algebras S and T. Following the conventions of Section 3.11 we equip $S \mathbin{\check{\otimes}} T$ and $S^{\#} \mathbin{\widehat{\otimes}} T$ with the τ-ring structures given by the endomorphisms which act as identity on S and as the q-power map on T.

Definition 1.1. Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \mathbin{\check{\otimes}} T$. The *germ cohomology* complex of $\mathcal{M}$ is the S-module complex

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} T,\mathcal{M}) = \Big[\mathrm{R}\Gamma(S \mathbin{\check{\otimes}} T,\mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} T,\mathcal{M})\Big].$$

The differential in this complex is induced by the natural map $S \mathbin{\check{\otimes}} T \to S^{\#} \mathbin{\widehat{\otimes}} T$ which is the completion of the continuous bijection $S \otimes_{\mathrm{ic}} T \to S^{\#} \otimes_{\mathrm{c}} T$. The n-th cohomology group of $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} T,\mathcal{M})$ is denoted $\mathrm{H}^n_g(S \mathbin{\check{\otimes}} T,\mathcal{M})$. As explained in the introduction we use the canonical representatives of Theorem 1.8.1 for the cohomology complexes $\mathrm{R}\Gamma(S \mathbin{\check{\otimes}} T,\mathcal{M})$ and $\mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} T,\mathcal{M})$. As a consequence $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} T,-)$ becomes a functor.

Proposition 1.2. *If $\mathcal{M}$ is a locally free shtuka on $S \mathbin{\check{\otimes}} T$ then the natural map*

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} T,\mathcal{M}) \to \mathrm{R}\Gamma\Big(S \mathbin{\check{\otimes}} T, \tfrac{\mathcal{M}(S^{\#} \widehat{\otimes} T)}{\mathcal{M}(S \check{\otimes} T)}\Big)[-1]$$

is a quasi-isomorphism.

Proof. Tensoring the short exact sequence

$$0 \to S \mathbin{\check{\otimes}} T \to S^{\#} \mathbin{\widehat{\otimes}} T \to \frac{S^{\#} \mathbin{\widehat{\otimes}} T}{S \mathbin{\check{\otimes}} T} \to 0$$

with a locally free $S \mathbin{\check{\otimes}} T$-module of finite rank we get a short exact sequence. As $\mathcal{M}$ is locally free the claim follows. □

Proposition 1.3. *Let $f\colon S_1 \to S_2$ and $g\colon T_1 \to T_2$ be continuous homomorphisms of locally compact $\mathbb{F}_q$-algebras. Let $\mathcal{M}$ be a locally free shtuka on $S_1 \mathbin{\check{\otimes}} T_1$. If f^* and g are local isomorphisms of topological $\mathbb{F}_q$-vector spaces then the natural map*

$$\mathrm{R}\Gamma_g(S_1 \mathbin{\check{\otimes}} T_1,\mathcal{M}) \to \mathrm{R}\Gamma_g(S_2 \mathbin{\check{\otimes}} T_2,\mathcal{M})$$

induced by $f \mathbin{\check{\otimes}} g$ is a quasi-isomorphism.

Proof. By Proposition 2.11.6 f,g induce a bijection

$$\frac{S_1^{\#} \mathbin{\widehat{\otimes}} T_1}{S_1 \mathbin{\check{\otimes}} T_1} \cong \frac{S_2^{\#} \mathbin{\widehat{\otimes}} T_2}{S_2 \mathbin{\check{\otimes}} T_2}.$$

As the shtuka $\mathcal{M}$ is locally free it follows that $f \mathbin{\check{\otimes}} g$ induces an isomorphism of shtukas

$$\frac{\mathcal{M}(S_1^{\#} \mathbin{\widehat{\otimes}} T_1)}{\mathcal{M}(S_1 \mathbin{\check{\otimes}} T_1)} \cong \frac{\mathcal{M}(S_2^{\#} \mathbin{\widehat{\otimes}} T_2)}{\mathcal{M}(S_2 \mathbin{\check{\otimes}} T_2)}.$$

The result now follows from Proposition 1.2. □

2. Local germ map

Fix a noetherian locally compact $\mathbb{F}_q$-algebra S. In applications this algebra will usually be a local field. Let K be a finite product of local fields. We use the notation and conventions of Section 3.5 regarding K. In particular K is supposed to contain $\mathbb{F}_q$, $\mathcal{O}_K \subset K$ stands for the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ denotes the Jacobson radical.

The ideal $\mathfrak{m}_K \subset \mathcal{O}_K$ is open so that we have a natural map $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K \to S \otimes \mathcal{O}_K/\mathfrak{m}_K$. Taking the completion of $S \otimes_{\mathrm{ic}} \mathcal{O}_K \to S \otimes_{\mathrm{ic}} \mathcal{O}_K/\mathfrak{m}_K$ we get a natural map $S \mathbin{\check{\otimes}} \mathcal{O}_K \to S \otimes \mathcal{O}_K/\mathfrak{m}_K$ since $S \otimes_{\mathrm{ic}} \mathcal{O}_K/\mathfrak{m}_K$ is discrete.

Proposition 2.1. *Let $\mathcal{M}$ be a locally free shtuka on $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then $\mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) = 0$.*

Proof. According to Proposition 3.6.6 the ring $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is noetherian and complete with respect to the ideal $S^{\#} \mathbin{\widehat{\otimes}} \mathfrak{m}_K$. By Proposition 3.6.4 the natural map $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K \to S \otimes \mathcal{O}_K/\mathfrak{m}_K$ is surjective with kernel $S^{\#} \mathbin{\widehat{\otimes}} \mathfrak{m}_K$. So the result follows from Proposition 1.9.4. □

Lemma 2.2. *If $\mathcal{M}$ is a locally free shtuka on $S \mathbin{\check{\otimes}} \mathcal{O}_K$ then the natural map $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M})$ is a quasi-isomorphism.*

Proof. The inclusion $\mathcal{O}_K \hookrightarrow K$ is a local isomorphism of topological $\mathbb{F}_q$-vector spaces. So the result is a consequence of Proposition 1.3. □

Let $\mathcal{M}$ be a locally free shtuka on $S \mathbin{\check{\otimes}} \mathcal{O}_K$. According to Definition 1.1

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) = \Big[\mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})\Big].$$

The projection to the first argument of the mapping fiber construction defines a natural map

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}).$$

Taking its composition with the quasi-isomorphism $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M}) \cong \mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M})$ of Lemma 2.2 we obtain a map

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M}) \to \mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}). \tag{2.1}$$

Lemma 2.3. *Let $\mathcal{M}$ be a locally free shtuka on $S \mathbin{\check{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then the natural map* (2.1) *is a quasi-isomorphism.*

Proof. By construction the natural map $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M})$ extends to a distinguished triangle

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \to [1].$$

Together with the quasi-isomorphism $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \cong \mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M})$ it gives us a distinguished triangle

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M}) \to \mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \to [1].$$

Proposition 2.1 shows that $\mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) = 0$, so the result follows. □

Definition 2.4. Let $\mathcal{M}$ be a locally free shtuka on $S \mathbin{\check{\otimes}} \mathcal{O}_K$ such that $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. By Lemma 2.1 the natural map (2.1) is a quasi-isomorphism. We define *the local germ map*

$$\mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M})$$

as its inverse. The adjective "local" signifies that it involves a shtuka defined over a semil-local ring $\mathcal{O}_K$. Observe that the local germ map is a quasi-isomorphism by construction.

Proposition 2.5. *Let $\mathcal{M}$ be a locally free shtuka on $S \mathbin{\check{\otimes}} \mathcal{O}_K$. If $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then $\mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M})$ and $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M})$ are concentrated in degree* 1.

Proof. The complex $\mathrm{R}\Gamma(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M})$ is concentrated in degrees 0 and 1 since $\mathrm{Spec}(S \mathbin{\check{\otimes}} \mathcal{O}_K)$ is affine. The complex $\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M})$ is concentrated in degrees 1 and 2 by Proposition 1.2. As these complexes are quasi-isomorphic via the local germ map the conclusion follows. □

Proposition 2.6. *Let $\mathcal{M} = [\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1]$ be a locally free shtuka on $S \mathbin{\check{\otimes}} \mathcal{O}_K$ and let $x \in \mathcal{M}_1$. Assume that $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent.*

(1) *There exists a unique $y \in \mathcal{M}_0(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ such that $(i-j)(y) = x$.*
(2) *Consider the composition*

$$\mathrm{H}^1(S \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \xrightarrow{\text{local}} \mathrm{H}^1_g(S \mathbin{\check{\otimes}} K, \mathcal{M}) \xrightarrow{\sim} \mathrm{H}^0\Big(S \mathbin{\check{\otimes}} K, \tfrac{\mathcal{M}(S^{\#} \widehat{\otimes} K)}{\mathcal{M}(S \mathbin{\check{\otimes}} K)}\Big)$$

of the local germ map and the natural isomorphism of Proposition 1.2. This composition sends the class of x to the image of y in the quotient $\mathcal{M}_0(S^{\#} \mathbin{\widehat{\otimes}} K)/\mathcal{M}_0(S \mathbin{\check{\otimes}} K)$.

