# Advances in computational methods for Quantum Field Theory calculations

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# A CUTVERTEX RULE FOR SCALAR DIAGRAMS

The cutvertex rule states that

$$\Delta(\gamma_1\gamma_2) = \Delta(\gamma_1)\Delta(\gamma_2)\,. \tag{307}$$

This statement can be proven by induction. We start by proving that the statement holds true for the trivial case, where both $\gamma_1$ and $\gamma_2$ contain no subdivergences. This can be proven as follows:

$$\begin{aligned}
\Delta(\gamma_1\gamma_2) &= -K\bar{R}(\gamma_1\gamma_2)\\
&= -K\big(\gamma_1\gamma_2 + \Delta(\gamma_1)\gamma_2 + \Delta(\gamma_2)\gamma_1\big)\\
&= -K\big((\gamma_1 + \Delta(\gamma_1))(\gamma_2 + \Delta(\gamma_2)) - \Delta(\gamma_1)\Delta(\gamma_2)\big)\\
&= -K\big(R(\gamma_1)R(\gamma_2) - \Delta(\gamma_1)\Delta(\gamma_2)\big)\\
&= K\big(\Delta(\gamma_1)\Delta(\gamma_2)\big)\\
&= \Delta(\gamma_1)\Delta(\gamma_2)\;.
\end{aligned} \tag{308}$$

Now we can prove inductively that the same holds for the general case, where we assume that both $\gamma_1$ and $\gamma_2$ have subdivergences. That is, we show that

$$\Delta(G_1G_2) = \Delta(G_1)\Delta(G_2) \tag{309}$$

holds, assuming the induction hypothesis $\Delta(\gamma_1\gamma_2) = \Delta(\gamma_1)\Delta(\gamma_2)$ where $\gamma_1$ and $\gamma_2$ are subgraphs of $G_1$ and $G_2$ respectively. Let us start with the definition:

$$\begin{aligned}
\Delta(G_1G_2) &= -K\bar{R}(G_1G_2)\\
&= -K\sum_{S\in\bar{W}(G_1G_2)}\Delta(S)*G_1G_2/S\,.
\end{aligned} \tag{310}$$

We will now use the fact that we can write

$$\bar{W}(G_1G_2) = W(G_1)\times W(G_2)\setminus\{\{G_1\},\{G_2\}\} \tag{311}$$

with $\times$ denoting the Cartesian product of two sets. This in turn implies

$$\Delta(G_1G_2) = -K\Bigg[\sum_{S_1\in W(G_1)}\sum_{S_2\in W(G_2)}\Delta(S_1S_2)*G_1G_2/S_1/S_2 \;-\Delta(G_1)\Delta(G_2)\Bigg] \tag{312}$$

Assuming the induction hypothesis $\Delta(S_1S_2) = \Delta(S_1)\Delta(S_2)$ we then get

$$\begin{aligned}
\Delta(G_1G_2) &= -K\Big[R(G_1)R(G_2) - \Delta(G_1)\Delta(G_2)\Big]\\
&= \Delta(G_1)\Delta(G_2)\,.
\end{aligned} \tag{313}$$

# B

# CUTVERTEX RULE FOR TENSOR DIAGRAMS

If weakly non-overlapping (no common edges) subgraphs $\gamma_1$ and $\gamma_2$ contain contracted Lorentz indices, one has in general

$$K\big(\Delta(\gamma_1)\Delta(\gamma_2)\big) \neq \Delta(\gamma_1)\Delta(\gamma_2)\,. \tag{314}$$

