Shtuka cohomology and special values of Goss L-functions

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CHAPTER 3

Topological rings and modules

In this chapter we use the tensor product and function space constructions of Chapter 2 to produce and study rings and modules over them.

We keep the conventions of Chapter 2. In particular we keep using the acronym "lth" and assume all locally compact vector spaces to be Hausdorff. We work with topological algebras over the fixed field $\mathbb{F}_q$ and with modules over such algebras.

A topological algebra is a topological vector space S equipped with a continuous multiplication map $S \times S \to S$ which makes S into a commutative associative unital $\mathbb{F}_q$-algebra. A topological module N over a topological algebra S is a topological vector space N equipped with a continuous S-action map $S \times N \to N$ which makes N into an S-module.

1. Overview

Let S, T be locally compact $\mathbb{F}_q$-algebras. Typical examples of such algebras relevant to our applications are

- the discrete algebra $\mathbb{F}_q[t]$,
- the locally compact algebra $\mathbb{F}_q((t^{-1}))$,
- the compact algebra $\mathbb{F}_q[[t^{-1}]]$.

The first goal of this chapter is to equip the tensor products $S \mathbin{\breve{\otimes}} T$ and $S \mathbin{\widehat{\otimes}} T$ with topological $\mathbb{F}_q$-algebra structures compatible with the dense subalgebra $S \otimes T$. In the case of $S \mathbin{\widehat{\otimes}} T$ it can be done only under certain assumptions on S or T (cf. Example 4.1). To handle this difficulty we work out some preliminaries in Sections 2 and 3.

The rings $S \mathbin{\breve{\otimes}} T$ and $S \mathbin{\widehat{\otimes}} T$ play a prominent role in this work. Some degree of familiarity with them will be assumed in the subsequent chapters. We discuss examples of such rings in Section 7.

Let N be a locally compact S-module and M a locally compact T-module. Another important goal of this chapter is to equip the function spaces $a(N, M)$, $b(N, M)$ and the germ space $g(N, M)$ with natural actions of tensor product rings:

- an $S \mathbin{\breve{\otimes}} T$-module structure on $a(N, M)$ and $g(N, M)$,
- an $S^{\#} \mathbin{\widehat{\otimes}} T$-module structure on $b(N, M)$.

In Section 10 we study $a(N, M)$ and $b(N, M)$ as modules in one particular case which is central to our applications. Let C be a smooth projective connected curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$ and set $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$. The local field F of C at ∞ is a locally compact $\mathbb{F}_q$-algebra. Its ring of integers $\mathcal{O}_F \subset F$ is a compact open subalgebra while $A \subset F$ is a discrete cocompact subalgebra. Serre duality for C implies important results for module structures on $a(N, T)$, $b(N, T)$ where N is one of the spaces

$$F, \quad F/A, \quad F/\mathcal{O}_F$$

and T is a locally compact $\mathbb{F}_q$-algebra. For example $a(F, T)$ is a free $F \breve{\otimes} T$-module of rank 1 while $b(F/A, T)$ is a locally free $A \widehat{\otimes} T$-module of rank 1.

In Section 11 we equip $S \breve{\otimes} T$ and $S \widehat{\otimes} T$ with τ-ring structures to facilitate applications in the context of shtukas. We provide the function spaces and the germ spaces with the structures of left modules over the corresponding τ-polynomial rings. As a result one can use them as arguments for the Hom shtuka construction of Section 1.12. A Hom shtuka with a function space or a germ space argument is one of the main constructions of this text.

The content of Sections 2 and 3 is well-known. The same applies to the rest of the chapter modulo the reservations we made on the tensor product $\breve{\otimes}$, the function spaces $a(N, M)$, $b(N, M)$ and the germ space $g(N, M)$ in Chapter 2.

As is the case for Chapter 2, this chapter was inspired by and owes much to Anderson's work [1], especially to §2 of that text where Anderson uses function spaces to compute certain Ext's for modules over τ-polynomial rings.

2. Completion

Lemma 2.1. *Let S be an lth vector space equipped with an $\mathbb{F}_q$-algebra structure.*

1. *$\widehat{S}$ admits at most one structure of an lth algebra such that the natural map $S \to \widehat{S}$ is a homomorphism.*
2. *Such a structure exists if and only if the multiplication map $S \times S \to S$ is continuous.*

Proof. (1) follows from the fact that $S \times S$ is dense in $\widehat{S} \times \widehat{S}$. (2) If the multiplication map $S \times S \to S$ is continuous then we get a continuous multiplication $\widehat{S} \times \widehat{S} \to \widehat{S}$ by completion. Assume that $\widehat{S}$ admits an lth algebra structure such that $S \to \widehat{S}$ is a homomorphism. We then have a commutative square

$$\begin{array}{ccc} S \times S & \longrightarrow & \widehat{S} \times \widehat{S} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \widehat{S} \end{array}$$

where the vertical arrows are the multiplication maps. If $U \subset S$ is an open subspace then Lemma 2.5.1 implies that the preimage of $\widehat{U} \subset \widehat{S}$ in S is U.

Hence if $s \in S$ is an arbitrary point then the preimage of $s + \widehat{U} \subset \widehat{S}$ in S is $s+U$. Since the right vertical arrow and the horizontal arrows in the diagram above are continuous we conclude that the preimage of $s+U$ in $S \times S$ is open. Hence the left vertical arrow is continuous. □

Lemma 2.2. *Let S be an lth algebra and let N be an lth vector space equipped with an S-module structure.*

(1) *$\widehat{N}$ admits at most one lth $\widehat{S}$-module structure such that the natural map $N \to \widehat{N}$ is an S-module homomorphism.*
(2) *Such a structure exists if and only if the S-multiplication on N is continuous.*

Proof. Follows by the same argument as Lemma 2.1. □

Lemma 2.3. *If $f\colon S \to T$ is a continuous homomorphism of lth algebras then there exists a unique continuous homomorphism $\widehat{f}\colon \widehat{S} \to \widehat{T}$ extending f.* □

Lemma 2.4. *Let N be an lth module of finite type over an lth algebra S. If N is a topological direct summand of $S^{\oplus n}$ for some n then the natural map $N \otimes_S \widehat{S} \to \widehat{N}$ is an $\widehat{S}$-module isomorphism.*

Proof. The claim reduces to the case when $N = S^{\oplus n}$ and this case is clear. □

3. Boundedness

Definition 3.1. Let S be a topological algebra and N a topological S-module.

(1) If $V \subset S, K \subset N$ are subsets then $V \cdot K$ denotes the set of products
$$\{s \cdot n \mid s \in V, n \in K\} \subset N.$$
(2) A subset $K \subset N$ is called *bounded* if for every open subspace $U \subset N$ there exists an open subspace $V \subset S$ such that $V \cdot K \subset U$.