Proof. (1) $\mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M})$ is represented by the complex

$$\Big[\mathcal{M}_0(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \xrightarrow{i-j} \mathcal{M}_1(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)\Big].$$

By Proposition 2.1 $\mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) = 0$. So the map $i - j$ in the complex above is a bijection and (1) follows.

(2) Consider the maps

$$\mathrm{H}^1_g(S \check{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{H}^1(S \check{\otimes} \mathcal{O}_K, \mathcal{M}),$$
$$\mathrm{H}^1_g(S \check{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{H}^0\Big(S \check{\otimes} \mathcal{O}_K, \tfrac{\mathcal{M}(S^\# \widehat{\otimes} \mathcal{O}_K)}{\mathcal{M}(S \check{\otimes} \mathcal{O}_K)}\Big)$$

determined by the natural maps of complexes

$$\mathrm{R}\Gamma_g(S \check{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S \check{\otimes} \mathcal{O}_K, \mathcal{M}),$$
$$\mathrm{R}\Gamma_g(S \check{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma\Big(S \check{\otimes} \mathcal{O}_K, \tfrac{\mathcal{M}(S^\# \widehat{\otimes} \mathcal{O}_K)}{\mathcal{M}(S \check{\otimes} \mathcal{O}_K)}\Big)[-1].$$

In order to prove (2) it is enough to produce a cohomology class $h \in \mathrm{H}^1_g(S \check{\otimes} \mathcal{O}_K, \mathcal{M})$ which maps to the class of x in $\mathrm{H}^1(S \check{\otimes} \mathcal{O}_K, \mathcal{M})$ and to the class of y in $\mathcal{M}_0(S^\# \widehat{\otimes} \mathcal{O}_K)/\mathcal{M}_0(S \check{\otimes} \mathcal{O}_K)$.

By definition $\mathrm{R}\Gamma_g(S \check{\otimes} \mathcal{O}_K, \mathcal{M})$ is represented by the total complex of the double complex

$$\begin{tikzcd}
\mathcal{M}_1(S \check{\otimes} \mathcal{O}_K) \arrow[r] & \mathcal{M}_1(S^\# \widehat{\otimes} \mathcal{O}_K) \\
\mathcal{M}_0(S \check{\otimes} \mathcal{O}_K) \arrow[r] \arrow[u, "i-j"] & \mathcal{M}_0(S^\# \widehat{\otimes} \mathcal{O}_K) \arrow[u, "j-i"']
\end{tikzcd}$$

The element $(x, y) \in \mathcal{M}_1(S \check{\otimes} \mathcal{O}_K) \oplus \mathcal{M}_0(S^\# \widehat{\otimes} \mathcal{O}_K)$ is a 1-cocyle in the total complex since $x + (j - i)(y) = 0$ by definition of y. By construction (x, y) maps to the class of x in $\mathrm{H}^1(S \check{\otimes} \mathcal{O}_K, \mathcal{M})$ and to the class of y in the quotient $\mathcal{M}_0(S^\# \widehat{\otimes} \mathcal{O}_K)/\mathcal{M}_0(S \check{\otimes} \mathcal{O}_K)$. Thus (2) follows. □

3. Global cohomology

So far we worked exclusively with affine τ-schemes. However for the proof of the class number formula it is necessary to consider schemes of a more general kind. In this section we give expressions for shtuka cohomology on general τ-schemes with τ a partial Frobenius endomorphism. The non-affine schemes appearing in our applications carry the τ-structures of this kind.

It is important to stress that we give these expressions for expository purposes only. Even though they will not be used in the rest of the text, one hopes that they lend more substance to the rather abstract theory of Chapter 1.

Lemma 3.1. *Let X be a τ-scheme. If $\tau\colon X \to X$ is the q-power map then for every $\mathcal{O}_X$-module shtuka $\mathcal{M} = [\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1]$ there is a natural quasi-isomorphism* $\mathrm{R}\Gamma(X, \mathcal{M}) \cong \mathrm{R}\Gamma(X, [\mathcal{M}_0 \xrightarrow{i-j} \mathcal{M}_1])$.

To make sense of the differential $i - j$ we use the fact that τ is the identity on the underlying topological space of X so that $\tau_* \mathcal{M}_1 = \mathcal{M}_1$ as abelian

sheaves. The functor $\mathrm{R}\Gamma(X,-)$ on the right hand side is the derived global sections functor for abelian sheaves on the topological space X.

In a way this formula justifies the use of notation $\mathrm{R}\Gamma$ for shtuka cohomology. With some precautions one may think of a shtuka as a two term complex of sheaves like the one above. Shtuka cohomology is then the usual sheaf cohomology of this complex.

Lemma 3.2. *Let S be an $\mathbb{F}_q$-algebra and X a scheme over $\mathbb{F}_q$. If the τ-structure on* $\operatorname{Spec} S\times X$ *is given by the endomorphism which acts as the identity on S and as the q-power map on X then for every quasi-coherent shtuka*

$$\mathcal{M} = \Big[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big]$$

on $\operatorname{Spec} S \times X$ *there is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma(\operatorname{Spec} S \times X, \mathcal{M}) \cong \mathrm{R}\Gamma(X, [\pi_*\mathcal{M}_0 \xrightarrow{i-j} \pi_*\mathcal{M}_1]).$$

Here $\pi\colon \operatorname{Spec} S \times X \to X$ *is the projection.*

4. Čech cohomology

Fix a *noetherian* $\mathbb{F}_q$-algebra S and a smooth projective curve X over $\mathbb{F}_q$. We call S the *coefficient algebra* and X the *base curve*. Consider the product $\operatorname{Spec} S\times X$. To simplify the notation we will write $S\times X$ instead of $\operatorname{Spec} S\times X$. We equip $S\times X$ with the τ-scheme structure given by the endomorphism which acts as the identity on S and as the q-power map on X. In this section we introduce a Čech method for computing shtuka cohomology on $S \times X$.

As in the introduction to this chapter we fix an affine open dense subscheme $\operatorname{Spec} R \subset X$. We denote K the product of local fields of X at the points in the complement of $\operatorname{Spec} R$. The notation and the terminology of Section 3.5 applies to K. In particular $\mathcal{O}_K \subset K$ stands for the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ denotes the Jacobson radical. The natural topology on $\mathcal{O}_K$ makes it into a compact open $\mathbb{F}_q$-subalgebra of a locally compact $\mathbb{F}_q$-algebra K.

In accordance with the conventions of Section 3.11 the τ-structures on the rings $S \otimes R$, $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K$ and $S^{\#} \mathbin{\widehat{\otimes}} K$ are given by endomorphisms which act as the identity on S and as the q-power map on the other factor.

Definition 4.1. Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times X$. The *Čech chomology complex* of $\mathcal{M}$ is the S-module complex

$$\mathrm{R}\check{\Gamma}(S \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(S \otimes R, \mathcal{M}) \oplus \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{M})\Big].$$

Here the differential is the difference of the natural maps. The n-th cohomology group of this complex is denoted $\check{\mathrm{H}}^n(S \times X, \mathcal{M})$.

The goal of this section is to construct a natural quasi-isomorphism $\mathrm{R}\check{\Gamma}(S\times X, \mathcal{M}) \cong \mathrm{R}\Gamma(S \times X, \mathcal{M})$. To do it we need some preparation.

Lemma 4.2. *The natural commutative square*

$$\begin{tikzcd} \operatorname{Spec}(S^{\#} \widehat{\otimes} K) \arrow[r, "\iota'"] \arrow[d, "f'"] & \operatorname{Spec}(S^{\#} \widehat{\otimes} \mathcal{O}_K) \arrow[d, "f"] \\ \operatorname{Spec}(S \otimes R) \arrow[r, "\iota"] & S \times X \end{tikzcd}$$

is cartesian. Furthermore $\operatorname{Spec}(S \otimes R)$ *and* $\operatorname{Spec}(S^{\#} \widehat{\otimes} \mathcal{O}_K)$ *form a flat covering of* $S \times X$.

Proof. Proposition 3.6.1 implies that the square is cartesian. The complement of $\operatorname{Spec}(S \otimes R)$ in $S \times X$ is $\operatorname{Spec}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ so the images of $\operatorname{Spec}(S \otimes R)$ and $\operatorname{Spec}(S^{\#} \widehat{\otimes} \mathcal{O}_K)$ cover $S \times X$. It remains to prove that $\operatorname{Spec}(S^{\#} \widehat{\otimes} \mathcal{O}_K)$ is flat over $S \times X$.

Pick an affine open subscheme $\operatorname{Spec} R' \subset X$ which contains $\operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K$. Shrinking $\operatorname{Spec} R'$ if necessary we can find an element $r' \in R'$ which is a uniformizer of $\mathcal{O}_K$. By Proposition 3.6.6 the ring $S^{\#} \widehat{\otimes} \mathcal{O}_K$ is complete with respect to the ideal $S^{\#} \widehat{\otimes} \mathfrak{m}_K$. This ideal is generated by $\mathfrak{m}_K$ according to Proposition 3.6.4. As r' is a generator of $\mathfrak{m}_K$ we deduce that $S^{\#} \widehat{\otimes} \mathcal{O}_K$ is the completion of $S \otimes R'$ with respect to $S \otimes r'R'$. Now the fact that $S \otimes R'$ is noetherian implies that $S^{\#} \widehat{\otimes} \mathcal{O}_K$ is flat over $S \otimes R'$ and therefore over $S \times X$. □

Let $\mathcal{F}$ be a quasi-coherent sheaf on $S \times X$. We define a complex of sheaves on $S \times X$:

$$\mathcal{C}(\mathcal{F}) = \left[\iota_* \iota^* \mathcal{F} \oplus f_* f^* \mathcal{F} \to g_* g^* \mathcal{F}\right]$$

where $g\colon \operatorname{Spec}(S^{\#} \widehat{\otimes} K) \to S \times X$ is the natural map and the differential is the difference of the natural maps as in the definition of $\mathrm{R}\check{\Gamma}$. The sum of adjunction units provides us with a natural morphism $\mathcal{F}[0] \to \mathcal{C}(\mathcal{F})$.

