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Univoque bases and Hausdorff dimension

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Abstract Given a positive integer M and a real number $q > 1$, a *q-expansion* of a real number x is a sequence $(c_i) = c_1c_2\ldots$ with $(c_i) \in \{0, \ldots, M\}^\infty$ such that

$$x = \sum_{i=1}^{\infty} c_i q^{-i}.$$

It is well known that if $q \in (1, M+1]$, then each $x \in I_q := [0, M/(q-1)]$ has a q-expansion. Let $\mathcal{U} = \mathcal{U}(M)$ be the set of *univoque bases* $q > 1$ for which 1 has a

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unique q-expansion. The main object of this paper is to provide new characterizations of $\mathcal{U}$ and to show that the Hausdorff dimension of the set of numbers $x \in I_q$ with a unique q-expansion changes the most if q "crosses" a univoque base. Denote by $\mathcal{B}_2 = \mathcal{B}_2(M)$ the set of $q \in (1, M+1]$ such that there exist numbers having precisely two distinct q-expansions. As a by-product of our results, we obtain an answer to a question of Sidorov (J Number Theory 129:741–754, 2009) and prove that

$$\dim_H(\mathcal{B}_2 \cap (q', q'+\delta)) > 0 \quad \text{for any} \quad \delta > 0,$$

where $q' = q'(M)$ is the Komornik–Loreti constant.

Keywords Univoque bases · Univoque sets · Hausdorff dimensions · Generalized Thue–Morse sequences

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1 Introduction

Non-integer base expansions have received much attention since the pioneering works of Rényi [25] and Parry [24]. Given a positive integer M and a real number $q \in (1, M+1]$, a sequence $(d_i) = d_1 d_2 \ldots$ with *digits* $d_i \in \{0, 1, \ldots, M\}$ is called a *q-expansion of x* or an *expansion of x in base q* if

$$x = \pi_q((d_i)) := \sum_{i=1}^{\infty} \frac{d_i}{q^i}.$$

It is well known that each $x \in I_q := [0, M/(q-1)]$ has a q-expansion. One such expansion—the *greedy q-expansion*—can be obtained by performing the so called *greedy algorithm* of Rényi which is defined recursively as follows: if $d_1, \ldots, d_{n-1}$ is already defined (no condition if $n = 1$), then d_n is the largest element of $\{0, \ldots, M\}$ satisfying $\sum_{i=1}^{n} d_i q^{-i} \le x$. Equivalently, (d_i) is the greedy q-expansion of $\sum_{i=1}^{\infty} d_i q^{-i}$ if and only if $\sum_{i=n+1}^{\infty} d_i q^{-i+n} < 1$ whenever $d_n < M, n = 1, 2, \ldots$. Hence if $1 < q < r \le M+1$, then the greedy q-expansion of a number $x \in I_q$ is also the greedy expansion in base r of a number in I_r.

Let $\mathcal{U}_q$ be the *univoque set* consisting of numbers $x \in I_q$ such that x has a unique q-expansion, and let $\mathcal{U}'_q$ be the set of corresponding expansions. Note that a sequence (c_i) belongs to $\mathcal{U}'_q$ if and only if both the sequences (c_i) and $(M - c_i) := (M - c_1)(M - c_2)\ldots$ are greedy q-expansions, hence $\mathcal{U}'_q \subseteq \mathcal{U}'_r$ whenever $1 < q < r \le M+1$. Many works are devoted to the univoque sets $\mathcal{U}_q$ (see, e.g., [10,11,14]). Recently, de Vries and Komornik investigated their topological properties in [8]. Komornik et al. considered their Hausdorff dimension in [19], and showed that the dimension function $D : q \mapsto \dim_H \mathcal{U}_q$ behaves like a Devil's staircase on $(1, M+1]$. For more information on the univoque set $\mathcal{U}_q$ we refer to the survey paper [15] and the references therein.

There is an intimate connection between the set $\mathcal{U}_q$ and the set of *univoque bases* $\mathcal{U} = \mathcal{U}(M)$ consisting of numbers $q > 1$ such that 1 has a unique q-expansion over the alphabet $\{0, 1, \ldots, M\}$. For instance, it was shown in [8] that $\mathcal{U}_q$ is closed if and only if q does not belong to the set $\overline{\mathcal{U}}$. It is well-known that $\mathcal{U}$ is a Lebesgue null set of full Hausdorff dimension (cf. [6,12,19]). Moreover, the smallest element of $\mathcal{U}$ is the *Komornik–Loreti constant* (cf. [16,17])

$$q' = q'(M),$$

while the largest element of $\mathcal{U}$ is (of course) $M + 1$. Recently, Komornik and Loreti showed in [18] that its closure $\overline{\mathcal{U}}$ is a *Cantor set* (see also, [9]), i.e., a nonempty closed set having neither isolated nor interior points. Writing the open set $(1, M + 1]\backslash\overline{\mathcal{U}} = (1, M + 1)\backslash\overline{\mathcal{U}}$ as the disjoint union of its connected components, i.e.,

$$(1, M + 1]\backslash\overline{\mathcal{U}} = (1, q') \cup \bigcup \left(q_0, q_0^*\right), \tag{1}$$

the left endpoints q_0 in (1) run over the whole set $\overline{\mathcal{U}}\backslash\mathcal{U}$, and the right endpoints q_0^* run through a subset of $\mathcal{U}$ (cf. [8]). Furthermore, each left endpoint q_0 is algebraic, while each right endpoint $q_0^* \in \mathcal{U}$ is transcendental (cf. [20]).

De Vries showed in [7], roughly speaking, that the sets $\mathcal{U}_q'$ change the most if we cross a univoque base. More precisely, it was shown that $q \in \mathcal{U}$ if and only if $\mathcal{U}_r'\backslash\mathcal{U}_q'$ is uncountable for each $r \in (q, M+1]$ and $r \in \overline{\mathcal{U}}$ if and only if $\mathcal{U}_r'\backslash\mathcal{U}_q'$ is uncountable for each $q \in (1, r)$.

The main object of this paper is to provide similar characterizations of $\mathcal{U}$ and $\overline{\mathcal{U}}$ in terms of the Hausdorff dimension of the sets $\mathcal{U}_r'\backslash\mathcal{U}_q'$ after a natural projection. Furthermore, we characterize the sets $\mathcal{U}$ and $\overline{\mathcal{U}}$ by looking at the Hausdorff dimensions of $\mathcal{U}$ and $\overline{\mathcal{U}}$ locally.