A finite subset is automatically bounded. For an example of an unbounded subset consider the lth algebra $\mathbb{F}_q(\!(t)\!)$. Since $\mathbb{F}_q(\!(t)\!)^\times$ acts on $\mathbb{F}_q(\!(t)\!)$ by automorphisms it follows that $\mathbb{F}_q(\!(t)\!)$ is not bounded.

Lemma 3.2. *Let N be an lth module over an lth algebra S. Every compact subset of N is bounded.*

Proof. Let K be a compact subset and $n \in K$ a point. The preimage of U in $S \times N$ under the multiplication map contains the point $(0, n)$. As this preimage is open there exist open subspaces $V_n \subset S, Y_n \subset N$ such that $V_n \cdot (n + Y_n) \subset U$. The translates $n + Y_n$ cover K. As K is compact we can choose finitely many such translates. Let V be the intersection of the corresponding V_n. By construction $V \cdot K \subset U$. □

A bounded subset need not be compact since every subset of a bounded subset is itself bounded.

4. Tensor products

Let S, T be locally compact algebras. Even though $S \otimes_c T$ is both an $\mathbb{F}_q$-algebra and an lth vector space it is not necessarily an lth algebra since its multiplication need not be continuous in the tensor product topology.

Example 4.1. Let us demonstrate that the multiplication on $\mathbb{F}_q(\!(t)\!) \otimes_c \mathbb{F}_q(\!(t)\!)$ is not continuous. Denote $F = \mathbb{F}_q(\!(t)\!)$ temporarily. We have

$$\lim_{n\to\infty} t^{-n} \otimes t^n = 0 = \lim_{n\to\infty} t^n \otimes t^{-n}$$

in $F \otimes_c F$. As a consequence

$$\lim_{n\to\infty} (t^{-n} \otimes t^n, t^n \otimes t^{-n}) = 0$$

in $(F \otimes_c F) \times (F \otimes_c F)$. The multiplication maps this sequence to the constant sequence $1 \otimes 1$. Hence it is not continuous.

We say that an lth algebra stucture on $S \mathbin{\widehat{\otimes}} T$ is *natural* if the canonical map $S \otimes T \to S \mathbin{\widehat{\otimes}} T$ is an $\mathbb{F}_q$-algebra homomorphism. According to Lemma 2.1 $S \mathbin{\widehat{\otimes}} T$ admits at most one natural lth algebra structure. We would like to give a sufficient condition for such a structure to exist. By Lemma 2.1 it exists if and only if the multiplication on $S \otimes_c T$ is continuous.

Lemma 4.2. *Let S, N be topological $\mathbb{F}_q$-vector spaces. A bilinear map $\mu\colon S \times N \to N$ is continuous if and only if*

1. *For every $s \in S$ the map $\mu(s, -)\colon N \to N$ is continuous.*
2. *For every $n \in N$ the map $\mu(-, n)\colon S \to N$ is continuous.*
3. *The map μ is continuous at $(0, 0)$.*

Proof. Assume (1), (2), (3). Let $(s, n) \in S \times N$ and let $s' \in S$, $n' \in N$. By bilinearity of μ we have

$$\mu(s + s', n + n') = \mu(s, n) + \mu(s, n') + \mu(s', n) + \mu(s', n').$$

From (1), (2), (3) it follows that the map $(s', n') \mapsto \mu(s+s', n+n')$ is continuous at $(0, 0)$. Hence μ is continuous. □

Lemma 4.3. *Let S be a compact algebra and N an lth S-module. Every compact open subspace $U \subset N$ contains an open submodule.*

Proof. Given $s \in S$ let $m_s\colon N \to N$ denote the multiplication by s map. As U is compact and open Lemma 3.2 provides us with an open subspace $V \subset S$ such that $V \cdot U \subset U$. Consider the intersection

$$U' = U \cap \bigcap_s m_s^{-1} U$$

where s runs over a set of representatives of the classes in S/V. The subspace U' is open since S/V is finite. By construction $S \cdot U' \subset U$ hence the submodule generated by U' is contained in U. □

Lemma 4.4. *Let N be an lth module over an lth algebra S. If N is bounded and admits a fundamental system of open submodules then the map $\mu\colon S \otimes_{\mathrm{c}} N \to N$ induced by the S-multiplication on N is continuous.*

Proof. Let $U \subset N$ be an open submodule. As N is bounded there exists an open subspace $V \subset S$ such that $V \cdot N \subset U$. Therefore $\mu(V \otimes N + S \otimes U) \subset U$. □

Lemma 4.5. *Let S, T be lth algebras, N an lth S-module and M an lth T-module. If the map $S \otimes_{\mathrm{c}} N \to N$ induced by the S-multiplication on N is continuous then the $S \otimes_{\mathrm{c}} T$-multiplication on $N \otimes_{\mathrm{c}} M$ is continuous.*

Proof. The $S \otimes_{\mathrm{c}} T$-multiplication on $N \otimes_{\mathrm{c}} M$ is a bilinear map which satisfies the conditions (1) and (2) of Lemma 4.2. It remains to show that the condition (3) holds. Let $U \subset N, V \subset M$ be open subspaces. As the map $S \otimes_{\mathrm{c}} N \to N$ is continuous there are open subspaces $O \subset S$ and $Y \subset N$ such that $O \cdot N \subset U$ and $S \cdot Y \subset U$. By continuity of T-multiplication on M there exist open subspaces $W \subset T$ and $E \subset M$ such that $W \cdot E \subset V$. We then have

$$\begin{gathered}(O \otimes T + S \otimes W) \cdot (Y \otimes M + N \otimes E) \subset \\ O \cdot Y \otimes M + O \cdot N \otimes M + S \cdot Y \otimes M + N \otimes W \cdot E \subset \\ U \otimes M + N \otimes V\end{gathered}$$

and the result follows. □

Proposition 4.6. *If S is a compact or discrete algebra and T an lth algebra then $S \mathbin{\widehat{\otimes}} T$ admits a natural lth algebra structure.*

Proof. We first prove that the multiplication map $S \otimes_{\mathrm{c}} S \to S$ is continuous. This is clear if S is discrete. If S is compact then it is bounded by Lemma 3.2 and admits a fundamental system of open S-submodules by Lemma 4.3. Hence the map $S \otimes_{\mathrm{c}} S \to S$ is continuous by Lemma 4.4. Now Lemma 4.5 implies that the multiplication map $(S \otimes_{\mathrm{c}} T) \times (S \otimes_{\mathrm{c}} T) \to (S \otimes_{\mathrm{c}} T)$ is continuous. Whence the result follows from Lemma 2.1. □

Proposition 4.7. *Let S, T be lth algebras, N an lth S-module and M an lth T-module. The vector space $N \mathbin{\widehat{\otimes}} M$ admits a natural lth $S \mathbin{\widehat{\otimes}} T$-module structure in either of the cases:*

(1) *S and N are discrete.*
(2) *S and N are compact.*

Proof. Follows from Lemma 2.2 by the same argument as the proposition above. □

As we observed above the completed tensor product $S \mathbin{\widehat{\otimes}} T$ of locally compact algebras S and T need not carry a natural lth algebra structure. The ind-complete tensor product $S \mathbin{\widetilde{\otimes}} T$ does not suffer from such a problem.

