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Torsional rigidity for cylinders with a Brownian fracture

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Abstract

We obtain bounds for the expected loss of torsional rigidity of a cylinder C_L of length L and planar cross-section Ω due to a Brownian fracture that starts at a random point in C_L and runs until the first time it exits C_L. These bounds are expressed in terms of the geometry of the cross-section $\Omega \subset \mathbb{R}^2$. It is shown that if Ω is a disc with radius R, then in the limit as $L \to \infty$ the expected loss of torsional rigidity equals cR^5 for some $c \in (0, \infty)$. We derive bounds for c in terms of the expected Newtonian capacity of the trace of a Brownian path that starts at the centre of a ball in $\mathbb{R}^3$ with radius 1, and runs until the first time it exits this ball.

1. *Introduction*

In Section 1.1 we formulate the problem, in Section 1.2 we recall some basic facts, in Section 1.3 we state our main theorems, and in Section 1.4 we discuss these theorems and provide an outline of the remainder of the paper.

1.1. *Background and motivation*

Let Λ be an open and bounded set in $\mathbb{R}^m$, with boundary $\partial\Lambda$ and Lebesgue measure $|\Lambda|$. Let Δ be the Laplace operator acting in $\mathcal{L}^2(\mathbb{R}^m)$. Let $(\bar\beta(s), s \geqslant 0; \bar{\mathbb{P}}_x, x \in \mathbb{R}^m)$ be Brownian motion in $\mathbb{R}^m$ with generator Δ. Denote the first exit time from Λ by

$$\bar\tau(\Lambda) = \inf\{s \geqslant 0\colon\ \bar\beta(s) \in \mathbb{R}^m - \Lambda\},$$

and the expected lifetime in Λ starting from x by

$$v_\Lambda(x) = \bar{\mathbb{E}}_x[\bar\tau(\Lambda)], \quad x \in \Lambda,$$

where $\bar{\mathbb{E}}_x$ denotes the expectation associated with $\bar{\mathbb{P}}_x$. The function v_Λ is the unique solution of the equation

$$-\Delta v = 1, \quad v \in H_0^1(\Lambda),$$

where the requirement $v \in H_0^1(\Lambda)$ imposes Dirichlet boundary conditions on $\partial\Lambda$. The function v_Λ is known as the *torsion function* and found its origin in elasticity theory (see, for example, [**17**]). The *torsional rigidity* $\mathcal{T}(\Lambda)$ of Λ is defined by

$$\mathcal{T}(\Lambda) = \int_\Lambda dx\, v_\Lambda(x).$$

Torsional rigidity plays a key role in many different parts of analysis. For example, the torsional rigidity of a cross-section of a beam appears in the computation of the angular change when a beam of a given length and a given modulus of rigidity is exposed to a twisting moment [**1**, **14**]. It also arises in the calculation of the heat content of sets with time-dependent

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boundary conditions [2], in the definition of gamma convergence [9], and in the study of minimal submanifolds [13]. Moreover, $\mathcal{T}(\Lambda)/|\Lambda|$ equals the expected lifetime of Brownian motion in Λ when averaged with respect to the uniform distribution over all starting points $x \in \Lambda$.

Consider a finite cylinder in $\mathbb{R}^3$ of the form

$$\Omega_L = (-L/2, L/2) \times \Omega,$$

where Ω is an open and bounded subset of $\mathbb{R}^2$, referred to as the cross-section. It follows from [6, Theorem 5.1] that

$$\mathcal{T}'(\Omega)L \geqslant \mathcal{T}(\Omega_L) = \mathcal{T}'(\Omega)L - 4\mathcal{H}^2(\Omega)\lambda_1'(\Omega)^{-3/2}, \tag{1.1}$$

where $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure, $\lambda_1'(\Omega)$ is the first eigenvalue of the two-dimensional Dirichlet Laplacian acting in $\mathcal{L}^2(\Omega)$, and $\mathcal{T}'(\Omega)$ is the two-dimensional torsional rigidity of the planar set Ω.

We observe that in (1.1) the leading term is extensive, that is, proportional to L, and that its coefficient $\mathcal{T}'(\Omega)$ depends on the torsional rigidity of the cross-section Ω. There is a substantial literature on the computation of the two-dimensional torsional rigidity for given planar sets Ω (see, for example, [16, 17]). The finiteness of the cylinder induces a correction that is at most $O(1)$.

Let $(\beta(s), s \geqslant 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be a Brownian motion, independent of $(\bar\beta(s), s \geqslant 0; \bar{\mathbb{P}}_x, x \in \mathbb{R}^m)$, and let

$$\tau(\Lambda) = \inf\{s \geqslant 0\colon\ \beta(s) \in \mathbb{R}^m - \Lambda\}. \tag{1.2}$$

Denote its trace in Λ up to the first exit time of Λ by

$$\mathfrak{B}(\Lambda) = \{\beta(s)\colon\ 0 \leqslant s \leqslant \tau(\Lambda)\}. \tag{1.3}$$

In this paper we investigate the effect of a Brownian fracture $\mathfrak{B}(\Omega_L)$ on the torsional rigidity of Ω_L. More specifically, we consider the random variable $\mathcal{T}(\Omega_L - \mathfrak{B}(\Omega_L))$, and we investigate the expected loss of torsional rigidity averaged over both the path $\mathfrak{B}(\Omega_L)$ and the starting point y, defined by

$$\mathfrak{T}(\Omega_L) = \frac{1}{|\Omega_L|}\int_{\Omega_L} dy\, \mathbb{E}_y\left[\mathcal{T}(\Omega_L) - \mathcal{T}(\Omega_L - \mathfrak{B}(\Omega_L))\right], \tag{1.4}$$

where $\mathbb{E}_y$ denotes the expectation associated with $\mathbb{P}_y$.

1.2. *Preliminaries*

It is well known that the rich interplay between elliptic and parabolic partial differential equations provides tools for linking various properties. See, for example, the monograph by Davies [10], and [3–6, 8] for more recent results. As both statements and proofs of Theorems 1.1, 1.2, and 1.3 rely on the connection between the torsion function, torsional rigidity, and heat content, we recall some basic facts.