Lemma 4.3. *If* $\mathcal{F}$ *is a quasi-coherent sheaf on* $S \times X$ *then the natural map* $\mathcal{F}[0] \to \mathcal{C}(\mathcal{F})$ *is a quasi-isomorphism.*

Proof. We first show that natural sequence

$$0 \to \mathcal{O}_{S \times X} \to \iota_* \iota^* \mathcal{O}_{S \times X} \oplus f_* f^* \mathcal{O}_{S \times X} \to g_* g^* \mathcal{O}_{S \times X} \to 0 \tag{4.1}$$

is exact. As the commutative diagram of Lemma 4.2 is cartesian and the morphism $f\colon \operatorname{Spec}(S^{\#} \widehat{\otimes} \mathcal{O}_K) \to S \times X$ is affine the pullback of (4.1) to $\operatorname{Spec}(S^{\#} \widehat{\otimes} \mathcal{O}_K)$ is

$$0 \to S^{\#} \widehat{\otimes} \mathcal{O}_K \xrightarrow{(1,\iota')} (S^{\#} \widehat{\otimes} \mathcal{O}_K) \oplus (S^{\#} \widehat{\otimes} K) \xrightarrow{(\iota',-1)} S^{\#} \widehat{\otimes} K \to 0.$$

This sequence is clearly exact. The same argument shows that the pullback of (4.1) to $\operatorname{Spec}(S \otimes R)$ is exact. As $\operatorname{Spec}(S \otimes R)$ and $\operatorname{Spec}(S^{\#} \widehat{\otimes} \mathcal{O}_K)$ form a flat covering of $S \times X$ it follows that (4.1) is exact.

Now let $\mathcal{F}$ be a quasi-coherent sheaf on $S \times X$. Consider the morphism $g\colon \operatorname{Spec}(S^{\#} \widehat{\otimes} K) \to S \times X$. As $S \times X$ is separated over $\mathbb{F}_q$ the morphism g is affine. Thus the natural map

$$(g_* \mathcal{O}_{S^{\#} \widehat{\otimes} K}) \otimes_{\mathcal{O}_{S \times X}} \mathcal{F} \to g_* g^* \mathcal{F}$$

is an isomorphism. The same argument applies to the maps ι and f. We conclude that

$$\mathcal{C}(\mathcal{O}_{S \times X}) \otimes_{\mathcal{O}_{S \times X}} \mathcal{F} = \mathcal{C}(\mathcal{F}).$$

Consider the distinguished triangle

$$\mathcal{O}_{S \times X}[0] \to \mathcal{C}(\mathcal{O}_{S \times X}) \to C \to [1]$$

extending the natural quasi-isomorphism $\mathcal{O}_{S \times X}[0] \to \mathcal{C}(\mathcal{O}_{S \times X})$. Applying the functor $- \otimes_{\mathcal{O}_{S \times X}} \mathcal{F}$ we obtain a distinguished triangle

$$\mathcal{F}[0] \to \mathcal{C}(\mathcal{F}) \to C \otimes_{\mathcal{O}_{S \times X}} \mathcal{F} \to [1]$$

where the first arrow is the natural map $\mathcal{F}[0] \to \mathcal{C}(\mathcal{F})$. By construction C is a bounded acyclic complex of flat $\mathcal{O}_{S \times X}$-modules. Hence the complex $C \otimes_{\mathcal{O}_{S \times X}} \mathcal{F}$ is acyclic and the first arrow in the triangle above is a quasi-isomorphism. □

Definition 4.4. Let $\mathcal{F}$ be a quasi-coherent sheaf on $S \times X$.

(1) The *Čech cohomology complex* of $\mathcal{F}$ is the S-module complex

$$\mathrm{R}\breve{\Gamma}(S \times X, \mathcal{F}) = \Gamma(S \times X, \mathcal{C}(\mathcal{F})).$$

(2) We define a natural map $\mathrm{R}\breve{\Gamma}(S \times X, \mathcal{F}) \to \mathrm{R}\Gamma(S \times X, \mathcal{F})$ as the composition

$$\Gamma(S \times X, \mathcal{C}(\mathcal{F})) \to \mathrm{R}\Gamma(S \times X, \mathcal{C}(\mathcal{F})) \xleftarrow{\sim} \mathrm{R}\Gamma(S \times X, \mathcal{F})$$

of the natural map $\Gamma \to \mathrm{R}\Gamma$ and the quasi-isomorphism provided by Lemma 4.3.

More explicitly

$$\mathrm{R}\breve{\Gamma}(S \times X, \mathcal{F}) = \Big[\Gamma(S \otimes R, \mathcal{F}) \oplus \Gamma(S^{\#} \widehat{\otimes} \mathcal{O}_K, \mathcal{F}) \to \Gamma(S^{\#} \widehat{\otimes} K, \mathcal{F})\Big].$$

The differential is as in the definition of $\mathrm{R}\breve{\Gamma}$ for shtukas.

To make the expressions in the rest of the section more legible we will generally omit the argument $S \times X$ of the functors Γ, $\mathrm{R}\breve{\Gamma}$ and $\mathrm{R}\Gamma$ for quasi-coherent sheaves and shtukas. The same applies to the associated complex functor Γ_{a}.

Theorem 4.5. *The natural map* $\mathrm{R}\breve{\Gamma}(\mathcal{F}) \to \mathrm{R}\Gamma(\mathcal{F})$ *is a quasi-isomorphism for every quasi-coherent sheaf* $\mathcal{F}$.

Proof. By construction $\mathcal{C}(\mathcal{F})$ sits in a distinguished triangle

$$\mathcal{C}(\mathcal{F}) \to (\iota_*\iota^*\mathcal{F} \oplus f_*f^*\mathcal{F})[0] \to g_*g^*\mathcal{F}[0] \to [1].$$

Applying $\Gamma(S\times X, -)$ and $\mathrm{R}\Gamma(S\times X, -)$ we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
\Gamma(\mathcal{C}(\mathcal{F})) \arrow[r] \arrow[d] & \Gamma(\iota_*\iota^*\mathcal{F} \oplus f_*f^*\mathcal{F})[0] \arrow[r] \arrow[d] & \Gamma(g_*g^*\mathcal{F})[0] \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{C}(\mathcal{F})) \arrow[r] & \mathrm{R}\Gamma(\iota_*\iota^*\mathcal{F} \oplus f_*f^*\mathcal{F}) \arrow[r] & \mathrm{R}\Gamma(g_*g^*\mathcal{F}) \arrow[r] & {[1]}
\end{tikzcd}$$

We will prove that the second and third vertical arrows in this diagram are quasi-isomorphisms. It follows that the first arrow is a quasi-isomorphism and so the lemma is proven.

Consider the third vertical arrow. The map g is affine so that $g_*g^*\mathcal{F}[0] = \mathrm{R}g_*g^*\mathcal{F}$. Hence $\mathrm{R}\Gamma(g_*g^*\mathcal{F}) = \mathrm{R}\Gamma(\mathrm{R}g_*g^*\mathcal{F}) = \mathrm{R}\Gamma(S^\# \widehat{\otimes} K, \mathcal{F})$. As $\mathrm{Spec}(S^\# \widehat{\otimes} K)$ is affine the natural map $\Gamma(S^\# \widehat{\otimes} K, \mathcal{F})[0] \to \mathrm{R}\Gamma(S^\# \widehat{\otimes} K, \mathcal{F})$ is a quasi-isomorphism. Hence the third vertical map in the diagram above is a quasi-isomorphism. The maps ι and f are also affine whence the same argument shows that the second vertical map is a quasi-isomorphism. □

Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times X$. Define a complex of shtukas on $S \times X$:

$$\mathcal{C}(\mathcal{M}) = \Big[\iota_*\iota^*\mathcal{M} \oplus f_*f^*\mathcal{M} \to g_*g^*\mathcal{M}\Big].$$

Here the differential is the difference of the natural maps as in the definition of $\mathrm{R}\check{\Gamma}(S \times X, \mathcal{M})$. The sum of the adjunction units gives a natural morphism $\mathcal{M}[0] \to \mathcal{C}(\mathcal{M})$.

Lemma 4.6. *If $\mathcal{M}$ is a quasi-coherent shtuka on $S \times X$ then* $\mathrm{R}\check{\Gamma}(\mathcal{M}) = \Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M}))$.