Lemma 4.8. *If S, T are lth algebras, N an lth S-module and M an lth T-module then the multiplication map $(S \otimes_{\mathrm{ic}} T) \times (N \otimes_{\mathrm{ic}} M) \to N \otimes_{\mathrm{ic}} M$ is continuous.*

Proof. The conditions (1) and (2) of Lemma 4.2 are clear and the condition (3) follows instantly from the definition of the ind-tensor product topology (Definition 2.7.1). □

Proposition 4.9. *If S, T are locally compact algebras then there exists a unique lth algebra structure on $S \mathbin{\breve{\otimes}} T$ such that the natural map $S \otimes T \to S \mathbin{\breve{\otimes}} T$ is a homomorphism.* □

Proposition 4.10. *If S, T are locally compact algebras, N a locally compact S-module and M a locally compact T-module then $N \mathbin{\breve{\otimes}} M$ admits a natural $S \mathbin{\breve{\otimes}} T$-module structure.* □

Next we study some natural maps of tensor product algebras.

Proposition 4.11. *Let S, T be lth algebras such that $S \mathbin{\widehat{\otimes}} T$ admits a natural lth algebra structure.*

1. *The natural map $\iota\colon S \mathbin{\widehat{\otimes}} T^{\#} \to S \mathbin{\widehat{\otimes}} T$ is an injective $\mathbb{F}_q$-algebra homomorphism.*
2. *If S is compact then ι is an $\mathbb{F}_q$-algebra isomorphism.*

Proof. As ι is the completion of the continuous algebra homomorphism $S \otimes_{\mathrm{ic}} T \to S^{\#} \otimes_{\mathrm{c}} T$ Lemma 2.3 shows that it is an $\mathbb{F}_q$-algebra homomorphism. The map ι is injective by Proposition 2.6.5 (1). If S is compact then ι is bijective by Proposition 2.6.5 (2). □

Proposition 4.12. *Let S, T be locally compact algebras.*

1. *The natural map $\iota\colon S \mathbin{\breve{\otimes}} T \to S^{\#} \mathbin{\widehat{\otimes}} T$ is an injective $\mathbb{F}_q$-algebra homomorphism.*
2. *If S is compact then ι is an isomorphism.*

Proof. As ι is the completion of the continuous algebra homomorphism $S \otimes_{\mathrm{ic}} T \to S^{\#} \otimes_{\mathrm{c}} T$ Lemma 2.3 shows that it is an $\mathbb{F}_q$-algebra homomorphism. The map ι is injective by Corollary 2.7.7. If S is compact then ι is bijective by Corollary 2.7.8. □

Proposition 4.13. *If S, T are compact algebras then the natural map $S \mathbin{\breve{\otimes}} T \to S \mathbin{\widehat{\otimes}} T$ is a $\mathbb{F}_q$-algebra isomorphism.*

Proof. The natural map in question is the composition of natural maps $S \mathbin{\breve{\otimes}} T \to S \mathbin{\widehat{\otimes}} T^{\#} \to S \mathbin{\widehat{\otimes}} T$. The first map is an isomorphism by Proposition 4.12 while the second map is an isomorphism by Proposition 4.11 (2). □

The natural maps of Propositions 4.11 and 4.12 are always continuous. However their inverses, if they exists, are not continuous in general. Similarly the natural map of Proposition 4.13 is a continuous bijection whose inverse is not continuous in general.

5. Finite products of local fields

In our context a local field always means a local field containing $\mathbb{F}_q$. Let $T = \prod_{i=1}^n T_i$ be a finite product of local fields. It will be convenient for us to treat such products in a uniform way independent of n. To do that we set up some notation and terminology.

Observe that T is a locally compact $\mathbb{F}_q$-algebra. It has a compact open subalgebra $\mathcal{O}_T = \prod_{i=1}^n \mathcal{O}_{T_i}$ which we call the *ring of integers* of T. We call an element $t \in \mathcal{O}_T$ a *uniformizer* if its projection to every $\mathcal{O}_{T_i}$ is a uniformizer. Observe that $T = \mathcal{O}_T[t^{-1}]$. By a slight abuse of notation we denote $\mathfrak{m}_T \subset \mathcal{O}_T$ the Jacobson radical of $\mathcal{O}_T$. It is the cartesian product of maximal ideals $\mathfrak{m}_{T_i} \subset \mathcal{O}_{T_i}$. Every uniformizer generates $\mathfrak{m}_T$. If T is not a single local field then $\mathfrak{m}_T$ is not maximal.

6. Algebraic properties

In this section we study the properties of $S \mathbin{\breve{\otimes}} T$, $S \mathbin{\widehat{\otimes}} T$ as commutative rings without topology. We are primarily interested in the case when S and T are finite products of local fields or the rings of integers in such finite products. We begin with some localization properties.

Proposition 6.1. *If S is an $\mathbb{F}_q$-algebra, T a finite product of local fields and $t \in \mathcal{O}_T$ a uniformizer then $(S^\# \mathbin{\widehat{\otimes}} \mathcal{O}_T)[t^{-1}] = S^\# \mathbin{\widehat{\otimes}} T$.*

Proof. The family of open subspaces $\{S^\# \otimes_{\mathrm{c}} t^{-n}\mathcal{O}_T\}_{n\geqslant 0}$ covers $S^\# \otimes_{\mathrm{c}} T$. Hence Proposition 2.5.4 demonstrates that the family $\{S^\# \mathbin{\widehat{\otimes}} t^{-n}\mathcal{O}_T\}_{n\geqslant 0}$ covers $S^\# \mathbin{\widehat{\otimes}} T$. Multiplication by t^n maps $S^\# \mathbin{\widehat{\otimes}} t^{-n}\mathcal{O}_T$ bijectively onto $S^\# \mathbin{\widehat{\otimes}} \mathcal{O}_T$ since the same is true with $\otimes_{\mathrm{c}}$ in place of $\mathbin{\widehat{\otimes}}$. The claim is now clear. □

Proposition 6.2. *If S is a compact $\mathbb{F}_q$-algebra, T a finite product of local fields and $t \in T$ a uniformizer then $(S \mathbin{\breve{\otimes}} \mathcal{O}_T)[t^{-1}] = S \mathbin{\breve{\otimes}} T$.*

