Shtuka cohomology and special values of Goss L-functions

Mornev, M.

Author: Mornev, M.
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CHAPTER 2

Topological vector spaces over finite fields

In this chapter we present some results on topological vector spaces over finite fields. Such a vector space over a finite field $\mathbb{F}_q$ of characteristic p can be viewed as a topological abelian p-torsion group equipped with a compatible extra structure of $\mathbb{F}_q$-multiplication.

The base field $\mathbb{F}_q$ is fixed throughout the chapter. It is assumed to carry the discrete topology. In the following a subspace of a vector space always means an $\mathbb{F}_q$-vector subspace, not an arbitrary topological subspace. As is usual in the theory of topological groups all our locally compact topological vector spaces are assumed to be Hausdorff. A topological vector space is said to be *linearly topologized* if every open neighbourhood of zero contains an open subspace. We mainly study linearly topologized Hausdorff vector spaces. Throughout this chapter we abbreviate "linearly topologized Hausdorff" as "*lth*".

1. Overview

This chapter is devoted to three groups of constructions: topological tensor products $\widehat{\otimes}$ and $\check{\otimes}$, function spaces and germ spaces. All of them figure prominently in this text. Past this chapter some degree of familiarity with them will be assumed.

Let V and W be locally compact vector spaces. The first goal of this chapter is to define and study two topological tensor products:

- the completed tensor product $V \mathbin{\widehat{\otimes}} W = \lim_{U,Y} V/U \otimes W/Y$.
- the ind-complete tensor product $V \mathbin{\check{\otimes}} W = \lim_{U,Y} (V \otimes W)/(U \otimes Y)$.

Here $U \subset V$ and $Y \subset W$ run over all open subspaces. Our main tool to study $V \mathbin{\check{\otimes}} W$ is the natural cartesian square

$$\begin{tikzcd} V \mathbin{\check{\otimes}} W \arrow[r] \arrow[d] & V \mathbin{\widehat{\otimes}} W^{\#} \arrow[d] \\ V^{\#} \mathbin{\widehat{\otimes}} W \arrow[r] & V \mathbin{\widehat{\otimes}} W \end{tikzcd}$$

in the category of topological vector spaces (cf. Proposition 7.6). Here $(-)^{\#}$ means the same vector space taken with the discrete topology.

The other important group of constructions is the function spaces:

- the space $c(V, W)$ of continuous $\mathbb{F}_q$-linear maps from V to W,

- the space $b(V,W)$ of bounded continuous $\mathbb{F}_q$-linear maps, i.e. the maps which have image in a compact subspace,
- the space $a(V,W)$ of locally constant bounded $\mathbb{F}_q$-linear maps.

The function spaces are equipped with topologies which make them into complete vector spaces. These topologies are carefully chosen to suit the applications. The space $c(V,W)$ carries the compact-open topology. The topologies on its subspaces $a(V,W)$ and $b(V,W)$ are finer than the induced ones.

To study the structure of the function spaces we will construct natural topological isomorphisms

$$\begin{array}{ll} c(V,W) \cong V^* \widehat{\otimes} W & \text{(Proposition 8.5)}, \\ b(V,W) \cong (V^*)^\# \widehat{\otimes} W & \text{(Proposition 9.4)}, \\ a(V,W) \cong V^* \check{\otimes} W & \text{(Proposition 10.3)} \end{array}$$

where V^* is the continuous $\mathbb{F}_q$-linear dual of V.

An important object related to function spaces is the space of germs $g(V,W)$. Its elements are equivalence classes of continuous $\mathbb{F}_q$-linear maps from V to W. Two such maps are equivalent if they restrict to the same map on an open subspace. The main property of $g(V,W)$ is invariance under local isomorphisms on the source V and the target W. This property will be indispensable for cohomological computations of Chapter 8 among others.

Much of the material in this chapter is largely well-known. However a reservation should be made about the ind-complete tensor product $V \check{\otimes} W$, the function spaces $a(V,W)$, $b(V,W)$ and the germ space $g(V,W)$. While these constructions are very natural and should have certainly appeared before, we are not aware of a reference for them in the literature.

Last but not least we should acknowledge our intellectual debt to G. W. Anderson. The inspiration for this chapter comes from his article [1], specifically from §2 of that text where he uses function spaces to compute what in retrospect is the cohomology of certain shtukas associated to Drinfeld modules.

2. Examples

To lend this discussion more substance let us give some examples. We begin with a few examples of locally compact vector spaces:

- $A = \mathbb{F}_q[t]$ with the discrete topology,
- $F = \mathbb{F}_q(\!(t^{-1})\!)$ with the locally compact topology,
- the compact open subspace $\mathcal{O}_F = \mathbb{F}_q[[t^{-1}]]$ in F.

The quotient F/A is naturally a compact $\mathbb{F}_q$-vector space.