Theorem 1.1 *Let $q \in (1, M + 1]$. The following statements are equivalent.*

(i) $q \in \mathcal{U}$.
(ii) $\dim_H \pi_{M+1}(\mathcal{U}_r'\backslash\mathcal{U}_q') > 0$ *for any* $r \in (q, M + 1]$.
(iii) $\dim_H \mathcal{U} \cap (q, r) > 0$ *for any* $r \in (q, M + 1]$.

Theorem 1.2 *Let $q \in (1, M + 1]$. The following statements are equivalent.*

(i) $q \in \overline{\mathcal{U}}\backslash(\bigcup \{q_0^*\} \cup \{q'\})$.
(ii) $\dim_H \pi_{M+1}(\mathcal{U}_q'\backslash\mathcal{U}_p') > 0$ *for any* $p \in (1, q)$.
(iii) $\dim_H \mathcal{U} \cap (p, q) > 0$ *for any* $p \in (1, q)$.

It follows at once from Theorems 1.1 and 1.2 that $\mathcal{U}$ (or, equivalently, $\overline{\mathcal{U}}$) does not contain isolated points.

We remark that the projection map π_{M+1} in Theorem 1.1 (ii) can be replaced by π_ρ for any $r \leq \rho \leq M + 1$. Similarly, the projection map π_{M+1} in Theorem 1.2 (ii) can also be replaced by π_ρ with $q \leq \rho \leq M + 1$. We also point out that Theorems 1.1 and 1.2 strengthen the main result of [7] where the cardinality of the sets $\mathcal{U}_q'\backslash\mathcal{U}_p'$ with $1 < p < q \leq M + 1$ was determined.

Let $\mathcal{B}_2$ be the set of bases $q \in (1, M + 1]$ for which there exists a number $x \in [0, M/(q-1)]$ having exactly two q-expansions. It was asked by Sidorov [26] whether

$\dim_H \mathcal{B}_2 \cap (q', q'+\delta) > 0$ for any $\delta > 0$, where q' is the Komornik–Loreti constant. Since $\mathcal{U} \subseteq \mathcal{B}_2$ (see [26, Lemma 3.1][1]), Theorem 1.1 answers this question in the affirmative.

Corollary 1 $\dim_H \mathcal{B}_2 \cap (q', q'+\delta) > 0$ *for any* $\delta > 0$.

The rest of the paper is arranged as follows. In Sect. 2 we recall some properties of unique q-expansions. The proof of Theorems 1.1 and 1.2 will be given in Sect. 3.

2 Preliminaries

In this section we recall some properties of the univoque set $\mathcal{U}_q$. Throughout this paper, a *sequence* $(d_i) = d_1 d_2 \ldots$ is an element of $\{0, \ldots, M\}^\infty$ with each digit d_i belonging to the *alphabet* $\{0, \ldots, M\}$. Moreover, for a *word* $\mathbf{c} = c_1 \ldots c_n$ we mean a finite string of digits with each digit c_i from $\{0, \ldots, M\}$. For two words $\mathbf{c} = c_1 \ldots c_n$ and $\mathbf{d} = d_1 \ldots d_m$ we denote by $\mathbf{cd} = c_1 \ldots c_n d_1 \ldots d_m$ the concatenation of the two words. For an integer $k \geq 1$ we denote by $\mathbf{c}^k$ the k-times concatenation of $\mathbf{c}$ with itself, and by $\mathbf{c}^\infty$ the infinite repetition of $\mathbf{c}$.

For a sequence (d_i) we denote its *reflection* by $\overline{(d_i)} := (M - d_1)(M - d_2) \ldots$. Accordingly, for a word $\mathbf{c} = c_1 \ldots c_n$ we denote its reflection by $\overline{\mathbf{c}} := (M - c_1) \ldots (M - c_n)$. If $c_n < M$ we denote by $\mathbf{c}^+ := c_1 \ldots c_{n-1}(c_n + 1)$. If $c_n > 0$ we write $\mathbf{c}^- := c_1 \ldots c_{n-1}(c_n - 1)$.

We will use systematically the lexicographic ordering $<, \leq, >$ and $\geq$ between sequences and between words. For two sequences $(c_i), (d_i) \in \{0, 1, \ldots, M\}^\infty$ we say that $(c_i) < (d_i)$ if there exists an integer $n \geq 1$ such that $c_1 \ldots c_{n-1} = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Furthermore, we write $(c_i) \leq (d_i)$ if $(c_i) < (d_i)$ or $(c_i) = (d_i)$. Similarly, we say $(c_i) > (d_i)$ if $(d_i) < (c_i)$, and $(c_i) \geq (d_i)$ if $(d_i) \leq (c_i)$. We extend this definition to words in the obvious way. For example, for two words $\mathbf{c}$ and $\mathbf{d}$ we write $\mathbf{c} < \mathbf{d}$ if $\mathbf{c}0^\infty < \mathbf{d}0^\infty$.

A sequence is called *finite* if it has a last nonzero element. Otherwise it is called *infinite*. So $0^\infty := 00\ldots$ is considered to be infinite. For $q \in (1, M+1]$ we denote by

$$\alpha(q) = (\alpha_i(q))$$

the *quasi-greedy* q-expansion of 1 (cf. [5]), i.e., the lexicographically largest *infinite* q-expansion of 1. Let $\beta(q) = (\beta_i(q))$ be the *greedy* q-expansion of 1 (cf. [24]), i.e., the lexicographically largest q-expansion of 1. For convenience, we set $\alpha(1) = 0^\infty$ and $\beta(1) = 10^\infty$, even though $\alpha(1)$ is not a 1-expansion of 1.

Moreover, we endow the set $\{0, \ldots, M\}$ with the discrete topology and the set of all possible sequences $\{0, 1, \ldots, M\}^\infty$ with the Tychonoff product topology.

The following properties of $\alpha(q)$ and $\beta(q)$ were established in [24], see also [3].

[1] This also follows directly from the observation that q^{-1} has exactly two q-expansions whenever $q \in \mathcal{U}$.

Lemma 2.1 (i) *The map* $q \mapsto \alpha(q)$ *is an increasing bijection from* $[1, M+1]$ *onto the set of all* infinite *sequences* (α_i) *satisfying*

$$\alpha_{n+1}\alpha_{n+2}\ldots \leq \alpha_1\alpha_2\ldots \quad \textit{whenever} \quad \alpha_n < M.$$

(ii) *The map* $q \mapsto \beta(q)$ *is an increasing bijection from* $[1, M+1]$ *onto the set of all sequences* (β_i) *satisfying*

$$\beta_{n+1}\beta_{n+2}\ldots < \beta_1\beta_2\ldots \quad \textit{whenever} \quad \beta_n < M.$$

Lemma 2.2 (i) $\beta(q)$ *is infinite if and only if* $\beta(q) = \alpha(q)$.
(ii) *If* $\beta(q) = \beta_1\ldots\beta_m 0^\infty$ *with* $\beta_m > 0$*, then* $\alpha(q) = (\beta_1\ldots\beta_m^-)^\infty$.
(iii) *The map* $q \mapsto \alpha(q)$ *is left-continuous, while the map* $q \mapsto \beta(q)$ *is right-continuous.*

In order to investigate the unique expansions we need the following lexicographic characterization of $\mathcal{U}_q'$ (cf. [3]).

