Shtuka cohomology and special values of Goss L-functions

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CHAPTER 1

Shtukas

In this chapter we present a theory of shtuka cohomology together with some supplementary constructions. By itself, shtuka cohomology is nothing new. It usually appears in the form of explicit complexes such as the one of Theorem 1.8.1 or the one of Lemma 4.3.2. By contrast the point of view we take in this chapter is rather abstract. Given a scheme X and an endomorphism τ we define an abelian category of shtukas on (X, τ), prove that it has enough injectives and define a shtuka cohomology functor as the right derived functor of a certain global sections functor. This theory is developed for an arbitrary scheme X over $\operatorname{Spec} \mathbb{Z}$ and an arbitrary endomorphism τ. Assumptions on X or τ are neither necessary nor will they make the theory simpler.

The main results of this theory are as follows:

- Theorem 4.6 relates shtuka cohomology to the Ext groups in the abelian category of shtukas.
- Theorem 5.6 provides a natural distinguished triangle which links shtuka cohomology with sheaf cohomology.
- Theorem 8.1 computes the cohomology of quasi-coherent shtukas on affine schemes.

Our treatment of shtuka cohomology was inspired by the article [17] of V. Lafforgue and the book [3] of G. Böckle and R. Pink.

The general theory of shtuka cohomology occupies the first eight sections of this chapter. Section 9 introduces the important notion of nilpotence borrowed from the theory of Böckle-Pink [3]. The construction of ζ-isomorphisms in Section 10 is due to V. Lafforgue [17]. The material of Section 11 is well-known. In Section 12 we study a Hom shtuka construction. Theorem 12.5 of that section relates the cohomology of the Hom shtuka to RHom in the category of left modules over a τ-polynomial ring. This result is of central importance to our computations of shtuka cohomology in the context of Drinfeld modules.

In reading this chapter a certain degree of familiarity with derived categories will be beneficial.

1. Basic definitions

Definition 1.1. A τ-ring is a pair (R, τ) consisting of a ring R and a ring endomorphism $\tau\colon R \to R$. A morphism of τ-rings $f\colon (R, \tau) \to (S, \sigma)$ is a ring homomorphism $f\colon R \to S$ such that $f\tau = \sigma f$.

A τ-scheme is a pair (X, τ) consisting of a scheme X and an endomorphism $\tau\colon X \to X$. A morphism of τ-schemes $f\colon (X,\tau) \to (Y,\sigma)$ is a morphism of schemes $f\colon X \to Y$ such that $f\tau = \sigma f$.

As we never work with more than one τ-ring structure on a given ring R we speak of a τ-ring R instead of (R,τ) and reserve the letter τ to denote the corresponding ring endomorphism. The same applies to τ-schemes.

A typical example of a τ-scheme appearing in this text is the following. Let $\mathbb{F}_q$ be a finite field with q elements, A an $\mathbb{F}_q$-algebra and X a smooth projective curve over $\mathbb{F}_q$. We equip the product $\operatorname{Spec} A \times_{\mathbb{F}_q} X$ with the τ-scheme structure given by the endomorphism which acts as the identity on $\operatorname{Spec} A$ and as the q-Frobenius on X.

Definition 1.2. Let X be a τ-scheme. An $\mathcal{O}_X$-module *shtuka* is a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

where $\mathcal{M}_0$, $\mathcal{M}_1$ are $\mathcal{O}_X$-modules and

$$\begin{aligned} i&\colon \mathcal{M}_0 \to \mathcal{M}_1, \\ j&\colon \mathcal{M}_0 \to \tau_*\mathcal{M}_1 \end{aligned}$$

are morphisms of $\mathcal{O}_X$-modules. A shtuka is called *quasi-coherent* if $\mathcal{M}_0$ and $\mathcal{M}_1$ are quasi-coherent $\mathcal{O}_X$-modules. It is called *locally free* if $\mathcal{M}_0$ and $\mathcal{M}_1$ are locally free $\mathcal{O}_X$-modules **of finite rank**.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-module shtukas given by diagrams

$$\mathcal{M} = \Big[\mathcal{M}_0 \underset{j_M}{\overset{i_M}{\rightrightarrows}} \mathcal{M}_1\Big], \quad \mathcal{N} = \Big[\mathcal{N}_0 \underset{j_N}{\overset{i_N}{\rightrightarrows}} \mathcal{N}_1\Big].$$

A morphism from $\mathcal{M}$ to $\mathcal{N}$ is a pair (f_0, f_1) where $f_n\colon \mathcal{M}_n \to \mathcal{N}_n$ are $\mathcal{O}_X$-module morphisms such that the diagrams

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, "f_0"] \arrow[d, "i_M"] & \mathcal{N}_0 \arrow[d, "i_N"] \\ \mathcal{M}_1 \arrow[r, "f_1"] & \mathcal{N}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}_0 \arrow[r, "f_0"] \arrow[d, "j_M"] & \mathcal{N}_0 \arrow[d, "j_N"] \\ \tau_*\mathcal{M}_1 \arrow[r, "\tau_*(f_1)"] & \tau_*\mathcal{N}_1 \end{tikzcd}$$

commute.

Our definition of a shtuka differs from the ones present in the literature in that we assume no restriction on $\mathcal{M}_0$, $\mathcal{M}_1$, i, j, X and even τ. This definition is the most convenient one for our purposes. We work with arbitrary $\mathcal{O}_X$-modules instead of just the quasi-coherent ones to make our definition of shtuka cohomology compatible with the cohomology of coherent sheaves. The latter relies on resolutions by injective $\mathcal{O}_X$-modules which are not quasi-coherent in general.

2. The category of shtukas

Let X be a τ-scheme. In the following we denote $\operatorname{Sht}\mathcal{O}_X$ the category of $\mathcal{O}_X$-module shtukas. Strictly speaking $\operatorname{Sht}\mathcal{O}_X$ depends not only on $\mathcal{O}_X$ but also on the endomorphism τ. We drop τ from the notation since we never work with more than one τ-structure on a given scheme X. In this section we establish basic properties of the category $\operatorname{Sht}\mathcal{O}_X$.

Lemma 2.1. *Let X be a τ-scheme. Let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{O}_X$-module shtukas defined by diagrams*

$$\mathcal{M} = \Big[\mathcal{M}_0 \underset{j_M}{\overset{i_M}{\rightrightarrows}} \mathcal{M}_1\Big], \quad \mathcal{N} = \Big[\mathcal{N}_0 \underset{j_N}{\overset{i_N}{\rightrightarrows}} \mathcal{N}_1\Big].$$

Denote

$$j_M^a \colon \tau^*\mathcal{M}_0 \to \mathcal{M}_1, \quad j_N^a \colon \tau^*\mathcal{N}_0 \to \mathcal{N}_1$$

the adjoints of

$$j_M \colon \mathcal{M}_0 \to \tau_*\mathcal{M}_1, \quad j_N \colon \mathcal{N}_0 \to \tau_*\mathcal{N}_1.$$

respectively.

Let $f_0 \colon \mathcal{M}_0 \to \mathcal{N}_0$ and $f_1 \colon \mathcal{M}_1 \to \mathcal{N}_1$ be morphisms of $\mathcal{O}_X$-modules. The pair (f_0, f_1) is a morphism of shtukas if and only if the squares

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, "f_0"] \arrow[d, "i_M"'] & \mathcal{N}_0 \arrow[d, "i_N"] \\ \mathcal{M}_1 \arrow[r, "f_1"] & \mathcal{N}_1 \end{tikzcd} \qquad \begin{tikzcd} \tau^*\mathcal{M}_0 \arrow[r, "\tau^*(f_0)"] \arrow[d, "j_M^a"'] & \tau^*\mathcal{N}_0 \arrow[d, "j_N^a"] \\ \mathcal{M}_1 \arrow[r, "f_1"] & \mathcal{N}_1 \end{tikzcd}$$

are commutative. □

Definition 2.2. Let X be a τ-scheme. We define functors from $\operatorname{Sht}\mathcal{O}_X$ to $\mathcal{O}_X$-modules:

$$\alpha_*[\mathcal{M}_0 \rightrightarrows \mathcal{M}_1] = \mathcal{M}_0,$$
$$\beta_*[\mathcal{M}_0 \rightrightarrows \mathcal{M}_1] = \mathcal{M}_1.$$

Proposition 2.3. *Let X be a τ-scheme.*

(1) $\operatorname{Sht}\mathcal{O}_X$ *is an abelian category.*
(2) *The functors α_* and β_* are exact.*
(3) *A sequence*

$$\mathcal{M} \to \mathcal{M}' \to \mathcal{M}''$$

of $\mathcal{O}_X$-module shtukas is exact if and only if the induced sequences

$$\alpha_*\mathcal{M} \to \alpha_*\mathcal{M}' \to \alpha_*\mathcal{M}'',$$
$$\beta_*\mathcal{M} \to \beta_*\mathcal{M}' \to \beta_*\mathcal{M}''.$$

are exact. □

Proof. $\operatorname{Sht}\mathcal{O}_X$ is clearly an additive category. As the functor τ_* is left exact it is straightforward to show that kernels in $\operatorname{Sht}\mathcal{O}_X$ exist and commute with α_*, β_*. In a similar way Lemma 2.1 and the fact that τ^* is right exact imply that cokernels exist and commute with α_*, β_*. A morphism of shtukas $f\colon \mathcal{M} \to \mathcal{N}$ is an isomorphism if and only if $\alpha_*(f)$ and $\beta_*(f)$ are isomorphisms. Therefore $\operatorname{Sht}\mathcal{O}_X$ is an abelian category. (2) and (3) are clear. □

Definition 2.4. Let X be a τ-scheme. We define functors α^*, β^* from the category of $\mathcal{O}_X$-modules to $\operatorname{Sht}\mathcal{O}_X$:

$$\alpha^*\mathcal{F} = \Big[\mathcal{F} \overset{(1,0)}{\underset{(0,\eta)}{\rightrightarrows}} \mathcal{F} \oplus \tau^*\mathcal{F}\Big],$$
$$\beta^*\mathcal{F} = \Big[0 \rightrightarrows \mathcal{F}\Big].$$

Here $\eta\colon \mathcal{F} \to \tau_*\tau^*\mathcal{F}$ is the adjunction unit.