Let K be a local field containing $\mathbb{F}_q$. An example of a function space which is particularly relevant to our study is $c(F/A,K)$, the space of continuous $\mathbb{F}_q$-linear maps from F/A to K. Proposition 8.5 provides us with a topological isomorphism

$$(F/A)^* \widehat{\otimes} K \cong c(F/A,K).$$

Let us show that this isomorphism gives a rather hands-on description of $c(F/A, K)$. Let $\Omega = \mathbb{F}_q[t]\,dt$ be the module of Kähler differentials of A over $\mathbb{F}_q$ equipped with the discrete topology and let

$$\mathrm{res}\colon \Omega \otimes_A F \to \mathbb{F}_q, \quad \sum_n a_n t^n \cdot dt \mapsto -a_{-1}$$

be the residue map at infinity. The map

$$\Omega \to (F/A)^*, \quad \omega \mapsto [x \mapsto \mathrm{res}(x\omega)]$$

is easily shown to be an isomorphism of topological vector spaces. As a consequence the isomorphism $\Omega \mathbin{\widehat{\otimes}} K \cong c(F/A, K)$ of Proposition 8.5 identifies $c(F/A, K)$ with the topological vector space of formal series à la Tate:

$$K\langle t\rangle\, dt = \Big\{ \sum_{n\geqslant 0} \alpha_n t^n dt \in K[[t]]dt \;\Big|\; \lim_{n\to\infty} \alpha_n = 0 \Big\}.$$

The topology on this space is induced by the norm

$$\Big| \sum_{n\geqslant 0} \alpha_n t^n \cdot dt \Big| = \sup_{n\geqslant 0} |\alpha_n|.$$

A series $\sum_{n\geqslant 0} \alpha_n t^n dt$ corresponds to the continuous function

$$F/A \to K, \quad x \mapsto \sum_{n\geqslant 0} \alpha_n \,\mathrm{res}(x t^n dt).$$

This function maps t^{-n} to $-\alpha_{n-1}$.

From this discussion one easily deduces that $g(F/A, K)$, the space of germs of continuous functions from F/A to K, is isomorphic to the quotient

$$\frac{\Omega \mathbin{\widehat{\otimes}} K}{\Omega \otimes K}.$$

Such quotients arise naturally in the cohomological computations of the subsequent chapters. The utility of $g(F/A, K)$ is that it gives them an accessible interpretation.

3. Basic properties

Lemma 3.1. *Every open embedding of topological vector spaces is continuously split.*

Proof. Let $j\colon U \hookrightarrow V$ be an open embedding. The quotient topology on V/U is discrete. So every $\mathbb{F}_q$-linear splitting $i\colon V/U \to V$ of the quotient map is continuous and the map $j \oplus i\colon U \oplus V/U \to V$ is a continuous bijection. If $U' \subset U$ is an open subset and $Y' \subset V/U$ any subset then the image of $U' \oplus Y'$ in V is a union of translates of U' whence open. Thus $j \oplus i$ is a topological isomorphism. $\square$

Corollary 3.2. *Let V, W be topological vector spaces, $U \subset V$ an open subspace. Every continuous $\mathbb{F}_q$-linear map $f\colon U \to W$ admits an extension to V with image $f(U)$.*

Proof. Take a splitting $V = U \oplus Y$ and extend f to Y by zero. □

Lemma 3.3. *A topological vector space is Hausdorff if and only if its zero point is closed.*

Proof. Let V be a topological vector space. Suppose that $0 \in V$ is closed. Consider the map $d\colon V \times V \to V$ which sends a pair (v, w) to $v - w$. The preimage of $0 \in V$ under d is the diagonal. Since d is continuous it follows that the diagonal is closed. Whence V is Hausdorff. □

Lemma 3.4. *Let V be an lth vector space, $V' \subset V$ a closed subspace. The quotient topology on V/V' is lth.* □

Lemma 3.5. *The category of lth vector spaces and continuous $\mathbb{F}_q$-linear maps is additive and has arbitrary limits. The limits commute with forgetful functors to $\mathbb{F}_q$-vector spaces (without topology) and to topological abelian groups.* □

4. Locally compact vector spaces

Proposition 4.1. *Every locally compact vector space is lth and contains a compact open subspace.*

Proof. Let V be a locally compact vector space. We first assume that $\mathbb{F}_q = \mathbb{F}_p$ for a prime p. So in effect we work with a locally compact p-torsion abelian group V. If V is connected then its Pontrjagin dual is trivial since the p-torsion subgroup of $\mathbb{C}^\times$ is disconnected. Hence V is itself trivial. The connected component of 0 for a general V is a connected locally compact subgroup so it is trivial. Translating by elements of V we conclude that V is totally disconnected. Now a theorem of van Dantzig [14, Ch. II, Theorem 7.7, p. 62] states that every open neighbourhood of $0 \in V$ contains a compact open subgroup.