Proof. Let $f\colon \mathcal{N} \to \mathcal{N}'$ be a morphism of shtukas and let $C = [\mathcal{N} \to \mathcal{N}']$ be its mapping fiber. The associated complex functor Γ_{a} is defined in such a way that $\Gamma_{\mathrm{a}}(C)$ is the mapping fiber of $\Gamma_{\mathrm{a}}(f)$. Applying this observation to $C = \mathcal{C}(\mathcal{M})$ we get the result. □

Suppose that a quasi-coherent shtuka $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

In Section 1.5 we constructed a natural distinguished triangle for the functor Γ_{a} (see Definition 1.5.2). The distinguished triangle of $\Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M}))$ looks as follows:

$$\Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M})) \to \Gamma(\mathcal{C}(\mathcal{M}_0)) \xrightarrow{i-j} \Gamma(\mathcal{C}(\mathcal{M}_1)) \to [1] \tag{4.2}$$

Definition 4.7. We define a natural distinguished triangle

$$\mathrm{R}\check{\Gamma}(\mathcal{M}) \to \mathrm{R}\check{\Gamma}(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\check{\Gamma}(\mathcal{M}_1) \to [1]$$

to be the triangle (4.2) where we identify $\Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M}))$ with $\mathrm{R}\check{\Gamma}(\mathcal{M})$, $\Gamma(\mathcal{C}(\mathcal{M}_0))$ with $\mathrm{R}\check{\Gamma}(\mathcal{M}_0)$ and $\Gamma(\mathcal{C}(\mathcal{M}_1))$ with $\mathrm{R}\check{\Gamma}(\mathcal{M}_1)$.

Lemma 4.8. *If $\mathcal{M}$ is a quasi-coherent shtuka on $S \times X$ then the natural map $\mathcal{M}[0] \to \mathcal{C}(\mathcal{M})$ is a quasi-isomorphism.*

Proof. Follows instantly from Lemma 4.3. □

Definition 4.9. We define a natural map $\mathrm{R}\check{\Gamma}(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M})$ as the composition

$$\Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M})) \to \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M})) \xleftarrow{\sim} \mathrm{R}\Gamma(\mathcal{M})$$

of the natural map $\Gamma_{\mathrm{a}} \to \mathrm{R}\Gamma$ and the quasi-isomorphism provided by Lemma 4.8.

Lemma 4.10. *If $\mathcal{M}$ is a quasi-coherent shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

then the natural maps of Definitions 4.9 and 4.4 form a morphism of natural distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\check{\Gamma}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd} \tag{4.3}$$

Here the top triangle is the one of Definition 4.7 and the bottom triangle is the one of Theorem 1.5.6.

Proof. According to Theorem 1.5.6 the natural diagram

$$\begin{tikzcd}
\Gamma_{\mathrm{a}}(\mathcal{C}(\mathcal{M})) \arrow[r] \arrow[d] & \Gamma(\mathcal{C}(\mathcal{M}_0)) \arrow[r, "i-j"] \arrow[d] & \Gamma(\mathcal{C}(\mathcal{M}_1)) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{C}(\mathcal{M})) \arrow[r] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_0)) \arrow[r, "i-j"] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_1)) \arrow[r] & {[1]}
\end{tikzcd}$$

is a morphism of distinguished triangles. Furthermore the distinguished triangles of Theorem 1.5.6 are natural. Hence the map $\mathcal{M}[0] \to \mathcal{C}(\mathcal{M})$ induces an isomorphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{C}(\mathcal{M})) \arrow[r] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_0)) \arrow[r, "i-j"] & \mathrm{R}\Gamma(\mathcal{C}(\mathcal{M}_1)) \arrow[r] & {[1]} \\
\mathrm{R}\Gamma(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{R}\Gamma(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[u] & \mathrm{R}\Gamma(\mathcal{M}_1) \arrow[r] \arrow[u] & {[1]}
\end{tikzcd}$$

By construction the second and third vertical arrows are induced by the natural quasi-isomorphisms $\mathcal{M}_0[0] \to \mathcal{C}(\mathcal{M}_0)$ and $\mathcal{M}_1[0] \to \mathcal{C}(\mathcal{M}_1)$. So the result follows. □

Theorem 4.11. *Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times X$.*

(1) *The natural map $\mathrm{R}\breve{\Gamma}(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M})$ is a quasi-isomorphism.*
(2) *The morphism of distinguished triangles* (4.3) *is an isomorphism.*

Proof. Theorem 4.5 shows that the natural map $\mathrm{R}\breve{\Gamma}(\mathcal{F}) \to \mathrm{R}\Gamma(\mathcal{F})$ is a quasi-isomorphism for every quasi-coherent sheaf $\mathcal{F}$ on $S \times X$. Hence the result follows from Lemma 4.10. □

Later in the text we will use the second assertion of Theorem 4.11 to control the distinguished triangle of $\mathrm{R}\Gamma(\mathcal{M})$.

5. Compactly supported cohomology

We continue using the notation and the conventions of Section 4. In this section we assume that the coefficient algebra S carries a structure of a locally compact $\mathbb{F}_q$-algebra. A typical example of S relevant to our applications is the discrete algebra $\mathbb{F}_q[t]$ and the locally compact algebra $\mathbb{F}_q(\!(t^{-1})\!)$.

Definition 5.1. Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \otimes R$. The *compactly supported cohomology complex* of $\mathcal{M}$ is the S-module complex

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M}) = \Big[\mathrm{R}\Gamma(S \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{M})\Big].$$

Here the differential is induced by the natural inclusion $S \otimes R \to S^{\#} \mathbin{\widehat{\otimes}} K$. The n-th cohomology group of $\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M})$ is denoted $\mathrm{H}^n_c(S \otimes R, \mathcal{M})$.

Let $\mathcal{M}$ be a quasi-coherent shtuka on $S \times X$. Recall that

$$\mathrm{R}\breve{\Gamma}(S \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(S \otimes R, \mathcal{M}) \oplus \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{M})\Big].$$

So the complex $\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M})$ embeds naturally into $\mathrm{R}\breve{\Gamma}(S \times X, \mathcal{M})$. Together with the quasi-isomorphism $\mathrm{R}\breve{\Gamma}(S \times X, \mathcal{M}) \cong \mathrm{R}\Gamma(S \times X, \mathcal{M})$ of Theorem 4.11 this embedding gives us a natural map

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma(S \times X, \mathcal{M}) \tag{5.1}$$

Proposition 5.2. *Let $\mathcal{M}$ be a locally free shtuka on $S \times X$. If $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then the natural map $\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma(S \times X, \mathcal{M})$ is a quasi-isomorphism.*

The condition that $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent may be interpreted as saying that $\mathcal{M}$ is an extension by zero of a shtuka on the open τ-subscheme $\mathrm{Spec}(S \otimes R) \subset S \times X$.

Proof of Proposition 5.2. The natural map $\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M}) \to \mathrm{R}\breve{\Gamma}(S \times X, \mathcal{M})$ extends to a distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M}) \to \mathrm{R}\breve{\Gamma}(S \times X, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) \to [1].$$

The result follows since $\mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K, \mathcal{M}) = 0$ by Proposition 2.1. □

Recall that according to Definition 1.1

$$\mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M}) = \big[\mathrm{R}\Gamma(S \mathbin{\check{\otimes}} K, \mathcal{M}) \to \mathrm{R}\Gamma(S^{\#} \mathbin{\widehat{\otimes}} K, \mathcal{M})\big].$$

The natural map $S \otimes R \to S \mathbin{\check{\otimes}} K$ thus defines a morphism

$$\mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M}) \tag{5.2}$$

Definition 5.3. Let $\mathcal{M}$ be a locally free shtuka on $S \times X$ such that $\mathcal{M}(S \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent. We define the *global germ map* in $D(S)$ as the composition

$$\mathrm{R}\Gamma(S \times X, \mathcal{M}) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{c}}(S \otimes R, \mathcal{M}) \xrightarrow{(5.2)} \mathrm{R}\Gamma_g(S \mathbin{\check{\otimes}} K, \mathcal{M})$$

where the first arrow is the quasi-isomorphism (5.1). The adjective "global" indicates that this map involves a shtuka on the whole $S \times X$ as opposed to $S \mathbin{\check{\otimes}} \mathcal{O}_K$.

6. Local-global compatibility

In this section we keep the conventions and the notation of Section 4. We assume that the coefficient algebra is a local field F. Its ring of integers will be denoted $\mathcal{O}_F$.

Let $\mathcal{M}$ be a locally free shtuka on $F \times X$. Under assumption that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent we have two maps from the cohomology of $\mathcal{M}$ to the germ cohomology $\mathrm{R}\Gamma_g(F \mathbin{\check{\otimes}} K, \mathcal{M})$:

- the local germ map of Definition 2.4,
- the global germ map of Definition 5.3.

They form a square in the derived category $D(F)$:

$$\begin{tikzcd} \mathrm{R}\Gamma(F \times X, \mathcal{M}) \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(F \mathbin{\check{\otimes}} K, \mathcal{M}) \arrow[d, equal] \\ \mathrm{R}\Gamma(F \mathbin{\check{\otimes}} \mathcal{O}_K, \mathcal{M}) \arrow[r, "\text{local}", "\sim"'] & \mathrm{R}\Gamma_g(F \mathbin{\check{\otimes}} K, \mathcal{M}) \end{tikzcd} \tag{6.1}$$

The left arrow in this square is the pullback map.