Lemma 2.5. *α^* is left adjoint to α_* and β^* is left adjoint to β_*.*

Proof. The first adjunction follows from Lemma 2.1. The second adjunction is clear. □

The following Theorem is of fundamental importance to our treatment of shtuka cohomology. Recall that an object U of an abelian category is called a generator if for every nonzero morphism $f\colon A \to B$ there is a morphism $g\colon U \to A$ such that the composition $f \circ g$ is nonzero.

Theorem 2.6. *Let X be a τ-scheme.*

(1) $\operatorname{Sht}\mathcal{O}_X$ *has all colimits and filtered colimits are exact.*
(2) $\operatorname{Sht}\mathcal{O}_X$ *admits a generator.*

It is a fundamental result of Grothendieck [11] that every abelian category satisfying (1) and (2) has enough injective objects.

Proof of Theorem 2.6. (1) Taking the direct sum of underlying $\mathcal{O}_X$-modules one concludes that $\operatorname{Sht}\mathcal{O}_X$ has arbitrary direct sums. As it is abelian it follows that it has all colimits. By construction the functors α_* and β_* commute with colimits. Applying α_* and β_* to a colimit of $\mathcal{O}_X$-module shtukas we deduce that filtered colimits are exact in $\operatorname{Sht}\mathcal{O}_X$ from the fact that they are exact in the category of $\mathcal{O}_X$-modules.

(2) Consider the $\mathcal{O}_X$-module

$$U = \bigoplus_{V \subset X} (\iota_V)_!\mathcal{O}_V$$

where $V \subset X$ runs over all open subsets and $\iota_V\colon V \hookrightarrow X$ denotes the corresponding open embedding. It is easy to see that U is a generator of the category of $\mathcal{O}_X$-modules.

We claim that $\alpha^*U \oplus \beta^*U$ is a generator of $\operatorname{Sht}\mathcal{O}_X$. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathcal{O}_X$-module shtukas. If $f \neq 0$ then either α_*f or β_*f is nonzero, say the first one. As U is a generator there exists a morphism $g\colon U \to \alpha_*\mathcal{M}$

such that $\alpha_* f \circ g \neq 0$. As a consequence the composition of the adjoint $g^a \colon \alpha^* U \to \mathcal{M}$ and f is nonzero. $\square$

Our treatment of shtuka cohomology relies on the notion of a K-injective complex. Recall that a complex C of objects in an abelian category is called K-injective if every morphism from an acyclic complex to C is zero up to homotopy. A bounded below complex of injective objects is K-injective. In general K-injective objects play the role of injective resolutions for unbounded complexes. The reader who does not want to bother with unbounded complexes can safely replace K-injective complexes with bounded below complexes of injective objects in all the statements of this chapter. However unbounded complexes play an essential role in some proofs.

In [20] Spaltenstein demonstrated that every complex of $\mathcal{O}_X$-modules on a ringed space X has a K-injective resolution. Serpé [19] generalized this result to an arbitrary abelian category which has the properties (1) and (2) of Theorem 2.6.

Corollary 2.7. *Let X be a τ-scheme. The category $\operatorname{Sht} \mathcal{O}_X$ has enough injectives. Every complex of $\mathcal{O}_X$-module shtukas has a K-injective resolution.*

Proof. By [079I] it follows from Theorem 2.6. $\square$

3. Injective shtukas

If $\mathcal{I}$ is an injective shtuka then, as we demonstrate below, $\beta_* \mathcal{I}$ is an injective sheaf of modules. On the contrary $\alpha_* \mathcal{I}$ need not be injective. Nevertheless we will show that it is good enough to compute derived pushforwards.

Lemma 3.1. *If $\mathcal{I}$ is a K-injective complex of $\mathcal{O}_X$-module shtukas over a τ-scheme X then $\beta_* \mathcal{I}$ is a K-injective complex of $\mathcal{O}_X$-modules.*

Proof. Immediate since β_* admits an exact left adjoint β^*. $\square$

Recall that a complex $\mathcal{F}$ of $\mathcal{O}_X$-modules on a ringed space X is called K-flat if the functor $\mathcal{F} \otimes_{\mathcal{O}_X} -$ preserves quasi-isomorphisms. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat. Spaltenstein [20] proved that every complex of $\mathcal{O}_X$-modules has a K-flat resolution.

In the following $K(\mathcal{O}_X)$ stands for the homotopy category of $\mathcal{O}_X$-module complexes and $D(\mathcal{O}_X)$ for the derived category. $K(\operatorname{Sht} \mathcal{O}_X)$ is the homotopy category of $\mathcal{O}_X$-module shtukas.

Lemma 3.2. *Let X be a τ-scheme. If $\mathcal{F}$ is a K-flat complex of $\mathcal{O}_X$-modules and $\mathcal{I}$ a K-injective complex of $\mathcal{O}_X$-module shtukas then $\operatorname{Hom}_{K(\mathcal{O}_X)}(\mathcal{F}, \alpha_* \mathcal{I}) = \operatorname{Hom}_{D(\mathcal{O}_X)}(\mathcal{F}, \alpha_* \mathcal{I})$.*

Proof. Assume that $\mathcal{F}$ is acyclic. By adjunction

$$\operatorname{Hom}_{K(\mathcal{O}_X)}(\mathcal{F}, \alpha_* \mathcal{I}) = \operatorname{Hom}_{K(\operatorname{Sht} \mathcal{O}_X)}(\alpha^* \mathcal{F}, \mathcal{I}).$$

The functor α^* is right exact whence $\alpha^* \mathcal{F}$ is acyclic and the Hom on the right side of the equation is zero. Now let $\mathcal{F}$ be an arbitrary K-flat complex and

$f\colon \mathcal{F}' \to \mathcal{F}$ a quasi-isomorphism of K-flat complexes. The cone of f is K-flat and acyclic. Applying $\mathrm{Hom}_{K(\mathcal{O}_X)}(-, \alpha_*\mathcal{I})$ to a distinguished triangle extending f we deduce that every map $g\colon \mathcal{F}' \to \alpha_*\mathcal{I}$ in $K(\mathcal{O}_X)$ factors through $\mathcal{F}$. As every $\mathcal{O}_X$-module complex admits a K-flat resolution [06YF] we conclude that $\mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{F}, \alpha_*\mathcal{I}) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{F}, \alpha_*\mathcal{I})$. □

Lemma 3.3. *Let X be a τ-scheme and $f\colon X \to Y$ a morphism of schemes. If $\mathcal{I}$ is a K-injective complex of $\mathcal{O}_X$-module shtukas then the natural map $f_*\alpha_*\mathcal{I} \to \mathrm{R}f_*\alpha_*\mathcal{I}$ is a quasi-isomorphism.*

Proof. Pick a K-injective resolution $\iota\colon \alpha_*\mathcal{I} \to \mathcal{J}$ and let C be the cone of ι so that we have a distinguished triangle

$$\alpha_*\mathcal{I} \xrightarrow{\iota} \mathcal{J} \to C \to [1]$$

in $K(\mathcal{O}_X)$. We need to prove that $f_*(\iota)$ is a quasi-isomorphism or equivalently that f_*C is acyclic. Let $\mathcal{F}$ be a K-flat $\mathcal{O}_Y$-module complex. Applying $\mathrm{Hom}_{K(\mathcal{O}_X)}(f^*\mathcal{F}, -)$ and $\mathrm{Hom}_{D(\mathcal{O}_X)}(f^*\mathcal{F}, -)$ to the triangle above we get a morphism of long exact sequences

$$\begin{array}{ccc}
\mathrm{Hom}_{K(\mathcal{O}_X)}(f^*\mathcal{F}, \alpha_*\mathcal{I}) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*\mathcal{F}, \alpha_*\mathcal{I}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{K(\mathcal{O}_X)}(f^*\mathcal{F}, \mathcal{J}) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*\mathcal{F}, \mathcal{J}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{K(\mathcal{O}_X)}(f^*\mathcal{F}, C) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*\mathcal{F}, C) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

The complex $f^*\mathcal{F}$ is K-flat so the top horizontal arrow in this diagram is an isomorphism by Lemma 3.2. The middle horizontal arrow is an isomorphism since $\mathcal{J}$ is K-injective. Thus the five lemma shows that the bottom horizontal arrow is an isomorphism. As C is acyclic we deduce that

$$0 = \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*\mathcal{F}, C) = \mathrm{Hom}_{K(\mathcal{O}_X)}(f^*\mathcal{F}, C) = \mathrm{Hom}_{K(\mathcal{O}_Y)}(\mathcal{F}, f_*C)$$

for an arbitrary K-flat complex $\mathcal{F}$. Since the complex f_*C admits a K-flat resolution $\mathcal{F} \to f_*C$ we conclude that f_*C is acyclic. □

4. Cohomology of shtukas

In this section we work over a fixed τ-scheme X.