For an open set Λ in $\mathbb{R}^m$ with boundary $\partial\Lambda$, we denote the Dirichlet heat kernel by $p_\Lambda(x, y; t),\ x, y \in \Lambda,\ t > 0$. The integral

$$u_\Lambda(x; t) = \int_\Lambda dy\, p_\Lambda(x, y; t), \quad x \in \Lambda,\ t > 0, \tag{1.5}$$

is the unique weak solution of the heat equation

$$\frac{\partial u}{\partial t}(x; t) = \Delta u(x; t), \quad x \in \Lambda,\ t > 0,$$

with initial condition

$$\lim_{t \downarrow 0} u(\,\cdot\,; t) = 1 \ \text{ in } \mathcal{L}^1(\Lambda),$$

and with Dirichlet boundary conditions

$$u(\,\cdot\,;t) \in H_0^1(\Lambda), \quad t > 0.$$

We denote the heat content of Λ at time t by

$$Q_\Lambda(t) = \int_\Lambda dx\, u_\Lambda(x;t) = \int_\Lambda dx \int_\Lambda dy\, p_\Lambda(x,y;t), \quad t > 0. \tag{1.6}$$

The heat content represents the amount of heat in Λ at time t when Λ has initial temperature 1 while $\partial\Lambda$ is kept at temperature 0 for all $t > 0$. Since the Dirichlet heat kernel is non-negative and is monotone in Λ, we have

$$0 \leqslant p_\Lambda(x,y;t) \leqslant p_{\mathbb{R}^m}(x,y;t) = (4\pi t)^{-m/2}\, e^{-|x-y|^2/(4t)}. \tag{1.7}$$

It follows from (1.5) and (1.7) that

$$0 \leqslant u_\Lambda(x;t) \leqslant 1, \quad x \in \Lambda,\ t > 0,$$

and that if $|\Lambda| < \infty$, then

$$0 \leqslant Q_\Lambda(t) \leqslant |\Lambda|, \quad t > 0. \tag{1.8}$$

In the latter case we also have an eigenfunction expansion for the Dirichlet heat kernel in terms of the Dirichlet eigenvalues $\lambda_1(\Lambda) \leqslant \lambda_2(\Lambda) \leqslant \cdots$, and a corresponding orthonormal set of eigenfunctions $\{\varphi_{\Lambda,1}, \varphi_{\Lambda,2}, \dots\}$, namely,

$$p_\Lambda(x,y;t) = \sum_{j=1}^\infty e^{-t\lambda_j(\Lambda)} \varphi_{\Lambda,j}(x)\varphi_{\Lambda,j}(y), \quad x,y \in \Lambda,\ t > 0.$$

We note that by [**10**, p. 63] the eigenfunctions are in $\mathcal{L}^p(\Lambda)$ for all $1 \leqslant p \leqslant \infty$. It follows from Parseval's formula that

$$\begin{aligned} Q_\Lambda(t) &= \sum_{j=1}^\infty e^{-t\lambda_j(\Lambda)} \left(\int_\Lambda dx\, \varphi_{\Lambda,j}(x)\right)^2 \leqslant e^{-t\lambda_1(\Lambda)} \sum_{j=1}^\infty \left(\int_\Lambda dx\, \varphi_{\Lambda,j}(x)\right)^2 \\ &= e^{-t\lambda_1(\Lambda)}\, |\Lambda|, \quad t > 0, \end{aligned} \tag{1.9}$$

which improves upon (1.8). Since the torsion function is given by

$$v_\Lambda(x) = \int_{[0,\infty)} dt\, u_\Lambda(x;t), \quad x \in \Lambda,$$

we have that

$$\mathcal{T}(\Lambda) = \int_{[0,\infty)} dt\, Q_\Lambda(t) = \sum_{j=1}^\infty \lambda_j(\Lambda)^{-1} \left(\int_\Omega dx\, \varphi_{\Lambda,j}(x)\right)^2. \tag{1.10}$$

1.3. *Main theorems*

To state our theorems, we introduce the following notation. Two-dimensional quantities, such as the heat content for the planar set Ω, carry a superscript $'$. The Newtonian capacity of a compact set $K \subset \mathbb{R}^3$ is denoted by $\mathrm{cap}(K)$. For $R, L > 0$ we define

$$\begin{aligned} D_R &= \{x' \in \mathbb{R}^2\colon\ |x'| < R\}, \\ C_{L,R} &= (-L/2, L/2) \times D_R, \\ C_R &= C_{R,\infty}. \end{aligned} \tag{1.11}$$

For $x \in \mathbb{R}^3$ and $r > 0$, we write $B(x;r) = \{y \in \mathbb{R}^3\colon\ |y-x| < r\}$.

THEOREM 1.1. *If $\Omega \subset \mathbb{R}^2$ is open and bounded, then*

(i)

$$0 \leqslant \mathcal{T}(\Omega_L) - \mathcal{T}'(\Omega)L + \frac{4}{\pi^{1/2}} \int_{[0,\infty)} dt\, t^{1/2} Q'_\Omega(t) \leqslant \frac{8}{L}\, \lambda'_1(\Omega)^{-1}\mathcal{T}'(\Omega), \quad L > 0, \tag{1.12}$$

(ii)

$$\mathfrak{T}(\Omega_L) \leqslant 6\lambda'_1(\Omega)^{-1/2}\mathcal{T}'(\Omega), \quad L > 0, \tag{1.13}$$

(iii)

$$\limsup_{L\to\infty} \mathfrak{T}(\Omega_L) \leqslant 4\lambda'_1(\Omega)^{-1/2}\mathcal{T}'(\Omega). \tag{1.14}$$

THEOREM 1.2. *If $\Omega = D_R$, then*

$$\lim_{L\to\infty} \mathfrak{T}(C_{L,R}) = cR^5, \quad R > 0, \tag{1.15}$$

with

$$\frac{67\,703\sqrt{79} - 582\,194}{5\,059\,848\,192}\,\kappa \leqslant c \leqslant \frac{\pi}{2j_0}, \tag{1.16}$$

where $j_0 = 2.4048\ldots$ is the first positive zero of the Bessel function J_0, and

$$\kappa = \mathbb{E}_0\left[\operatorname{cap}\left(\mathfrak{B}(B(0;1))\right)\right].$$