The definitions of the local and the global germ map have nothing in common so there is no a priori reason for (6.1) to be commutative. Nevertheless we will prove that (6.1) commutes under the additional assumption that $\mathcal{M}$ extends as a locally free shtuka to $\mathcal{O}_F \times X$.

Theorem 6.1. *If $\mathcal{M}$ is a locally free shtuka on $\mathcal{O}_F \times X$ such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent then the square* (6.1) *is commutative.*

Later in this chapter we will show that the left arrow in (6.1) is a quasi-isomorphism provided $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent (Theorem 7.11). The local germ map is a quasi-isomorphism by construction. The commutativity of (6.1) then implies that the global germ map is a quasi-isomorphism, a property which is in no way evident from its definition.

Proof of Theorem 6.1. Take H^1 of (6.1) and extend it to the left as follows:

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{O}_F \times X, \mathcal{M}) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{H}^1(F \times X, \mathcal{M}) \arrow[r] \arrow[d] & \mathrm{H}^1_g(F \check{\otimes} K, \mathcal{M}) \arrow[d, equal] \\
\mathrm{H}^1(\mathcal{O}_F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[r, dashed] & \mathrm{H}^1(F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[r] & \mathrm{H}^1_g(F \check{\otimes} K, \mathcal{M})
\end{tikzcd} \tag{6.2}$$

The three additional maps are the pullback morphisms. We proceed to prove that the outer rectangle of (6.2) commutes.

Theorem 4.11 equips us with natural isomorphisms

$$\mathrm{H}^1(\mathcal{O}_F \times X, \mathcal{M}) \cong \check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M}),$$
$$\mathrm{H}^1(F \times X, \mathcal{M}) \cong \check{\mathrm{H}}^1(F \times X, \mathcal{M}),$$

while Proposition 1.2 provides a natural isomorphism

$$\mathrm{H}^1_g(F \check{\otimes} K, \mathcal{M}) \cong \mathrm{H}^0(F \check{\otimes} K, \mathcal{Q})$$

where

$$\mathcal{Q} = \frac{\mathcal{M}(F^\# \widehat{\otimes} K)}{\mathcal{M}(F \check{\otimes} K)}.$$

Using them we rewrite (6.2) as

$$\begin{tikzcd}
\check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M}) \arrow[r] \arrow[d] & \check{\mathrm{H}}^1(F \times X, \mathcal{M}) \arrow[r] & \mathrm{H}^0(F \check{\otimes} K, \mathcal{Q}) \arrow[d, equal] \\
\mathrm{H}^1(\mathcal{O}_F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[r] & \mathrm{H}^1(F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[r] & \mathrm{H}^0(F \check{\otimes} K, \mathcal{Q})
\end{tikzcd} \tag{6.3}$$

The middle arrow is omitted since it is not easy to describe in terms of Čech cohomology.

Let the shtuka $\mathcal{M}$ be given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Let S be either $\mathcal{O}_F$ or F. By definition $\mathrm{R}\check{\Gamma}(S \times X, \mathcal{M})$ is the total complex of the double complex

$$\begin{tikzcd}
\mathcal{M}_1(S \otimes R) \oplus \mathcal{M}_1(S^\# \widehat{\otimes} \mathcal{O}_K) \arrow[r, "\text{difference}"] & \mathcal{M}_1(S^\# \widehat{\otimes} K) \\
\mathcal{M}_0(S \otimes R) \oplus \mathcal{M}_0(S^\# \widehat{\otimes} \mathcal{O}_K) \arrow[r, "\text{difference}"] \arrow[u, "i-j"] & \mathcal{M}_0(S^\# \widehat{\otimes} K) \arrow[u, "j-i"']
\end{tikzcd}$$

So a cohomology class in $\check{\mathrm{H}}^1(S \times X, \mathcal{M})$ is represented by a triple

$$(a, x, b) \in \mathcal{M}_1(S \otimes R) \oplus \mathcal{M}_1(S^\# \widehat{\otimes} \mathcal{O}_K) \oplus \mathcal{M}_0(S^\# \widehat{\otimes} K)$$

satisfying $a - x + (j - i)(b) = 0$.

Fix a cohomology class $h \in \check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M})$. We want to compute its image under the composition

$$\check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M}) \to \check{\mathrm{H}}^1(F \times X, \mathcal{M}) \to \mathrm{H}^0(F \check{\otimes} K, \mathcal{Q})$$

of the two top arrows in (6.3). Let (a, x, b) be a triple representing h. The image of h in $\check{\mathrm{H}}^1(F \times X, \mathcal{M})$ is represented by the same triple (a, x, b). From Definition 5.3 it follows that the map $\check{\mathrm{H}}^1(F \times X, \mathcal{M}) \to \mathrm{H}^1_g(F \check{\otimes} K, \mathcal{M})$ of (6.3) is a composition

$$\check{\mathrm{H}}^1(F \times X, \mathcal{M}) \xleftarrow{\sim} \mathrm{H}^1_c(F \otimes R, \mathcal{M}) \to \mathrm{H}^0(F \check{\otimes} K, \mathcal{Q}) \tag{6.4}$$

By construction $\mathrm{R}\Gamma_c(F \otimes R, \mathcal{M})$ is the total complex of the double complex

$$\begin{tikzcd} \mathcal{M}_1(F \otimes R) \arrow[r] & \mathcal{M}_1(F^\# \widehat{\otimes} K) \\ \mathcal{M}_0(F \otimes R) \arrow[r] \arrow[u, "i-j"] & \mathcal{M}_0(F^\# \widehat{\otimes} K) \arrow[u, "j-i"'] \end{tikzcd}$$

So a cohomology class in $\mathrm{H}^1_c(F \otimes R, \mathcal{M})$ is represented by a pair

$$(a', b') \in \mathcal{M}_1(F \otimes R) \oplus \mathcal{M}_0(F^\# \widehat{\otimes} K)$$

such that $a' + (j - i)(b') = 0$.

The isomorphism $\mathrm{H}^1_c(F \otimes R, \mathcal{M}) \cong \check{\mathrm{H}}^1(F \times X, \mathcal{M})$ of (6.4) sends (a', b') to $(a', 0, b')$. Thus in order to compute the image of h in $\mathrm{H}^1_c(F \otimes R, \mathcal{M})$ we need to replace (a, x, b) with a cohomologous triple of the form $(a', 0, b')$. A triple is a coboundary if and only if it has the form

$$\big((i - j)(a'), (i - j)(y), a' - y\big)$$

where $a' \in \mathcal{M}_0(F \otimes R)$ and $y \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$. By Proposition 2.6 (1) there is a unique $y \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ such that $(i - j)(y) = x$. The triple $(0, x, -y)$ is then a coboundary so that (a, x, b) is cohomologous to $(a, 0, b + y)$ and the image of h in $\mathrm{H}^1_c(F \otimes R, \mathcal{M})$ is represented by $(a, b + y)$.

We are finally ready to compute the image of $h \in \check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M})$ under the composition (6.4):

$$\check{\mathrm{H}}^1(F \times X, \mathcal{M}) \xleftarrow{\sim} \mathrm{H}^1_c(F \otimes R, \mathcal{M}) \to \mathrm{H}^0(F \check{\otimes} K, \mathcal{Q})$$

By definition of $\mathcal{Q}$ we have

$$\mathrm{R}\Gamma(F \check{\otimes} K, \mathcal{Q}) = \left[\frac{\mathcal{M}_0(F^\# \widehat{\otimes} K)}{\mathcal{M}_0(F \check{\otimes} K)} \xrightarrow{i-j} \frac{\mathcal{M}_1(F^\# \widehat{\otimes} K)}{\mathcal{M}_1(F \check{\otimes} K)}\right].$$

The second arrow in (6.4) sends a pair (a', b') representing a class in $\mathrm{H}^1_c(F \otimes R, \mathcal{M})$ to the equivalence class $[b']$ of b' in the quotient $\mathcal{M}_0(F^\# \widehat{\otimes} K)/\mathcal{M}_0(F \check{\otimes} K)$. Above we demonstrated that the image of h in $\mathrm{H}^1_c(F \otimes R, \mathcal{M})$ is represented by the pair $(a, b + y)$. Hence the image of h in $\mathrm{H}^0(F \check{\otimes} K, \mathcal{Q})$ is given by the equivalence class $[b + y]$.

The key observation in this proof is that $[b+y]=[y]$. Indeed the left arrow in the natural commutative square

$$\begin{tikzcd} \mathcal{O}_F \check{\otimes} K \arrow[r] \arrow[d] & F \check{\otimes} K \arrow[d] \\ \mathcal{O}_F^{\#} \widehat{\otimes} K \arrow[r] & F^{\#} \widehat{\otimes} K. \end{tikzcd}$$

is an isomorphism by Proposition 3.4.12. Hence the homomorphism $\mathcal{O}_F^{\#} \widehat{\otimes} K \to F^{\#} \widehat{\otimes} K$ factors through $F \check{\otimes} K$ and the natural map $\mathcal{M}_0(\mathcal{O}_F^{\#} \widehat{\otimes} K) \to \mathcal{M}_0(F^{\#} \widehat{\otimes} K)/\mathcal{M}_0(F \check{\otimes} K)$ is zero. As $b \in \mathcal{M}_0(\mathcal{O}_F^{\#} \widehat{\otimes} K)$ by construction we conclude that $[b+y]=[y]$.