Lemma 2.3 *Let* $q \in (1, M+1]$. *Then* $(d_i) \in \mathcal{U}_q'$ *if and only if*

$$\begin{cases} d_{n+1}d_{n+2}\ldots < \alpha_1(q)\alpha_2(q)\ldots & \textit{whenever} \;\; d_n < M, \\ d_{n+1}d_{n+2}\ldots > \overline{\alpha_1(q)\alpha_2(q)\ldots} & \textit{whenever} \;\; d_n > 0. \end{cases}$$

Note that $q \in \mathcal{U}$ if and only if $\alpha(q)$ is the unique q-expansion of 1. Then Lemma 2.3 yields a characterization of $\mathcal{U}$ (see also, [11,17]).

Lemma 2.4 *Let* $q \in (1, M+1)$. *Then* $q \in \mathcal{U}$ *if and only if* $\alpha(q) = (\alpha_i(q))$ *satisfies*

$$\overline{\alpha(q)} < \alpha_{n+1}(q)\alpha_{n+2}(q)\ldots < \alpha(q) \quad \textit{for all} \quad n \geq 1.$$

Consider a connected component (q_0, q_0^*) of $(q', M+1)\setminus\overline{\mathcal{U}}$ as in (1). Then there exists a (unique) word $\mathbf{t} = t_1\ldots t_p$ such that (cf. [8,20])

$$\alpha(q_0) = \mathbf{t}^\infty \quad \text{and} \quad \alpha\left(q_0^*\right) = \lim_{n\to\infty} g^n(\mathbf{t}),$$

where $g^n = \underbrace{g \circ \cdots \circ g}_{n}$ denotes the n-fold composition of g with itself, and

$$g(\mathbf{c}) := \mathbf{c}^+\overline{\mathbf{c}^+} \quad \text{for any word} \quad \mathbf{c} = c_1\ldots c_k \text{ with } c_k < M. \tag{2}$$

We point out that the word $\mathbf{t} = t_1\ldots t_p$ in the definitions of $\alpha(q_0)$ and $\alpha(q_0^*)$ is called an *admissible block* in [20, Definition 2.1] which satisfies the following lexicographical inequalities: $t_p < M$ and for any $1 \leq i \leq p$ we have

$$\overline{t_1\ldots t_p} \leq t_i\ldots t_p t_1\ldots t_{i-1} \quad \text{and} \quad t_i\ldots t_p\,\overline{t_1\ldots t_{i-1}} \leq t_1\ldots t_p^+.$$

We also mention that the limit $\lim_{n\to\infty} g^n(\mathbf{t})$ stands for the infinite sequence beginning with $\mathbf{t}^+\overline{\mathbf{t}}\,\overline{\mathbf{t}^+}\mathbf{t}^+\ \overline{\mathbf{t}^+}\mathbf{t}\,\mathbf{t}^+\overline{\mathbf{t}}\ldots$, and the existence of this limit was shown by Allouche [2].

In this case (q_0, q_0^*) is called the *connected component generated by* $\mathbf{t}$. The closed interval $[q_0, q_0^*]$ is the so called *admissible interval generated by* $\mathbf{t}$ (see [20, Definition 2.4]). Furthermore, the sequence

$$\alpha\left(q_0^*\right) = \lim_{n\to\infty} g^n(\mathbf{t}) = \mathbf{t}^+\,\overline{\mathbf{t}}\,\overline{\mathbf{t}^+}\,\mathbf{t}^+\ \overline{\mathbf{t}^+}\,\mathbf{t}\,\mathbf{t}^+\,\overline{\mathbf{t}}\ldots$$

is a generalized Thue–Morse sequence (cf. [20, Definition 2.2], see also [1]).

The following lemma for the generalized Thue–Morse sequence $\alpha(q_0^*)$ was established in [20, Lemma 4.2].

Lemma 2.5 *Let $(q_0, q_0^*) \subset (q', M+1)\setminus\overline{\mathcal{U}}$ be a connected component generated by $t_1 \ldots t_p$. Then the sequence $(\theta_i) = \alpha(q_0^*)$ satisfies*

$$\overline{\theta_1 \ldots \theta_{2^n p - i}} < \theta_{i+1} \ldots \theta_{2^n p} \leq \theta_1 \ldots \theta_{2^n p - i}$$

for any $n \geq 0$ and any $0 \leq i < 2^n p$.

Finally, we recall some topological properties of $\mathcal{U}$ and $\overline{\mathcal{U}}$ which were essentially established in [8,18] (see also, [9]).

Lemma 2.6 (i) *If $q \in \mathcal{U}$, then there exists a* decreasing *sequence (r_n) of elements in $\bigcup\left\{q_0^*\right\}$ that converges to q as $n \to \infty$;*
(ii) *If $q \in \overline{\mathcal{U}}\setminus(\bigcup\left\{q_0^*\right\} \cup \left\{q'\right\})$, then there exists an* increasing *sequence (p_n) of elements in $\bigcup\left\{q_0^*\right\}$ that converges to q as $n \to \infty$.*

We remark here that the bases q_0^* are called *de Vries–Komornik numbers* which were shown to be transcendental in [20]. By Lemma 2.6 it follows that the set of de Vries–Komornik numbers is dense in $\overline{\mathcal{U}}$.

3 Proofs of Theorems 1.1 and 1.2

3.1 Proof of Theorem 1.1 for (i) ⇔ (ii).

For each connected component (q_0, q_0^*) of $(q', M+1)\setminus\overline{\mathcal{U}}$ we construct a sequence of bases (r_n) in $\mathcal{U}$ strictly decreasing to q_0^*.

Lemma 3.1 *Let $(q_0, q_0^*) \subset (q', M+1)\setminus\overline{\mathcal{U}}$ be a connected component generated by $t_1 \ldots t_p$, and let $(\theta_i) = \alpha(q_0^*)$. Then for each $n \geq 1$, the number $r_n \in \mathcal{U}$ determined by*

$$\alpha(r_n) = \beta(r_n) = \theta_1 \ldots \theta_{2^n p}\left(\theta_{2^n p+1} \ldots \theta_{2^{n+1} p}\right)^\infty,$$

belongs to $\mathcal{U}$. Furthermore, (r_n) is a strictly decreasing sequence that converges to q_0^*.

