Shtuka cohomology and special values of Goss L-functions

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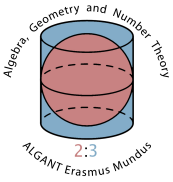

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Shtuka cohomology and special values of Goss L-functions

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Introduction

Assuming everywhere good reduction we generalize the class number formula of Taelman [25] to Drinfeld modules over arbitrary coefficient rings A. In order to prove this formula we develop a theory of shtukas and their cohomology.

1. A class number formula for Drinfeld modules

Fix a finite field $\mathbb{F}_q$. In the following all morphisms, fiber and tensor products will be over $\mathbb{F}_q$ unless indicated otherwise. Let C be a smooth projective connected curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$. The $\mathbb{F}_q$-algebra

$$A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$$

will be called the *coefficient ring*. Fix an A-algebra R (the base ring). We denote $\iota\colon A \to R$ the natural map.

Consider the group scheme $\mathbb{G}_a$ over R. It is well-known that every $\mathbb{F}_q$-linear endomorphism of $\mathbb{G}_a$ can be uniquely written in the form of a *τ-polynomial*

$$r_0 + r_1\tau + \ldots + r_n\tau^n$$

where $r_0, \ldots, r_n \in R$, $r_n \neq 0$ and $\tau\colon \mathbb{G}_a \to \mathbb{G}_a$ denotes the q-Frobenius.

Recall that an A-module scheme is an abelian group scheme equipped with an action of A.

Definition. A Drinfeld module over R with coefficients in A is an A-module scheme E over R which has the following properties:

(1) The underlying additive group scheme of E is Zariski-locally isomorphic to $\mathbb{G}_a$.
(2) For every element $a \in A$ the induced endomorphism of the Lie algebra scheme Lie_E is the multiplication by $\iota(a)$.
(3) There exists an element $a \in A$, a faithfully flat R-algebra S and a group scheme isomorphism $\mu\colon E_S \to \mathbb{G}_{a,S}$ such that the endomorphism $\mu a \mu^{-1}$ of $\mathbb{G}_{a,S}$ is given by a τ-polynomial of positive degree and with top coefficient a unit.

Example. Let $A = \mathbb{F}_q[t]$, $R = \mathbb{F}_q[\theta]$ and let $\iota\colon A \to R$ be the isomorphism which sends t to θ. An example of a Drinfeld A-module over R is the *Carlitz*

module E. Its underlying additive group scheme is $\mathbb{G}_a$. The action of $t \in A$ on E is given by the τ-polynomial

$$\theta + \tau.$$

The Frobenius τ induces the zero endomorphism on Lie_E. Hence t acts on Lie_E as the multiplication by $\theta = \iota(t)$. It follows that the condition (2) holds for E. The conditions (1) and (3) are clear.

From now on we assume that R is a domain and that it is finite flat over A. We denote K the fraction field of R. The generic fiber of a Drinfeld module over R is a Drinfeld module over K. However, not every Drinfeld module over K extends to a Drinfeld module over R. The ones which do are said to have *good reduction everywhere*.

Drinfeld modules behave in a way similar to elliptic curves. For the latter the role of the coefficient ring A is played by $\mathbb{Z}$. Given an elliptic curve E over a number field and a prime $(p) \subset \mathbb{Z}$ one can consider its p-adic Tate module

$$T_p E = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, E(\overline{\mathbb{Q}})).$$

Much in the same way for a Drinfeld module E over the function field K and a prime $\mathfrak{p} \subset A$ one has the $\mathfrak{p}$-adic Tate module

$$T_{\mathfrak{p}} E = \mathrm{Hom}_A(F_{\mathfrak{p}}/A_{\mathfrak{p}}, E(K^{\mathrm{sep}}))$$

where K^{sep} denotes a separable closure of K, $A_{\mathfrak{p}}$ is the completion of A at $\mathfrak{p}$ and $F_{\mathfrak{p}}$ is the field of fractions of $A_{\mathfrak{p}}$.