Next let $\mathbb{F}_q$ be an arbitrary finite field. We prove that every open subgroup $U \subset V$ contains an open $\mathbb{F}_q$-vector subspace. Indeed the subgroup

$$U' = \bigcap_{\alpha \in \mathbb{F}_q{}^\times} \alpha U$$

is open as an intersection of finitely many opens and is stable under the $\mathbb{F}_q$-action. □

Lemma 4.2. *Let V be a locally compact vector space. For every compact subset $K \subset V$ there exists a compact open subspace of V containing K.*

Proof. Let $U \subset V$ be a compact open subspace which exists by Proposition 4.1. The quotient V/U is discrete so the image of K in it is finite. Let $K' \subset U/V$ be a finite $\mathbb{F}_q$-vector subspace containing the image of K. The preimage of K' in V is compact open and contains K by construction. □

Definition 4.3. For a locally compact vector space V we define V^* to be the space of continuous $\mathbb{F}_q$-linear functions $V \to \mathbb{F}_q$ equipped with the compact-open topology.

Let $\mathbb{F}_p \subset \mathbb{F}_q$ be the prime subfield. For an lth $\mathbb{F}_q$-vector space V we denote $V_{\mathbb{F}_p}$ the restriction of scalars to $\mathbb{F}_p$. Consequently $(V_{\mathbb{F}_p})^*$ stands for the continuous $\mathbb{F}_p$-linear dual of V. Observe that we have a trace map $\mathrm{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$.

Lemma 4.4. *If V is a locally compact vector space then the natural map $V^* \to (V_{\mathbb{F}_p})^*$ given by composition with $\mathrm{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ is a topological isomorphism.*

Proof. As the extension $\mathbb{F}_p \subset \mathbb{F}_q$ is separable the map

$$\mathrm{Hom}_{\mathbb{F}_q}(V, \mathbb{F}_q) \xrightarrow{\mathrm{tr}\,\circ -} \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)$$

is an isomorphism. It identifies V^* and $(V_{\mathbb{F}_p})^*$ since the topology on V and $V_{\mathbb{F}_p}$ is the same. Hence the natural map of this lemma is an isomorphism. It remains to prove that it is a homeomorphism.

For a compact open subspace $U \subset V$ we view $(V/U)^*$ as the subspace of V^* consisting of functions which vanish on U. Let $\mathcal{U}$ be the family of all compact open subspaces in V. Lemma 4.2 shows that the family $\{(V/U)^* \mid U \in \mathcal{U}\}$ is a fundamental system in V^* while Proposition 4.1 implies that it covers V^*. The same argument shows that the subspaces $(V/U)^*_{\mathbb{F}_p} \subset (V_{\mathbb{F}_p})^*$ form a fundamental system which covers $(V_{\mathbb{F}_p})^*$. As the natural isomorphism $V^* \cong (V_{\mathbb{F}_p})^*$ identifies $(V/U)^*$ with $(V/U)^*_{\mathbb{F}_p}$ it is in fact a homeomorphism. □

In our context the duality theorem of Pontrjagin takes the following form:

Theorem 4.5. *Let V be a locally compact vector space.*

(1) *V^* is locally compact. Moreover:*
   (a) *V is discrete if and only if V^* is compact.*
   (b) *V is compact if and only if V^* is discrete.*
(2) *The natural map $V \to V^{**}$ is a topological isomorphism.*

Proof. Lemma 4.4 reduces the problem to the case $\mathbb{F}_q = \mathbb{F}_p$. In this case we can invoke the usual Pontrjagin duality for locally compact abelian groups as follows. A choice of a primitive p-th root of unity determines a topological isomorphism of $(\mathbb{F}_p, +)$ and the p-torsion subgroup $\mu_p(\mathbb{C}) \subset \mathbb{C}^\times$. Every character of V as a locally compact p-torsion abelian group has the image in $\mu_p(\mathbb{C})$. Thus the chosen isomorphism $(\mathbb{F}_p, +) \cong \mu_p(\mathbb{C})$ identifies V^* with the Pontrjagin dual of V. The result is now clear. □

5. Completion

Let V be an lth vector space. The completion of V is the lth vector space

$$\widehat{V} = \lim_{U} V/U$$

where $U \subset V$ runs over all open subspaces. It is enough to take the limit over a fundamental system of open subspaces. V is called complete if the natural map $V \to \widehat{V}$ is a topological isomorphism. Every continuous $\mathbb{F}_q$-linear map from V to a complete lth vector space factors uniquely over $\widehat{V}$.

Lemma 5.1. *If U is an open subspace in an lth vector space V then the natural sequence*

$$0 \to \widehat{U} \to \widehat{V} \to V/U \to 0$$

is exact. In particular the natural map $\widehat{U} \to \widehat{V}$ is an open embedding. □

Lemma 5.2. *Let V be an lth vector space.*

(1) *$\widehat{V}$ is complete.*
(2) *The natural map $V \to \widehat{V}$ is injective with dense image.*
(3) *If $V \to \widehat{V}$ is bijective then it is a topological isomorphism.* □

Lemma 5.3. *A locally compact vector space is complete.*

Proof. Let V be a compact vector space. The natural map $V \to \widehat{V}$ is injective with dense image so a topological isomorphism. Thus a compact space is complete. If a space V admits a complete open subspace then it is complete. In particular every locally compact space is complete. □

Proposition 5.4. *If $\{U_i\}$ is a covering of an lth vector space V by open subspaces then $\{\widehat{U}_i\}$ covers $\widehat{V}$.*

Proof. According to Lemma 5.1 the natural maps $\widehat{U}_i \to \widehat{V}$ are open embeddings. Let W be the union of $\widehat{U}_i$ inside of $\widehat{V}$. By construction W contains V and so is dense. Being an open subspace W is automatically closed. Thus $W = \widehat{V}$. □

6. Completed tensor product

Recall that according to our convention a tensor product $\otimes$ without subscript means a tensor product over $\mathbb{F}_q$.