Proof. As S is compact the natural maps $S \mathbin{\breve{\otimes}} \mathcal{O}_T \to S^\# \mathbin{\widehat{\otimes}} \mathcal{O}_T$ and $S \mathbin{\breve{\otimes}} T \to S^\# \mathbin{\widehat{\otimes}} T$ are isomorphisms by Proposition 4.12. Hence the claim follows from Proposition 6.1. □

Proposition 6.3. *If S, T are finite products of local fields with uniformizers $s \in \mathcal{O}_S$, $t \in \mathcal{O}_T$ then $(\mathcal{O}_S \mathbin{\breve{\otimes}} \mathcal{O}_T)[(st)^{-1}] = S \mathbin{\breve{\otimes}} T$.*

Proof. The family of open subspaces $\{s^{-n}\mathcal{O}_S \otimes_{\mathrm{ic}} t^{-n}\mathcal{O}_T\}_{n\geqslant 0}$ covers $S \otimes_{\mathrm{ic}} T$. Hence the family $\{s^{-n}\mathcal{O}_S \mathbin{\breve{\otimes}} t^{-n}\mathcal{O}_T\}_{n\geqslant 1}$ covers $S \mathbin{\breve{\otimes}} T$ by Proposition 2.5.4. Multiplication by $(st)^n$ maps $s^{-n}\mathcal{O}_S \mathbin{\breve{\otimes}} t^{-n}\mathcal{O}_T$ bijectively onto $\mathcal{O}_S \mathbin{\breve{\otimes}} \mathcal{O}_T$ since the same is true with $\otimes_{\mathrm{ic}}$ in place of $\mathbin{\breve{\otimes}}$. The claim is now clear. □

Next we study quotients of tensor product algebras. Observe that an ideal $I \subset \mathcal{O}_T$ is open if and only if it projects to nonzero ideals in all factors of $\mathcal{O}_T = \prod_{i=1}^n \mathcal{O}_{T_i}$. If S is an $\mathbb{F}_q$-algebra and $I \subset \mathcal{O}_T$ an open ideal then we have a natural map $S^\# \mathbin{\widehat{\otimes}} \mathcal{O}_T \to S \otimes \mathcal{O}_T/I$.

Proposition 6.4. *Let S be an $\mathbb{F}_q$-algebra and T a finite product of local fields. If $I \subset \mathcal{O}_T$ is an open ideal then the following holds:*

1. *The sequence $0 \to S^{\#} \mathbin{\widehat{\otimes}} I \to S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T \to S \otimes \mathcal{O}_T/I \to 0$ is exact.*
2. *The natural map $(S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T) \otimes_{\mathcal{O}_T} I \to S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ is injective with image $S^{\#} \mathbin{\widehat{\otimes}} I$.*

Proof. (1) Indeed the sequence $0 \to S^{\#} \otimes_{\mathrm{c}} I \to S^{\#} \otimes_{\mathrm{c}} \mathcal{O}_T \to S^{\#} \otimes_{\mathrm{c}} \mathcal{O}_T/I \to 0$ is clearly exact and the first map in it is an open embedding. Hence the result follows from Lemma 2.5.1. (2) Observe that I is a free $\mathcal{O}_T$-module of rank 1. Let $x \in I$ be a generator. Multiplication by x identifies $S^{\#} \otimes_{\mathrm{c}} \mathcal{O}_T$ with $S^{\#} \otimes_{\mathrm{c}} I$. Taking completion we get the result. □

If $I \subset \mathcal{O}_T$ is an open ideal then the quotient $\mathcal{O}_T/I$ is discrete. As a consequence $S \otimes_{\mathrm{ic}} \mathcal{O}_T/I$ is discrete. Taking the completion of $S \otimes_{\mathrm{ic}} \mathcal{O}_T \to S \otimes_{\mathrm{ic}} \mathcal{O}_T/I$ we get a natural map $S \mathbin{\breve{\otimes}} \mathcal{O}_T \to S \otimes \mathcal{O}_T/I$.

Proposition 6.5. *Let S be an lth $\mathbb{F}_q$-algebra, T a finite product of local fields. If $I \subset \mathcal{O}_T$ is an open ideal then then the following holds:*

1. *The sequence $0 \to S \mathbin{\breve{\otimes}} I \to S \mathbin{\breve{\otimes}} \mathcal{O}_T \to S \otimes \mathcal{O}_T/I \to 0$ is exact.*
2. *The natural map $(S \mathbin{\breve{\otimes}} \mathcal{O}_T) \otimes_{\mathcal{O}_T} I \to S \mathbin{\breve{\otimes}} \mathcal{O}_T$ is injective with image $S \mathbin{\breve{\otimes}} I$.*

Proof. Follows by the same argument as Proposition 6.4. □

Finally we discuss some structural properties of tensor product algebras.

Proposition 6.6. *Let S be a noetherian $\mathbb{F}_q$-algebra and T a finite product of local fields.*

1. *$S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ is noetherian and complete with respect to the ideal $S^{\#} \mathbin{\widehat{\otimes}} \mathfrak{m}_T$.*
2. *$S^{\#} \mathbin{\widehat{\otimes}} T$ is noetherian.*

Proof. Without loss of generality we assume that T is a local field. In this case $T \cong k(\!(t)\!)$ for some finite field extension k of $\mathbb{F}_q$. (1) By definition of the completed tensor product

$$S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T = \lim_{n \geqslant 1} S^{\#} \otimes \mathcal{O}_T/\mathfrak{m}_T^n.$$

Therefore $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ is the completion of the ring $(S \otimes k)[t]$ at the ideal (t). The ring $S \otimes k$ is of finite type over S and so is noetherian. Thus $(S \otimes k)[t]$ is noetherian and so is its completion $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$. The completion of the ideal $(t) \subset (S \otimes k)[t]$ is clearly $S^{\#} \mathbin{\widehat{\otimes}} \mathfrak{m}_T$ so $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ is complete with respect to $S^{\#} \mathbin{\widehat{\otimes}} \mathfrak{m}_T$. (2) follows from (1) in view of Proposition 6.1. □

Proposition 6.7. *Let S, T be finite products of local fields, $s \in \mathcal{O}_S$ and $t \in \mathcal{O}_T$ uniformizers.*

1. *$\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathcal{O}_T$ is a finite product of complete regular 2-dimensional local rings.*
2. *The maximal ideals of $\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathcal{O}_T$ are precisely the prime ideals containing s and t.*

Proof. It is enough to assume that S and T are local fields. In this case $S \cong k_1[\![s]\!]$ and $T \cong k_2[\![t]\!]$ for some finite field extensions k_1 and k_2 of $\mathbb{F}_q$. Therefore

$$\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathcal{O}_T = \lim_{n,m\geqslant 0} (k_1 \otimes k_2)[s,t]/(s^n, t^m) = (k_1 \otimes k_2)[\![s,t]\!].$$

Observe that $k_1 \otimes k_2$ is a finite product of finite fields. (1) and (2) are now clear. □