Proof Using (2) one may verify that the sequence (θ_i) satisfies

$$\theta_{2^n p+k} = \overline{\theta_k} \quad \text{for all} \quad 1 \le k < 2^n p; \quad \theta_{2^{n+1} p} = \overline{\theta_{2^n p}}^{\,+}$$

for all $n \ge 0$. Now fix $n \ge 1$. We claim that

$$\sigma^i \left(\theta_1 \dots \theta_{2^n p} (\theta_{2^n p+1} \dots \theta_{2^{n+1} p})^\infty\right) < \theta_1 \dots \theta_{2^n p} (\theta_{2^n p+1} \dots \theta_{2^{n+1} p})^\infty \tag{3}$$

for all $i \ge 1$, where σ is the left shift on $\{0, \dots, M\}^\infty$ defined by $\sigma((c_i)) = (c_{i+1})$. By periodicity it suffices to prove (3) for $0 < i < 2^{n+1} p$. We distinguish between the following three cases: (I) $0 < i < 2^n p$; (II) $i = 2^n p$; (III) $2^n p < i < 2^{n+1} p$.

Case (I). $0 < i < 2^n p$. Then by Lemma 2.5 it follows that

$$\theta_{i+1} \dots \theta_{2^n p} \le \theta_1 \dots \theta_{2^n p-i}$$

and

$$\theta_{2^n p+1} \dots \theta_{2^n p+i} = \overline{\theta_1 \dots \theta_i} < \theta_{2^n p-i+1} \dots \theta_{2^n p}.$$

This implies (3) for $0 < i < 2^n p$.

Case (II). $i = 2^n p$. Note by [17] that $\alpha_1(q') = [M/2] + 1$ (see also, [4]), where $[y]$ denotes the integer part of a real number y. Then by using $q_0^* > q'$ in Lemma 2.1 we have

$$\theta_1 = \alpha_1 \left(q_0^*\right) \ge \alpha_1(q') > \overline{\alpha_1(q')} \ge \overline{\theta_1}.$$

This, together with $n \ge 1$, implies

$$\theta_{2^n p+1} \dots \theta_{2^{n+1} p} = \overline{\theta_1 \dots \theta_{2^n p}}^{\,+} < \theta_1 \dots \theta_{2^n p}.$$

So, (3) holds true for $i = 2^n p$.

Case (III). $2^n p < i < 2^{n+1} p$. Write $j = i - 2^n p$. Then $0 < j < 2^n p$. Once again, we infer from Lemma 2.5 that

$$\theta_{i+1} \dots \theta_{2^{n+1} p} = \overline{\theta_{j+1} \dots \theta_{2^n p}}^{\,+} \le \theta_1 \dots \theta_{2^n p-j}$$

and

$$\theta_{2^n p+1} \dots \theta_{2^n p+j} = \overline{\theta_1 \dots \theta_j} < \theta_{2^n p-j+1} \dots \theta_{2^n p}.$$

This yields (3) for $2^n p < i < 2^{n+1} p$.

Note by Lemma 2.5 that

$$\sigma^i \left(\theta_1 \dots \theta_{2^n p} (\theta_{2^n p+1} \dots \theta_{2^{n+1} p})^\infty\right) > \overline{\theta_1 \dots \theta_{2^n p} (\theta_{2^n p+1} \dots \theta_{2^{n+1} p})^\infty}$$

for any $i \geq 0$. Then by (3) and Lemma 2.4 it follows that there exists $r_n \in \mathcal{U}$ such that

$$\alpha(r_n) = \beta(r_n) = \theta_1 \ldots \theta_{2^n p}(\theta_{2^n p+1} \ldots \theta_{2^{n+1} p})^\infty.$$

In the following we prove $r_n \searrow q_0^*$ as $n \to \infty$. For $n \geq 1$ we observe that

$$\begin{aligned} \beta(r_{n+1}) &= \theta_1 \ldots \theta_{2^{n+1} p}(\theta_{2^{n+1} p+1} \ldots \theta_{2^{n+2} p})^\infty \\ &= \theta_1 \ldots \theta_{2^n p}\overline{\theta_1 \ldots \theta_{2^n p}}^{+}\overline{\theta_1 \ldots \theta_{2^n p}} \ldots \\ &< \theta_1 \ldots \theta_{2^n p} \left(\overline{\theta_1 \ldots \theta_{2^n p}}^{+}\right)^\infty = \beta(r_n). \end{aligned}$$

Then by Lemma 2.1 (ii) we have $r_{n+1} < r_n$. Note that $\beta(q_0^*) = \alpha(q_0^*) = (\theta_i)$, and

$$\beta(r_n) \to (\theta_i) = \beta\left(q_0^*\right) \quad \text{as} \quad n \to \infty.$$

Hence, we conclude from Lemma 2.2 (iii) that $r_n \searrow q_0^*$ as $n \to \infty$. □

Lemma 3.2 *Let $(q_0, q_0^*) \subset (q', M+1)\backslash\overline{\mathcal{U}}$ be a connected component generated by $t_1 \ldots t_p$, and let $(\theta_i) = \alpha(q_0^*)$. Then for any $n \geq 1$ and any $0 \leq i < 2^n p$ we have*

$$\begin{aligned} \overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} &< \sigma^i(\xi_n\overline{\xi_n}) < \theta_1 \ldots \theta_{2^{n+1} p-i}, \\ \overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} &< \sigma^i(\xi_n\overline{\xi_n^-}) \leq \theta_1 \ldots \theta_{2^{n+1} p-i}, \\ \overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} &< \sigma^i(\xi_n^-\xi_n) < \theta_1 \ldots \theta_{2^{n+1} p-i}, \end{aligned} \tag{4}$$

and thus (by symmetry),

$$\begin{aligned} \overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} &< \sigma^i(\overline{\xi_n}\xi_n) < \theta_1 \ldots \theta_{2^{n+1} p-i}, \\ \overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} &\leq \sigma^i(\overline{\xi_n}\xi_n^-) < \theta_1 \ldots \theta_{2^{n+1} p-i}, \\ \overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} &< \sigma^i(\overline{\xi_n^-}\,\overline{\xi_n}) < \theta_1 \ldots \theta_{2^{n+1} p-i}, \end{aligned}$$

where $\xi_n := \theta_1 \ldots \theta_{2^n p}$.

Proof By symmetry it suffices to prove (4).