We obtain better estimates when the Brownian fracture starts on the axis of the cylinder $C_{L,R}$, with a uniformly distributed starting point. Let

$$\mathfrak{C}(C_{L,R}) = \frac{1}{L}\int_{(-L/2,L/2)} dy_1\, \mathbb{E}_{(y_1,0)}\left[\mathcal{T}(C_{L,R}) - \mathcal{T}\left(C_{L,R} - \mathfrak{B}(C_{L,R})\right)\right]. \tag{1.17}$$

THEOREM 1.3. *If $\Omega = D_R$, then*

$$\lim_{L\to\infty} \mathfrak{C}(C_{L,R}) = c'R^5, \quad R > 0, \tag{1.18}$$

with

$$\frac{2867\sqrt{61} - 21\,773}{303\,750}\,\kappa \leqslant c' \leqslant \frac{\pi}{4}\left(1 + \frac{1}{j_0}\right). \tag{1.19}$$

1.4. *Discussion and outline*

Theorem 1.1(i) is a refinement of (1.1), while Theorems 1.1(ii) and 1.1(iii) provide upper bounds for the expected loss of torsional rigidity. Theorem 1.2 gives a formula for the expected loss of torsional rigidity in the special case where Ω is a disc with radius R. Theorem 1.3 does the same when the fracture starts on the axis of the cylinder, with a uniformly distributed starting point.

Computing the bounds in (1.16) numerically, we find that the upper bound is 0.653 and the lower bound is approximately $0.386 \times 10^{-5}\kappa$. Since κ is bounded from above by $\operatorname{cap}(B(0;1)) = 4\pi$, the left-hand side is at most 0.485×10^{-4}. Thus, the bounds are at least 4 orders of magnitude apart. It is not clear what the correct order of c should be. The bounds for c' in Theorem 1.3 are at least 2 orders of magnitude apart.

The remainder of this paper is organised as follows. The proof of Theorem 1.1 is given in Section 2, and uses the spectral representation of the heat kernel in Section 1.2. The proofs of Theorems 1.2 and 1.3 are given in Section 4, and rely on a key proposition, stated and proved in Section 3, that provides a representation of the constants c and c'.

2. *Proof of Theorem 1.1*

Proof of Theorem 1.1(i). We use separation of variables, and write $x = (x_1, x')$, $y = (y_1, y')$, $x_1, y_1 \in \mathbb{R}$, $x', y' \in \mathbb{R}^2$. Since the heat kernel factorises, we have

$$p_{\Omega_L}(x, y; t) = p^{(1)}_{(-L/2,L/2)}(x_1, y_1; t)\, p'_\Omega(x', y'; t), \quad x, y \in \Omega_L,\ t > 0,$$

where $p^{(1)}_{(-L/2,L/2)}(x_1, y_1; t)$ is the one-dimensional Dirichlet heat kernel for the interval $(-L/2, L/2)$, and $p'_\Omega(x', y'; t)$ is the two-dimensional Dirichlet heat kernel for the planar set Ω. By integrating over Ω_L, we see that the heat content also factorises,

$$Q_{\Omega_L}(t) = Q^{(1)}_{(-L/2,L/2)}(t)\, Q'_\Omega(t), \quad t > 0, \tag{2.1}$$

where $Q^{(1)}_{(-L/2,L/2)}$ is the one-dimensional heat content for the interval $(-L/2, L/2)$, and Q'_Ω is the two-dimensional heat content for the planar set Ω. In [**6**] it was shown that

$$L - \frac{4t^{1/2}}{\pi^{1/2}} \leqslant Q^{(1)}_{(-L/2,L/2)}(t) \leqslant L - \frac{4t^{1/2}}{\pi^{1/2}} + \frac{8t}{L}, \quad t > 0. \tag{2.2}$$

Combining (1.10), (2.1), and (2.2), we have

$$\begin{aligned} \mathcal{T}(\Omega_L) &= \int_{[0,\infty)} dt\; Q_{\Omega_L}(t) \leqslant \int_{[0,\infty)} dt\; \left(L - \frac{4t^{1/2}}{\pi^{1/2}} + \frac{8t}{L}\right) Q'_\Omega(t) \\ &= L\mathcal{T}'(\Omega) - \frac{4}{\pi^{1/2}} \int_{[0,\infty)} dt\; t^{1/2} Q'_\Omega(t) + \frac{8}{L} \int_{[0,\infty)} dt\; t\, Q'_\Omega(t). \end{aligned} \tag{2.3}$$

To bound the third term in the right-hand side of (2.3), we use the identities in (1.9) and (1.10) to obtain

$$\begin{aligned} \int_{[0,\infty)} dt\; t\; Q'_\Omega(t) &= \int_{[0,\infty)} dt\; t \sum_{j=1}^{\infty} e^{-t\lambda'_j(\Omega)} \left(\int_\Omega dx\, \varphi_{\Omega,j}(x)\right)^2 = \sum_{j=1}^{\infty} \lambda'_j(\Omega)^{-2} \left(\int_\Omega dx\, \varphi_{\Omega,j}(x)\right)^2 \\ &\leqslant \lambda'_1(\Omega)^{-1} \sum_{j=1}^{\infty} \lambda'_j(\Omega)^{-1} \left(\int_\Omega dx\, \varphi_{\Omega,j}(x)\right)^2 = \lambda'_1(\Omega)^{-1} \mathcal{T}'(\Omega). \end{aligned} \tag{2.4}$$

This completes the proof of the right-hand side of (1.12). The left-hand side of (1.12) follows from (1.10), (2.1), and the first inequality in (2.2). □

Proof of Theorem 1.1(ii). Since $\Omega_L \subset \mathbb{R} \times \Omega$, we have that $v_{\Omega_L}(x_1, x') \leqslant v_{\mathbb{R}\times\Omega}(x_1, x') = v'_\Omega(x')$. Hence

$$\mathcal{T}(\Omega_L) \leqslant \int_{(-L/2,L/2)} dx_1 \int_\Omega dx'\; v'_\Omega(x') = L\mathcal{T}'(\Omega). \tag{2.5}$$

To prove the upper bound in (1.13), we recall (1.4) and combine (2.5) with a lower bound for $\mathbb{E}_y[(\mathcal{T}(\Omega_L - \mathfrak{B}(\Omega_L))]$. We observe that, for the Brownian motion defining $\mathfrak{B}(\Omega_L)$ (recall (1.2) and (1.3)) with starting point $\beta(0) = (\beta_1(0), \beta'(0))$,

$$\tau(\Omega_L) \leqslant \tau'(\Omega) = \inf\{s \geqslant 0\colon\ \beta'(s) \notin \Omega\}.$$