This means that the proof for the factorisation of the counterterm operation $\Delta$ given in appendix A breaks down. As a result, it is rather difficult to derive a corresponding generalised "cut-vertex rule" for the case of contracted tensor subgraphs that does not result in a change of renormalisation scheme. However, when one is interested only in computing the poles of a factorised Feynman graph $G_1G_2$ via the use of the identity

$$KG = -K\,\delta RG\,, \tag{315}$$

we will show that the following cutvertex rule still holds:

$$\Delta(G_1G_2) \to \Delta(G_1)\Delta(G_2)\,. \tag{316}$$

We can actually prove this statement rather easily by noting that the $R$-operation computed with eq. (316) results in the following replacement:

$$R(G_1G_2) \to R(G_1)R(G_2)\,. \tag{317}$$

We can now write

$$\delta R(G_1G_2) = R(G_1G_2) - G_1G_2 = R(G_1)R(G_2) - G_1G_2 + \xi\,, \tag{318}$$

where $\xi$ denotes the "error" one makes by computing with eq. (316). From this it follows that

$$\xi = R(G_1G_2) - R(G_1)R(G_2)\,. \tag{319}$$

Given that $\xi$ is manifestly finite, we obtain:

$$K\xi = 0 \Rightarrow K\delta R(G_1G_2) = KR(G_1)R(G_2) - KG_1G_2\,. \tag{320}$$

This completes the proof that the poles of a factorised graph can be computed by consistently applying eq. (316), even though the UV counterterm is in a different renormalisation scheme.

B

# C

# IR SUBGRAPH SEARCH

One question that remains is how to find all IR subgraphs. Since the IR graphs could be disconnected, it is not as straightforward as for the UV. Below we describe a method to find the complete IR spinney at once.

In section 5.1 the contracted IR subgraph $\tilde{\gamma}$ was defined by contracting the remaining graph (or quotient) graph $\bar{\gamma} = G \setminus \gamma'$ to a point in $G$, i.e.,

$$\tilde{\gamma} = G/\bar{\gamma}\,. \tag{321}$$

In fact this observation generalizes further to the case of IR spinneys $S'$:

$$\tilde{S} = G/\bar{S}, \qquad \bar{S} = G \setminus S', \qquad \tilde{S} = \prod_i \tilde{\gamma}_i\,. \tag{322}$$

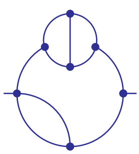
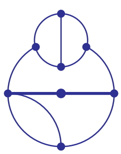
The different $\tilde{\gamma}_i$ are then only connected through cut-vertices in $\tilde{S}$. This dual description of contracted IR spinneys offers the possibility for an alternative IR search procedure by searching instead for valid remaining graphs. An easy identification of valid remaining graphs can be obtained from the contracted massless vacuum graph $G_c$ of the graph $G$ itself, which is defined by contracting in $G$ all the external lines in a single vertex and contracting all massive lines into points.

All valid remaining graphs can then be identified with all spinneys of $G_c$, which include the formerly external lines. More precisely, we have the relation:

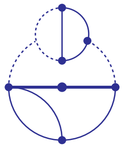
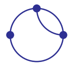
$$W'(G) = \{\tilde{S}\} = \{\, G_c/S | S \in W(G_c)\,,\, l_E(G) \subset S\,, \tilde{\omega}(G_c/S) \geq 0\,\}\,, \tag{323}$$

where $l_E(G)$ is the set of external lines of $G$. This allows one to construct a simple algorithm to find all IR spinneys by finding and combining 1PI subgraphs, similar to the construction of the UV spinney. A further advantage of this method is that disconnected IR subgraphs, such as the example we gave in eq.(124), are automatically included in this alternative search method.

It is instructive to see how this works in an example. Consider the following graph and its associated contracted vacuum graph:

$G =$  $\Rightarrow$ $G_c =$  . (324)

Here we have indicated the contracted external lines in $G_c$ with a thicker line. An example for a UV spinney in $G_c$ and its associated IR spinney (in this case consisting of a single IR subgraph) is given by

$S =$  $\quad \Rightarrow \quad \tilde{S} = G_c/S =$  . (325)

Here we used dashed lines to indicate those lines not contained in the spinney $S$. These dashed lines become the IR spinney after shrinking the disconnected components of $S$ to points in $G_c$.