Definition 4.1. Observe that $\tau\colon X \to X$ induces a ring endomorphism of $\mathcal{O}_X(X)$. We define the *ring of invariants* $\mathcal{O}_X(X)^{\tau=1}$ to be the subring $\{s \mid \tau(s) = s\} \subset \mathcal{O}_X(X)$.

Consider an $\mathcal{O}_X$-module shtuka

$$\mathcal{M} = \Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big].$$

The arrows of $\mathcal{M}$ determine natural maps

$$i, j \colon \Gamma(X, \mathcal{M}_0) \to \Gamma(X, \mathcal{M}_1)$$

with the same source and target. In the case of j we identify $\Gamma(X, \tau_*\mathcal{M}_1)$ with $\Gamma(X, \mathcal{M}_1)$ using the fact that $\tau^{-1}X = X$. Observe that j is only $\mathcal{O}_X(X)^{\tau=1}$-linear since the natural identification $\Gamma(X, \mathcal{M}_1) = \Gamma(X, \tau_*\mathcal{M}_1)$ is only $\mathcal{O}_X(X)^{\tau=1}$-linear.

Definition 4.2. Let an $\mathcal{O}_X$-module shtuka $\mathcal{M}$ be given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

We define

$$\Gamma(X, \mathcal{M}) = \{s \in \Gamma(X, \mathcal{M}_0) \mid i(s) = j(s)\}$$

and call $\Gamma(X, \mathcal{M})$ the module of *global sections* of $\mathcal{M}$. The construction $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ defines a functor from $\mathcal{O}_X$-module shtukas to $\mathcal{O}_X(X)^{\tau=1}$-modules.

Definition 4.3. The functor $\Gamma(X, -)$ on the category $\operatorname{Sht} \mathcal{O}_X$ is left exact. We define the *derived global sections* functor $\mathrm{R}\Gamma(X, -)$ as its right derived functor.

We call $\mathrm{R}\Gamma(X, \mathcal{M})$ the *cohomology complex* of $\mathcal{M}$ or simply the cohomology of $\mathcal{M}$. The n-th cohomology module of $\mathrm{R}\Gamma(X, \mathcal{M})$ is denoted $\mathrm{H}^n(X, \mathcal{M})$.

Definition 4.4. The *unit shtuka* $\mathbb{1}_X$ is defined by the diagram

$$\mathcal{O}_X \overset{1}{\underset{\tau^\sharp}{\rightrightarrows}} \mathcal{O}_X$$

where $\tau^\sharp \colon \mathcal{O}_X \to \tau_*\mathcal{O}_X$ is the homomorphism of sheaves of rings determined by τ.

Lemma 4.5. *For every $\mathcal{O}_X$-module shtuka $\mathcal{M}$ there is a natural isomorphism*

$$\operatorname{Hom}(\mathbb{1}_X, \mathcal{M}) \cong \Gamma(X, \mathcal{M}).$$

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

A morphism $f \colon \mathbb{1}_X \to \mathcal{M}$ is a pair of maps

$$f_0 \colon \mathcal{O}_X \to \mathcal{M}_0, \quad f_1 \colon \mathcal{O}_X \to \mathcal{M}_1$$

such that

$$i \circ f_0 = f_1, \quad j \circ f_0 = \tau_*(f_1) \circ \tau^\sharp.$$

The pair (f_0, f_1) is determined by the section $s = f_0(1)$ of $\Gamma(X, \mathcal{M}_0)$ which satisfies the equation $i(s) = j(s)$. Such sections are precisely the elements of $\Gamma(X, \mathcal{M})$. $\square$

Theorem 4.6. *For every complex of $\mathcal{O}_X$-module shtukas $\mathcal{M}$ there is a natural quasi-isomorphism*

$$\mathrm{RHom}(\mathbb{1}_X, \mathcal{M}) \cong \mathrm{R}\Gamma(X, \mathcal{M}).$$

Proof. Follows instantly from Lemma 4.5. $\square$

5. Associated complex

It will be convenient for us to view the functor $\mathrm{R}\Gamma$ on shtukas not as the derived functor of the global sections functor Γ but as the derived functor of the so-called *associated complex* functor Γ_{a}. This functor sends an $\mathcal{O}_X$-module shtuka $\mathcal{M}$ to a two-term complex of modules over the invariant ring $\mathcal{O}_X(X)^{\tau=1}$. It has two equally important applications. First, it gives rise to a natural distinguished triangle of the form

$$\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \alpha_* \mathcal{M}) \xrightarrow{i-j} \mathrm{R}\Gamma(X, \beta_* \mathcal{M}) \to [1].$$

Second, it provides a canonical representative for the cohomology complex of a quasi-coherent shtuka on an affine τ-scheme (Theorem 8.1).

Definition 5.1. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. Denote $\mathcal{M}^n$ the shtuka in degree n. The arrows of $\mathcal{M}^n$ determine natural maps

$$i, j \colon \Gamma(X, \alpha_* \mathcal{M}^n) \to \Gamma(X, \beta_* \mathcal{M}^n),$$

We define the *associated complex* $\Gamma_{\mathrm{a}}(X, \mathcal{M})$ as the total complex of the double complex

$$\begin{tikzcd}
 & \\
\Gamma(X, \alpha_* \mathcal{M}^{n+1}) \arrow[u] \arrow[r, "(-1)^{n+1}(i-j)"] & \Gamma(X, \beta_* \mathcal{M}^{n+1}) \arrow[u] \\
\Gamma(X, \alpha_* \mathcal{M}^{n}) \arrow[u] \arrow[r, "(-1)^{n}(i-j)"] & \Gamma(X, \beta_* \mathcal{M}^{n}) \arrow[u] \\
 \arrow[u] & \arrow[u]
\end{tikzcd}$$

The vertical maps are the differentials of $\Gamma(X, \alpha_* \mathcal{M})$ respectively $\Gamma(X, \beta_* \mathcal{M})$. The object $\Gamma(X, \alpha_* \mathcal{M}^n)$ is placed in the bidegree $(n, 0)$ while $\Gamma(X, \beta_* \mathcal{M}^n)$ is in the bidegree $(n, 1)$. By construction we have a natural inclusion of complexes $\Gamma(X, \mathcal{M}) \hookrightarrow \Gamma_{\mathrm{a}}(X, \mathcal{M})$.

Example. If an $\mathcal{O}_X$-module shtuka $\mathcal{M}$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

then regarding $\mathcal{M}$ as a complex of shtukas concentrated in degree 0 we have

$$\Gamma_{\mathrm{a}}(X, \mathcal{M}) = \Big[\Gamma(X, \mathcal{M}_0) \xrightarrow{i-j} \Gamma(X, \mathcal{M}_1)\Big].$$

The square brackets denote the mapping fiber complex of Chapter "Notation and conventions".

The following proposition is very important for our theory. It identifies $\mathrm{R}\Gamma$ as the derived functor of Γ_{a}. In essence it is due to V. Lafforgue [17, Section 4].

Proposition 5.2. *If $\mathcal{I}$ is a K-injective complex of $\mathcal{O}_X$-module shtukas then the natural inclusion $\Gamma(X, \mathcal{I}) \hookrightarrow \Gamma_{\mathrm{a}}(X, \mathcal{I})$ is a quasi-isomorphism.*

Proof. Consider the shtukas

$$\mathbb{1}_X = \Big[\mathcal{O}_X \underset{\tau^\sharp}{\overset{1}{\rightrightarrows}} \mathcal{O}_X\Big],$$

$$\alpha^* \mathcal{O}_X = \Big[\mathcal{O}_X \underset{(0,\tau^\sharp)}{\overset{(1,0)}{\rightrightarrows}} \mathcal{O}_X \oplus \mathcal{O}_X\Big],$$

$$\beta^* \mathcal{O}_X = \Big[0 \rightrightarrows \mathcal{O}_X\Big].$$

They form a short exact sequence

$$0 \to \beta^* \mathcal{O}_X \xrightarrow{d} \alpha^* \mathcal{O}_X \xrightarrow{q} \mathbb{1}_X \to 0 \tag{5.1}$$

where d is given by the map $(-1, 1)\colon \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{O}_X$ and q is given the summation map $\mathcal{O}_X \oplus \mathcal{O}_X \to \mathcal{O}_X$ and the identity map $\mathcal{O}_X \to \mathcal{O}_X$.