So far we have demonstrated the following. Let $h \in \check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M})$ be a cohomology class. If h is represented by a triple

$$(a,x,b) \in \mathcal{M}_1(\mathcal{O}_F \otimes R) \oplus \mathcal{M}_1(\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K) \oplus \mathcal{M}_0(\mathcal{O}_F^{\#} \widehat{\otimes} K)$$

then the image of h under the composition

$$\check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M}) \to \check{\mathrm{H}}^1(F \times X, \mathcal{M}) \to \mathrm{H}^0(F \check{\otimes} K, \mathcal{Q})$$

of the two top arrows in the square (6.3) is given by the equivalence class

$$[y] \in \mathcal{M}_0(F^{\#} \widehat{\otimes} K)/\mathcal{M}_0(F \check{\otimes} K)$$

where $y \in \mathcal{M}_0(F^{\#} \widehat{\otimes} \mathcal{O}_K)$ is the unique element satisfying $(i-j)(y)=x$. We are now in position to prove that the square (6.3) is commutative.

The cohomology classes in $\mathrm{H}^1(\mathcal{O}_F \check{\otimes} \mathcal{O}_K, \mathcal{M})$ are represented by elements of $\mathcal{M}_1(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$. By Proposition 3.4.12 the natural map $\mathcal{O}_F \check{\otimes} \mathcal{O}_K \to \mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K$ is an isomorphism. Hence we can identify $\mathcal{M}_1(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$ with $\mathcal{M}_1(\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K)$. The left arrow $\check{\mathrm{H}}^1(\mathcal{O}_F \times X, \mathcal{M}) \to \mathrm{H}^1(\mathcal{O}_F \check{\otimes} \mathcal{O}_K, \mathcal{M})$ of the square (6.3) sends the cocycle (a,x,b) to $x \in \mathcal{M}_1(\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K) = \mathcal{M}_1(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$. Now Proposition 2.6 (2) implies that the image of x under the composition of the two bottom arrows in (6.3) is $[y]$. Therefore the square (6.3) is commutative.

We deduce that the outer rectangle of (6.2) is commutative. Since the F-linear extension

$$F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{O}_F \times X, \mathcal{M}) \to \mathrm{H}^1(F \times X, \mathcal{M})$$

of the top horizontal map in this square is an isomorphism the right square of (6.2) is commutative too. By Proposition 2.5 the complex $\mathrm{R}\Gamma_g(F \check{\otimes} K, \mathcal{M})$ is concentrated in degree 1. Therefore commutativity of the right square of (6.2) implies commutativity of the main diagram (6.1) in the derived category $D(F)$. $\square$

7. Completed Čech cohomology

We keep the notation and the conventions of Section 4. Let F be a local field with the ring of integers $\mathcal{O}_F$. In this section we work with the coefficient algebra $\mathcal{O}_F$. The τ-structures on the tensor product rings below are as in Section 3.11.

We present a refined version of the Čech method for computing the cohomology of *coherent* shtukas on $\mathcal{O}_F \times X$. In essense it is the Čech method of Section 4 developed in the setting of formal schemes over $\operatorname{Spec} \mathcal{O}_F$. However we avoid the language of formal schemes to spare the reader technical difficulties.

Using the Čech method of this section we deduce Theorem 7.11 which captures the important cohomology concentration phenomenon for locally free shtukas $\mathcal{M}$ on $\mathcal{O}_F \times X$. This result will play a significant role in Chapter 6.

Definition 7.1. Let $\mathcal{M}$ be a coherent shtuka on $\mathcal{O}_F \times X$. The *completed Čech chomology complex* of $\mathcal{M}$ is

$$\mathrm{R}\widehat{\Gamma}(\mathcal{O}_F \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{M}) \oplus \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} K, \mathcal{M})\Big].$$

Here the differential is the difference of the natural maps.

Recall that the Čech complex of $\mathcal{M}$ is

$$\mathrm{R}\check{\Gamma}(\mathcal{O}_F \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\mathcal{O}_F \otimes R, \mathcal{M}) \oplus \mathrm{R}\Gamma(\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F^{\#} \widehat{\otimes} K, \mathcal{M})\Big].$$

We thus have a natural map

$$\mathrm{R}\check{\Gamma}(\mathcal{O}_F \times X, \mathcal{M}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{O}_F \times X, \mathcal{M}).$$

In the following we will prove that this map is a quasi-isomorphism provided the shtuka $\mathcal{M}$ is coherent. We will derive this result from the corresponding statement for coherent sheaves.

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{O}_F \times X$. Recall that

$$\mathrm{R}\check{\Gamma}(\mathcal{O}_F \times X, \mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \otimes R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F^{\#} \widehat{\otimes} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F^{\#} \widehat{\otimes} K, \mathcal{F})\Big].$$

We set

$$\mathrm{R}\widehat{\Gamma}(\mathcal{O}_F \times X, \mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \widehat{\otimes} K, \mathcal{F})\Big]$$

with the same differentials as in the definition of $\mathrm{R}\widehat{\Gamma}$ for shtukas. To improve the legibility we will generally omit the argument $\mathcal{O}_F \times X$ of the functors $\mathrm{R}\check{\Gamma}$ and $\mathrm{R}\widehat{\Gamma}$. By construction we have a natural map $\mathrm{R}\check{\Gamma}(\mathcal{F}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{F})$.

For technical reasons it will be more convenient for us to work with different presentations of the complexes $\mathrm{R}\breve{\Gamma}(\mathcal{F})$ and $\mathrm{R}\widehat{\Gamma}(\mathcal{F})$. We define the complexes

$$B(\mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \otimes R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \breve{\otimes} K, \mathcal{F})\Big],$$
$$\widehat{B}(\mathcal{F}) = \Big[\Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{F}) \oplus \Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K^{\#}, \mathcal{F}) \to \Gamma(\mathcal{O}_F \widehat{\otimes} K^{\#}, \mathcal{F})\Big]$$

with the same differentials as $\mathrm{R}\breve{\Gamma}(\mathcal{F})$ and $\mathrm{R}\widehat{\Gamma}(\mathcal{F})$.

Lemma 7.2. *The natural map $B(\mathcal{F}) \to \mathrm{R}\breve{\Gamma}(\mathcal{F})$ is an isomorphism.*

Proof. Follows from Proposition 3.4.12 since $\mathcal{O}_F$ is compact. □

Lemma 7.3. *The natural map $\widehat{B}(\mathcal{F}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{F})$ is an isomorphism.*

Proof. Follows from Proposition 3.4.11 since $\mathcal{O}_F$ is compact. □

Lemma 7.4. *For every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{O}_F \times X$ there exists a natural quasi-isomorphism $B(\mathcal{F}) \cong \mathrm{R}\Gamma(\mathcal{O}_F \times X, \mathcal{F})$.*

Proof. The natural map $B(\mathcal{F}) \to \mathrm{R}\breve{\Gamma}(\mathcal{F})$ is an isomorphism by Lemma 7.2. So the result is a consequence of Theorem 4.5. □

Lemma 7.5. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{O}_F \times X$. If $\mathfrak{m}_F^n \mathcal{F} = 0$ for some $n \gg 0$ then the natural map $B(\mathcal{F}) \to \widehat{B}(\mathcal{F})$ is an isomorphism.*

Proof. Consider the natural diagram

$$\Gamma(\mathcal{O}_F \breve{\otimes} K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \widehat{\otimes} K^{\#}, \mathcal{F}) \to \Gamma(\mathcal{O}_F/\mathfrak{m}_F^n \otimes K, \mathcal{F}).$$

By Proposition 3.6.4 the second arrow in this diagram is the reduction modulo $\mathfrak{m}_F^n$. The composite arrow is the reduction modulo $\mathfrak{m}_F^n$ by Propostion 3.6.5. Both arrows are isomorphisms since $\mathfrak{m}_F^n \mathcal{F} = 0$. Hence so is the first arrow. The same argument shows that the natural maps $\Gamma(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K, \mathcal{F}) \to \Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K^{\#}, \mathcal{F})$ and $\Gamma(\mathcal{O}_F \otimes R, \mathcal{F}) \to \Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{F})$ are isomorphisms. □

Lemma 7.6. *If $\mathcal{F}$ is a coherent sheaf on $\mathcal{O}_F \times X$ then the natural map $\widehat{B}(\mathcal{F}) \to \lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)$ is an isomorphism.*

Proof. Proposition 3.6.6 shows that the ring $\mathcal{O}_F \widehat{\otimes} K^{\#}$ is noetherian and complete with respect to the ideal $\mathfrak{m}_F \widehat{\otimes} K^{\#}$. According to Proposition 3.6.4 this ideal is generated by $\mathfrak{m}_F$. The $\mathcal{O}_F \widehat{\otimes} K^{\#}$-module $\Gamma(\mathcal{O}_F \widehat{\otimes} K^{\#}, \mathcal{F})$ is finitely generated. As a consequence it is complete with respect to $\mathfrak{m}_F(\mathcal{O}_F \widehat{\otimes} K^{\#})$. Hence the natural map

$$\Gamma(\mathcal{O}_F \widehat{\otimes} K^{\#}, \mathcal{F}) \to \lim_n \Gamma(\mathcal{O}_F \widehat{\otimes} K^{\#}, \mathcal{F}/\mathfrak{m}_F^n) = \lim_n \Gamma(\mathcal{O}_F \widehat{\otimes} K, \mathcal{F})/\mathfrak{m}_F^n$$

is an isomorphism. The same argument applies to the rings $\mathcal{O}_F \widehat{\otimes} R$ and $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K^{\#}$. □

The next two lemmas use the derived limit functor Rlim for abelian groups. We will use the Stacks Project [07KV] as a reference for Rlim.