The Tate module $T_{\mathfrak{p}} E$ is a finitely generated free $A_{\mathfrak{p}}$-module. Its rank can be any integer greater than zero. This rank does not depend on $\mathfrak{p}$ and is called the *rank of* E. The Tate module $T_{\mathfrak{p}} E$ is naturally a continuous representation of the Galois group $G(K^{\mathrm{sep}}/K)$ unramified at almost all primes $\mathfrak{m} \subset R$.

Let us fix a Drinfeld module E over R. Let $\mathfrak{p} \subset A$ and $\mathfrak{m} \subset R$ be primes such that $\mathfrak{p} \neq \iota^{-1}(\mathfrak{m})$. Since the generic fiber of E extends to a Drinfeld module over R the Tate module $T_{\mathfrak{p}} E$ is unramified at $\mathfrak{m}$. It thus makes sense to consider the inverse characteristic polynomial of the geometric Frobenius element at $\mathfrak{m}$ acting on $T_{\mathfrak{p}} E$. This polynomial has coefficients in the fraction field of A and is independent of the choice of $\mathfrak{p}$. We denote it $P_{\mathfrak{m}}(T)$.

Definition. Let F_∞ be the local field of the curve C at ∞. We define $L(E^*, 0) \in F_\infty$ by the formula

$$L(E^*, 0) = \prod_{\mathfrak{m}} \frac{1}{P_{\mathfrak{m}}(1)}$$

where the product ranges over all primes $\mathfrak{m} \subset R$.

It is not difficult to show that this product converges. The resulting element $L(E^*, 0) \in F_\infty$ is indeed a value of a certain function, the Goss L-function of the strictly compatible family of Galois representations given by the Tate modules $T_{\mathfrak{p}} E$. We use the notation $L(E^*, 0)$ instead of $L(E, 0)$ since the usual Goss L-function of E is given by the family of dual Tate modules $(T_{\mathfrak{p}} E)^*$.

For the experts we remark that our $L(E^*, 0)$ coincides with the special value considered by Taelman in [25].

Example. Let us examine the Carlitz module E. For every prime $\mathfrak{m} \subset R = \mathbb{F}_q[\theta]$ there exists a unique monic irreducible polynomial $f \in \mathbb{F}_q[t]$ such that

$$\mathfrak{m} = \iota(f)R.$$

Let $g \in \mathbb{F}_q[t]$ be another monic irreducible polynomial, $g \neq f$. The Tate module $T_g E$ is free of rank 1. One can show that the *arithmetic* Frobenius at $\mathfrak{m}$ acts on $T_g E$ by multiplication by f. Therefore the inverse charateristic polynomial of the *geometric* Frobenius at $\mathfrak{m}$ is

$$P_{\mathfrak{m}}(T) = 1 - f^{-1}T.$$

We conclude that the special value $L(E^*, 0)$ is given by the Euler product

$$L(E^*, 0) = \prod_f \frac{1}{1 - \frac{1}{f}}$$

where $f \in \mathbb{F}_q[t]$ runs over monic irreducible polynomials. Expanding this product we get the formula

$$L(E^*, 0) = \sum_h \frac{1}{h}$$

where $h \in \mathbb{F}_q[t]$ runs over all monic polynomials.

Let $K_\infty = R \otimes_A F_\infty$. The exponential map of the Drinfeld module E is the unique map

$$\exp\colon \mathrm{Lie}_E(K_\infty) \to E(K_\infty)$$

satisfying the following conditions:

(1) exp is a homomorphism of A-modules,
(2) exp is an analytic function with derivative 1 at zero in the following sense. Fix an $\mathbb{F}_q$-linear isomorphism of group schemes $E \cong \mathbb{G}_a$ defined over K_∞. It identifies $E(K_\infty)$ with K_∞ while its differential identifies $\mathrm{Lie}_E(K_\infty)$ with K_∞. We demand that the resulting map $\exp\colon K_\infty \to K_\infty$ is given by an everywhere convergent power series

$$\exp(z) = z + a_1 z^q + a_2 z^{q^2} + \dots$$

with coefficients in K_∞.