7. Examples of tensor product algebras

We are now in position to discuss some examples of tensor product algebras. Consider the locally compact algebras

$$\begin{aligned} S &= \mathbb{F}_q(\!(s)\!), \quad \mathcal{O}_S = \mathbb{F}_q[[s]] \\ T &= \mathbb{F}_q(\!(t)\!), \quad \mathcal{O}_T = \mathbb{F}_q[[t]]. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{O}_S \mathbin{\overset{\vee}{\otimes}} \mathcal{O}_T &= \mathbb{F}_q[[s,t]], \\ \mathcal{O}_S \mathbin{\widehat{\otimes}} \mathcal{O}_T &= \mathbb{F}_q[[s,t]]. \end{aligned}$$

However the topologies on $\mathcal{O}_S \mathbin{\overset{\vee}{\otimes}} \mathcal{O}_T$ and $\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathcal{O}_T$ are different. The topology on $\mathcal{O}_S \mathbin{\overset{\vee}{\otimes}} \mathcal{O}_T$ is given by the powers of the ideal (st) while the topology on $\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathcal{O}_T$ is given by the powers of the ideal (s,t). In particular $\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathcal{O}_T$ is compact while $\mathcal{O}_S \mathbin{\overset{\vee}{\otimes}} \mathcal{O}_T$ is not even locally compact.

The completed tensor products with a discrete factor look as follows:

$$\begin{aligned} \mathcal{O}_S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T &= \mathbb{F}_q[[s,t]], \\ \mathcal{O}_S^{\#} \mathbin{\widehat{\otimes}} T &= \mathbb{F}_q[[s]](\!(t)\!), \\ S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T &= \mathbb{F}_q(\!(s)\!)[[t]], \\ S^{\#} \mathbin{\widehat{\otimes}} T &= \mathbb{F}_q(\!(s)\!)(\!(t)\!). \end{aligned}$$

The topologies on $\mathcal{O}_S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ and $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ are given by powers of the ideals (t). The topologies on $\mathcal{O}_S^{\#} \mathbin{\widehat{\otimes}} T$ and $S^{\#} \mathbin{\widehat{\otimes}} T$ are determined by open subalgebras $\mathcal{O}_S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ and $S^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_T$ respectively.

The ind-complete tensor products with a compact factor has the following form:

$$S \mathbin{\overset{\vee}{\otimes}} \mathcal{O}_T = \mathbb{F}_q[[t]](\!(s)\!).$$

Its topology is defined by the open subalgebra $\mathcal{O}_S \mathbin{\overset{\vee}{\otimes}} \mathcal{O}_T$. The ind-complete tensor product of S and T is

$$S \mathbin{\overset{\vee}{\otimes}} T = \mathbb{F}_q[[s,t]][(st)^{-1}]$$

with the topology given by the open subalgebra $\mathcal{O}_S \mathbin{\overset{\vee}{\otimes}} \mathcal{O}_T$. As we demonstrated in Example 4.1 the tensor product $S \mathbin{\widehat{\otimes}} T$ makes no sense as an lth algebra.

Another important example is $S \mathbin{\widehat{\otimes}} \mathbb{F}_q[t]$. It is topologically isomorphic to the algebra of Tate series

$$S\langle t\rangle = \Big\{ \sum_{n\geqslant 0} \alpha_n t^n \in S[[t]] \;\Big|\; \lim_{n\to\infty} \alpha_n = 0 \Big\}.$$

For the sake of completeness let us describe the algebra $\mathcal{O}_S \mathbin{\widehat{\otimes}} T$. It can be identified with the algebra of power series

$$\sum_{n\in\mathbb{Z}} \alpha_n t^n,\ \alpha_n \in \mathcal{O}_S,\ \lim_{n\to-\infty} \alpha_n = 0.$$

For every nonzero ideal $I \subset \mathcal{O}_S$ and every integer $m \in \mathbb{Z}$ the subspace

$$\Big\{ \sum_{n\in\mathbb{Z}} \alpha_n t^n \mid \alpha_n \in I \text{ for all } n \leqslant m \Big\} \subset \mathcal{O}_S \mathbin{\widehat{\otimes}} T$$

is open. Such subspaces form a fundamental system.

8. Function spaces as modules

Let S, T be locally compact algebras, N a locally compact S-module and M a locally compact T-module. The function spaces $a(N, M)$ and $b(N, M)$ carry an action of S on the right and T on the left by functoriality. Since S and T are commutative we get an $S\otimes T$-action on $a(N, M)$ and $b(N, M)$. According to our convention $S \otimes T$ carries no topology so its action is not supposed to be continuous. Nonetheless we demonstrate below that this action extends uniquely to an lth $S \mathbin{\widecheck{\otimes}} T$-module structure on $a(N, M)$ and an lth $S^{\#} \mathbin{\widehat{\otimes}} T$-module structure on $b(N, M)$. We also study the resulting modules.

Definition 8.1. Let S be an lth algebra and N an lth S-module. We equip N^*, the continuous $\mathbb{F}_q$-linear dual of N, with the S-module structure given by the action of S on N.

Lemma 8.2. *If N is locally compact then the S-action map $S \times N^* \to N^*$ is continuous.*

Proof. We will deduce the result from Lemma 4.2. To do it we need to check the following conditions:

(1) For every $s \in S$ the induced map $N^* \to N^*$, $f \mapsto fs$ is continuous.
(2) For every $f \in N^*$ the induced map $S \to N^*$, $s \mapsto fs$ is continuous.
(3) The S-action map $S \times N^* \to N^*$ is continuous at $(0, 0)$.

The condition (1) holds by functoriality of N^*. Let us check (3). For a compact open subset $U \subset N$ let $[U] \subset N^*$ be the subspace of functions which vanish on U. By Lemma 3.2 there exists an open $\mathbb{F}_q$-vector subspace $V \subset S$ such that $V \cdot U \subset U$. As a consequence $V \cdot [U] \subset [U]$ so the condition (3) holds. Given $f \in N^*$ there exists a compact open $U \subset N$ such that f vanishes on U. As before we can find an open $V \subset S$ with the property that $V \cdot U \subset U$. Hence $V \cdot f \subset [U]$ and the condition (2) holds as well. □

Lemma 8.3. *Let S, T be locally compact algebras, N a locally compact S-module and M a locally compact T-module.*

(1) *$(N^*)^\# \otimes_{\mathrm{c}} M$ is an lth $S^\# \otimes_{\mathrm{c}} T$-module.*
(2) *The natural map*

$$(N^*)^\# \otimes_{\mathrm{c}} M \to b(N, M)$$
$$f \otimes m \mapsto (n \mapsto f(n)m)$$

is continuous, $S \otimes T$-linear, and identifies $b(N, M)$ as the completion of $(N^*)^\# \otimes_{\mathrm{c}} M$.