We denote C the cone of the morphism $d\colon \beta^* \mathcal{O}_X \to \alpha^* \mathcal{O}_X$. Let $\delta\colon C \to \mathbb{1}_X[0]$ be the morphism given by the map $q\colon \alpha^* \mathcal{O}_X \to \mathbb{1}_X$ in degree 0. It is a quasi-isomorphism since the sequence (5.1) is exact.

Let an $\mathcal{O}_X$-module shtuka $\mathcal{M}$ be given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

Lemma 4.5 shows that

$$\mathrm{Hom}(\mathbb{1}_X, \mathcal{M}) = \Gamma(X, \mathcal{M}).$$

In a similar way one easily proves that

$$\mathrm{Hom}(\alpha^* \mathcal{O}_X, \mathcal{M}) = \Gamma(X, \mathcal{M}_0),$$
$$\mathrm{Hom}(\beta^* \mathcal{O}_X, \mathcal{M}) = \Gamma(X, \mathcal{M}_1).$$

Under these identifications the morphism $d\colon \beta^* \mathcal{O}_X \to \alpha^* \mathcal{O}_X$ induces the map

$$\Gamma(X, \mathcal{M}_0) \xrightarrow{j-i} \Gamma(X, \mathcal{M}_1)$$

while $q\colon \alpha^*\mathcal{M} \to \mathbb{1}_X$ induces the natural inclusion $\Gamma(X,\mathcal{M}) \hookrightarrow \Gamma(X,\mathcal{M}_0)$.

Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. The observations above imply that

$$\operatorname{Hom}(\mathbb{1}_X,\mathcal{M}) = \Gamma(X,\mathcal{M}),$$
$$\operatorname{Hom}(C,\mathcal{M}) = \Gamma_{\mathrm{a}}(X,\mathcal{M}).$$

Here we use the definition of the Hom complex as in [0A8H]. Under the identifications above the morphism $\delta\colon C \to \mathbb{1}_X[0]$ induces the natural inclusion $\Gamma(X,\mathcal{M}) \hookrightarrow \Gamma_{\mathrm{a}}(X,\mathcal{M})$. Now if $\mathcal{I}$ is a K-injective complex of shtukas then $\operatorname{Hom}(-,\mathcal{I})$ preserves quasi-isomorphisms. Since δ is a quasi-isomorphism we conclude that the natural inclusion $\Gamma(X,\mathcal{I}) \hookrightarrow \Gamma_{\mathrm{a}}(X,\mathcal{I})$ is a quasi-isomorphism. □

Definition 5.3. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. Pick a K-injective resolution $\mathcal{I}$ of $\mathcal{M}$. We define the natural morphism

$$\Gamma_{\mathrm{a}}(X,\mathcal{M}) \to \mathrm{R}\Gamma(X,\mathcal{M})$$

as the composition

$$\Gamma_{\mathrm{a}}(X,\mathcal{M}) \to \Gamma_{\mathrm{a}}(X,\mathcal{I}) \xleftarrow{\sim} \Gamma(X,\mathcal{I}) \xrightarrow{\sim} \mathrm{R}\Gamma(X,\mathcal{M})$$

where the second map is the quasi-isomorphism of Proposition 5.2.

Our next goal is to construct a natural distinguished triangle of the form

$$\mathrm{R}\Gamma(X,\mathcal{M}) \to \mathrm{R}\Gamma(X,\alpha_*\mathcal{M}) \xrightarrow{i-j} \mathrm{R}\Gamma(X,\beta_*\mathcal{M}) \to [1].$$

To do it we first construct a similar triangle for $\Gamma_{\mathrm{a}}(X,\mathcal{M})$.

Definition 5.4. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. We define a natural triangle

$$\Gamma_{\mathrm{a}}(X,\mathcal{M}) \xrightarrow{p} \Gamma(X,\alpha_*\mathcal{M}) \xrightarrow{i-j} \Gamma(X,\beta_*\mathcal{M}) \xrightarrow{-\iota} \Gamma_{\mathrm{a}}(X,\mathcal{M})[1] \tag{5.2}$$

as follows. Denote $\mathcal{M}^n$ the shtuka in degree n. According to Definition 5.1 the object of $\Gamma_{\mathrm{a}}(X,\mathcal{M})$ in degree n is

$$\Gamma(X,\alpha_*\mathcal{M}^n) \oplus \Gamma(X,\beta_*\mathcal{M}^{n-1}).$$

The morphism p is the projection $(a,b) \mapsto a$. The morphism ι is defined by the formula $b \mapsto (0,(-1)^n b)$ in degree n. The morphism $i-j$ is the difference of the natural maps induced by the arrows of the shtukas $\mathcal{M}^n$.

Lemma 5.5. *The triangle* (5.2) *is distinguished.*

Proof. The sequence

$$0 \to \Gamma(X,\beta_*\mathcal{M})[-1] \xrightarrow{\iota[-1]} \Gamma_{\mathrm{a}}(X,\mathcal{M}) \xrightarrow{p} \Gamma(X,\alpha_*\mathcal{M}) \to 0$$

is exact and is termwise split. Such a sequence determines a distinguished triangle in the following way. Let π be the splitting of $\iota[-1]$ given by the formula $(a,b) \mapsto (-1)^n b$ in degree $n+1$ and let s be the splitting of p given by the formula $a \mapsto (a, 0)$. Let $\delta = \pi \circ d \circ s$ where d is the differential of $\Gamma_{\mathrm{a}}(X, \mathcal{M})$. The triangle

$$\Gamma(X, \beta_* \mathcal{M})[-1] \xrightarrow{\iota[-1]} \Gamma_{\mathrm{a}}(X, \mathcal{M}) \xrightarrow{p} \Gamma(X, \alpha_* \mathcal{M}) \xrightarrow{\delta} \Gamma(X, \beta_* \mathcal{M})$$

is distinguished [014Q]. An easy computation reveals that $\delta = i - j$. Rotating this triangle we conclude that (5.2) is distinguished. $\square$

Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas. The arrows of the shtukas of $\mathcal{M}$ determine natural maps

$$i, j \colon \mathrm{R}\Gamma(X, \alpha_* \mathcal{M}) \to \mathrm{R}\Gamma(X, \beta_* \mathcal{M}),$$

with the same source and target. The first map is induced by the i-arrows. The j-arrows induce a map $\mathrm{R}\Gamma(X, \alpha_* \mathcal{M}) \to \mathrm{R}\Gamma(X, \tau_* \beta_* \mathcal{M})$. Taking its composition with the natural map $\mathrm{R}\Gamma(X, \tau_* \beta_* \mathcal{M}) \to \mathrm{R}\Gamma(X, \mathrm{R}\tau_* \beta_* \mathcal{M})$ and using the identity $\mathrm{R}\Gamma(X, \beta_* \mathcal{M}) = \mathrm{R}\Gamma(X, \mathrm{R}\tau_* \beta_* \mathcal{M})$ we get a map of the desired form.

Theorem 5.6. *For every complex $\mathcal{M}$ of $\mathcal{O}_X$-module shtukas there exists a natural distinguished triangle*

$$\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \alpha_* \mathcal{M}) \xrightarrow{i-j} \mathrm{R}\Gamma(X, \beta_* \mathcal{M}) \to [1]$$

with the following properties:

(1) *The natural diagram*

$$\begin{tikzcd}
\Gamma_{\mathrm{a}}(X, \mathcal{M}) \arrow[r, "p"] \arrow[d] & \Gamma(X, \alpha_* \mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \Gamma(X, \beta_* \mathcal{M}) \arrow[r, "-\iota"] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(X, \mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(X, \alpha_* \mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X, \beta_* \mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd}$$

is a morphism of distinguished triangles.