Definition. The *complex of units* of E is the A-module complex

$$U_E = \Big[\mathrm{Lie}_E(K_\infty) \xrightarrow{\exp} \frac{E(K_\infty)}{E(R)} \Big]$$

where $\mathrm{Lie}_E(K_\infty)$ is placed in degree 0.

Observe that

$$\mathrm{H}^0(U_E) = \exp^{-1} E(R),$$
$$\mathrm{H}^1(U_E) = \frac{E(K_\infty)}{\exp(K_\infty) + E(R)}.$$

The cohomology modules of U_E have individual names: $\mathrm{H}^0(U_E)$ is the module of units and $\mathrm{H}^1(U_E)$ is the class module.

Definition. Consider the morphism of complexes $U_E \to \mathrm{Lie}_E(K_\infty)[0]$ given by the identity in degree zero. The *regulator*

$$\rho\colon F_\infty \otimes_A U_E \to \mathrm{Lie}_E(K_\infty)[0]$$

is the F_∞-linear extension of this morphism.

Theorem 1.1 (Taelman [23]). *The A-module complex U_E is perfect and the regulator ρ is a quasi-isomorphism.*

This theorem is usually stated in a different form: $\mathrm{H}^1(U_E)$ is finite and $\mathrm{H}^0(U_E)$ is a lattice in $\mathrm{Lie}_E(K_\infty)$. Recall that an A-submodule Λ in a finite-dimensional F_∞-vector space V is called a *lattice* if one of the following equivalent conditions is satisfied:

- Λ is discrete and cocompact.
- The natural map $F_\infty \otimes_A \Lambda \to V$ is an isomorphism.

A lattice is automatically a finitely generated projective A-module.

One may interpret Taelman's theorem as saying that U_E is a lattice in $\mathrm{Lie}_E(K_\infty)$ in a derived sense. $\mathrm{Lie}_E(K_\infty)$ contains one more natural lattice: the integral Lie algebra $\mathrm{Lie}_E(R)$. We would like to determine the relative position of U_E and $\mathrm{Lie}_E(R)$.

Since U_E is perfect one can apply the theory of Knudsen-Mumford determinants [16] to define an invertible A-module $\det_A U_E$ and an isomorphism of one-dimensional F_∞-vector spaces

$$\det_{F_\infty}(\rho)\colon F_\infty \otimes_A \det_A U_E \cong \det_{F_\infty} \mathrm{Lie}_E(K_\infty).$$

We identify $\det_A U_E$ with its image under this isomorphism. Now we are ready to state our main result.

Theorem 1.2. *If E is a Drinfeld module over R then*

$$\det_A U_E = L(E^*, 0) \cdot \det_A \mathrm{Lie}_E(R)$$

as invertible A-submodules in the one-dimensional F_∞-vector space

$$\det_{F_\infty} \mathrm{Lie}_E(K_\infty).$$

Remark. Theorem 1.2 implies that the invertible A-modules $\det_A U_E$ and $\det_A \mathrm{Lie}_E(R)$ are isomorphic. This is by no means immediate if the class group of the coefficient ring A is not zero. In fact, it was not previously known apart from the trivial case $\mathrm{Pic}\, A = 0$.

Remark. It is important to realize that Theorem 1.2 pinpoints the element $L(E^*, 0) \in F_\infty$. A priori the condition

$$\det_A U_E = x \cdot \det_A \mathrm{Lie}_E(R)$$

determines $x \in F_\infty$ up to a unit of A. However one can prove that $L(E^*, 0)$ is a 1-unit in F_∞. The only unit of A which is also a 1-unit of F_∞ is the element 1. Hence the condition above determines a 1-unit x uniquely. Consequently, one can regard Theorem 1.2 as a formula for $L(E^*, 0)$.

Remark. The statement of Theorem 1.2 goes back to the fundamental work of Taelman [25] where he established a formula for $L(E^*, 0)$ under assumption that the coefficient ring A is $\mathbb{F}_q[t]$. Unlike our Theorem 1.2 the result of Taelman applies to Drinfeld modules with arbitrary reduction type.