Definition 6.1. Let V, W be lth vector spaces. Define the *tensor product topology* on $V \otimes W$ by the fundamental system of subspaces $U \otimes W + V \otimes Y$ where $U \subset V, Y \subset W$ run over all open subspaces. The $\mathbb{F}_q$-vector space $V \otimes W$ equipped with this topology is denoted $V \otimes_{\mathrm{c}} W$. We reserve the tensor product $V \otimes W$ without decorations to indicate the corresponding vector space without topology.

In general a continuous bilinear map $U \times V \to W$ does not induce a continuous map $U \otimes_{\mathrm{c}} V \to W$.

Lemma 6.2. *If V and W are lth vector spaces then $V \otimes_{\mathrm{c}} W$ is lth.*

Proof. We need to prove that $V \otimes_{\mathrm{c}} W$ is Hausdorff. According to Lemma 3.3 it suffices to prove that $0 \in V \otimes_{\mathrm{c}} W$ is closed. Assume W is discrete. If $U \subset V$ is an open subspace then $U \otimes W \subset V \otimes_{\mathrm{c}} W$ is open and hence closed. Letting U run over all open subspaces of V we conclude that $0 = \bigcap_U U \otimes W$ is closed. Now let W be arbitrary. Fix an open subspace $Y \subset W$. The natural map $V \otimes_{\mathrm{c}} W \to V \otimes_{\mathrm{c}} W/Y$ is continuous. As the latter space is Hausdorff it follows that $V \otimes Y$ is closed. The intersection $\bigcap_Y V \otimes Y = 0$ is thus closed. □

Definition 6.3. Let V, W be lth vector spaces. We define the *completed tensor product* $V \mathbin{\widehat{\otimes}} W$ as the completion of $V \otimes_{\mathrm{c}} W$. In other words

$$V \mathbin{\widehat{\otimes}} W = \lim_{U,Y} V/U \otimes W/Y$$

where $U \subset V, Y \subset W$ run over all open subspaces and the tensor products in the limit diagram are equipped with the discrete topology. If $f\colon V_1 \to V_2$ and $g\colon W_1 \to W_2$ are continuous $\mathbb{F}_q$-linear maps then $f \mathbin{\widehat{\otimes}} g\colon V_1 \mathbin{\widehat{\otimes}} W_1 \to V_2 \mathbin{\widehat{\otimes}} W_2$ is defined as the completion of $f \otimes g$.

A completed tensor product of two compact spaces is compact. However a completed tensor product of an infinite discrete and an infinite compact space is never locally compact.

Definition 6.4. For a vector space V we define $V^{\#}$ to be this space equipped with the discrete topology.

Proposition 6.5. *Let V, W be lth vector spaces. Consider the natural map $\iota\colon V^{\#} \mathbin{\widehat{\otimes}} W \to V \mathbin{\widehat{\otimes}} W$.*

(1) *The map ι is injective with dense image.*
(2) *If V is complete and W compact then ι is a bijection.*

Proof. For every open subspace $Y \subset W$ let $\iota_Y\colon V \otimes_{\mathrm{c}} W/Y \to \lim_U(V/U \otimes_{\mathrm{c}} W/Y)$ be the completion map. At the level of $\mathbb{F}_q$-vector spaces without topology the map ι is the limit of ι_Y over all open subspaces $Y \subset W$.

(1) The space $V \otimes_{\mathrm{c}} W/Y$ is Hausdorff by Lemma 6.2. Hence ι_Y is injective by Lemma 5.2. It follows that ι is injective. The density statement is clear.

(2) The space W/Y is finite since W is compact. Hence the natural map

$$\lim_U(V/U \otimes W/Y) \to \lim_U(V/U) \otimes W/Y$$

is an isomorphism of $\mathbb{F}_q$-vector spaces without topology. Since V is complete we conclude that ι_Y is bijective. As a consequence ι is a bijection. □

7. Ind-complete tensor product

In this section we introduce a different topology on $V \otimes W$ which is better for some purposes than the usual tensor product topology.

Definition 7.1. Let V, W be lth vector spaces. We define the *ind-tensor product topology* on $V \otimes W$ by the fundamental system of open subspaces $U \otimes Y$ where $U \subset V$, $Y \subset W$ run over all open subspaces. We denote $V \otimes_{\mathrm{ic}} W$ the tensor product $V \otimes W$ equipped with this topology.