(2) *If $\mathcal{M}$ is K-injective then the morphism of the distinguished triangles above is an isomorphism.*

Moreover the property (1) *uniquely characterizes the collection of all such triangles.*

Proof. Let $\mathcal{I}$ be a K-injective resolution of $\mathcal{M}$. We have a natural morphism of distinguished triangles

$$\begin{tikzcd}
\Gamma_{\mathrm{a}}(X, \mathcal{M}) \arrow[r, "p"] \arrow[d] & \Gamma(X, \alpha_* \mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \Gamma(X, \beta_* \mathcal{M}) \arrow[r, "-\iota"] \arrow[d] & {[1]} \\
\Gamma_{\mathrm{a}}(X, \mathcal{I}) \arrow[r, "p"] & \Gamma(X, \alpha_* \mathcal{I}) \arrow[r, "i-j"] & \Gamma(X, \beta_* \mathcal{I}) \arrow[r, "-\iota"] & {[1].}
\end{tikzcd}$$

Proposition 5.2 identifies $\Gamma_{\mathrm{a}}(X, \mathcal{I})$ with $\mathrm{R}\Gamma(X, \mathcal{M})$. Applying Lemma 3.3 to the structure map $f \colon X \to \operatorname{Spec} \mathbb{Z}$ we deduce that $\Gamma(X, \alpha_* \mathcal{I}) = \mathrm{R}\Gamma(X, \alpha_* \mathcal{M})$. According to Lemma 3.1 the complex $\beta_* \mathcal{I}$ is K-injective so that $\Gamma(X, \beta_* \mathcal{I}) =$

$\mathrm{R}\Gamma(X, \beta_*\mathcal{M})$. Therefore the second row of the diagram above forms a distinguished triangle

$$\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \to \mathrm{R}\Gamma(X, \beta_*\mathcal{M}) \to [1].$$

We leave it to the reader to check that the second map in this triangle is the difference of maps i and j as described above. $\square$

6. Pushforward

Definition 6.1. Let $f\colon X \to Y$ be a morphism of τ-schemes and let

$$\mathcal{M} = \Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big]$$

be an $\mathcal{O}_X$-module shtuka. Define

$$f_*\mathcal{M} = \Big[f_*\mathcal{M}_0 \overset{f_* i}{\underset{f_* j}{\rightrightarrows}} f_*\mathcal{M}_1\Big].$$

Here we use the natural isomorphism $f_*\tau_*\mathcal{M}_1 = \tau_* f_*\mathcal{M}_1$ to interpret $f_* j$ as the map to $\tau_* f_*\mathcal{M}_1$.

Definition 6.2. Let $f\colon X \to Y$ be a morphism of τ-schemes. The functor f_* on the category of $\mathcal{O}_X$-module shtukas is left exact. We define $\mathrm{R}f_*$ as its right derived functor.

Lemma 6.3. *If $f\colon X \to Y$ is a morphism of τ-schemes then the following holds:*

(1) *The natural map $\alpha_*\mathrm{R}f_* \to \mathrm{R}f_*\alpha_*$ is a quasi-isomorphism.*
(2) *The natural map $\beta_*\mathrm{R}f_* \to \mathrm{R}f_*\beta_*$ is a quasi-isomorphism.*

Proof. Let $\mathcal{M}$ be a complex of $\mathcal{O}_X$-module shtukas and let $\mathcal{I}$ be its K-injective resolution. Observe that $f_*\alpha_*\mathcal{I} = \alpha_* f_*\mathcal{I}$ and $f_*\beta_*\mathcal{I} = \beta_* f_*\mathcal{I}$ by construction of f_*. Lemma 3.3 shows that the natural map $f_*\alpha_*\mathcal{I} \to \mathrm{R}f_*\alpha_*\mathcal{I}$ is a quasi-isomorphism so that we get (1). According to Lemma 3.1 the complex $\beta_*\mathcal{I}$ is K-injective whence (2) follows. $\square$

Let $f\colon X \to Y$ be a morphism of τ-schemes. For every complex $\mathcal{M}$ of $\mathcal{O}_X$-module shtukas we have a natural isomorphism $\Gamma(X, \mathcal{M}) = \Gamma(Y, f_*\mathcal{M})$. It induces a natural map $\mathrm{R}\Gamma(X, \mathcal{M}) \to \mathrm{R}\Gamma(Y, \mathrm{R}f_*\mathcal{M})$. Furthermore we have a natural quasi-isomorphism

$$\mathrm{R}\Gamma(X, \alpha_*\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(Y, \mathrm{R}f_*\alpha_*\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(Y, \alpha_*\mathrm{R}f_*\mathcal{M})$$

where the second arrow is the quasi-isomorphism of Lemma 6.3. In a similar way we have a natural quasi-isomorphism $\mathrm{R}\Gamma(X, \beta_*\mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(Y, \beta_*\mathrm{R}f_*\mathcal{M})$.

Proposition 6.4. *Let $f\colon X \to Y$ be a morphism of τ-schemes. For every complex $\mathcal{M}$ of $\mathcal{O}_X$-module shtukas the natural diagram*

(6.1)
$$\begin{tikzcd}
\mathrm{R}\Gamma(X,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(X,\alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\Gamma(X,\beta_*\mathcal{M}) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(Y,\mathrm{R}f_*\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(Y,\alpha_*\mathrm{R}f_*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(Y,\beta_*\mathrm{R}f_*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd}$$

is an isomorphism of distinguished triangles.

Proof. Let $\mathcal{I}$ be a K-injective resolution of $\mathcal{M}$ and let $\mathcal{J}$ be a K-injective resolution of $f_*\mathcal{I}$. We have a natural morphism of distinguished triangles

$$\begin{tikzcd}
\Gamma_{\mathrm{a}}(Y,f_*\mathcal{I}) \arrow[r] \arrow[d] & \Gamma(Y,\alpha_*f_*\mathcal{I}) \arrow[r, "i-j"] \arrow[d] & \Gamma(Y,\beta_*f_*\mathcal{I}) \arrow[r] \arrow[d] & {[1]} \\
\Gamma_{\mathrm{a}}(Y,\mathcal{J}) \arrow[r] & \Gamma(Y,\alpha_*\mathcal{J}) \arrow[r, "i-j"] & \Gamma(Y,\beta_*\mathcal{J}) \arrow[r] & {[1]}
\end{tikzcd} \tag{6.2}$$

The top distinguished triangle coincides with the distinguished triangle

$$\Gamma_{\mathrm{a}}(X,\mathcal{I}) \to \Gamma(X,\alpha_*\mathcal{I}) \xrightarrow{i-j} \Gamma(X,\beta_*\mathcal{I}) \to [1].$$

So Theorem 5.6 identifies the diagram (6.2) with the diagram (6.1). Whence the result. □

7. Pullback

Definition 7.1. Let $f\colon X \to Y$ be a morphism of τ-schemes and let

$$\mathcal{M} = \Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big]$$

be an $\mathcal{O}_Y$-module shtuka. Define

$$f^*\mathcal{M} = \Big[f^*\mathcal{M}_0 \overset{f^*i}{\underset{\mu\circ f^*j}{\rightrightarrows}} f^*\mathcal{M}_1\Big]$$

where μ is the base change map $f^*\tau_*\mathcal{M}_1 \to \tau_* f^*\mathcal{M}_1$ arising from the commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\tau"] & Y \arrow[d, "\tau"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Given morphism of τ-schemes $f\colon X \to Y$ and a shtuka $\mathcal{M}$ on Y we will often denote $\mathcal{M}(X)$ the pullback of $\mathcal{M}$ to X. Similarly if $f\colon R \to S$ is a morphism of τ-rings and $\mathcal{M}$ is an R-module shtuka then we will denote $\mathcal{M}(S)$ the pullback of $\mathcal{M}$ to S.

Lemma 7.2. *Let $f\colon X \to Y$ be a morphism of τ-schemes. There exists a unique adjunction*

$$\mathrm{Hom}_{\mathrm{Sht}\,\mathcal{O}_X}(f^*-,-) \cong \mathrm{Hom}_{\mathrm{Sht}\,\mathcal{O}_Y}(-,f_*-)$$

which is compatible with the adjunction

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*-,-) \cong \mathrm{Hom}_{\mathcal{O}_Y}(-,f_*-)$$

through the natural maps given by functors α_* and β_*. □

Definition 7.3. Let $f\colon X \to Y$ be a morphism of τ-schemes and $\mathcal{M}$ an $\mathcal{O}_Y$-module shtuka. Set

$$\mathrm{R}\Gamma(X,\mathcal{M}) = \mathrm{R}\Gamma(X, f^*\mathcal{M}).$$

We define the *pullback map*

$$\mathrm{R}\Gamma(Y,\mathcal{M}) \xrightarrow{f^*} \mathrm{R}\Gamma(X,\mathcal{M})$$

in the following way. Let $\eta\colon \mathcal{M} \to f_*f^*\mathcal{M}$ be the adjunction unit. Taking its composition with the natural map $f_*f^*\mathcal{M} \to \mathrm{R}f_*f^*\mathcal{M}$ and applying $\mathrm{R}\Gamma(Y,-)$ we obtain a map from $\mathrm{R}\Gamma(Y,\mathcal{M})$ to $\mathrm{R}\Gamma(Y,\mathrm{R}f_*f^*\mathcal{M})$. Proposition 6.4 identifies $\mathrm{R}\Gamma(Y,\mathrm{R}f_*f^*\mathcal{M})$ with $\mathrm{R}\Gamma(X,f^*\mathcal{M}) = \mathrm{R}\Gamma(X,\mathcal{M})$. The resulting map from $\mathrm{R}\Gamma(Y,\mathcal{M})$ to $\mathrm{R}\Gamma(X,\mathcal{M})$ is the pullback map.