Fang [9] extended the result of Taelman to Anderson modules [1] which are a higher-dimensional generalization of Drinfeld modules. He also considered the coefficient ring $A = \mathbb{F}_q[t]$ only.

Debry [5] was the first to generalize this formula to coefficient rings A different from $\mathbb{F}_q[t]$. His result only applies to coefficient rings with trivial class group. In our Theorem 1.2 the coefficient ring A can be arbitrary but the Drinfeld module E is assumed to have good reduction everywhere.

Remark. One can describe the image of $\det_A U_E$ under $\det_{F_\infty}(\rho)$ as follows. Theorem 1.1 implies that the A-submodule

$$\exp^{-1} E(R) \subset \mathrm{Lie}_E(K_\infty)$$

is a lattice so that its top exterior power

$$\det_A \exp^{-1} E(R)$$

is an invertible A-submodule in the determinant of $\mathrm{Lie}_E(K_\infty)$. The image of $\det_A U_E$ is the A-submodule

$$I \cdot \det_A \exp^{-1} E(R)$$

where I is the 0-th Fitting ideal of the class module $\mathrm{H}^1(U_E)$. This ideal has the following explicit description. The A-module $\mathrm{H}^1(U_E)$ can be written as a finite direct sum

$$\bigoplus_n A/I_n$$

where $I_n \subset A$ are ideals. The 0-th Fitting ideal of $\mathrm{H}^1(U_E)$ is

$$I = \prod_n I_n.$$

Example. Let us show how Theorem 1.2 works for the Carlitz module E. In this case $F_\infty = \mathbb{F}_q(\!(t^{-1})\!)$ and $K_\infty = \mathbb{F}_q(\!(\theta^{-1})\!)$. The exponential map

$\exp\colon K_\infty \to K_\infty$ of the Carlitz module admits a local inverse around 0, the Carlitz logarithm map. It is given by the power series

$$\log z = z - \frac{z^q}{\theta^q - \theta} + \frac{z^{q^2}}{(\theta^q - \theta)(\theta^{q^2} - \theta)} - \dots$$

The series converges for z such that $|z| \leqslant q^{\frac{q}{q-1}}$. In particular the image of the exponential map contains the unit ball $\mathbb{F}_q[[\theta^{-1}]] \subset K_\infty$. As a consequence the class module

$$\mathrm{H}^1(U_E) = \frac{\mathbb{F}_q(\!(\theta^{-1})\!)}{\exp(K_\infty) + \mathbb{F}_q[\theta]}$$

is zero.

The F_∞-vector space $\mathrm{Lie}_E(K_\infty)$ is of dimension 1. Hence Theorem 1.1 implies that $\mathrm{H}^0(U_E)$ is a free A-module of rank 1. By construction the element

$$\widetilde{\pi} = \log(1)$$

belongs to $\mathrm{H}^0(U_E) = \exp^{-1} E(R)$. A priori it generates an A-submodule of finite index. However it is easy to show that a nonconstant element of A can not divide $1 \in E(R)$. Therefore

$$\mathrm{H}^0(U_E) = A \cdot \widetilde{\pi}.$$

The A-module $\mathrm{H}^0(U_E)$ coincides with its determinant since it is free of rank 1. Now $\mathrm{H}^1(U_E) = 0$ so Theorem 1.2 implies that

$$L(E^*, 0) = \alpha \iota^{-1}(\widetilde{\pi})$$

for some $\alpha \in A^\times$. Here $\iota\colon F_\infty \cong K_\infty$ is the natural isomorphism. As observed before, one can show that $L(E^*, 0)$ is a 1-unit of F_∞. Since $\widetilde{\pi}$ is a 1-unit by construction it follows that

$$L(E^*, 0) = \iota^{-1}(\widetilde{\pi}).$$

Expanding the definitions we obtain a formula

$$\sum_{\substack{h \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{1}{h} = 1 - \frac{1}{t^q - t} + \frac{1}{(t^q - t)(t^{q^2} - t)} - \dots$$

in $\mathbb{F}_q(\!(t^{-1})\!)$. Observe that the series on the right hand side converges much faster than the series on the left hand side. This formula was discovered by Carlitz [4] in the 1930-ies.