Note that $\xi_n\overline{\xi_n} = \theta_1 \ldots \theta_{2^{n+1} p}^-$ and $\xi_n\overline{\xi_n^-} = \theta_1 \ldots \theta_{2^{n+1} p}$. Then by Lemma 2.5 it follows that

$$\overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} < \sigma^i(\xi_n\overline{\xi_n}) < \theta_1 \ldots \theta_{2^{n+1} p-i}$$

and

$$\overline{\theta_1 \ldots \theta_{2^{n+1} p-i}} < \sigma^i(\xi_n\overline{\xi_n^-}) \leq \theta_1 \ldots \theta_{2^{n+1} p-i}$$

for any $0 \leq i < 2^n p$.

So, it suffices to prove the inequalities

$$\overline{\theta_1 \ldots \theta_{2^{n+1}p-i}} < \sigma^i(\theta_1 \ldots \theta_{2^n p}^- \theta_1 \ldots \theta_{2^n p}) < \theta_1 \ldots \theta_{2^{n+1}p-i} \tag{5}$$

for any $0 \leq i < 2^n p$. By Lemma 2.5 it follows that for any $0 \leq i < 2^n p$ we have

$$\overline{\theta_1 \ldots \theta_{2^n p-i}} \leq \theta_{i+1} \ldots \theta_{2^n p}^- < \theta_1 \ldots \theta_{2^n p-i}$$

and

$$\theta_1 \ldots \theta_i > \overline{\theta_{2^n p-i+1} \ldots \theta_{2^n p}}.$$

This proves (5). ◻

Lemma 3.3 *Let* $(q_0, q_0^*) \subset (q', M+1)\backslash\overline{\mathcal{U}}$ *be a connected component generated by* $t_1 \ldots t_p$. *Then* $\dim_H \pi_{M+1}(\mathcal{U}_r' \backslash \mathcal{U}_{q_0^*}') > 0$ *for any* $r \in (q_0^*, M+1]$.

Proof Take $r \in (q_0^*, M+1]$. By Lemma 3.1 there exists $n \geq 1$ such that

$$r_n \in \left(q_0^*, r\right) \cap \mathcal{U}.$$

Write $(\theta_i) = \alpha(q_0^*)$ and let $\xi_n = \theta_1 \ldots \theta_{2^n p}$. Denote by $X_A^{(n)}$ the subshift of finite type over the states $\left\{\xi_n, \xi_n^-, \overline{\xi_n}, \overline{\xi_n^-}\right\}$ with adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $\alpha(r_n) = \theta_1 \ldots \theta_{2^n p}(\theta_{2^n p+1} \ldots \theta_{2^{n+1}p})^\infty$. Then by Lemmas 3.2 and 2.3 it follows that

$$X_A^{(n)} \subseteq \mathcal{U}_{r_n}' \subseteq \mathcal{U}_r'. \tag{6}$$

Furthermore, note that

$$\begin{aligned} \xi_n \overline{\xi_n^-} (\overline{\xi_n} \xi_n)^3 &= \theta_1 \ldots \theta_{2^{n+1}p} \left(\overline{\theta_1 \ldots \theta_{2^{n+1}p}}^+\right)^3 \\ &= \theta_1 \ldots \theta_{2^{n+2}p} \left(\overline{\theta_1 \ldots \theta_{2^{n+1}p}}^+\right)^2 \\ &> \theta_1 \ldots \theta_{2^{n+2}p} \overline{\theta_1 \ldots \theta_{2^{n+1}p} \theta_{2^{n+1}p+1} \ldots \theta_{2^{n+2}p}}^+ \\ &= \theta_1 \ldots \theta_{2^{n+2}p} \theta_{2^{n+2}p+1} \ldots \theta_{2^{n+3}p}. \end{aligned}$$

Then by Lemmas 2.3 and 3.1 it follows that any sequence starting at

$$\mathbf{c} := \xi_n^- \xi_n \overline{\xi_n^-} (\overline{\xi_n} \xi_n)^3$$

can not belong to $\mathcal{U}'_{r_{n+2}}$. Therefore, by (6) we obtain

$$X_A^{(n)}(\mathbf{c}) := \left\{(d_i) \in X_A^{(n)} : d_1 \ldots d_{(2^{n+3}+2^n)p} = \mathbf{c}\right\} \subseteq X_A^{(n)} \setminus \mathcal{U}'_{r_{n+2}} \subset \mathcal{U}'_r \setminus \mathcal{U}'_{q_0^*}. \quad (7)$$

Note that the subshift of finite type $X_A^{(n)}$ is irreducible (cf. [22]), and the image $\pi_{M+1}(X_A^{(n)})$ is a graph-directed set satisfying the open set condition (cf. [23]). Then by (7) it follows that

$$\begin{aligned}\dim_H \pi_{M+1}\left(\mathcal{U}'_r \setminus \mathcal{U}'_{q_0^*}\right) &\geq \dim_H \pi_{M+1}(X_A^{(n)}(\mathbf{c})) \\ &= \dim_H \pi_{M+1}(X_A^{(n)}) = \frac{\log\left((1+\sqrt{5})/2\right)}{2^n p \log(M+1)} > 0.\end{aligned}$$

□

The following lemma can be shown in a way which resembles closely the analysis in [21, pp. 2829–2830]. For the sake of completeness we include a sketch of its proof.

Lemma 3.4 *Let* $(q_0, q_0^*) \subset (q', M+1) \setminus \overline{\mathcal{U}}$ *be a connected component. Then* $\dim_H \pi_{M+1}(\mathcal{U}'_{q_0^*} \setminus \mathcal{U}'_{q_0}) = 0$.

Proof (Sketch of the proof) Suppose that (q_0, q_0^*) is a connected component generated by $\mathbf{t} = t_1 \ldots t_p$. Then

$$\alpha(q_0) = \mathbf{t}^\infty \quad \text{and} \quad \alpha\left(q_0^*\right) = \lim_{n\to\infty} g^n(\mathbf{t}) = \mathbf{t}^+ \,\overline{\mathbf{t}}\, \overline{\mathbf{t}^+}\, \mathbf{t}^+ \ldots, \quad (8)$$

where $g(\cdot)$ is defined in (2).