Hence

$$\mathfrak{B}(\Omega_L) \subset \left[\max\left\{-\frac{L}{2}, \min_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s)\right\}, \min\left\{\frac{L}{2}, \max_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s)\right\}\right] \times \Omega.$$

Therefore $\Omega_L - \mathfrak{B}(\Omega_L)$ is contained in the union of at most two cylinders with cross-section Ω and with lengths $(L/2 + \min_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s))_+$ and $(L/2 - \max_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s))_+$, respectively. For each of these cylinders we apply the lower bound in Theorem 1.1(i), taking into account that the total length of these cylinders is bounded from below by $L - (\max_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s) - \min_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s))$. This gives

$$\begin{aligned}\mathcal{T}(\Omega_L - \mathfrak{B}(\Omega_L)) \geqslant & \left(L - \left(\max_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s) - \min_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s)\right)\right)\mathcal{T}'(\Omega) \\ & - \frac{8}{\pi^{1/2}} \int_{[0,\infty)} dt\, t^{1/2} Q'_\Omega(t). \end{aligned} \tag{2.6}$$

With obvious abbreviations, by the independence of the Brownian motions B_1 and B', we have that $\mathbb{E}_{(y_1,y')} = \mathbb{E}_{y_1} \otimes \mathbb{E}_{y'}$. For the expected range of one-dimensional Brownian motion it is known that (see, for example, [**11**])

$$\mathbb{E}_{y_1}\left[\max_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s) - \min_{0\leqslant s\leqslant \tau'(\Omega)} \beta_1(s)\right] = \frac{4\tau'(\Omega)^{1/2}}{\pi^{1/2}}. \tag{2.7}$$

Furthermore,

$$\begin{aligned}\mathbb{E}_{y'}\left[\tau'(\Omega)^{1/2}\right] &= \int_{[0,\infty)} d\tau\, \tau^{1/2}\, \mathbb{P}_{y'}(\tau'(\Omega) \in d\tau) = -\int_{[0,\infty)} d\tau\, \tau^{1/2}\left(\frac{d}{d\tau}\mathbb{P}_{y'}(\tau'(\Omega) > \tau)\right) \\ &= \frac{1}{2}\int_{[0,\infty)} d\tau\, \tau^{-1/2}\, \mathbb{P}_{y'}(\tau'(\Omega) > \tau) = \frac{1}{2}\int_{[0,\infty)} d\tau\, \tau^{-1/2} \int_\Omega dz'\, p'_\Omega(y', z'; \tau).\end{aligned} \tag{2.8}$$

Therefore, by (1.6) and Tonelli's theorem,

$$\int_\Omega dy'\, \mathbb{E}_{y'}\left[\tau'(\Omega)^{1/2}\right] = \frac{1}{2}\int_{[0,\infty)} d\tau\, \tau^{-1/2} Q'_\Omega(\tau). \tag{2.9}$$

So with $|\Omega_L|/L = \mathcal{H}^2(\Omega)$,

$$\frac{1}{|\Omega_L|}\int_{\Omega_L} dy\, \mathbb{E}_y\left[\tau'(\Omega)^{1/2}\right] = \frac{1}{2\mathcal{H}^2(\Omega)}\int_{[0,\infty)} d\tau\ \tau^{-1/2} Q'_\Omega(\tau). \tag{2.10}$$

Combining (1.4), (2.5), (2.6), and (2.10), we obtain

$$\mathfrak{T}(\Omega_L) \leqslant \frac{8}{\pi^{1/2}}\int_{[0,\infty)} dt\ t^{1/2} Q'_\Omega(t) + \left(\frac{2}{\pi^{1/2}\mathcal{H}^2(\Omega)}\int_{[0,\infty)} d\tau\ \tau^{-1/2} Q'_\Omega(\tau)\right)\mathcal{T}'(\Omega). \tag{2.11}$$

The second integral in the right-hand side of (2.11) can be bounded from above using (1.9). This gives that

$$\frac{2}{\pi^{1/2}\mathcal{H}^2(\Omega)}\int_{[0,\infty)} d\tau\ \tau^{-1/2} Q'_\Omega(\tau) \leqslant \frac{2}{\pi^{1/2}}\int_{[0,\infty)} d\tau\, \tau^{-1/2}\, e^{-\tau\lambda'_1(\Omega)} = 2\lambda'_1(\Omega)^{-1/2}. \tag{2.12}$$

Via a calculation similar to the one in (2.4), we obtain that

$$\frac{8}{\pi^{1/2}}\int_{[0,\infty)} dt\ t^{1/2} Q'_\Omega(t) \leqslant 4\lambda'_1(\Omega)^{-1/2}\,\mathcal{T}'(\Omega). \tag{2.13}$$

Combining (2.11), (2.12), and (2.13), we arrive at (1.13). □

Proof of Theorem 1.1(iii). If we use the upper bound in (1.12) instead of the upper bound in (2.5), then we obtain that

$$\mathfrak{T}(\Omega_L) \leqslant 4\lambda_1'(\Omega)^{-1/2}\,\mathcal{T}'(\Omega) + 8L^{-1}\lambda_1'(\Omega)^{-1}\,\mathcal{T}'(\Omega).$$

This in turn implies (1.14). □

3. *Key proposition*

The proofs of Theorems 1.2 and 1.3 rely on the following proposition which states formulae for the constants c in (1.15) and c' in (1.18), respectively. We recall definitions (1.4), (1.11), and (1.17).