Remark. The complex of units U_E has an interesting analog in the context of number fields. For the moment, let K be a number field and let $\mathcal{O}_K \subset K$ be its ring of integers. Set $K_\infty = \mathbb{R} \otimes_{\mathbb{Q}} K$ and consider the complex

$$U_K = \Big[\mathrm{Lie}_{\mathbb{G}_\mathrm{m}}(K_\infty) \xrightarrow{\exp} \frac{\mathbb{G}_\mathrm{m}(K_\infty)}{\mathbb{G}_\mathrm{m}(\mathcal{O}_K)} \Big]$$

where exp is the exponential of the Lie group $\mathbb{G}_{\mathrm{m}}(K_\infty)$. In this setting the regulator

$$\rho\colon \mathbb{R}\otimes_{\mathbb{Z}} U_K \to \mathrm{Lie}_{\mathbb{G}_{\mathrm{m}}}(K_\infty)$$

is the $\mathbb{R}$-linear extension of the natural morphism $U_K \to \mathrm{Lie}_{\mathbb{G}_{\mathrm{m}}}(K_\infty)[0]$ given by the identity in degree zero. The Dirichlet's unit theorem for K is equivalent to the following statement:

Theorem 1.0 (Dirichlet). *The $\mathbb{Z}$-module complex U_K is perfect and* $\mathrm{cone}(\rho)$ *is quasi-isomorphic to* $\mathbb{R}[0]$.

2. Overview of the proof

To prepare the ground for the proof of Theorem 1.2 we develop a theory of shtukas and their cohomology. While retaining some features of the works of Taelman [25] and Fang [9], our approach differs from them in an essential way. Certain aspects of this approach were envisaged by Taelman in [24]. The central idea of using Anderson trace formula [2] to study special values of shtukas is due to V. Lafforgue [17]. In general, the ideas of Anderson [1, 2] play an important role in this text.

Our cohomology theory for shtukas was heavily motivated by the works of Böckle-Pink [3] and V. Lafforgue [17]. The notion of a nilpotent τ-sheaf from [3] figures prominently in it. Our intellectual debt to Drinfeld [7, 8] is obvious.

We begin with an overview of shtuka theory relevant to the proof of Theorem 1.2. Let us first describe the setting. The finite flat A-algebra R is a Dedekind domain of finite type over $\mathbb{F}_q$. To such an algebra R one can functorially associate a smooth connected projective curve X over $\mathbb{F}_q$ together with an open embedding $\operatorname{Spec} R \subset X$. Consider the scheme $\operatorname{Spec} A \times X$. Let $\tau\colon \operatorname{Spec} A\times X \to \operatorname{Spec} A\times X$ be the endomorphism which acts as the identity on A and as the q-Frobenius on X.

Definition. A *shtuka* $\mathcal{M}$ on $\operatorname{Spec} A \times X$ is given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

where $\mathcal{M}_0$, $\mathcal{M}_1$ are coherent sheaves on $\operatorname{Spec} A \times X$ and

$$i\colon \mathcal{M}_0 \to \mathcal{M}_1,$$
$$j\colon \mathcal{M}_0 \to \tau_*\mathcal{M}_1$$

are morphisms of coherent sheaves. Morphisms of shtukas are given by morphisms of underlying coherent sheaves which commute with i and j (cf. Definition 1.1.2).

An example of a shtuka is the *unit shtuka*

$$\mathbb{1} = \Big[\mathcal{O} \underset{\tau^\#}{\overset{1}{\rightrightarrows}} \mathcal{O}\Big]$$

where $\mathcal{O}$ is the structure sheaf of $\operatorname{Spec} A \times X$ and $\tau^{\#}\colon \mathcal{O} \to \tau_*\mathcal{O}$ is the map defined by the endomorphism τ. Shtukas on $\operatorname{Spec} A \times X$ form an abelian category.