For $n \geq 0$ let $\omega_n := g^n(\mathbf{t})^+$. Take $(d_i) \in \mathcal{U}'_{q_0^*} \setminus \mathcal{U}'_{q_0}$. Then by using (8) and Lemma 2.3 it follows that there exists $m \geq 1$ such that

$$\mathbf{t}^\infty = \alpha(q_0) \leq d_{m+1} d_{m+2} \ldots < \alpha\left(q_0^*\right) = \mathbf{t}^+ \overline{\mathbf{t}} \ldots, \quad (9)$$

or symmetrically,

$$\mathbf{t}^\infty = \alpha(q_0) \leq \overline{d_{m+1} d_{m+2} \ldots} < \alpha\left(q_0^*\right) = \mathbf{t}^+ \overline{\mathbf{t}} \ldots. \quad (10)$$

Suppose $(d_{m+i}) \neq \mathbf{t}^\infty$ and $(d_{m+i}) \neq \overline{\mathbf{t}^\infty}$. Then there exists $u \geq m$ such that

$$d_{u+1} \ldots d_{u+p} = \mathbf{t}^+ = \omega_0 \quad \text{or} \quad d_{u+1} \ldots d_{u+p} = \overline{\mathbf{t}^+} = \overline{\omega_0}.$$

- If $d_{u+1} \ldots d_{u+p} = \omega_0 = \mathbf{t}^+$, then by (9) and Lemma 2.3 it follows that

$$d_{u+p+1} \ldots d_{u+2p} = \overline{\mathbf{t}^+} \quad \text{or} \quad d_{u+p+1} \ldots d_{u+2p} = \overline{\mathbf{t}}.$$

This implies $d_{u+1} \ldots d_{u+2p} = \mathbf{t}^+ \overline{\mathbf{t}^+} = \omega_0 \overline{\omega_0}$ or $d_{u+1} \ldots d_{u+2p} = \mathbf{t}^+ \overline{\mathbf{t}} = \omega_1$.

– If $d_{u+1}\dots d_{u+p}=\overline{\omega_0}=\overline{\mathbf{t}^+}$, then by (10) and Lemma 2.3 it follows that

$$d_{u+p+1}\dots d_{u+2p}=\mathbf{t}^+\quad\text{or}\quad d_{u+p+1}\dots d_{u+2p}=\mathbf{t}.$$

This yields that $d_{u+1}\dots d_{u+2p}=\overline{\omega_0}\,\omega_0$ or $d_{u+1}\dots d_{u+2p}=\overline{\omega_1}$.

Note that for each $n\ge 0$ the word $g^n(\mathbf{t})^+\,\overline{g^n(\mathbf{t})}$ is a prefix of $\alpha(q_0^*)$. By iteration of the above arguments, one can show that if $d_{v+1}\dots d_{v+2^n p}=\omega_n$, then $d_{v+1}\dots d_{v+2^{n+1}p}=\omega_n\overline{\omega_n}$ or ω_{n+1}. Symmetrically, if $d_{v+1}\dots d_{v+2^n p}=\overline{\omega_n}$, then $d_{v+1}\dots d_{v+2^{n+1}p}=\overline{\omega_n}\omega_n$ or $\overline{\omega_{n+1}}$.

Hence, we conclude that (d_i) must end with

$$\mathbf{t}^*(\omega_{i_0}\overline{\omega_{i_0}})^*(\omega_{i_0}\overline{\omega_{j_0}})^{s_0}(\omega_{i_1}\overline{\omega_{i_1}})^*(\omega_{i_1}\overline{\omega_{j_1}})^{s_1}\dots(\omega_{i_n}\overline{\omega_{i_n}})^*(\omega_{i_n}\overline{\omega_{j_n}})^{s_n}\dots$$

or its reflections, where $s_n\in\{0,1\}$ and

$$0=i_0<j_0\le i_1<j_1\le i_2<\dots\le i_n<j_n\le i_{n+1}<\cdots.$$

Here $*$ is an element of the set $\{0,1,2,\dots\}\cup\{\infty\}$.

Since the length of $\omega_n=g^n(\mathbf{t})^+$ grows exponentially fast as $n\to\infty$, we conclude that $\dim_H\pi_{M+1}(\mathcal{U}'_{q_0^*}\backslash\mathcal{U}'_{q_0})=0$. □

Proof of Theorem 1.1 for (i) ⇔ (ii) First we prove (i) ⇒ (ii). If $q=q_0^*$ is the right endpoint of a connected component of $(q',M+1)\backslash\overline{\mathcal{U}}$, then by Lemma 3.3 we have

$$\dim_H\pi_{M+1}(\mathcal{U}'_r\backslash\mathcal{U}'_q)>0\quad\text{for any}\quad r\in(q,M+1].$$

Clearly, it is trivial when $q=M+1$. Now we take $q\in(\mathcal{U}\backslash\{M+1\})\backslash\bigcup\{q_0^*\}$ and take $r\in(q,M+1]$. By Lemma 2.6 (i) one can find $q_0^*\in(q,r)$, and therefore by Lemma 3.3 we obtain

$$\dim_H\pi_{M+1}\left(\mathcal{U}'_r\backslash\mathcal{U}'_q\right)\ge\dim_H\pi_{M+1}\left(\mathcal{U}'_r\backslash\mathcal{U}'_{q_0^*}\right)>0.$$

Now we prove (ii) ⇒ (i). Take $q\in(1,M+1]\backslash\mathcal{U}$. We will show that $\dim_H\pi_{M+1}(\mathcal{U}'_r\backslash\mathcal{U}'_q)=0$ for some $r\in(q,M+1]$. Note that $\bigcup\{q_0\}=\overline{\mathcal{U}}\backslash\mathcal{U}$. Then by (1) it follows that

$$q\in(1,q')\cup\bigcup\left[q_0,q_0^*\right).$$

Therefore, it suffices to prove $\dim_H\pi_{M+1}(\mathcal{U}'_r\backslash\mathcal{U}'_q)=0$ for some $r\in(q,M+1]$. We distinct the following two cases.

Case (I). $q\in(1,q')$. Then for any $r\in(q,q')$ we have

$$\dim_H\pi_{M+1}(\mathcal{U}'_r\backslash\mathcal{U}'_q)\le\dim_H\pi_{M+1}(\mathcal{U}'_r)=0,$$

where the last equality follows by [21, Theorem 4.6] (see also [4,14]).

Case (II). $q \in [q_0, q_0^*)$. Then for any $r \in (q, q_0^*)$ we have by Lemma 3.4 that

$$\dim_H \pi_{M+1}\left(\mathcal{U}_r' \backslash \mathcal{U}_q'\right) \leq \dim_H \pi_{M+1}\left(\mathcal{U}_{q_0^*}' \backslash \mathcal{U}_{q_0}'\right) = 0.$$

□

3.2 Proof of Theorem 1.1 for (i) ⇔ (iii)

The following property for the Hausdorff dimension is well-known (cf. [13, Proposition 2.3]).

Lemma 3.5 *Let $f : (X, d_1) \to (Y, d_2)$ be a map between two metric spaces . If there exist constants $C > 0$ and $\lambda > 0$ such that*

$$d_2(f(x), f(y)) \leq C d_1(x, y)^\lambda$$

for any $x, y \in X$, then $\dim_H X \geq \lambda \dim_H f(X)$.