PROPOSITION 3.1. *If* $\Omega = D_R$, *then*

$$\lim_{L\to\infty} \mathfrak{T}(C_{L,R}) = cR^5, \quad \lim_{L\to\infty} \mathfrak{C}(C_{L,R}) = c'R^5, \quad R>0, \tag{3.1}$$

with

$$\begin{aligned} c &= \frac{1}{\pi}\int_{D_1} dy'\,\mathbb{E}_{(0,y')}\left[\int_{C_1} dx\ \left(v_{C_1}(x) - v_{C_1-\mathfrak{B}(C_1)}(x)\right)\right], \\ c' &= \mathbb{E}_{(0,0)}\left[\int_{C_1} dx\ \left(v_{C_1}(x) - v_{C_1-\mathfrak{B}(C_1)}(x)\right)\right]. \end{aligned} \tag{3.2}$$

Proof. The proof for $\mathfrak{T}(C_{L,R})$ comes in 10 steps.
(1) By (1.4),

$$\mathfrak{T}(C_{L,R}) = \frac{1}{\pi R^2 L}\int_{C_{L,R}} dy\ \mathbb{E}_y\left[\int_{C_{L,R}} dx\ \left(v_{C_{L,R}}(x) - v_{C_{L,R}-\mathfrak{B}(C_{L,R})}(x)\right)\right]. \tag{3.3}$$

We observe that $x \mapsto v_{C_{L,R}}(x) - v_{C_{L,R}-\mathfrak{B}(C_{L,R})}(x)$ is harmonic on $C_{L,R} - \mathfrak{B}(C_{L,R})$, is non-negative, and equals 0 for $x \in \partial C_{L,R}$. By Lemma A.1 in Appendix, $N \mapsto v_{C_{N,R}}(x) - v_{C_{N,R}-\mathfrak{B}(C_{L,R})}(x)$ is increasing on $[L,\infty)$, and bounded by $\frac{1}{4}R^2$ uniformly in x. Therefore

$$\begin{aligned} v_{C_{L,R}}(x) - v_{C_{L,R}-\mathfrak{B}(C_{L,R})}(x) &\leqslant \lim_{N\to\infty}\left(v_{C_{N,R}}(x) - v_{C_{N,R}-\mathfrak{B}(C_{L,R})}(x)\right) \\ &= \lim_{N\to\infty} v_{C_{N,R}}(x) - \lim_{N\to\infty} v_{C_{N,R}-\mathfrak{B}(C_{N,R})}(x) \\ &= v_{C_R}(x) - v_{C_R-\mathfrak{B}(C_{L,R})}(x) \\ &\leqslant v_{C_R}(x) - v_{C_R-\mathfrak{B}(C_R)}(x), \quad x \in C_{L,R} - \mathfrak{B}(C_{L,R}). \end{aligned} \tag{3.4}$$

The last inequality in (3.4) follows from the domain monotonicity of the torsion function. Inserting (3.4) into (3.3), we get

$$\mathfrak{T}(C_{L,R}) \leqslant \frac{1}{\pi R^2 L}\int_{C_{L,R}} dy \int_{C_R} dx\ \mathbb{E}_y\left[\left(v_{C_R}(x) - v_{C_R-\mathfrak{B}(C_R)}(x)\right)\right]. \tag{3.5}$$

Since $v_{C_R}(x)$ is independent of x_1, we have $v_{C_R}(x) = v_{C_R}(x-(y_1,0))$ and so

$$\mathbb{E}_y\left[v_{C_R}(x)\right] = \mathbb{E}_{(0,y')}\left[v_{C_R}(x-(y_1,0))\right]. \tag{3.6}$$

Since the stopping time $\tau(C_R - \mathfrak{B}(C_R))$ is independent of y_1, we also see that

$$\mathbb{E}_y \left[v_{C_R - \mathfrak{B}(C_R)}(x) \right] = \mathbb{E}_{(0,y')} \left[v_{C_R - \mathfrak{B}(C_R)}(x - (y_1, 0)) \right]. \tag{3.7}$$

Combining (3.5), (3.6), and (3.7), we obtain

$$\begin{aligned}
\mathfrak{T}(C_{L,R}) &\leqslant \frac{1}{\pi R^2 L} \int_{C_{L,R}} dy\, \mathbb{E}_{(0,y')} \left[\int_{C_R} dx\, \left(v_{C_R}(x - (y_1, 0)) - v_{C_R - \mathfrak{B}(C_R)}(x - (y_1, 0)) \right) \right] \\
&= \frac{1}{\pi R^2 L} \int_{C_{L,R}} dy\, \mathbb{E}_{(0,y')} \left[\int_{C_R} dx\, \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x) \right) \right] \\
&= \frac{1}{\pi R^2} \int_{D_R} dy'\, \mathbb{E}_{(0,y')} \left[\int_{C_R} dx\, \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x) \right) \right].
\end{aligned}$$

We conclude that

$$\limsup_{L \to \infty} \mathfrak{T}(C_{L,R}) \leqslant \frac{1}{\pi R^2} \int_{D_R} dy'\, \mathbb{E}_{(0,y')} \left[\int_{C_R} dx\, \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x) \right) \right].$$

Scaling each of the space variables y' and x by a factor R, we gain a factor R^5 for the respective integrals with respect to y' and x. Furthermore, scaling the torsion functions v_{C_R} and $v_{C_R - \mathfrak{B}(C_R)}$, we gain a further factor R^2. This completes the proof of the upper bound for c.

(2) To obtain the lower bound for c, we define $\tilde{L} = \{x \in \mathbb{R}^3 \colon\ x_1 = \pm L/2\}$ and

$$\tilde{C}_{L,R} = \left\{ (x_1, x') \in C_R \colon\ -\frac{L}{2} + (RL)^{1/2} < x_1 < \frac{L}{2} - (RL)^{1/2} \right\}, \quad L \geqslant 4R.$$

Then, with $\mathbf{1}$ denoting the indicator function, we have that

$$\begin{aligned}
\mathfrak{T}(C_{L,R}) &\geqslant \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\int_{C_{L,R}} dx\, \left(v_{C_{L,R}}(x) - v_{C_{L,R} - \mathfrak{B}(C_{L,R})}(x) \right) \right] \\
&\geqslant \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_{L,R}) \cap \tilde{L} = \emptyset\}} \int_{C_{L,R}} dx\, \left(v_{C_{L,R}}(x) - v_{C_{L,R} - \mathfrak{B}(C_{L,R})}(x) \right) \right] \\
&= \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_{L,R}) \cap \tilde{L} = \emptyset\}} \int_{C_{L,R}} dx\, \left(v_{C_{L,R}}(x) - v_{C_{L,R} - \mathfrak{B}(C_R)}(x) \right) \right] \\
&= \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\int_{C_{L,R}} dx\, \left(v_{C_{L,R}}(x) - v_{C_{L,R} - \mathfrak{B}(C_R)}(x) \right) \right] - A_1,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
A_1 &= \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_{L,R}) \cap \tilde{L} \neq \emptyset\}} \int_{C_{L,R}} dx\, \left(v_{C_{L,R}}(x) - v_{C_{L,R} - \mathfrak{B}(C_{L,R})}(x) \right) \right] \\
&\leqslant \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\, \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_{L,R}) \cap \tilde{L} \neq \emptyset\}} \right] \int_{C_{L,R}} dx\, v_{C_{L,R}}(x) \\
&\leqslant \frac{R^2}{8} \int_{\tilde{C}_{L,R}} dy\, \mathbb{P}_y \left(\mathfrak{B}(C_{L,R}) \cap \tilde{L} \neq \emptyset \right) \\
&\leqslant \frac{\pi R^4 L}{8} \sup_{y \in \tilde{C}_{L,R}} \mathbb{P}_y \left(\theta(\tilde{L}) \leqslant \tau(C_R) \right),
\end{aligned} \tag{3.9}$$