Definition. Let $\mathcal{M}$ be a shtuka on $A \times X$. The *cohomology complex of* $\mathcal{M}$ is the A-module complex

$$\mathrm{R}\Gamma(\mathcal{M}) = \mathrm{RHom}(\mathbb{1}, \mathcal{M})$$

where RHom on the right hand side is computed in the derived category of shtukas.

Remark. To avoid confusion we should stress that in the actual theory (Chapter 1) shtukas and their cohomology are defined in a more general and flexible way. The definitions in this section are simplified for expository purposes.

Definition. We define the *linearization functor* ∇ from the category of shtuka to itself in the following way:

$$\nabla\Bigl[\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Bigr] = \Bigl[\mathcal{M}_0 \underset{0}{\overset{i}{\rightrightarrows}} \mathcal{M}_1\Bigr].$$

The cohomology of $\nabla\mathcal{M}$ is often easier to compute than the cohomology $\mathcal{M}$. Even though the complexes $\mathrm{R}\Gamma(\mathcal{M})$ and $\mathrm{R}\Gamma(\nabla\mathcal{M})$ are usually quite different, there is a subtle link between them. Let $\mathcal{M}$ be a shtuka given by a diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

One can show that the cohomology complex $\mathrm{R}\Gamma(\mathcal{M})$ fits into a natural distinguished triangle

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1].$$

Here we write $\mathrm{R}\Gamma(\mathcal{M}_n)$ instead of $\mathrm{R}\Gamma(\operatorname{Spec} A \times X, \mathcal{M}_n)$ to simplify the notation. The sheaves $\mathcal{M}_0$ and $\mathcal{M}_1$ are coherent by assumption. As a consequence the A-module complexes $\mathrm{R}\Gamma(\mathcal{M}_0)$ and $\mathrm{R}\Gamma(\mathcal{M}_1)$ are perfect. The distinguished triangle now implies that $\mathrm{R}\Gamma(\mathcal{M})$ is a perfect A-module complex. So we can apply the theory of Knudsen-Mumford determinants to $\mathrm{R}\Gamma(\mathcal{M})$.

Definition. We define the *ζ-isomorphism*

$$\zeta_{\mathcal{M}}\colon \det_A \mathrm{R}\Gamma(\mathcal{M}) \cong \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$$

as the composition of the isomorphisms

$$\det_A \mathrm{R}\Gamma(\mathcal{M}) \cong \det_A \mathrm{R}\Gamma(\mathcal{M}_0) \otimes_A \det_A^{-1} \mathrm{R}\Gamma(\mathcal{M}_1) \cong \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$$

induced by the natural distinguished triangles

$$\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i-j} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1],$$

$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{M}_0) \xrightarrow{i} \mathrm{R}\Gamma(\mathcal{M}_1) \to [1].$$

The ζ-isomorphisms are named by analogy with the ζ-elements of Kato [15]. Such isomorphisms first appeared in [17] of V. Lafforgue.

Now we are in position to describe the main steps in the proof of Theorem 1.2. To a Drinfeld module E over R we associate a certain shtuka called *a model of* E. Its precise definition is a bit technical and is not necessary to understand the following. The construction of a model proceeds roughly as follows. We take the Anderson motive of E and dualize it to obtain a shtuka on $\operatorname{Spec} A \otimes R$. We then extend it to $\operatorname{Spec} A \times X$ using the functor of extension by zero from the theory of Böckle-Pink [3].

Every Drinfeld module E admits many different shtuka models. However all the models share important properties. The underlying coherent sheaves of a model are locally free. Their rank coincides with the rank of E. As it turns out the cohomology of a model captures important arithmetic invariants of E.