One can prove that a continuous bilinear map $U \times V \to W$ induces a continuous map $U \otimes_{\mathrm{ic}} V \to W$. On the downside the ind-tensor product topology has some counterintuitive properties. For example $V \otimes_{\mathrm{ic}} \mathbb{F}_q = V^{\#}$ so $\mathbb{F}_q$ is not a tensor unit for $\otimes_{\mathrm{ic}}$.

Lemma 7.2. *If V, W are lth vector spaces then the natural map $V \otimes_{\mathrm{ic}} W \to V^{\#} \otimes_{\mathrm{c}} W$ is continuous.*

Proof. Indeed if $Y \subset W$ is an open subspace then $V \otimes Y$ is open both in $V \otimes_{\mathrm{ic}} W$ and in $V^{\#} \otimes_{\mathrm{c}} W$. The result follows since the subspaces $V \otimes Y$ form a fundamental system in $V^{\#} \otimes_{\mathrm{c}} W$. □

Lemma 7.3. *If V, W are lth vector spaces then $V \otimes_{\mathrm{ic}} W$ is lth.*

Proof. The space $V^{\#} \otimes_{\mathrm{c}} W$ is Hausdorff by Lemma 6.2. As the natural bijection $V \otimes_{\mathrm{ic}} W \to V^{\#} \otimes_{\mathrm{c}} W$ is continuous the point $0 \in V \otimes_{\mathrm{ic}} W$ is closed. So $V \otimes_{\mathrm{ic}} W$ is Hausdorff by Lemma 3.3. □

Lemma 7.4. *If $f\colon V_1 \to V_2$ and $g\colon W_1 \to W_2$ are continuous $\mathbb{F}_q$-linear maps of lth vector spaces then the map $f \otimes g\colon V_1 \otimes_{\mathrm{ic}} W_1 \to V_2 \otimes_{\mathrm{ic}} W_2$ is continuous.* □

Definition 7.5. Let V, W be lth vector spaces. We define the *ind-complete tensor product* $V \check{\otimes} W$ as the completion of $V \otimes_{\mathrm{ic}} W$. If $f\colon V_1 \to V_2$ and $g\colon W_1 \to W_2$ are continuous $\mathbb{F}_q$-linear maps of lth vector spaces then $f \check{\otimes} g\colon V_1 \check{\otimes} W_1 \to V_2 \check{\otimes} W_2$ is defined as the completion of $f \otimes g$.

Lemma 7.2 equips us with natural maps $V \check{\otimes} W \to V^{\#} \widehat{\otimes} W$ and $V \check{\otimes} W \to V \widehat{\otimes} W^{\#}$.

Proposition 7.6. *If V, W are lth vector spaces then the natural square*

$$\begin{tikzcd} V \check{\otimes} W \arrow[r] \arrow[d] & V \widehat{\otimes} W^{\#} \arrow[d] \\ V^{\#} \widehat{\otimes} W \arrow[r] & V \widehat{\otimes} W \end{tikzcd}$$

is cartesian in the category of topological vector spaces.

Proof. The proposition claims that the map

$$f\colon V \check{\otimes} W \to (V^{\#} \widehat{\otimes} W) \times_{V \widehat{\otimes} W} (V \widehat{\otimes} W^{\#})$$

defined by the diagram above is a topological isomorphism.

Given open subspaces $U \subset V$, $Y \subset W$ let us temporarily denote

$$\begin{aligned} [U, Y] &= V/U \otimes_{\mathrm{c}} W/Y, \\ \langle U, Y \rangle &= (V^{\#} \otimes_{\mathrm{c}} W/Y) \times_{[U,Y]} (V/U \otimes_{\mathrm{c}} W^{\#}). \end{aligned}$$

As limits commute with limits the target of the map f is $\lim_{U,Y} \langle U, Y \rangle$ where U, Y range over all open subspaces. Hence f is defined by the natural projections

$$f_{U,Y}\colon V \otimes_{\mathrm{ic}} W \to \langle U, Y \rangle.$$

A choice of splittings $V \cong U \oplus V/U$, $W \cong Y \oplus W/Y$ induces isomorphisms

$$\begin{aligned} \langle U, Y \rangle &\cong (U^{\#} \otimes_{\mathrm{c}} W/Y) \times (V/U \otimes_{\mathrm{c}} Y^{\#}) \times [U, Y], \\ V \otimes_{\mathrm{ic}} W &\cong (U \otimes_{\mathrm{ic}} Y) \times (U \otimes_{\mathrm{ic}} W/Y) \times (V/U \otimes_{\mathrm{ic}} W) \times [U, Y] \end{aligned}$$

which identify $f_{U,Y}$ with the projection to the last three factors. Hence $f_{U,Y}$ is onto with the kernel $U \otimes_{\mathrm{ic}} Y$. The resulting continuous bijection

$$(V \otimes_{\mathrm{ic}} W)/(U \otimes_{\mathrm{ic}} Y) \to \langle U, Y \rangle$$

is a topological isomorphism since its target and source are both discrete. Taking the limit over all U, Y we deduce the desired result. $\square$

Corollary 7.7. *If V, W are lth vector spaces then the natural maps $V \check{\otimes} W \to V^{\#} \widehat{\otimes} W$ and $V \check{\otimes} W \to V \widehat{\otimes} W^{\#}$ are injective.*