Lemma 3.6 *Let $q \in \mathcal{U} \backslash \{M+1\}$. Then for any $r \in (q, M+1)$ we have*

$$\dim_H \mathcal{U} \cap (q, r) \geq \dim_H \pi_{M+1}\left(\{\alpha(p) : p \in \mathcal{U} \cap (q, r)\}\right).$$

Proof Fix $q \in \mathcal{U} \backslash \{M+1\}$ and $r \in (q, M+1)$. Then Lemma 2.6 yields that $\mathcal{U} \cap (q, r)$ contains infinitely many elements. Take $p_1, p_2 \in \mathcal{U} \cap (q, r)$ with $p_1 < p_2$. Then by Lemma 2.1 we have $\alpha(p_1) < \alpha(p_2)$. So, there exists $n \geq 1$ such that

$$\alpha_1(p_1) \ldots \alpha_{n-1}(p_1) = \alpha_1(p_2) \ldots \alpha_{n-1}(p_2) \quad \text{and} \quad \alpha_n(p_1) < \alpha_n(p_2). \tag{11}$$

This implies

$$\begin{aligned} \pi_{M+1}(\alpha(p_2)) - \pi_{M+1}(\alpha(p_1)) &= \sum_{i=1}^{\infty} \frac{\alpha_i(p_2) - \alpha_i(p_1)}{(M+1)^i} \\ &\leq \sum_{i=n}^{\infty} \frac{M}{(M+1)^i} = (M+1)^{1-n}. \end{aligned} \tag{12}$$

Note that $r < M+1$. By Lemma 2.1 we have $\alpha(r) < \alpha(M+1) = M^\infty$. Then there exists $N \geq 1$ such that

$$\alpha_1(r) \ldots \alpha_N(r) < \underbrace{M \ldots M}_{N}.$$

Therefore, by (11) and Lemma 2.3 we obtain

$$\sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_1^i} \geq \sum_{i=1}^{\infty} \frac{\alpha_i(p_1)}{p_1^i} = 1 = \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)}{p_2^i} > \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} + \frac{1}{p_2^{n+N}}.$$

Note that p_1, p_2 are elements of $\mathcal{U}$. Then $p_2 > p_1 \geq q'$. This implies

$$\frac{1}{(M+1)^{n+N}} < \frac{1}{p_2^{n+N}} < \sum_{i=1}^{n}\left(\frac{\alpha_i(p_2)}{p_1^i} - \frac{\alpha_i(p_2)}{p_2^i}\right)$$
$$\leq \sum_{i=1}^{\infty}\left(\frac{M}{p_1^i} - \frac{M}{p_2^i}\right) = \frac{M(p_2-p_1)}{(p_1-1)(p_2-1)} \leq \frac{M(p_2-p_1)}{(q'-1)^2}.$$

Therefore, by (12) it follows that

$$\pi_{M+1}(\alpha(p_2)) - \pi_{M+1}(\alpha(p_1)) \leq (M+1)^{1-n} \leq \frac{(M+1)^{2+N}}{(q'-1)^2}(p_2-p_1).$$

Furthermore, by Lemma 2.1 it follows that $\pi_{M+1}(\alpha(p_2)) - \pi_{M+1}(\alpha(p_1)) \geq 0$. Hence, by using

$$f = \pi_{M+1} \circ \alpha : \mathcal{U} \cap (q,r) \to \pi_{M+1}(\{\alpha(p) : p \in \mathcal{U} \cap (q,r)\})$$

in Lemma 3.5 we establish the lemma. □

Lemma 3.7 *Let (q_0, q_0^*) be a connected component of $(q', M+1)\backslash\overline{\mathcal{U}}$. Then* $\dim_H \mathcal{U} \cap (q_0^*, r) > 0$ *for any* $r \in (q_0^*, M+1]$.

Proof Suppose that (q_0, q_0^*) is a connected component generated by $t_1 \ldots t_p$. Let $(\theta_i) = \alpha(q_0^*)$. For $n \geq 2$ we write $\xi_n = \theta_1 \ldots \theta_{2^n p}$, and denote by

$$\Gamma_n' := \left\{(d_i) : d_1 \ldots d_{2^{n+1}p} = \xi_{n-1}\left(\overline{\xi_{n-1}}^{+}\right)^3, \quad (d_{2^{n+1}p+i}) \in X_A^{(n)}(\overline{\xi_n})\right\}.$$

Here $X_A^{(n)}(\overline{\xi_n})$ is the follower set of $\overline{\xi_n}$ in the subshift of finite type $X_A^{(n)}$ defined in (7). Now we claim that any sequence $(d_i) \in \Gamma_n'$ satisfies

$$\overline{(d_i)} < \sigma^j((d_i)) < (d_i) \quad \text{for all} \quad j \geq 1. \tag{13}$$

Take $(d_i) \in \Gamma_n'$. Then we deduce by the definition of Γ_n' that

$$d_1 \ldots d_{2^{n+1}p+2^{n-1}p} = \theta_1 \ldots \theta_{2^{n-1}p}\left(\overline{\theta_1 \ldots \theta_{2^{n-1}p}}^{+}\right)^3 \overline{\theta_1 \ldots \theta_{2^n p}}. \tag{14}$$

We will split the proof of (13) into the following five cases.

(a) $1 \leq j < 2^{n-1}p$. By (14) and Lemma 2.5 it follows that

$$\overline{\theta_1 \ldots \theta_{2^{n-1}p-j}} < d_{j+1} \ldots d_{2^{n-1}p} = \theta_{j+1} \ldots \theta_{2^{n-1}p} \leq \theta_1 \ldots \theta_{2^{n-1}p-j},$$

and

$$d_{2^{n-1}p+1} \ldots d_{2^{n-1}p+j} = \overline{\theta_1 \ldots \theta_j} < \theta_{2^{n-1}p-j+1} \ldots \theta_{2^{n-1}p}.$$

This implies that (13) holds for all $1 \le j < 2^{n-1}p$.