Theorem 2.1. *For every shtuka model $\mathcal{M}$ of E there are natural quasi-isomorphisms*

$$\mathrm{R}\Gamma(\nabla\mathcal{M}) \cong \operatorname{Lie}_E(R)[-1],$$
$$\mathrm{R}\Gamma(\mathcal{M}) \cong U_E[-1].$$

Recall that the units complex U_E is defined in terms of an analytic map, the exponential of E. Theorem 2.1 provides an algebraic description of U_E. One important application of it is the following:

Corollary. $\det_A U_E \cong \det_A \operatorname{Lie}_E(R)$.

Proof. As $\mathcal{M}$ is a shtuka we have a ζ-isomorphism $\zeta_{\mathcal{M}}\colon \det_A \mathrm{R}\Gamma(\mathcal{M}) \cong \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$. □

Remark. The first quasi-isomorphism in Theorem 2.1 is easy to construct. In contrast there is no obvious natural map between the complexes $\mathrm{R}\Gamma(\mathcal{M})$ and $U_E[-1]$. The construction of the quasi-isomorphism $\mathrm{R}\Gamma(\mathcal{M}) \cong U_E[-1]$ is rather intricate.

Remark. Taelman [24] established Theorem 2.1 for the Carlitz module E over an arbitrary finite flat A-algebra R, $A = \mathbb{F}_q[t]$. He constructed shtuka models of E in an ad hoc manner. His result was generalized by Fang [9] to Anderson modules with coefficients in $A = \mathbb{F}_q[t]$. The construction of shtuka models in [9] is also ad hoc.

Remark. It is necessary to mention that our proof of Theorem 2.1 extends without change to arbitrary Anderson A-modules, including the non-uniformizable ones. In this text we limit the exposition to Drinfeld modules since other important parts of the theory still depend on their special properties.

Remark. Our proof of Theorem 2.1 was inspired by the article [1] of Anderson. In [1, §2] he proves a vanishing statement for Ext^1 which in retrospect can

be viewed as a statement on cohomology of certain shtukas related to Drinfeld modules.

As we mentioned above the cohomology complexes of a shtuka $\mathcal{M}$ and its linearization $\nabla\mathcal{M}$ are quite different in general. The ζ-isomorphism $\zeta_{\mathcal{M}}$ relates their determinants. A more direct link is given by the *regulator*

$$\rho_{\mathcal{M}}\colon F_\infty \otimes_A \mathrm{R}\Gamma(\mathcal{M}) \to F_\infty \otimes_A \mathrm{R}\Gamma(\nabla\mathcal{M}).$$

The regulator is a quasi-isomorphism and is natural in $\mathcal{M}$. It is defined for *elliptic shtukas*, a natural class of shtukas which generalize shtuka models of Drinfeld modules.

The central result about the regulator is the trace formula which expresses $\zeta_{\mathcal{M}}$ in terms of $\rho_{\mathcal{M}}$ and an explicit numerical invariant $L(\mathcal{M}) \in F_\infty$. This invariant is a product of local factors, one for each prime $\mathfrak{m} \subset R$. The local factor at $\mathfrak{m}$ depends only on the restriction of $\mathcal{M}$ to $A \otimes R/\mathfrak{m}$.

Theorem 2.2 (Trace formula) *If $\mathcal{M}$ is an elliptic shtuka then*

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_{F_\infty}(\rho_{\mathcal{M}})$$

as maps from $F_\infty \otimes_A \det_A \mathrm{R}\Gamma(\mathcal{M})$ to $F_\infty \otimes_A \det_A \mathrm{R}\Gamma(\nabla\mathcal{M})$.

The following important lemma is very easy to prove:

Lemma. *If $\mathcal{M}$ is a shtuka model of E then $L(\mathcal{M}) = L(E^*, 0)$.*

Remark. Theorem 2.2 is basically the trace formula of Anderson [2] in disguise.

Remark. In general the invariant $L(\mathcal{M}) \in F_\infty$ is transcendental over $A \subset F_\infty$. Its inherent complexity reflects in the construction of the regulator making it rather involved.

Remark. To the best of our knowledge the regulator $\rho_{\mathcal{M}}$ does not appear in previous work even in the context of shtukas related to Drinfeld modules.