Lemma 7.7. *If $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{O}_F \times X$ then the natural map $\lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n) \to \mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}^n)$ is a quasi-isomorphism.*

Proof. Let us denote

$$\begin{aligned} A_n &= \Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{F}/\mathfrak{m}_F^n), \\ B_n &= \Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K^\#, \mathcal{F}/\mathfrak{m}_F^n), \\ C_n &= \Gamma(\mathcal{O}_F \widehat{\otimes} K^\#, \mathcal{F}/\mathfrak{m}_F^n). \end{aligned}$$

The natural map in question extends to a morphism of distinguished triangles

$$\begin{tikzcd} \lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n) \arrow[r] \arrow[d] & \lim_n A_n \oplus B_n \arrow[r] \arrow[d] & \lim_n C_n \arrow[r] \arrow[d] & {[1]} \\ \mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n) \arrow[r] & \mathrm{Rlim}_n A_n \oplus B_n \arrow[r] & \mathrm{Rlim}_n C_n \arrow[r] & {[1]} \end{tikzcd}$$

So in order to show that the first vertical arrow is a quasi-isomorphism it is enough to prove that so are the second and the third vertical arrows.

The transition maps in the projective system $\{B_n\}_{n \geqslant 1}$ are surjective by construction. Hence this system satisfies the Mittag-Leffler condition [02N0]. As a consequence $\mathrm{R}^1\lim_n B_n = 0$ [07KW]. The natural map $\lim_n B_n \to \mathrm{Rlim}_n B_n$ is thus a quasi-isomorphism. The same argument applies to $\{A_n\}$ and $\{C_n\}$. □

Lemma 7.8. *If $\mathcal{F}$ is a coherent sheaf on $\mathcal{O}_F \times X$ then the natural map $\mathrm{H}^i(\widehat{B}(\mathcal{F})) \to \lim_n \mathrm{H}^i(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ is an isomorphism for every i.*

Proof. Lemma 7.6 implies that the map $\mathrm{H}^i(\widehat{B}(\mathcal{F})) \to \mathrm{H}^i(\lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ is an isomorphism for every i. At the same time the map $\mathrm{H}^i(\lim_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)) \to \mathrm{H}^i(\mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ is an isomorphism by Lemma 7.7. The cohomology group $\mathrm{H}^i(\mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ sits in a natural short exact sequence [07KY]

$$\begin{aligned} 0 \to \mathrm{R}^1\lim_n \mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)) \to \mathrm{H}^i(\mathrm{Rlim}_n \widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)) \to \\ \to \lim_n \mathrm{H}^i(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n) \to 0. \end{aligned}$$

We thus need to prove that the first term in this sequence vanishes.

Lemma 7.4 provides us with natural isomorphisms $\mathrm{H}^{i-1}(\mathcal{O}_F \times X, \mathcal{F}/\mathfrak{m}_F^n) \cong \mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$. As $\mathcal{O}_F \times X$ is proper over $\mathcal{O}_F$ it follows that the cohomology groups $\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ are finitely generated $\mathcal{O}_F/\mathfrak{m}_F^n$-modules. Thus the image of $\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^m))$ in $\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))$ is independent of m for $m \gg n$. In other words the projective system

$$\{\mathrm{H}^{i-1}(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n))\}_{n \geqslant 1}$$

satisfies the Mittag-Leffler condition [02N0]. Hence its first derived limit $\mathrm{R}^1\lim$ is zero [07KW]. □

Lemma 7.9. *If $\mathcal{F}$ is a coherent sheaf on $\mathcal{O}_F \times X$ then the natural map $\mathrm{R}\check{\Gamma}(\mathcal{F}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{F})$ is a quasi-isomorphism.*

Proof. In view of Lemmas 7.2 and 7.3 it is enough to prove that the natural map $B(\mathcal{F}) \to \widehat{B}(\mathcal{F})$ is a quasi-isomorphism. Let $i \in \mathbb{Z}$. We have a natural commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i(B(\mathcal{F})) \arrow[r] \arrow[d] & \mathrm{H}^i(\widehat{B}(\mathcal{F})) \arrow[d] \\
\lim_n \mathrm{H}^i(B(\mathcal{F}/\mathfrak{m}_F^n)) \arrow[r] & \lim_n \mathrm{H}^i(\widehat{B}(\mathcal{F}/\mathfrak{m}_F^n)).
\end{tikzcd}$$

The right arrow is an isomorphism by Lemma 7.8 while the bottom arrow is an isomorphism by Lemma 7.5. Thus in order to prove that the top arrow is an isomorphism it is enough to show that the left arrow is so. This arrow fits into a natural commutative square

$$\begin{tikzcd}
\mathrm{H}^i(\mathcal{O}_F \times X, \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^i(B(\mathcal{F})) \arrow[d] \\
\lim_n \mathrm{H}^i(\mathcal{O}_F \times X, \mathcal{F}/\mathfrak{m}_F^n) \arrow[r] & \lim_n \mathrm{H}^i(B(\mathcal{F}/\mathfrak{m}_F^n))
\end{tikzcd}$$

where the horizontal arrows are the natural isomorphisms of Lemma 7.4. According to the Theorem on formal functions [02OC] the left arrow in this square is an isomorphism. Whence the result follows. □

Theorem 7.10. *If $\mathcal{M}$ is a coherent shtuka on $\mathcal{O}_F \times X$ then the natural map $\mathrm{R}\check{\Gamma}(\mathcal{M}) \to \mathrm{R}\widehat{\Gamma}(\mathcal{M})$ is a quasi-isomorphism.*

Proof. Let the shtuka $\mathcal{M}$ be given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

We have a natural morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\check{\Gamma}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\check{\Gamma}(\mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\widehat{\Gamma}(\mathcal{M}) \arrow[r] & \mathrm{R}\widehat{\Gamma}(\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\widehat{\Gamma}(\mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd}$$

The second and third vertical arrows are quasi-isomorphisms by Lemma 7.9 so we are done. □

Theorem 7.11. *Let $\mathcal{M}$ be a locally free shtuka on $\mathcal{O}_F \times X$. If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent then the natural map $\mathrm{R}\Gamma(\mathcal{O}_F \times X, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K, \mathcal{M})$ is a quasi-isomorphism.*

Proof. Due to Theorem 7.10 it is enough to prove that the natural map

$$\mathrm{R}\widehat{\Gamma}(\mathcal{O}_F \times X, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K, \mathcal{M})$$

is a quasi-isomorphism. By definition

$$\mathrm{R}\widehat{\Gamma}(\mathcal{O}_F \times X, \mathcal{M}) = \Big[\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{M}) \oplus \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} K, \mathcal{M})\Big].$$

Hence it is enough to show that the complexes $\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{M})$ and $\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} K, \mathcal{M})$ are acyclic.

According to Proposition 3.6.6 the ring $\mathcal{O}_F \widehat{\otimes} R$ is noetherian and complete with respect to the ideal $\mathfrak{m}_F \widehat{\otimes} R$. By Proposition 3.6.4 the natural map $\mathcal{O}_F \widehat{\otimes} R \to \mathcal{O}_F/\mathfrak{m}_F \otimes R$ is surjective with kernel $\mathfrak{m} \widehat{\otimes} R$. Thus $\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} R, \mathcal{M}) = 0$ by Proposition 1.9.4. Applying the same argument to $\mathcal{O}_F \widehat{\otimes} K^{\#}$ we deduce that $\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} K^{\#}, \mathcal{M}) = 0$. The natural map $\mathcal{O}_F \widehat{\otimes} K^{\#} \to \mathcal{O}_F \widehat{\otimes} K$ is an isomorphism by Proposition 3.4.11 whence $\mathrm{R}\Gamma(\mathcal{O}_F \widehat{\otimes} K, \mathcal{M}) = 0$. □

8. Change of coefficients

Fix a noetherian $\mathbb{F}_q$-algebra S. In this section we study how the cohomology of shtukas on $S \times X$ changes under the pullback to $T \times X$ where T is an S-algebra.