where

$$\theta(K) = \inf\{s \geqslant 0\colon\ \beta(s) \in K\}$$

denotes the first entrance time of K. The penultimate inequality in (3.9) uses the two bounds $\int_{C_{L,R}} dx\, v_{C_{L,R}}(x) \leqslant \int_{C_{L,R}} dx\, v_{C_R}(x) = \frac{1}{8}\pi R^4 L$ and $|\tilde{C}_{L,R}| \leqslant \pi R^2 L$.

(3) The following lemma gives a decay estimate for the supremum in the right-hand side of (3.9) and implies that $\lim_{L\to\infty} A_1 = 0$.

Lemma 3.2.

$$\sup_{y\in\tilde{C}_{L,R}} \mathbb{P}_y\left(\theta(\tilde{L}) \leqslant \tau(C_R)\right) \leqslant (j_0+1)\pi^{1/2}\, e^{-j_0 L^{1/2}/(2R^{1/2})}, \quad L \geqslant 4R. \tag{3.10}$$

Proof. First observe that the distance of y to $\tilde{L}$ is bounded from below by $(LR)^{1/2}$. Therefore

$$\mathbb{P}_y\left(\theta(\tilde{L}) \leqslant \tau(C_R)\right) \leqslant \mathbb{P}_{(0,y')}\left(\max_{0\leqslant s\leqslant\tau'(C_R)} |\beta_1(s)| \geqslant (LR)^{1/2}\right). \tag{3.11}$$

By [**7**, (6.3), Corollary 6.4],

$$\mathbb{P}_0^{(1)}\left(\max_{0\leqslant s\leqslant t} |\beta_1(s)| \geqslant R\right) \leqslant 2^{3/2} e^{-R^2/(8t)}. \tag{3.12}$$

Combining (3.11) and (3.12) with the independence of β_1 and β', we obtain via an integration by parts,

$$\begin{aligned}\mathbb{P}_y\left(\theta(\tilde{L}) \leqslant \tau(C_R)\right) &\leqslant 2^{3/2}\int_{[0,\infty)} d\tau\, \left(\frac{\partial}{\partial\tau}\mathbb{P}_{y'}\left(\tau'(D_R) > \tau\right)\right)\, e^{-LR/(8\tau)}\\ &= \frac{LR}{2^{3/2}}\int_{[0,\infty)} \frac{d\tau}{\tau^2}\,\mathbb{P}_{y'}\left(\tau'(D_R) > \tau\right)\, e^{-LR/(8\tau)}.\end{aligned} \tag{3.13}$$

By the Cauchy–Schwarz inequality, the semigroup property of the heat kernel, the eigenfunction expansion of the heat kernel, and the domain monotonicity of the heat kernel, we have that

$$\begin{aligned}\mathbb{P}_{y'}\left(\tau'(D_R) > \tau\right) &= \int_{D_R} dz'\, p'_{D_R}(z', y'; \tau)\\ &\leqslant (\pi R^2)^{1/2}\left(\int_{D_R} dz'\, (p'_{D_R}(z', y'; \tau))^2\right)^{1/2}\\ &= (\pi R^2)^{1/2}\left(p'_{D_R}(y', y'; 2\tau)\right)^{1/2}\\ &= (\pi R^2)^{1/2}\left(\sum_{j=1}^{\infty} e^{-2\tau\lambda'_j(D_R)}\left(\varphi'_{D_R,j}(y')\right)^2\right)^{1/2}\\ &\leqslant (\pi R^2)^{1/2} e^{-\tau\lambda'_1(D_R)/2}\left(\sum_{j=1}^{\infty} e^{-\tau\lambda'_j(D_R)}\left(\varphi'_{D_R,j}(y')\right)^2\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&= (\pi R^2)^{1/2} e^{-\tau\lambda_1'(D_R)/2} \big(p'_{D_R}(y',y';\tau)\big)^{1/2}\\
&\leqslant (\pi R^2)^{1/2} e^{-\tau\lambda_1'(D_R)/2} \big(p'_{\mathbb{R}^2}(y',y';\tau)\big)^{1/2}\\
&= \frac{Re^{-j_0^2\tau/(2R^2)}}{(4\tau)^{1/2}}.
\end{aligned}
\tag{3.14}
$$

Combining (3.13) and (3.14), and changing variables twice, we arrive at

$$
\begin{aligned}
\mathbb{P}_y\left(\theta(\tilde{L}) \leqslant \tau(C_R)\right) &\leqslant \frac{LR^2}{2^{5/2}} \int_{[0,\infty)} \frac{d\tau}{\tau^{5/2}}\, e^{-j_0^2\tau/(2R^2) - LR/(8\tau)}\\
&= \frac{j_0^{3/2} L^{1/4}}{2R^{1/4}} \int_{[0,\infty)} \frac{d\tau}{\tau^{5/2}}\, e^{-j_0 L^{1/2}(\tau+\tau^{-1})/(4R^{1/2})}\\
&= \frac{j_0^{3/2} L^{1/4}}{R^{1/4}} \int_{[0,\infty)} \frac{d\tau}{\tau^{4}}\, e^{-j_0 L^{1/2}(\tau^2+\tau^{-2})/(4R^{1/2})}\\
&= \pi^{1/2} j_0 \left(1 + \frac{2R^{1/2}}{j_0 L^{1/2}}\right) e^{-j_0 L^{1/2}/(2R^{1/2})}.
\end{aligned}
\tag{3.15}
$$

The last equality follows from [**12**, 3.472.4]. This proves (3.10) because $L \geqslant 4R$. □