Observe that $\alpha_*f^*\mathcal{M} = f^*\alpha_*\mathcal{M}$ and $\beta_*f^*\mathcal{M} = f^*\beta_*\mathcal{M}$ by construction. We thus have natural pullback maps $\mathrm{R}\Gamma(Y,\alpha_*\mathcal{M}) \to \mathrm{R}\Gamma(X,\alpha_*f^*\mathcal{M})$ and $\mathrm{R}\Gamma(Y,\beta_*\mathcal{M}) \to \mathrm{R}\Gamma(X,\beta_*f^*\mathcal{M})$.

Proposition 7.4. *If $f\colon X \to Y$ is a morphism of τ-schemes then for every complex $\mathcal{M}$ of $\mathcal{O}_Y$-module shtukas the natural diagram*

$$\begin{tikzcd}
\mathrm{R}\Gamma(Y,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(Y,\alpha_*\mathcal{M}) \arrow[r, "i-j"] \arrow[d] & \mathrm{R}\Gamma(Y,\beta_*\mathcal{M}) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(X,f^*\mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(X,\alpha_*f^*\mathcal{M}) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X,\beta_*f^*\mathcal{M}) \arrow[r] & {[1]}
\end{tikzcd} \tag{7.1}$$

is a morphism of distinguished triangles.

Proof. The natural map $\mathcal{M} \to \mathrm{R}f_* f^*\mathcal{M}$ induces a morphism of distinguished triangles

$$\begin{tikzcd} \mathrm{R}\Gamma(Y,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(Y,\mathrm{R}f_* f^*\mathcal{M}) \arrow[d] \\ \mathrm{R}\Gamma(Y,\alpha_*\mathcal{M}) \arrow[r] \arrow[d, "i-j"'] & \mathrm{R}\Gamma(Y,\alpha_*\mathrm{R}f_* f^*\mathcal{M}) \arrow[d, "i-j"] \\ \mathrm{R}\Gamma(Y,\beta_*\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(Y,\beta_*\mathrm{R}f_* f^*\mathcal{M}) \arrow[d] \\ {[1]} & {[1]} \end{tikzcd} \tag{7.2}$$

At the same time Proposition 6.4 states that the natural diagram

$$\begin{tikzcd} \mathrm{R}\Gamma(X,f^*\mathcal{M}) \arrow[r, "\sim"] \arrow[d] & \mathrm{R}\Gamma(X,\mathrm{R}f_* f^*\mathcal{M}) \arrow[d] \\ \mathrm{R}\Gamma(X,\alpha_* f^*\mathcal{M}) \arrow[r, "\sim"] \arrow[d, "i-j"'] & \mathrm{R}\Gamma(X,\alpha_*\mathrm{R}f_* f^*\mathcal{M}) \arrow[d, "i-j"] \\ \mathrm{R}\Gamma(X,\beta_* f^*\mathcal{M}) \arrow[r, "\sim"] \arrow[d] & \mathrm{R}\Gamma(X,\beta_*\mathrm{R}f_* f^*\mathcal{M}) \arrow[d] \\ {[1]} & {[1]} \end{tikzcd} \tag{7.3}$$

is an isomorphism of distinguished triangles. A quick inspection shows that the composition of (7.2) and the inverse of (7.3) gives the diagram (7.1). ☐

8. Shtukas over affine schemes

It follows from Definition 1.2 that a quasi-coherent shtuka $\mathcal{M}$ on an affine τ-scheme $X = \operatorname{Spec} R$ is given by a diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1$$

where M_0, M_1 are R-modules, $i\colon M_0 \to M_1$ an R-module homomorphism and $j\colon M_0 \to M_1$ a τ-linear R-module homomorphism: for all $r \in R$ and $m \in M_0$ one has $j(rm) = \tau(r)j(m)$. The associated complex of $\mathcal{M}$ is

$$\Gamma_{\mathrm{a}}(X,\mathcal{M}) = \Big[M_0 \xrightarrow{i-j} M_1\Big].$$

We will show that this complex computes the cohomology of $\mathcal{M}$.

Theorem 8.1. *If $\mathcal{M}$ is a quasi-coherent shtuka over an affine τ-scheme X then the natural map $\Gamma_{\mathrm{a}}(X, \mathcal{M}) \to \mathrm{R}\Gamma(X, \mathcal{M})$ is a quasi-isomorphism.*

Proof. The natural map in question extends to a morphism of distinguished triangles

$$\begin{tikzcd}
\Gamma_{\mathrm{a}}(X, \mathcal{M}) \arrow[r] \arrow[d] & \Gamma(X, \mathcal{M}_0) \arrow[r, "i-j"] \arrow[d] & \Gamma(X, \mathcal{M}_1) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{R}\Gamma(X, \mathcal{M}) \arrow[r] & \mathrm{R}\Gamma(X, \mathcal{M}_0) \arrow[r, "i-j"] & \mathrm{R}\Gamma(X, \mathcal{M}_1) \arrow[r] & {[1]}
\end{tikzcd}$$

As $\mathcal{M}_0$ and $\mathcal{M}_1$ are quasi-coherent $\mathcal{O}_X$-modules over an affine scheme X the complexes $\mathrm{R}\Gamma(X, \mathcal{M}_0)$ and $\mathrm{R}\Gamma(X, \mathcal{M}_1)$ are concentrated in degree 0 [01XB]. Hence the second and third vertical maps in the diagram above are quasi-isomorphism. It follows that so is the first map. $\square$

To make the expressions more legible we will often write $\mathrm{R}\Gamma(R, \mathcal{M})$ instead of $\mathrm{R}\Gamma(\operatorname{Spec} R, \mathcal{M})$. If there is no ambiguity in the choice of R then we further shorten it to $\mathrm{R}\Gamma(\mathcal{M})$. For a quasi-coherent shtuka $\mathcal{M}$ we identify $\mathrm{R}\Gamma(\mathcal{M})$ with $\Gamma_{\mathrm{a}}(X, \mathcal{M})$ using the Theorem above.

9. Nilpotence

The notion of nilpotence for shtukas is crucial to this work. It first appeared in the book of Böckle-Pink [3] in the context of τ-sheaves.

Definition 9.1. Let X be a τ-scheme. An $\mathcal{O}_X$-module shtuka

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

is called *nilpotent* if i is an isomorphism and the composition

$$\mathcal{M}_0 \xrightarrow{\tau_*(i^{-1})\circ j} \tau_*\mathcal{M}_0 \xrightarrow{\tau_*^2(i^{-1})\circ\tau_*(j)} \ldots \to \tau_*^n\mathcal{M}_0$$

is zero for some $n \geqslant 1$.

Proposition 9.2. *Let $f\colon X \to Y$ be a morphism of τ-schemes and let $\mathcal{M}$ be an $\mathcal{O}_Y$-module shtuka. If $\mathcal{M}$ is nilpotent then $f^*\mathcal{M}$ is nilpotent.*

Proof. Without loss of generality we assume that $\mathcal{M}$ is given by a diagram of the form

$$\mathcal{M}_0 \underset{j}{\overset{1}{\rightrightarrows}} \mathcal{M}_0.$$

Let $j^a\colon \tau^*\mathcal{M}_0 \to \mathcal{M}_0$ be the adjoint of j. It is easy to show that $\mathcal{M}$ is nilpotent if and only if the composition

$$\tau^{*n}(\mathcal{M}_0) \xrightarrow{\tau^{*(n-1)}(j^a)} \tau^{*(n-1)}(\mathcal{M}_0) \to \ldots \to \tau^*(\mathcal{M}_0) \xrightarrow{j^a} \mathcal{M}_0$$

is zero for some $n \geqslant 1$. Taking the pullback along f and using the fact that $\tau \circ f = f \circ \tau$ we get the result. $\square$

Let R be a τ-ring. Observe that an R-module shtuka

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1$$

is nilpotent if and only if i is an isomorphism and the endomorphism $i^{-1}j$ of M_0 is nilpotent.

Proposition 9.3. *Let R be a τ-ring and $\mathcal{M}$ an R-module shtuka. If $\mathcal{M}$ is nilpotent then* $\mathrm{R}\Gamma(\mathcal{M}) = 0$.

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

The endomorphism $1 - i^{-1}j$ of M_0 is an isomorphism since $i^{-1}j$ is nilpotent. Thus $i - j = i(1 - i^{-1}j)$ is an isomorphism and the result follows from Theorem 8.1. □

The following proposition is our main tool to deduce vanishing of cohomology.

Proposition 9.4. *Let R be a Noetherian τ-ring complete with respect to an ideal $I \subset R$. Assume that $\tau(I) \subset I$ so that τ descends to the quotient R/I. Let $\mathcal{M}$ be a locally free R-module shtuka. If $\mathcal{M}(R/I)$ is nilpotent then the following holds:*

(1) $\mathrm{R}\Gamma(\mathcal{M}) = 0$.
(2) *For every $n > 0$ the shtuka $\mathcal{M}(R/I^n)$ is nilpotent.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1.$$

Observe that I is in the Jacobson radical of R. Hence Nakayama's lemma implies that i is surjective. The kernel of i is automatically flat. It is also of finite type as R is Noetherian. Applying Nakayama's lemma again we deduce that the kernel is zero. Whence i is an isomorphism.