Proof. (1) We need to prove that the $S^\# \otimes_{\mathrm{c}} T$-multiplication on $(N^*)^\# \otimes_{\mathrm{c}} M$ is continuous. The S-action map $S^\# \otimes_{\mathrm{c}} (N^*)^\# \to (N^*)^\#$ is continuous since $S^\# \otimes_{\mathrm{c}} (N^*)^\#$ is discrete. Whence the result follows from Lemma 4.5. (2) $S \otimes T$-linearity is clear. By Proposition 2.9.4 this map induces a topological isomorphism $(N^*)^\# \mathbin{\widehat{\otimes}} M \cong b(N, M)$ so the result follows. □

Proposition 8.4. *If S, T are locally compact algebras, N a locally compact S-module and M a locally compact T-module then the $S \otimes T$-module structure on $b(N, M)$ extends uniquely to an lth $S^\# \mathbin{\widehat{\otimes}} T$-module structure.*

Proof. By Lemma 8.3 we can identify $b(N, M)$ with the completion of an lth $S^\# \otimes_{\mathrm{c}} T$-module $(N^*)^\# \otimes_{\mathrm{c}} M$. The result now follows from Lemma 2.2. □

Definition 8.5. Let S, T be locally compact algebras, N a locally compact S-module and M a locally compact T-module. We define the *natural $S^\# \mathbin{\widehat{\otimes}} T$-module structure on $b(N, M)$* to be the one of Proposition 8.4. From now on we work only with this $S^\# \mathbin{\widehat{\otimes}} T$-module structure.

Lemma 8.6. *Let S, T be locally compact algebras and N a locally compact S-module. If N^* is projective of finite type as an S-module without topology then the natural map*

$$N^* \otimes_S (S^\# \mathbin{\widehat{\otimes}} T) \to b(N, T)$$

is an $S^\# \mathbin{\widehat{\otimes}} T$-module isomorphism.

Proof. By Lemma 8.3 we can identify $b(N, M)$ with the completion of an lth $S^\# \otimes_{\mathrm{c}} T$-module $(N^*)^\# \otimes_{\mathrm{c}} M$. The assumption on N^* implies that $(N^*)^\# \otimes_{\mathrm{c}} T$ is a topological direct summand of $(S^\# \otimes_{\mathrm{c}} T)^{\oplus n}$ for some n. Hence the result follows from Lemma 2.4. □

Lemma 8.7. *Let S, T be locally compact algebras, N a locally compact S-module and M a locally compact T-module.*

(1) *$N^* \otimes_{\mathrm{ic}} M$ is an lth $S \otimes_{\mathrm{ic}} T$-module.*
(2) *The natural map*

$$N^* \otimes_{\mathrm{ic}} M \to a(N, M)$$
$$f \otimes m \mapsto (n \mapsto f(n)m)$$

is continuous, $S \otimes T$-linear, and identifies $a(N, M)$ as the completion of $N^* \otimes_{\mathrm{ic}} M$.

Proof. (1) Follows from Lemma 4.8. (2) $S \otimes T$-linearity is clear. By Proposition 2.10.3 this map induces a topological isomorphism $N^* \mathbin{\breve{\otimes}} M \cong a(N, M)$ so the result follows. □

Proposition 8.8. *If S, T are locally compact algebras, N a locally compact S-module and M a locally compact T-module then the $S \otimes T$-module structure on $a(N, M)$ extends uniquely to an lth $S \mathbin{\breve{\otimes}} T$-module structure.*

Proof. By Lemma 8.7 we can identify $a(N, M)$ with the completion of an lth $S \otimes_{\mathrm{ic}} T$-module $N^* \otimes_{\mathrm{ic}} M$. The result now follows from Lemma 2.2. □

Definition 8.9. Let S, T be locally compact algebras, N a locally compact S-module and M a locally compact T-module. We define the *natural $S \mathbin{\breve{\otimes}} T$-module structure on $a(N, M)$* to be the one of Proposition 8.8. From now on we work only with this $S \mathbin{\breve{\otimes}} T$-module structure.

Lemma 8.10. *Let S, T be locally compact algebras and N a locally compact S-module. If N^* is a topological direct summand of a free S-module of finite rank then the natural map*

$$N^* \otimes_S (S \mathbin{\breve{\otimes}} T) \to a(N, T)$$

is an $S \mathbin{\breve{\otimes}} T$-module isomorphism.

Proof. By Lemma 8.7 we can identify $a(N, T)$ with the completion of an lth $S \otimes_{\mathrm{ic}} T$-module $N^* \otimes_{\mathrm{ic}} T$. Due to our assumption on N^* the module $N^* \otimes_{\mathrm{ic}} T$ is a topological direct summand of $(S \otimes_{\mathrm{ic}} T)^{\oplus n}$ for some n. So the result is a consequence of Lemma 2.4. □

Lemma 8.11. *If S, T are locally compact algebras, N a locally compact S-module and M a locally compact T-module then the natural map $a(N, M) \to b(N, M)$ is an $S \mathbin{\breve{\otimes}} T$-module homomorphism.*

Proof. Indeed the map is continuous, $S \otimes T$-linear and $S \otimes T$ is dense in $S \mathbin{\breve{\otimes}} T$. □

9. Germ spaces as modules

Definition 9.1. Let S, T be locally compact $\mathbb{F}_q$-algebras, N a locally compact S-module and M a locally compact T-module. We equip the germ space $g(N, M)$ with an $S \mathbin{\breve{\otimes}} T$-module structure in the following way. Consider the short exact sequence of Proposition 2.11.2:

$$0 \to a(N, M) \to b(N, M) \to g(N, M) \to 0.$$

According to Lemma 8.11 the first arrow in this sequence is an $S \mathbin{\breve{\otimes}} T$-module homomorphism. We equip $g(N, M)$ with the resulting $S \mathbin{\breve{\otimes}} T$-module structure.

10. Residue and duality

In this section we show that in one special case the function spaces $a(N, M)$ and $b(N, M)$ have particularly nice module structures. It is exactly the case which appears in our applications.

Let C be a smooth projective connected curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$. Let F be the local field of C at ∞, $\mathcal{O}_F \subset F$ the ring of integers and $A = \Gamma(C - \{\infty\}, \mathcal{O})$ where $\mathcal{O}$ is the structure sheaf of C. The natural topology on F makes it into a locally compact $\mathbb{F}_q$-algebra with a compact open subalgebra $\mathcal{O}_F \subset F$ and a discrete cocompact subalgebra $A \subset F$.