Now we have almost all the tools to prove Theorem 1.2. Thanks to Theorem 2.1 the shtuka-theoretic regulator $\rho_{\mathcal{M}}$ of a model $\mathcal{M}$ induces a quasi-isomorphism

$$F_\infty \otimes_A U_E \to \mathrm{Lie}_E(K_\infty)[0].$$

However there is no a priori reason for it to coincide with the arithmetic regulator

$$\rho_E\colon F_\infty \otimes_A U_E \to \mathrm{Lie}_E(K_\infty)[0]$$

which is defined purely in terms of the Drinfeld module E.

Theorem 2.3. *If $\mathcal{M}$ is a shtuka model of E then the following square is commutative*

$$\begin{array}{ccc} F_\infty \otimes_A \mathrm{R}\Gamma(\mathcal{M}) & \xrightarrow{\rho_{\mathcal{M}}} & F_\infty \otimes_A \mathrm{R}\Gamma(\nabla\mathcal{M}) \\ \downarrow\wr & & \downarrow\wr \\ F_\infty \otimes_A U_E[-1] & \xrightarrow{\rho_E[-1]} & \mathrm{Lie}_E(K_\infty)[-1] \end{array}$$

Here the vertical arrows are the quasi-isomorphisms of Theorem 2.1.

As we observed above, shtuka models $\mathcal{M}$ of E exist and have the property that $L(\mathcal{M}) = L(E^*, 0)$. Hence Theorems 2.1, 2.2 and 2.3 imply Theorem 1.2 for E.

Remark. Our theory of shtukas is very sensitive to reduction properties of Drinfeld modules. Its extension to the bad reduction case is not at all straightforward and may be difficult. Such an extension is a subject of current research. We also work on an extension of our theory to Anderson modules [1].

Notation and conventions

1. References to the Stacks Project

We use the Stacks Project [27] as the reference for the theory of schemes and homological algebra. Since the order and numeration of items in the Stacks Project is subject to change we refer to them by tags as explained at the page `http://stacks.math.columbia.edu/tags`. The reference to a tag has the form [$wxyz$] where "$wxyz$" is a combination of four letters and numbers. The corresponding item of the Stacks Project can be accessed by the URI `http://stacks.math.columbia.edu/tag/`$wxyz$.

2. Ground field

Throughout the text we fix a finite field $\mathbb{F}_q$. Correspondingly the letter q stands for its cardinality. Apart from Chapter 1 we work over $\mathbb{F}_q$. The tensor product $\otimes$ and the fiber product $\times$ without subscripts mean the products over $\mathbb{F}_q$.

3. Mapping fiber

Definition 3.1. Let $f\colon A \to B$ be a morphism in an abelian category. The *mapping fiber of* f is the complex

$$\big[A \xrightarrow{f} B\big]$$

where A is placed in degree 0 and B in degree 1. It coincides with $\mathrm{cone}(f)[-1]$ up to sign.

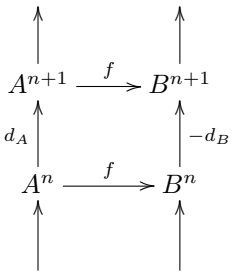
We extend this definition to a morphism $f\colon A \to B$ of complexes in an abelian category in the following way. The mapping fiber complex

$$\big[A \xrightarrow{f} B\big]$$

has the object $A^n \oplus B^{n-1}$ in degree n and the differential is given by the matrix

$$\begin{pmatrix} d_A & 0 \\ f & -d_B \end{pmatrix}$$

where d_A and d_B are differentials of A respectively B. Alternatively one can describe the mapping fiber complex as the total complex of the double complex



where A^n is placed in bidegree $(n, 0)$ and B^n in bidegree $(n, 1)$.

Denoting the mapping fiber complex C we get a natural distinguished triangle

$$C \xrightarrow{p} A \xrightarrow{f} B \xrightarrow{-i} C[1]$$

where p is the natural projection and i is the natural embedding. The sign change for i is necessary to make the triangle distinguished.