Proof. According to Proposition 6.5 both natural maps $V^{\#} \widehat{\otimes} W \to V \widehat{\otimes} W$ and $V \widehat{\otimes} W^{\#} \to V \widehat{\otimes} W$ are injective. So the result follows from Proposition 7.6. $\square$

Corollary 7.8. *If V is a compact vector space and W a complete lth vector space then the natural map $V \check{\otimes} W \to V^{\#} \widehat{\otimes} W$ is a continuous bijection.*

Proof. Indeed $V \widehat{\otimes} W^{\#} \to V \widehat{\otimes} W$ is a bijection by Proposition 6.5 (2) whence the result follows from Proposition 7.6. $\square$

8. Continuous functions

Definition 8.1. Let V, W be topological vector spaces. We define $c(V, W)$ to be the space of continuous $\mathbb{F}_q$-linear maps $V \to W$ equipped with the compact-open topology.

Lemma 8.2. *Let V be a topological vector space. If W is an lth vector space then so is $c(V, W)$.* $\square$

Lemma 8.3. *If an $\mathbb{F}_q$-linear map $V \to W$ is continuous then so are the induced natural transformations $c(W, -) \to c(V, -)$ and $c(-, V) \to c(-, W)$.* $\square$

The natural map $c(U \otimes_{\mathrm{c}} V, W) \to c(U, c(V, W))$ is not surjective in general and so does not define an adjunction of $- \otimes_{\mathrm{c}} V$ and $c(V, -)$.

Definition 8.4. Let V, W be topological vector spaces. Define $\sigma_{V,W} \colon V^* \otimes W \to c(V, W)$ to be the map which sends a tensor $f \otimes w$ to the function $v \mapsto f(v)w$.

Proposition 8.5. *If V is a locally compact vector space and W a complete lth vector space then there exists a unique topological isomorphism*

$$V^* \widehat{\otimes} W \to c(V, W)$$

extending $\sigma_{V,W}$ on $V^* \otimes W$.

Proof. Step 1. Assume V is compact and W discrete. The space V^* is discrete by Theorem 4.5 whence $V^* \widehat{\otimes} W = V^* \otimes_{\mathrm{c}} W$ is discrete. As a consequence

$$V^* \widehat{\otimes} W = \bigcup_{W' \subset W} V^* \otimes_{\mathrm{c}} W'$$

where $W' \subset W$ ranges over all finite subspaces.

The space $c(V,W)$ is discrete since $V \subset V$ is compact and $\{0\} \subset W$ is open. As V is compact and W discrete the image of every continuous $\mathbb{F}_q$-linear map $V \to W$ is finite. Therefore

$$c(V,W) = \bigcup_{W' \subset W} c(V,W')$$

where $W' \subset W$ again ranges over all finite subspaces. Hence it is enough to consider the case when W is finite. This case instantly reduces to the case $W \cong \mathbb{F}_q$ which is clear.

Step 2. Assume that V is locally compact and W is complete lth. For an open subspace $Y \subset W$ let $q_Y : W \to W/Y$ be the quotient map. For a compact open subspace $U \subset V$ let $\rho_{U,Y} : c(V,W) \to c(U,W/Y)$ be the map given by restriction to U and composition with q_Y. The subspace $\ker \rho_{U,Y} \subset c(V,W)$ consists of functions which send the compact subset U to the open subset Y. As a consequence it is open. Lemma 4.2 implies that the collection of all the subspaces $\ker \rho_{U,Y}$ is a fundamental system of open subspaces in $c(V,W)$. Every $\rho_{U,Y}$ is surjective by Corollary 3.2. Therefore the limit map

$$\rho\colon c(V,W) \to \lim_{U,Y} c(U,W/Y).$$

defined by the $\rho_{U,Y}$ is the completion map $c(V,W) \to c(V,W)\widehat{}$. The map ρ is bijective since V is the union of its compact open subspaces and $W = \lim_Y W/Y$. As ρ is the completion map of $c(V,W)$ it is in fact a topological isomorphism.

Let $\rho_U : V^* \to U^*$ be the restriction map. Arguing as in the case of $\rho_{U,Y}$ above we conclude that the collection of subspaces $\ker \rho_U$ is a fundamental system in V^* and that every map ρ_U is surjective. As a consequence the limit map

$$\psi\colon V^* \otimes_{\mathrm{c}} W \to \lim_{U,Y} (U^* \otimes W/Y).$$

defined by the $\rho_U \otimes q_Y$ is the completion map of $V^* \otimes_{\mathrm{c}} W$. Altogether we obtain a commutative diagram

$$\begin{tikzcd}
V^* \otimes_{\mathrm{c}} W \arrow[r, "\sigma_{V,W}"] \arrow[d, "\psi"] & c(V,W) \arrow[d, "\rho"] \\
\lim_{U,Y} (U^* \otimes_{\mathrm{c}} W/Y) \arrow[r, "\lim \sigma_{U,W/Y}"] & \lim_{U,Y} c(U,W/Y).
\end{tikzcd}$$