Let ω be the coherent sheaf of Kähler differentials of C over $\mathbb{F}_q$. We use the following notation:

$$\begin{aligned} \omega(A) &= \Gamma(\operatorname{Spec} A, \omega), \\ \omega(\mathcal{O}_F) &= \Gamma(\operatorname{Spec} \mathcal{O}_F, \omega), \\ \omega(F) &= \Gamma(\operatorname{Spec} F, \omega). \end{aligned}$$

The F-module $\omega(F)$ carries a natural locally compact topology with $\omega(\mathcal{O}_F) \subset \omega(F)$ a compact open $\mathcal{O}_F$-submodule and $\omega(A) \subset \omega(F)$ a discrete cocompact A-submodule. The module $\omega(F)$ comes equipped with a residue map $\omega(F) \to k$ where k is the residue field at ∞. We denote $\operatorname{res}\colon \omega(F) \to \mathbb{F}_q$ its composition with the trace map $\operatorname{tr}\colon k \to \mathbb{F}_q$. In our study we need the following duality theorem for res:

Theorem 10.1. *The pairing $\omega(F) \times F \to \mathbb{F}_q$, $(\omega, x) \mapsto \operatorname{res}(x\omega)$ induces the following topological isomorphisms:*

$$\begin{aligned} \omega(A) &\cong (F/A)^*, \\ \omega(\mathcal{O}_F) &\cong (F/\mathcal{O}_F)^*, \\ \omega(F) &\cong F^*. \end{aligned}$$

Proof. The result is well-known. Still we sketch a proof for the reader's convenience. Let $\mathcal{O}$ be the structure sheaf of C, $\mathcal{O}(1)$ the Serre twists of $\mathcal{O}$ by the divisor ∞. Let $n \in \mathbb{Z}$. A Čech computation shows that

$$\begin{aligned} \mathrm{R}\Gamma(C, \mathcal{O}(n)) &= \Big[A \oplus z^{-n}\mathcal{O}_F \to F\Big], \\ \mathrm{R}\Gamma(C, \omega(-n)) &= \Big[\omega(A) \oplus z^{n}\omega(\mathcal{O}_F) \to \omega(F)\Big] \end{aligned}$$

where $z \in \mathcal{O}_F$ is a uniformizer and the differentials send (x, y) to $x - y$. The residue pairing $\omega(F) \times F \to \mathbb{F}_q$ induces the following perfect pairings:

$$\begin{aligned} \mathrm{H}^1(C, \omega(-n)) \times \mathrm{H}^0(C, \mathcal{O}(n)) &\to \mathbb{F}_q, \\ \mathrm{H}^0(C, \omega(-n)) \times \mathrm{H}^1(C, \mathcal{O}(n)) &\to \mathbb{F}_q. \end{aligned}$$

Using the explicit descriptions of cohomology groups provided by the complexes above we rewrite these pairings as

$$\frac{\omega(F)}{\omega(A)+z^n\omega(\mathcal{O}_F)}\times(A\cap z^{-n}\mathcal{O}_F)\to\mathbb{F}_q, \tag{10.1}$$

$$[\omega(A)\cap z^n\omega(\mathcal{O}_F)]\times\frac{F}{A+z^{-n}\mathcal{O}_F}\to\mathbb{F}_q. \tag{10.2}$$

The open subspaces $(A+z^{-n}\mathcal{O}_F)/A$ form a fundamental system which covers F/A. Taking the limit of (10.2) as $n\to-\infty$ we conclude that the residue pairing induces a topological isomorphism $\omega(A)\cong(F/A)^*$. It remains to deduce the topological isomorphisms $\omega(F)\cong F^*$ and $\omega(\mathcal{O}_F)\cong(F/\mathcal{O}_F)^*$.

Let us denote $\rho\colon\omega(F)\to\mathrm{Hom}_{\mathbb{F}_q}(F,\mathbb{F}_q)$ the map defined by the residue pairing. A priori we do not even know whether its image is contained in $F^*\subset\mathrm{Hom}_{\mathbb{F}_q}(F,\mathbb{F}_q)$. First we prove that ρ sends $z^n\omega(\mathcal{O}_F)$ to $(F/z^{-n}\mathcal{O}_F)^*\subset F^*$. As ρ is F-linear it is enough to treat the case $n=0$. In this case (10.1) implies that $\mathrm{res}(\eta)=0$ for every $\eta\in\omega(\mathcal{O}_F)$. Hence $\mathrm{res}(x\eta)=0$ for all $x\in\mathcal{O}_F$ and $\eta\in\omega(\mathcal{O}_F)$. We conclude that $\rho(\eta)\in(F/\mathcal{O}_F)^*\subset F^*$.

Our next step is to prove that for every $n\in\mathbb{Z}$ the induced map

$$\rho\colon\frac{z^n\omega(\mathcal{O}_F)}{z^{n+1}\omega(\mathcal{O}_F)}\to\Big(\frac{z^{-(n+1)}\mathcal{O}_F}{z^{-n}\mathcal{O}_F}\Big)^* \tag{10.3}$$

is injective. Since $\rho(\eta)(x)=\mathrm{res}(x\eta)$ it is enough to prove this for a single $n\in\mathbb{Z}$. As the divisor $\infty\in C$ is ample there exists an $n\gg0$ such that $\mathrm{H}^1(C,\mathcal{O}(n))=0$. Now (10.2) implies that $\omega(A)\cap z^n\omega(\mathcal{O}_F)=0$. If $\eta\in z^n\omega(\mathcal{O}_F)$ is such that $\mathrm{res}(z^{-(n+1)}x\eta)=0$ for any $x\in\mathcal{O}_F^\times$ then the pairing (10.1) implies that $\eta\in\omega(A)+z^{n+1}\omega(\mathcal{O}_F)$. Since $\omega(A)\cap z^n\omega(\mathcal{O}_F)=0$ we conclude that $\eta\in z^{n+1}\omega(\mathcal{O}_F)$. Whence (10.3) is injective.

At the same time (10.3) is a morphism of one-dimensional $\mathcal{O}_F/z$-vector spaces. It is therefore an isomorphism. We conclude that for every $n>0$ the induced map

$$\rho\colon\frac{\omega(\mathcal{O}_F)}{z^n\omega(\mathcal{O}_F)}\to\Big(\frac{z^{-n}\mathcal{O}_F}{\mathcal{O}_F}\Big)^*$$

is an isomorphism. As $\omega(\mathcal{O}_F)$ is complete it follows that $\rho\colon\omega(\mathcal{O}_F)\to(F/\mathcal{O}_F)^*$ is a topological isomorphism. Since the open subspaces $z^n\omega(\mathcal{O}_F)$ cover $\omega(F)$ we deduce that $\rho\colon\omega(F)\to F^*$ is a topological isomorphism. □

Corollary 10.2. *Let T be a locally compact algebra.*

1. *$a(F/A,T)$ is a locally free $A\otimes T$-module of rank 1.*
2. *$b(F/A,T)$ is a locally free $A\mathbin{\widehat{\otimes}}T$-module of rank 1.*
3. *The natural map $a(F/A,T)\otimes_{A\otimes T}(A\mathbin{\widehat{\otimes}}T)\to b(F/A,T)$ is an isomorphism.*