(4) We write the double integral in the right-hand side of (3.8) as $B_1 + B_2$, where

$$
B_1 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_R)\cap \hat{L} = \emptyset\}} \int_{C_{L,R}} dx\ \left(v_{C_{L,R}}(x) - v_{C_{L,R}-\mathfrak{B}(C_R)}(x)\right)\right], \tag{3.16}
$$

$$
B_2 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_{L,R})\cap \hat{L} \neq \emptyset\}} \int_{C_{L,R}} dx\ \left(v_{C_{L,R}}(x) - v_{C_{L,R}-\mathfrak{B}(C_R)}(x)\right)\right],
$$

with

$$
\hat{L} = \pm\frac{L}{2} \mp \frac{(RL)^{1/2}}{2}.
$$

We have that

$$
\begin{aligned}
B_2 &\leqslant \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\, \mathbb{P}_y\left(\mathfrak{B}(C_R)\cap \hat{L} \neq \emptyset\right) \int_{C_{L,R}} dx\, v_{C_R}(x)\\
&\leqslant \frac{\pi R^4 L}{8} \sup_{y\in \tilde{C}_{L,R}} \mathbb{P}_y\left(\tau(\hat{L}) \leqslant \tau(C_R)\right).
\end{aligned}
\tag{3.17}
$$

The distance from any $y \in \tilde{C}_{L,R}$ to $\hat{L}$ is bounded from below by $(RL)^{1/2}/8$. Following the argument leading from (3.13) to (3.15) with $(RL/4)^{1/2}$ replacing $(RL)^{1/2}$, we find that

$$
\mathbb{P}_y\left(\tau(\hat{L}) \leqslant \tau(C_R)\right) \leqslant \pi^{1/2} j_0 \left(1 + \frac{4R^{1/2}}{j_0 L^{1/2}}\right) e^{-j_0 L^{1/2}/(4R^{1/2})}. \tag{3.18}
$$

This, together with (3.17), shows that $\lim_{L\to\infty} B_2 = 0$. It remains to obtain the asymptotic behaviour of B_1.

(5) We write $B_1 = B_3 + B_4 + B_5$, where

$$
B_3 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\ \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_R)\cap \hat{L} = \emptyset\}} \int_{C_{L,R}} dx\ \left(v_{C_R}(x) - v_{C_R-\mathfrak{B}(C_R)}(x)\right)\right],
$$

$$B_4 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_R)\cap \hat{L}=\emptyset\}} \int_{C_{L,R}} dx \; \left(v_{C_{L,R}}(x) - v_{C_R}(x) \right) \right],$$

$$B_5 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_R)\cap \hat{L}=\emptyset\}} \int_{C_{L,R}} dx \; \left(v_{C_R - \mathfrak{B}(C_R)}(x) - v_{C_{L,R}-\mathfrak{B}(C_R)}(x) \right) \right]. \quad (3.19)$$

We have that

$$\begin{aligned} B_4 &= \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{P}_y \left(\{\mathfrak{B}(C_R) \cap \hat{L} = \emptyset\} \right) \int_{C_{L,R}} dx \; \left(v_{C_{L,R}}(x) - v_{C_R}(x) \right) \\ &\geqslant \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy \; \left(\mathcal{T}(C_{L,R}) - L\mathcal{T}'(D_R) \right) \\ &\geqslant -\frac{4}{\pi^{1/2}} \int_{[0,\infty)} dt \; t^{1/2} Q'_{D_R}(t), \end{aligned} \quad (3.20)$$

where we have used the lower bound in (1.12) for $\Omega = D_R$. Furthermore,

$$B_3 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{E}_y \left[\int_{C_R} dx \; \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x) \right) \right] - A_2 - A_3, \quad (3.21)$$

where

$$A_2 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_R)\cap \hat{L}=\emptyset\}} \int_{C_R - C_{L,R}} dx \; \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x) \right) \right],$$

$$A_3 = \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{E}_y \left[\mathbf{1}_{\{\mathfrak{B}(C_R)\cap \hat{L}\neq\emptyset\}} \int_{C_R} dx \; \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x) \right) \right]. \quad (3.22)$$

(6) To bound A_2 we note that $x \mapsto v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x)$ is harmonic on $C_R - \mathfrak{B}(C_R)$, equals 0 for $x \in \partial C_R$, and equals $\frac{1}{4}(R^2 - |x'|^2)$ for $x \in \mathfrak{B}(C_R)$. Therefore

$$v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x) \leqslant \frac{R^2}{4} \bar{\mathbb{P}}_x \left(\bar{\tau}(\mathfrak{B}(C_R)) \leqslant \bar{\tau}(C_R) \right).$$

On the set $\{\mathfrak{B}(C_R) \cap \hat{L} = \emptyset\}$ we have that $\bar{\tau}(\hat{L}) \leqslant \bar{\tau}(\mathfrak{B}(C_R))$. Hence

$$\begin{aligned} A_2 &\leqslant \frac{1}{4\pi L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{E}_y \left[1_{\{\mathfrak{B}(C_R)\cap \hat{L}=\emptyset\}} \int_{C_R - C_{L,R}} dx \, \bar{\mathbb{P}}_x \left(\bar{\tau}(\hat{L}) \leqslant \bar{\tau}(C_R) \right) \right] \\ &\leqslant \frac{1}{4\pi L} \int_{\tilde{C}_{L,R}} dy \; \mathbb{E}_y \left[\int_{C_R - C_{L,R}} dx \, \bar{\mathbb{P}}_x \left(\bar{\tau}(\hat{L}) \leqslant \bar{\tau}(C_R) \right) \right] \\ &= \frac{R^2}{4} \left(1 - \frac{2R^{1/2}}{L^{1/2}} \right) \int_{C_R - C_{L,R}} dx \, \bar{\mathbb{P}}_x \left(\bar{\tau}(\hat{L}) \leqslant \bar{\tau}(C_R) \right). \end{aligned} \quad (3.23)$$

Recall that $\bar{\tau}(\hat{L})$ equals the first hitting time of $\hat{L}$ by $\bar{\beta}_1$, and that $\bar{\tau}(C_R)$ is the first exit time of D_R by $\bar{\beta}'$. Furthermore, for $x \in C_R - C_{L,R}$ the distance from x to $\hat{L}$ is equal to $(RL/4)^{1/2} + x_1$. By (3.14),

$$\bar{\mathbb{P}}_{x'} \left(\bar{\tau}'(D_R) > \tau \right) \leqslant \frac{R \, e^{-j_0^2 \tau/(2R^2)}}{(4\tau)^{1/2}}.$$