We begin with a general remark. Let T be an S-algebra. Consider the derived categories $\mathrm{D}(S)$ and $\mathrm{D}(T)$. The functor

$$- \otimes_S^{\mathbf{L}} T \colon \mathrm{D}(S) \to \mathrm{D}(T)$$

has a right adjoint, the restriction of scalars functor. We denote it $\iota \colon \mathrm{D}(T) \to \mathrm{D}(S)$ temporarily. A morphism of complexes $M \to \iota(N)$ in $\mathrm{D}(S)$ determines a morphism $M \otimes_S^{\mathbf{L}} T \to N$ in $\mathrm{D}(T)$ by adjunction. The adjunction is functorial with respect to commutative squares

$$\begin{array}{ccc} M_1 & \longrightarrow & \iota(N_1) \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & \iota(N_2). \end{array}$$

Definition 8.1. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_{S\times X}$-modules. We define a natural morphism

$$\mathrm{R}\Gamma(S \times X, \mathcal{E}) \otimes_S^{\mathbf{L}} T \to \mathrm{R}\Gamma(T \times X, \mathcal{E})$$

as the adjoint of the pullback morphism $\mathrm{R}\Gamma(S \times X, \mathcal{E}) \to \mathrm{R}\Gamma(T \times X, \mathcal{E})$. Here we identify $\mathrm{R}\Gamma(T \times X, \mathcal{E})$ with its image in $\mathrm{D}(S)$ under $\iota \colon \mathrm{D}(T) \to \mathrm{D}(S)$.

Definition 8.2. Let $\mathcal{M}$ be a $\mathcal{O}_{S\times X}$-module shtuka. We define a natural morphism

$$\mathrm{R}\Gamma(S \times X, \mathcal{M}) \otimes_S^{\mathbf{L}} T \to \mathrm{R}\Gamma(T \times X, \mathcal{M})$$

as the adjoint of the pullback morphism $\mathrm{R}\Gamma(S \times X, \mathcal{M}) \to \mathrm{R}\Gamma(T \times X, \mathcal{M})$.

Lemma 8.3. *Let T be an S-algebra. If $\mathcal{M}$ is an $\mathcal{O}_{S\times X}$-module shtuka given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

then the natural diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma(S\times X,\mathcal{M})\otimes^{\mathbf{L}}_S T \arrow[r] \arrow[d] & \mathrm{R}\Gamma(T\times X,\mathcal{M}) \arrow[d] \\
\mathrm{R}\Gamma(S\times X,\mathcal{M}_0)\otimes^{\mathbf{L}}_S T \arrow[r] \arrow[d, "i-j"'] & \mathrm{R}\Gamma(T\times X,\mathcal{M}_0) \arrow[d, "i-j"] \\
\mathrm{R}\Gamma(S\times X,\mathcal{M}_1)\otimes^{\mathbf{L}}_S T \arrow[r] \arrow[d] & \mathrm{R}\Gamma(T\times X,\mathcal{M}_1) \arrow[d] \\
{[1]} & {[1]}
\end{tikzcd} \tag{8.1}$$

is a morphism of distinguished triangles. Here the left column is the image under $-\otimes^{\mathbf{L}}_S T$ of the triangle of Theorem 1.5.6 for $\mathcal{M}$ and the right column is the triangle of Theorem 1.5.6 for the pullback of $\mathcal{M}$ to $T\times X$.

Proof. Indeed Proposition 1.7.4 tells us that the pullback maps form a morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma(S\times X,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(S\times X,\mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\Gamma(S\times X,\mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(T\times X,\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(T\times X,\mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\Gamma(T\times X,\mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd}$$

Taking the adjoints of the vertical arrows we get the result. □

Proposition 8.4. *Let $\mathcal{M}$ be an $\mathcal{O}_{S\times X}$-module shtuka given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

If $\mathcal{M}_0$, $\mathcal{M}_1$ are coherent and flat over S then the following holds:

1. $\mathrm{R}\Gamma(S\times X,\mathcal{M})$ *is a perfect S-module complex.*
2. *For every S-algebra T the natural map*

$$\mathrm{R}\Gamma(S\times X,\mathcal{M})\otimes^{\mathbf{L}}_S T \to \mathrm{R}\Gamma(T\times X,\mathcal{M})$$

is a quasi-isomorphism. Moreover the natural diagram (8.1) *is an isomorphism of distinguished triangles.*

Proof. Since $\mathcal{M}_0$ is coherent and flat over S the base change theorem for coherent cohomology [07VK] shows that $\mathrm{R}\Gamma(S \times X, \mathcal{M}_0)$ is a perfect S-module complex and the natural map

$$\mathrm{R}\Gamma(S \times X, \mathcal{M}_0) \otimes_S^{\mathbf{L}} T \to \mathrm{R}\Gamma(T \times X, \mathcal{M}_0)$$

is a quasi-isomorphism. The same applies to $\mathcal{M}_1$. As $\mathrm{R}\Gamma(S \times X, \mathcal{M})$ fits to a distinguished triangle

$$\mathrm{R}\Gamma(S \times X, \mathcal{M}) \to \mathrm{R}\Gamma(S \times X, \mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(S \times X, \mathcal{M}_1) \to [1]$$

we conclude that it is a perfect S-module complex. Finally Lemma 8.3 implies that (8.1) is an isomorphism of distinguished triangles. $\square$

9. ζ-isomorphisms

Let S be a noetherian $\mathbb{F}_q$-algebra. In this section we study ζ-isomorphisms for shtukas over $S \times X$. We will prove that under suitable conditions they are stable under change of S.

Let $\mathcal{M}$ be an $\mathcal{O}_{S \times X}$-module shtuka given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Assume that the S-modules $\mathrm{H}^n(\mathcal{M})$, $\mathrm{H}^n(\mathcal{M}_0)$ and $\mathrm{H}^n(\mathcal{M}_1)$ are perfect for all $n \geqslant 0$ and zero for $n \gg 0$. In this situation we have a ζ-isomorphism

$$\zeta_{\mathcal{M}} \colon \det_S \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_S \mathrm{R}\Gamma(\nabla \mathcal{M}).$$

According to Definition 1.10.2 it is the composition

$$\det_S \mathrm{R}\Gamma(\mathcal{M}) \xrightarrow{\sim} \det_S \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_S \det_S^{-1} \mathrm{R}\Gamma(\mathcal{M}_1) \xrightarrow{\sim} \det_S \mathrm{R}\Gamma(\nabla \mathcal{M})$$

of natural isomorphisms induced by the distinguished triangles

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1],$$

$$\mathrm{R}\Gamma(\nabla \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1].$$

of Theorem 1.5.6.

Proposition 9.1. *Assume that S is a regular $\mathbb{F}_q$-algebra. If $\mathcal{M}$ is a coherent shtuka on $S \times X$ then the ζ-isomorphism is defined for $\mathcal{M}$.*

We will also need ζ-isomorphisms for coefficient rings S which are not regular. The example of such an S relevant to our study is a local artinian ring which is not a field.

Proof of Proposition 9.1. Suppose that $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Grothendieck vanishing theorem [02UZ] shows that the S-modules $\mathrm{H}^n(\mathcal{M}_0)$ and $\mathrm{H}^n(\mathcal{M}_1)$ are zero for $n > 1$. By [02O5] they are finitely generated for

$n = 0, 1$. Thus $\mathrm{H}^n(\mathcal{M})$ is zero for $n > 2$ and finitely generated for $n = 0, 1, 2$. The ring S has finite global dimension since it is regular [00O7]. Whence $\mathrm{H}^n(\mathcal{M})$, $\mathrm{H}^n(\mathcal{M}_0)$ and $\mathrm{H}^n(\mathcal{M}_1)$ are perfect S-modules. □

Proposition 9.2. *Let T be an S-algebra. Let $\mathcal{M}$ be a shtuka on $S \times X$ given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Let $\mathcal{M}_T$ be the pullback of $\mathcal{M}$ to $T \times X$. Assume that

(1) *the ζ-isomorphisms are defined for $\mathcal{M}$ and $\mathcal{M}_T$,*
(2) *$\mathcal{M}_0$ and $\mathcal{M}_1$ are coherent and flat over S.*

Then the following holds:

(1) *The natural maps*

$$\mathrm{R}\Gamma(\mathcal{M}) \otimes^{\mathbf{L}}_S T \to \mathrm{R}\Gamma(\mathcal{M}_T),$$
$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \otimes^{\mathbf{L}}_S T \to \mathrm{R}\Gamma(\nabla\mathcal{M}_T)$$

of Definition 8.2 are quasi-isomorphisms.

(2) *The natural square*

$$\begin{tikzcd}
\det_S \mathrm{R}\Gamma(\mathcal{M}) \otimes_S T \arrow[r, "\zeta_{\mathcal{M}}"] \arrow[d, "\wr"'] & \det_S \mathrm{R}\Gamma(\nabla\mathcal{M}) \otimes_S T \arrow[d, "\wr"] \\
\det_T \mathrm{R}\Gamma(\mathcal{M}_T) \arrow[r, "\zeta_{\mathcal{M}_T}"] & \det_T \mathrm{R}\Gamma(\nabla\mathcal{M}_T)
\end{tikzcd}$$

is commutative. Here the vertical arrows are induced by the quasi-isomorphisms of (1).

Proof. The natural isomorphisms of determinants

$$\det_S \mathrm{R}\Gamma(\mathcal{M}) \to \det_S \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_S \det_S^{-1} \mathrm{R}\Gamma(\mathcal{M}_1),$$
$$\det_S \mathrm{R}\Gamma(\nabla\mathcal{M}) \to \det_S \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_S \det_S^{-1} \mathrm{R}\Gamma(\mathcal{M}_1)$$

induced by the triangles of $\mathcal{M}$ and $\nabla\mathcal{M}$ are stable under the pullback to T by construction (see the proof of Corollary 2 after Theorem 2 in [16]). So the result follows from Proposition 8.4. □