The bottom arrow is a topological isomorphism as a limit of topological isomorphisms $\sigma_{U,W/Y}$. Hence $\sigma_{V,W}$ factors through a topological isomorphism $V^* \mathbin{\widehat{\otimes}} W \to c(V,W)$. As $V^* \otimes W$ is dense in $V^* \mathbin{\widehat{\otimes}} W$ this is the topological isomorphism σ we need to construct. $\square$

9. Bounded functions

Definition 9.1. Let V, W be locally compact vector spaces. A continuous $\mathbb{F}_q$-linear map $f\colon V \to W$ is said to be bounded if its image is contained in a compact subspace. We define $b(V,W) \subset c(V,W)$ to be the space of all bounded maps. The topology on $b(V,W)$ is given by the fundamental system of subspaces $c(V,Y)$ where $Y \subset W$ ranges over all compact open subspaces.

Lemma 9.2. *The inclusion $b(V,W) \subset c(V,W)$ is continuous.* □

Lemma 9.3. *If an $\mathbb{F}_q$-linear map $V \to W$ is continuous then so are the induced natural transformations $b(W,-) \to b(V,-)$ and $b(-,V) \to b(-,W)$.* □

Observe that the map $\sigma_{V,W}\colon V^* \otimes W \to c(V,W)$ of Definition 8.4 has image in $b(V,W)$.

Proposition 9.4. *If V, W are locally compact vector spaces then there exists a unique topological isomorphism*

$$(V^*)^{\#} \widehat{\otimes} W \to b(V,W)$$

extending $\sigma_{V,W}$ on $V^* \otimes W$.

Proof. The composition

$$(V^*)^{\#} \widehat{\otimes} W \xrightarrow{\iota} V^* \widehat{\otimes} W \xrightarrow{\sigma} c(V,W)$$

of the natural inclusion ι and the topological isomorphism σ of Proposition 8.5 is continuous and restricts to $\sigma_{V,W}$ on $V^* \otimes W$. Hence our claim follows if $\sigma\iota$ is a homeomorphism onto $b(V,W)$.

Let $\mathcal{U}$ be the family of all compact open subspaces of W. $\mathcal{U}$ is a fundamental system in W by Proposition 4.1. It covers W by Lemma 4.2. Since $(V^*)^{\#}$ is discrete it follows that the family

$$\{(V^*)^{\#} \otimes Y \mid Y \in \mathcal{U}\}$$

is a fundamental system which covers $(V^*)^{\#} \otimes_{\mathrm{c}} W$. As a consequence the family

$$\{(V^*)^{\#} \widehat{\otimes} Y \mid Y \in \mathcal{U}\}$$

is a fundamental system in $(V^*)^{\#} \widehat{\otimes} W$. It covers $(V^*)^{\#} \widehat{\otimes} W$ by Proposition 5.4.

As $Y \in \mathcal{U}$ is compact the map ι identifies $(V^*)^{\#} \widehat{\otimes} Y$ with $(V^*) \widehat{\otimes} Y$ by Proposition 6.5 (2). The map σ sends the latter subspace isomorphically onto $c(V,Y) \subset b(V,W)$. The subspaces $c(V,Y)$ form a fundamental system in $b(V,W)$. This system covers $b(V,W)$ as a consequence of Lemma 4.2. We conclude that $\iota\sigma$ is a homeomorphism onto $b(V,W)$. □

10. Bounded locally constant functions

Definition 10.1. Let V, W be locally compact vector spaces. A continuous $\mathbb{F}_q$-linear map $f\colon V \to W$ is called bounded locally constant if it is bounded and its kernel is open. We define $a(V, W) \subset b(V, W)$ to be the space of all bounded locally constant maps. The space $a(V, W)$ is equipped with the minimal topology such that the inclusions $a(V, W) \subset b(V, W)$ and $a(V, W) \subset c(V, W^{\#})$ are continuous.

One can describe $a(V, W)$ set-theoretically as the intersection

$$a(V, W) = b(V, W) \cap c(V, W^{\#}) \subset c(V, W).$$

By construction the topology on $a(V, W)$ is that of the fibre product

$$b(V, W) \times_{c(V,W)} c(V, W^{\#}).$$

In general both inclusions

$$b(V, W^{\#}) \subset a(V, W) \subset b(V, W)$$

are proper.

Lemma 10.2. *If an $\mathbb{F}_q$-linear map $V \to W$ is continuous then so are the induced natural transformations $a(W, -) \to a(V, -)$ and $a(-, V) \to a(-, W)$.* $\square$

Observe that the map $\sigma_{V,W}\colon V^* \otimes W \to c(V, W)$ of Definition 8.4 has image in $a(V, W)$.

Proposition 10.3. *If V, W are locally compact vector spaces then there exists a unique topological isomorphism*

$$V^* \check{\otimes} W \to a(V, W)$$

extending $\sigma_{V,W}$ on $V^* \otimes W$.