(b) $2^{n-1}p \le j < 2^n p$. Let $k = j - 2^{n-1}p$. Then $0 \le k < 2^{n-1}p$. Clearly, if $k = 0$, then by using $\theta_1 > \overline{\theta_1}$ and $n \ge 2$ it yields that

$$\overline{\theta_1 \ldots \theta_{2^{n-1}p}} < d_{j+1} \ldots d_{2^n p} = \overline{\theta_1 \ldots \theta_{2^{n-1}p}}^+ < \theta_1 \ldots \theta_{2^{n-1}p}.$$

Now we assume $1 \le k < 2^{n-1}p$. Then by (14) and Lemma 2.5 it follows that

$$\overline{\theta_1 \ldots \theta_{2^{n-1}p-k}} < d_{j+1} \ldots d_{2^n p} = \overline{\theta_{k+1} \ldots \theta_{2^{n-1}p}}^+ \le \theta_1 \ldots \theta_{2^{n-1}p-k},$$

and

$$d_{2^n p+1} \ldots d_{2^n p+k} = \overline{\theta_1 \ldots \theta_k} < \theta_{2^{n-1}p-k+1} \ldots \theta_{2^{n-1}p}.$$

Therefore, (13) holds for all $2^{n-1}p \le j < 2^n p$.

(c) $2^n p \le j < 2^n p + 2^{n-1}p$. Let $k = j - 2^n p$. Then in a similar way as in Case (b) one can prove (13).

(d) $2^n p + 2^{n-1}p \le j < 2^{n+1}p$. Let $k = j - 2^n p - 2^{n-1}p$. Again by the same arguments as in Case (b) we obtain (13).

(e) $j \ge 2^{n+1}p$. Note that

$$d_1 \ldots d_{2^{n+1}p} = \theta_1 \ldots \theta_{2^{n-1}p} \left(\overline{\theta_1 \ldots \theta_{2^{n-1}p}}^+\right)^3 > \theta_1 \ldots \theta_{2^{n+1}p}.$$

Then (13) follows by Lemma 3.2.

Therefore, by (13) and Lemma 2.4 it follows that any sequence in Γ_n' corresponds to a unique base $q \in \mathcal{U}$. Furthermore, by (14) and Lemma 3.1 each sequence $(d_i) \in \Gamma_n'$ satisfies

$$\alpha\left(q_0^*\right) = (\theta_i) < (d_i) < \theta_1 \ldots \theta_{2^{n-1}p} \left(\overline{\theta_1 \ldots \theta_{2^{n-1}p}}^+\right)^\infty = \alpha(r_{n-1}).$$

Then by Lemma 2.1 it follows that

$$\alpha(q) \in \Gamma_n' \Longrightarrow q \in \mathcal{U} \cap \left(q_0^*, r_{n-1}\right).$$

Fix $r > q_0^*$. So by Lemma 3.1 there exists a sufficiently large integer $n \ge 2$ such that

$$\Gamma_n' \subset \left\{\alpha(q) : q \in \mathcal{U} \cap \left(q_0^*, r\right)\right\}. \tag{15}$$

Note by the proof of Lemma 3.3 that $X_A^{(n)}$ is an irreducible subshift of finite type over the states $\left\{\xi_n, \xi_n^-, \overline{\xi_n}, \overline{\xi_n^-}\right\}$. Hence, by (15) and Lemma 3.6 it follows that

$$\begin{aligned}\dim_H \mathcal{U} \cap \left(q_0^*, r\right) \ge \dim_H \pi_{M+1}(\Gamma_n') &= \dim_H \pi_{M+1}(X_A^{(n)}) \\ &= \frac{\log\left((1+\sqrt{5})/2\right)}{2^n p \log(M+1)} > 0.\end{aligned}$$

□

Proof of Theorem 1.1 *for (i)* $\Leftrightarrow$ *(iii)* First we prove (i) $\Rightarrow$ (iii). Excluding the trivial case $q = M+1$ we take $q \in \mathcal{U} \setminus \{M+1\}$. Suppose that $r \in (q, M+1]$. If $q = q_0^*$, then by Lemma 3.7 we have $\dim_H \mathcal{U} \cap (q, r) > 0$.

If $q \in (\mathcal{U} \setminus \{M+1\}) \setminus \bigcup \{q_0^*\}$, then by Lemma 2.6 (i) there exists $q_0^* \in (q, r)$. So, by Lemma 3.7 we have

$$\dim_H \mathcal{U} \cap (q, r) \geq \dim_H \mathcal{U} \cap \left(q_0^*, r\right) > 0.$$

Now we prove (iii) $\Rightarrow$ (i). Suppose on the contrary that $q \in (1, M+1] \setminus \mathcal{U}$. We will show that $\mathcal{U} \cap (q, r) = \emptyset$ for some $r \in (q, M+1]$. Take $q \in (1, M+1] \setminus \mathcal{U}$. By (1) it follows that

$$q \in (1, q') \cap \bigcup [q_0, q_0^*).$$

This implies that $\mathcal{U} \cap (q, r) = \emptyset$ for $r \in (q, M+1]$ sufficiently close to q. □

3.3 Proof of Theorem 1.2

Proof of Theorem 1.2 (i) $\Rightarrow$ *(ii)* Take $q \in \overline{\mathcal{U}} \setminus (\bigcup \{q_0^*\} \cup \{q'\})$ and $p \in (1, q)$. By Lemma 2.6 (ii) there exists $q_0^* \in (p, q)$. Hence, by Lemma 3.3 it follows that

$$\dim_H \pi_{M+1} \left(\mathcal{U}_q' \setminus \mathcal{U}_p'\right) \geq \dim_H \pi_{M+1} \left(\mathcal{U}_q' \setminus \mathcal{U}_{q_0^*}'\right) > 0.$$

(ii) $\Rightarrow$ (i). Suppose on the contrary that $q \notin \overline{\mathcal{U}} \setminus (\bigcup \{q_0^*\} \cup \{q'\})$. Then by (1) we have

$$q \in (1, q'] \cup \bigcup \left(q_0, q_0^*\right].$$

By using Lemma 3.4 it follows that for $p \in (1, q)$ sufficiently close to q we have $\dim_H \pi_{M+1}(\mathcal{U}_q' \setminus \mathcal{U}_p') = 0$.

(i) $\Rightarrow$ (iii). Take $q \in \overline{\mathcal{U}} \setminus (\bigcup \{q_0^*\} \cup \{q'\})$ and $p \in (1, q)$. By Lemma 2.6 (ii) there exists $q_0^* \in (p, q)$. Hence, by Lemma 3.7 it follows that

$$\dim_H \mathcal{U} \cap (p, q) \geq \dim_H \mathcal{U} \cap \left(q_0^*, q\right) > 0.$$

(iii) $\Rightarrow$ (i). Suppose $q \notin \overline{\mathcal{U}} \setminus (\bigcup \{q_0^*\} \cup \{q'\})$. Then by (1) we have $q \in (1, q'] \cup \bigcup (q_0, q_0^*]$. So, for $p \in (1, q)$ sufficiently close to q we have $\mathcal{U} \cap (p, q) = \emptyset$. □

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