The endomorphism $i^{-1}j$ of M_0 preserves the filtration by powers of I. Furthermore $(i^{-1}j)^m M_0 \subset IM_0$ for some $m \geqslant 0$ since $\mathcal{M}(R/I)$ is nilpotent. As a consequence $(i^{-1}j)^{mn} M_0 \subset I^n M_0$ and we get (2). Moreover (2) implies that $1 - i^{-1}j$ is an isomorphism modulo every power of I. Since M_0 is I-adically complete we deduce that $1 - i^{-1}j$ is an isomorphism. As i is an isomorphism the claim (1) now follows from Theorem 8.1. □

10. The linearization functor and ζ-isomorphisms

The constructions of this section are due to V. Lafforgue [17] but the terminology is our own. The notion of a ζ-isomorphism is at the heart of our approach to the class number formula.

Definition 10.1. Let X be a τ-scheme. We define the *linearization functor* ∇ from $\operatorname{Sht}\mathcal{O}_X$ to $\operatorname{Sht}\mathcal{O}_X$ as follows:

$$\nabla\Big[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big] = \Big[\mathcal{M}_0 \underset{0}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Big].$$

We say that an $\mathcal{O}_X$-module shtuka $[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1]$ is *linear* if $j = 0$.

The complex $\mathrm{R\Gamma}(X, \nabla\mathcal{M})$ is often easier to compute than $\mathrm{R\Gamma}(X, \mathcal{M})$. Even though the complexes $\mathrm{R\Gamma}(X, \mathcal{M})$ and $\mathrm{R\Gamma}(X, \nabla\mathcal{M})$ are very different in general, a link beween them exists under some natural assumptions on $\mathcal{M}$.

Fix a subring $A \subset \mathcal{O}_X(X)^{\tau=1}$. If $\mathrm{R\Gamma}(X, \mathcal{M})$ is a perfect complex of A-modules then the theory of Knudsen-Mumford [16] associates to it an invertible A-module[1] $\det_A \mathrm{R\Gamma}(X, \mathcal{M})$. This determinant is functorial in quasi-isomorphisms. Theorem 5.6 provides us with a natural distinguished triangle

$$\mathrm{R\Gamma}(X, \mathcal{M}) \to \mathrm{R\Gamma}(X, \mathcal{M}_0) \xrightarrow{i-j} \mathrm{R\Gamma}(X, \mathcal{M}_1) \to [1].$$

If the cohomology modules $\mathrm{H}^n(X, \mathcal{M})$, $\mathrm{H}^n(X, \mathcal{M}_0)$ and $\mathrm{H}^n(X, \mathcal{M}_1)$ are perfect for all $n \geqslant 0$ and zero for $n \gg 0$ then this distinguished triangle determines a natural A-module isomorphism

$$\det_A \mathrm{R\Gamma}(X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R\Gamma}(X, \mathcal{M}_0) \otimes_A \det_A^{-1} \mathrm{R\Gamma}(X, \mathcal{M}_1)$$

[16, Corollary 2 after Theorem 2].

Definition 10.2. Let X be a τ-scheme and let $A \subset \mathcal{O}_X(X)^{\tau=1}$ be a subring. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module shtuka given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

We say that the *ζ-isomorphism is defined for* $\mathcal{M}$ if $\mathrm{H}^n(X, \mathcal{M})$, $\mathrm{H}^n(X, \mathcal{M}_0)$ and $\mathrm{H}^n(X, \mathcal{M}_1)$ are perfect A-modules for all $n \geqslant 0$ and zero for $n \gg 0$. Under this assumption we define the *ζ-isomorphism*

$$\zeta_{\mathcal{M}} \colon \det_A \mathrm{R\Gamma}(X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R\Gamma}(X, \nabla\mathcal{M})$$

as the composition

$$\det_A \mathrm{R\Gamma}(X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R\Gamma}(X, \mathcal{M}_0) \otimes_A \det_A^{-1} \mathrm{R\Gamma}(X, \mathcal{M}_1) \xrightarrow{\sim} \det_A \mathrm{R\Gamma}(X, \nabla\mathcal{M})$$

of isomorphisms determined by the distinguished triangles

$$\mathrm{R\Gamma}(X, \mathcal{M}) \to \mathrm{R\Gamma}(X, \mathcal{M}_0) \xrightarrow{i-j} \mathrm{R\Gamma}(X, \mathcal{M}_1) \to [1],$$

$$\mathrm{R\Gamma}(X, \nabla\mathcal{M}) \to \mathrm{R\Gamma}(X, \mathcal{M}_0) \xrightarrow{i} \mathrm{R\Gamma}(X, \mathcal{M}_1) \to [1]$$

of Theorem 5.6.

[1] Strictly speaking the determinant is a pair (L, α) consisting of an invertible A-module L and a continuous function $\alpha\colon \operatorname{Spec} A \to \mathbb{Z}$. This function is not important for the following discussion so we ignore it.

11. τ-polynomials

In this section we work with a fixed τ-ring R.

Definition 11.1. We define the ring $R\{\tau\}$ as follows. Its elements are formal polynomials

$$r_0 + r_1\tau + r_2\tau^2 + \dots r_n\tau^n,$$

$r_0, \dots r_n \in R$, $n \geqslant 0$. The multiplication in $R\{\tau\}$ is subject to the following identity: for every $r \in R$

$$\tau \cdot r = r^\tau \cdot \tau$$

where $r^\tau = \tau(r)$ is the image of r under $\tau \colon R \to R$.

Unlike all the other rings in this text the ring $R\{\tau\}$ is not commutative in general. Still it is associative and has the multiplicative unit 1. Left $R\{\tau\}$-modules are directly related to R-module shtukas.

Definition 11.2. Let M be a left $R\{\tau\}$-module. The *R-module shtuka associated to M* is

$$M \underset{\tau}{\overset{1}{\rightrightarrows}} M.$$

Here $\tau \colon M \to M$ is the τ-multiplication map. It is tautologically τ-linear so that the diagram above indeed defines a shtuka.

The next lemma is for the reader's convenience. Its easy proof is omitted since we do not use it.

Lemma 11.3. *The functor which sends a left $R\{\tau\}$-module M to the R-module shtuka*

$$M \underset{\tau}{\overset{1}{\rightrightarrows}} M$$

is exact and fully faithful. Its essential image consists of shtukas

$$M_0 \underset{j}{\overset{i}{\rightrightarrows}} M_1$$

such that i is an isomorphism. □

Our next goal is to construct and describe a natural resolution for left $R\{\tau\}$-modules.

Definition 11.4. Let M be a left $R\{\tau\}$-module. In this section we denote $a \colon R\{\tau\} \otimes_R M \to M$ the map which sends a tensor $\varphi \otimes m$ to $\varphi \cdot m$. The letter a stands for "action".

Lemma 11.5. *If M is a left $R\{\tau\}$-module then the sequence of left $R\{\tau\}$-modules*

$$0 \to R\{\tau\}\tau \otimes_R M \xrightarrow{d} R\{\tau\} \otimes_R M \xrightarrow{a} M \to 0$$

is exact. Here $d(\varphi\tau \otimes m) = \varphi \otimes \tau \cdot m - \varphi\tau \otimes m$.

Proof. It is clear that $a \circ d = 0$ and a is surjective. Let us verify the injectivity of d. The modules $R\{\tau\}$, $R\{\tau\}\tau$ carry filtrations by degree of τ-polynomials. The map d is compatible with the induced filtrations on $R\{\tau\}\tau \otimes_R M$ and $R\{\tau\} \otimes_R M$ and is injective on subquotients. It is therefore injecitve.