Proof. Consider the commutative diagram

$$(*) \qquad \begin{tikzcd} (V^*)^{\#} \widehat{\otimes} W \arrow[r] \arrow[d] & V^* \widehat{\otimes} W \arrow[d] & V^* \widehat{\otimes} W^{\#} \arrow[l] \arrow[d] \\ b(V, W) \arrow[r] & c(V, W) & c(V, W^{\#}) \arrow[l] \end{tikzcd}$$

where the horizontal arrows are the natural maps, the left vertical arrow is the topological isomorphism of Proposition 9.4 and the other two vertical arrows are the topological isomorphisms provided by Propositon 8.5. According to Proposition 7.6

$$V^* \check{\otimes} W = ((V^*)^{\#} \widehat{\otimes} W) \times_{V^* \widehat{\otimes} W} (V^* \widehat{\otimes} W^{\#}).$$

At the same time

$$a(V, W) = b(V, W) \times_{c(V,W)} c(V, W^{\#}).$$

Thus $(*)$ defines a topological isomorphism $V^* \check{\otimes} W \to a(V,W)$. It extends $\sigma_{V,W}$ since the vertical maps in $(*)$ do so. $\square$

11. Germ spaces

Definition 11.1. Let V, W be lth vector spaces. The $\mathbb{F}_q$-vector *space of germs* $g(V,W)$ is

$$g(V,W) = \operatorname*{colim}_U c(U,W)$$

where $U \subset V$ runs over all open subspaces and the transition maps are restrictions. We do not equip $g(V,W)$ with a topology. The image of $f \in c(U,W)$ under the natural map $c(U,W) \to g(V,W)$ is called the germ of f.

An element of $g(V,W)$ can be represented by a pair (U,f) where $U \subset V$ is an open subspace and $f\colon U \to W$ a continuous $\mathbb{F}_q$-linear map. Two such pairs (U_1,f_1), (U_2,f_2) represent the same element of $g(V,W)$ if there exists an open subspace $U \subset U_1 \cap U_2$ such that $f_1|_U = f_2|_U$.

Proposition 11.2. *If V, W are locally compact vector spaces then the natural sequence*

$$0 \to a(V,W) \to b(V,W) \to g(V,W) \to 0$$

is exact.

Proof. The sequence is clearly left exact. We need to prove that $b(V,W) \to g(V,W)$ is surjective. Let $U \subset V$ be an open subspace. As V is locally compact there exists a compact open subspace $U' \subset U$. According to Corollary 3.2 the restriction map $c(U,W) \to c(U',W)$ is onto. Furthermore $c(U',W) = b(U',W)$ since U' is compact. The map $b(V,W) \to b(U',W)$ is surjective by Corollary 3.2 again. Hence the map $b(V,W) \to g(V,W)$ is surjective. $\square$

Definition 11.3. Let V, W be lth vector spaces. A continuous $\mathbb{F}_q$-linear map $f\colon V \to W$ is called a *local isomorphism* if there exists an open subspace $U \subset V$ such that $f(U) \subset W$ is open and the restriction $f|_U\colon U \to f(U)$ is a topological isomorphism.

Proposition 11.4. *If $f\colon V \to W$ is a local isomorphism of lth vector spaces then the induced natural transformations $g(W,-) \to g(V,-)$ and $g(-,V) \to g(-,W)$ are isomorphisms.* $\square$

Lemma 11.5. *For every pair of locally compact vector spaces V, W the natural map $V \check{\otimes} W \to V^{\#} \hat{\otimes} W$ extends to a natural short exact sequence*

$$0 \to V \check{\otimes} W \to V^{\#} \hat{\otimes} W \to g(V^*,W) \to 0.$$

Proof. Consider the short exact sequence

$$0 \to a(V^*,W) \to b(V^*,W) \to g(V^*,W) \to 0$$

of Proposition 11.2. The isomorphisms

$$b(V^*, W) \cong (V^{**})^{\#} \mathbin{\widehat{\otimes}} W,$$
$$a(V^*, W) \cong V^{**} \mathbin{\check{\otimes}} W$$

of Propositions 9.4, 10.3 and Pontrjagin duality of Theorem 4.5 transform it to

$$0 \to V \mathbin{\check{\otimes}} W \to V^{\#} \mathbin{\widehat{\otimes}} W \to g(V^*, W) \to 0$$

and the result follows. □

Proposition 11.6. *Let $f_V : V_1 \to V_2$ and $f_W : W_1 \to W_2$ be continuous $\mathbb{F}_q$-linear maps of locally compact vector spaces. If f_V^* and f_W are local isomorphisms then the induced map*

$$\frac{V_1^{\#} \mathbin{\widehat{\otimes}} W_1}{V_1 \mathbin{\check{\otimes}} W_1} \to \frac{V_2^{\#} \mathbin{\widehat{\otimes}} W_2}{V_2 \mathbin{\check{\otimes}} W_2}$$

is an isomorphism.

Proof. The induced map $g(V_1^*, W_1) \to g(V_2^*, W_2)$ is an isomorphism by Proposition 11.4. Hence the result follows from Lemma 11.5. □