Proof. Observe that $A\otimes T=A\mathbin{\breve{\otimes}}T$ since A is discrete. Theorem 10.1 tells us that $\omega(A)\cong(F/A)^*$. The module $\omega(A)$ is discrete and invertible as an A-module without topology. As a consequence it is a topological direct

summand of a free A-module of finite type. Now Lemma 8.10 shows that $(F/A)^* \otimes_A (A \mathbin{\breve{\otimes}} T) \cong a(F/A, T)$ so (1) holds. Similarly Lemma 8.6 tells us that $(F/A)^* \otimes_A (A \mathbin{\widehat{\otimes}} T) \cong b(F/A, T)$ and in particular (2) holds. Combining these natural isomorphisms we get (3). □

Corollary 10.3. *Let T be a locally compact algebra.*

(1) *$a(F, T)$ is a free $F \mathbin{\breve{\otimes}} T$-module of rank 1.*
(2) *$b(F, T)$ is a free $F^\# \mathbin{\widehat{\otimes}} T$-module of rank 1.*
(3) *The natural map $a(F, T) \otimes_{F \breve{\otimes} T} (F^\# \mathbin{\widehat{\otimes}} T) \to b(F, T)$ is an isomorphism.*

Proof. Theorem 10.1 implies that $F^* \cong F$. Hence the natural map $F^* \otimes_F (F \mathbin{\breve{\otimes}} T) \to a(F, T)$ is an isomorphism by Lemma 8.10 while the natural map $F^* \otimes_F (F^\# \mathbin{\widehat{\otimes}} T) \to b(F, T)$ is an isomorphism by Lemma 8.6. (1), (2) and (3) follow instantly. □

Corollary 10.4. *If T is a locally compact algebra then $a(F/\mathcal{O}_F, T)$ is a free $\mathcal{O}_F \mathbin{\breve{\otimes}} T$-module of rank 1.*

Proof. Theorem 10.1 shows that $(F/\mathcal{O}_F)^* \cong \mathcal{O}_F$ as a topological $\mathcal{O}_F$-module. Lemma 8.10 then implies that the natural map $(F/\mathcal{O}_F)^* \otimes_{\mathcal{O}_F} (\mathcal{O}_F \mathbin{\breve{\otimes}} T) \to a(F/\mathcal{O}_F, T)$ is an isomorphism. □

Corollary 10.5. *If T is a locally compact algebra then the natural maps*

$$a(F/A, T) \otimes_{A \otimes T} (F \mathbin{\breve{\otimes}} T) \to a(F, T)$$
$$b(F/A, T) \otimes_{A \widehat{\otimes} T} (F^\# \mathbin{\widehat{\otimes}} T) \to b(F, T)$$
$$a(F/\mathcal{O}_F, T) \otimes_{\mathcal{O}_F \breve{\otimes} T} (F \mathbin{\breve{\otimes}} T) \to a(F, T)$$

are isomorphisms.

Proof. Theorem 10.1 identifies $\omega(A)$ with $(F/A)^*$, $\omega(\mathcal{O}_F)$ with $(F/\mathcal{O}_F)^*$ and $\omega(F)$ with F^*. In particular $(F/A)^*$ is an invertible discrete A-module, $(F/\mathcal{O}_F)^*$ is a free topological $\mathcal{O}_F$-module of rank 1 and F^* is a free topological F-module of rank 1. Hence Lemma 8.10 shows that the natural maps

$$(F/A)^* \otimes_A (A \otimes T) \to a(F/A, T)$$
$$(F/\mathcal{O}_F)^* \otimes_{\mathcal{O}_F} (\mathcal{O}_F \mathbin{\breve{\otimes}} T) \to a(F/\mathcal{O}_F, T)$$
$$(F^*) \otimes_F (F \mathbin{\breve{\otimes}} T) \to a(F, T)$$

are isomorphisms, and Lemma 8.6 shows that the natural maps

$$(F/A)^* \otimes_A (A \mathbin{\widehat{\otimes}} T) \to b(F/A, T)$$
$$(F^*) \otimes_F (F^\# \mathbin{\widehat{\otimes}} T) \to b(F, T)$$

are isomorphisms. Now the natural maps $\omega(A) \otimes_A F \to \omega(F)$ and $\omega(\mathcal{O}_F) \otimes_{\mathcal{O}_F} F \to \omega(F)$ are isomorphisms whence the result. □

11. τ-ring and τ-module structures

Recall from Definition 1.1.1 that a τ-ring is a ring R equipped with an endomorphism $\tau\colon R \to R$. We would like to fix τ-ring structures on algebras of the form $S \mathbin{\breve{\otimes}} T$, $S \mathbin{\widehat{\otimes}} T$.

Definition 11.1. Let S, T be locally compact $\mathbb{F}_q$-algebras. Let $\sigma\colon T \to T$ be the q-power map.

(1) We equip $S \mathbin{\breve{\otimes}} T$ with the τ-ring structure given by the endomorphism $1 \mathbin{\breve{\otimes}} \sigma$.
(2) Assuming $S \mathbin{\widehat{\otimes}} T$ admits a natural lth algebra structure we equip it with the τ-ring structure given by the endomorphism $1 \mathbin{\widehat{\otimes}} \sigma$.

Lemma 11.2. *Let S, T be locally compact $\mathbb{F}_q$-algebras and N a locally compact S-module. Let $\sigma\colon T \to T$ be the q-power map. If $a(N,T)$, $b(N,T)$ and $g(N,T)$ are equipped with endomorphisms τ given by composition with σ then the following is true:*

(1) *$a(N,T)$ is a left $S \mathbin{\breve{\otimes}} T\{\tau\}$-module.*
(2) *$b(N,T)$ is a left $S^{\#} \mathbin{\widehat{\otimes}} T\{\tau\}$-module.*
(3) *$g(N,T)$ is a left $S \mathbin{\breve{\otimes}} T\{\tau\}$-module.*

In all cases the τ-ring structures are as in Definition 11.1.

Definition 11.3. Under assumptions of Lemma 11.2 we equip the spaces $a(N,T)$, $b(N,T)$ and $g(N,T)$ with the τ-module structures as described above. From now on we work with only these τ-module structures.

Proof of Lemma 11.2. (1) Let $f \in a(N,T)$, $x \in S \mathbin{\breve{\otimes}} T$. We need to prove that

$$\sigma \circ (x \cdot f) = \tau(x) \cdot (\sigma \circ f).$$

This is clear if $x \in S \otimes T$. As $S \otimes T \subset S \mathbin{\breve{\otimes}} T$ is dense and $\tau\colon S \mathbin{\breve{\otimes}} T \to S \mathbin{\breve{\otimes}} T$ is continuous the general statement follows. (2) follows in the same manner. (3) follows from (1), (2) and the short exact sequence of Proposition 2.11.2. $\square$