Let us verify the exactness of the sequence at $R\{\tau\} \otimes_R M$. Consider the quotient of $R\{\tau\} \otimes_R M$ by the image of d. In this quotient we have the identity $r\tau^{n+1} \otimes m \equiv r\tau^n \otimes \tau m$ for all $r \in R$, $m \in M$, $n \geqslant 0$. As a consequence $\varphi \otimes m \equiv \varphi \cdot m$ for every $\varphi \in R\{\tau\}$. Hence every element $y \in R\{\tau\} \otimes_R M$ is equivalent to $1 \otimes a(y)$. If $a(y) = 0$ then $y \equiv 0$ or in other words y is in the image of d. □

Remark 11.6. Let M be an R-module. We denote τ^*M the R-module $R^\tau \otimes_R M$ where R^τ is R with the R-algebra structure given by the homomorphism $\tau\colon R \to R$. We write the elements of τ^*M as sums of pure tensors $r \otimes m$, $r \in R^\tau$, $m \in M$. If $r, r_1 \in R$ and $m \in M$ then

$$r \otimes r_1 m = r\tau(r_1) \otimes m.$$

The ring R acts on $\tau^*M = R^\tau \otimes_R M$ via the factor $R^\tau = R$. If $r, r_1 \in R$ and $m \in M$ then

$$r_1 \cdot (r \otimes m) = r_1 r \otimes m.$$

Lemma 11.7. *Let M be an R-module. The maps*

$$R\{\tau\}\tau \otimes_R M \to R\{\tau\} \otimes_R \tau^*M$$
$$\varphi\tau \otimes m \mapsto \varphi \otimes (1 \otimes m)$$

and

$$R\{\tau\} \otimes_R \tau^*M \to R\{\tau\}\tau \otimes_R M$$
$$\varphi \otimes (r \otimes m) \mapsto \varphi r\tau \otimes m$$

are mutually inverse isomorphisms of left $R\{\tau\}$-modules. □

Proposition 11.8. *If M is a left $R\{\tau\}$-module then the sequence of left $R\{\tau\}$-modules*

$$0 \to R\{\tau\} \otimes_R \tau^*M \xrightarrow{1\otimes\tau^a - \eta} R\{\tau\} \otimes_R M \xrightarrow{a} M \to 0$$

is exact. Here $\tau^a\colon \tau^*M \to M$ is the adjoint of the τ-multiplication map $M \to \tau_*M$ and η is the map given by the formula

$$\eta\colon \varphi \otimes (r \otimes m) \mapsto \varphi r\tau \otimes m.$$

Proof. Using the isomorphism $R\{\tau\}\tau \otimes_R M \cong R\{\tau\} \otimes_R \tau^*M$ of Lemma 11.7 we rewrite the short sequence in question as

$$0 \to R\{\tau\}\tau \otimes_R M \xrightarrow{d} R\{\tau\} \otimes_R M \xrightarrow{a} M \to 0.$$

An easy computation shows that

$$d(\varphi\tau \otimes m) = \varphi \otimes \tau \cdot m - \varphi\tau \otimes m.$$

The result thus follows from Lemma 11.5. □

12. The Hom shtuka

Let R be a τ-ring and let M and N be R-module shtukas. In this section we construct the Hom shtuka $\mathcal{H}\mathrm{om}_R(M,N)$. To some extent it behaves like an internal Hom in the category of shtukas. It is literally the internal Hom for shtukas which come from left $R\{\tau\}$-modules. Even if both M and N are left $R\{\tau\}$-modules, $\mathcal{H}\mathrm{om}_R(M,N)$ is in general a genuine shtuka which does not come from a left $R\{\tau\}$-module. Apart from the Drinfeld construction of Chapter 7 the $\mathcal{H}\mathrm{om}$ construction is the main source of nontrivial shtukas in the present work.

Definition 12.1. Let R be a τ-ring. Let

$$M = \Big[M_0 \overset{i_M}{\underset{j_M}{\rightrightarrows}} M_1\Big], \quad N = \Big[N_0 \overset{i_N}{\underset{j_N}{\rightrightarrows}} N_1\Big]$$

be R-module shtukas. The *Hom shtuka* $\mathcal{H}\mathrm{om}_R(M,N)$ is given by a diagram

$$\mathrm{Hom}_R(M_1,N_0) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_R(\tau^* M_0, N_1)$$

where i and j are defined as follows. Let $j_M^a \colon \tau^* M_0 \to M_1$ and $j_N^a \colon \tau^* N_0 \to N_1$ be the adjoint maps. For $f \in \mathrm{Hom}_R(M_1,N_0)$ we define

$$\begin{aligned} i(f) &= i_N \circ f \circ j_M^a, \\ j(f) &= j_N^a \circ \tau^*(f) \circ \tau^*(i_M). \end{aligned}$$

Observe that the pullback map $\mathrm{Hom}_R(M_0,N_1) \to \mathrm{Hom}_R(\tau^* M_0, \tau^* N_1)$ is τ-linear so that j is τ-linear too.

If M and N are left $R\{\tau\}$-modules then $\mathcal{H}\mathrm{om}_R(M,N)$ means $\mathcal{H}\mathrm{om}_R$ applied to the R-module shtukas associated to M and N as in Definition 11.2. The Hom shtukas we work with are typically of this sort. We will also need $\mathcal{H}\mathrm{om}_R(M,N)$ in the case when M is an R-module shtuka which does not come from a left $R\{\tau\}$-module (cf. Section 9.6).

Let M be an R-module. In the rest of this section we use the notation of Remark 11.6 for the elements of $\tau^* M$.

Lemma 12.2. *Let R be a τ-ring. If M and N are left $R\{\tau\}$-modules then*

$$\mathcal{H}\mathrm{om}_R(M,N) = \Big[\mathrm{Hom}_R(M,N) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}_R(\tau^* M, N)\Big]$$

where

$$\begin{aligned} i(f) &= f \circ \tau_M^a, \\ j(f) &= \tau_N^a \circ \tau^*(f), \end{aligned}$$

$\tau_M^a\colon \tau^*M \to M$ and $\tau_N^a\colon \tau^*N \to N$ are the adjoints of the τ-multiplication maps. In other words

$$i(f)\colon r\otimes m \mapsto f(r\tau\cdot m),$$
$$j(f)\colon r\otimes m \mapsto r\tau\cdot f(m)$$

for all $r\in R$, $m\in M$.

Next we describe the cohomology of $\mathcal{H}\mathrm{om}_R(M,N)$ in the case when M and N are left $R\{\tau\}$-modules.

Proposition 12.3. *Let R be a τ-ring. If M and N are left $R\{\tau\}$-modules then*

$$\mathrm{H}^0(\mathcal{H}\mathrm{om}_R(M,N)) = \mathrm{Hom}_{R\{\tau\}}(M,N)$$

as abelian subgroups of $\mathrm{Hom}_R(M,N)$.

Proof. Let i and j be the arrows of $\mathcal{H}\mathrm{om}_R(M,N)$. Let $f\in\mathrm{Hom}_R(M,N)$. Lemma 12.2 implies that $i(f)=j(f)$ if and only if f commutes with τ. □

Lemma 12.4. *Let R be a τ-ring. Let M be an R-module and N a left $R\{\tau\}$-module. The functor $\mathrm{Hom}_{R\{\tau\}}(-,N)$ transforms the map*

$$\eta\colon R\{\tau\}\otimes_R \tau^*M \to R\{\tau\}\otimes_R M,$$
$$\varphi\otimes(r\otimes m)\mapsto \varphi r\tau\otimes m$$

to the map

$$\mathrm{Hom}_R(M,N)\to\mathrm{Hom}_R(\tau^*M,N)$$
$$f\mapsto [r\otimes m\mapsto r\tau\cdot f(m)]$$

Proof. This simple observation is quite important so we spell out the details. Let $f\in\mathrm{Hom}_R(M,N)$. The induced map $f^a\colon R\{\tau\}\otimes_R M\to N$ is given by the formula $\varphi\otimes m\mapsto\varphi\cdot f(m)$. Therefore

$$f^a\circ\eta\colon \varphi\otimes(r\otimes m)\mapsto \varphi r\tau\cdot f(m).$$

We conclude that $\mathrm{Hom}_{R\{\tau\}}(-,N)(\eta)$ sends f to the map $r\otimes m\mapsto r\tau\cdot f(m)$. □

Theorem 12.5. *Let R be a τ-ring. Let M and N be left $R\{\tau\}$-modules. If M is projective as an R-module then there exists a natural quasi-isomorphism*

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}_R(M,N))\cong\mathrm{RHom}_{R\{\tau\}}(M,N).$$

Proof. By Proposition 11.8 we have a short exact sequence

$$0\to R\{\tau\}\otimes_R\tau^*M\xrightarrow{1\otimes\tau_M^a-\eta} R\{\tau\}\otimes_R M\xrightarrow{a} M\to 0. \tag{12.1}$$

If M is a projective R-module then so is τ^*M. As a consequence $R\{\tau\}\otimes_R M$ and $R\{\tau\}\otimes_R\tau^*M$ are projective left $R\{\tau\}$-modules. Thus (12.1) is a projective resolution of M as a left $R\{\tau\}$-module. Applying $\mathrm{Hom}_{R\{\tau\}}(-,N)$ to (3) we conclude that

$$\mathrm{RHom}_{R\{\tau\}}(M,N) = \Big[\mathrm{Hom}_R(M,N)\xrightarrow{(1\otimes\tau_M^a)^*-\eta^*}\mathrm{Hom}_R(\tau^*M,N)\Big]. \tag{12.2}$$

where * indicates the induced maps. Recall that

$$\mathcal{H}\mathrm{om}_R(M,N) = \Big[\operatorname{Hom}_R(M,N) \overset{i}{\underset{j}{\rightrightarrows}} \operatorname{Hom}_R(\tau^* M, N)\Big].$$

According to Lemma 12.2 the maps i and $(1 \otimes \tau_M^a)^*$ coincide. Lemma 12.4 in combination with Lemma 12.2 implies that $\eta^* = j$. Therefore (12.2) computes $\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}_R(M,N))$ by Theorem 8.1. ☐