It is well known that

$$\bar{\mathbb{P}}_0^{(1)}\left(\max_{0\leqslant s\leqslant\tau}\bar{\beta}_1(s)>R\right)=(\pi\tau)^{-1/2}\int_{[R,\infty)}d\xi\,e^{-\xi^2/(4\tau)}\leqslant 2^{1/2}\,e^{-R^2/(8\tau)}.$$

Hence

$$\bar{\mathbb{P}}_0^{(1)}\left(\max_{0\leqslant s\leqslant\tau}\bar{\beta}_1(s)>(RL/4)^{1/2}+x_1\right)\leqslant 2^{1/2}e^{-(RL+4x_1^2)/(32\tau)}.$$

By the independence of $\bar{\beta}_1$ and $\bar{\beta}'$ we have, similarly to (3.13),

$$\begin{aligned}
\bar{\mathbb{P}}_x\left(\bar{\tau}(\hat{L})\leqslant\bar{\tau}(C_R)\right)&\leqslant 2^{1/2}\int_{[0,\infty)}d\tau\,\left(\frac{\partial}{\partial\tau}\bar{\mathbb{P}}_{x'}\left(\bar{\tau}'(D_R)>\tau\right)\right)e^{-(RL+4x_1^2)/(32\tau)}\\
&\leqslant\frac{R(RL+4x_1^2)}{2^{11/2}}\int_{[0,\infty)}\frac{d\tau}{\tau^{5/2}}\,e^{-j_0^2\tau/(2R^2)-(RL+4x_1^2)/(32\tau)}\\
&=\frac{R(RL+4x_1^2)}{2^{11/2}}\int_{[0,\infty)}d\tau\,\tau^{1/2}\,e^{-j_0^2/(2R^2\tau)-(RL+4x_1^2)\tau/32}\\
&=\frac{R(RL+4x_1^2)}{2^{9/2}}\int_{[0,\infty)}d\tau\,\tau^{2}\,e^{-j_0^2/(2R^2\tau^2)-(RL+4x_1^2)\tau^2/32}\\
&=2\pi^{1/2}R(RL+4x_1^2)^{-1/2}\left(1+\frac{j_0(RL+4x_1^2)^{1/2}}{4R}\right)e^{-j_0(RL+4x_1^2)^{1/2}/(4R)}\\
&\leqslant 2\pi^{1/2}\left(\frac{R^{1/2}}{L^{1/2}}+\frac{j_0}{4}\right)e^{-(j_0^2L/(32R))^{1/2}-(j_0x_1^2/(32R^2))^{1/2}},
\end{aligned}$$

where we have used [**12**, 3.472.2]. Integration of the above over $x\in C_R-C_{L,R}$, together with (3.23), gives

$$A_2=O\left(e^{-(L/(6R))^{1/2}}\right),\quad L\to\infty. \tag{3.24}$$

(7) To bound A_3 in (3.22), we use the Cauchy–Schwarz inequality to estimate

$$A_3\leqslant\frac{1}{\pi R^2L}\int_{\tilde{C}_{L,R}}dy\,\left(\mathbb{P}_y\left(\theta(\hat{L})\leqslant\tau(C_R)\right)\right)^{1/2}\left(\mathbb{E}_y\left[\int_{C_R}dx\,\left(v_{C_R}(x)-v_{C_R-\mathfrak{B}(C_R)}(x)\right)\right]^2\right)^{1/2}. \tag{3.25}$$

The probability in (3.25) decays subexponentially fast in $(L/R)^{1/2}$ by (3.18). Hence it remains to show that the expectation in (3.25) is finite. Define

$$\hat{\mathfrak{B}}(C_R)=\left\{x\in C_R\colon\min_{0\leqslant s\leqslant\tau(C_R)}\beta_1(s)<x_1<\max_{0\leqslant s\leqslant\tau(C_R)}\beta_1(s)\right\}.$$

Then $\mathfrak{B}(C_R)\subset\hat{\mathfrak{B}}(C_R)$, and

$$\mathbb{E}_y\left[\int_{C_R}dx\,\left(v_{C_R}(x)-v_{C_R-\mathfrak{B}(C_R)}(x)\right)\right]^2\leqslant\mathbb{E}_y\left[\int_{C_R}dx\,\left(v_{C_R}(x)-v_{C_R-\hat{\mathfrak{B}}(C_R)}(x)\right)\right]^2.$$

For $x\in\hat{\mathfrak{B}}(C_R)$ we have $v_{C_R}(x)\leqslant R^2/4$ and $v_{C_R-\hat{\mathfrak{B}}(C_R)}(x)=0$. Furthermore,

$$v_{C_R}(x)-v_{C_R-\mathfrak{B}(C_R)}(x)\leqslant\frac{R^2}{4}\bar{\mathbb{P}}_x\left(\bar{\tau}(\hat{\mathfrak{B}}(C_R))\leqslant\bar{\tau}(C_R)\right),\quad x\in C_R-\hat{\mathfrak{B}}(C_R),$$

and hence

$$\mathbb{E}_y\left[\int_{C_R} dx\,\left(v_{C_R}(x)-v_{C_R-\hat{\mathfrak{B}}(C_R)}(x)\right)\right]^2$$

$$\leqslant \frac{R^4}{8}\mathbb{E}_y\left[|\hat{\mathfrak{B}}(C_R)|^2+\left(\int_{C_R-\hat{\mathfrak{B}}(C_R)} dx\,\bar{\mathbb{P}}_x\left(\bar{\tau}(\hat{\mathfrak{B}}(C_R))\leqslant\bar{\tau}(C_R)\right)\right)^2\right]. \tag{3.26}$$

The probability distribution of the range of one-dimensional Brownian motion is known (see, for example, [**11**, equation (19)]). This gives

$$\mathbb{E}_{y'}\left[\max_{0\leqslant s\leqslant\tau'(D_R)}\beta_1(s)-\min_{0\leqslant s\leqslant\tau'(D_R)}\beta_1(s)\right]^2=\frac{64\log 2}{\pi^{1/2}}\,\tau'(D_R). \tag{3.27}$$

By a calculation similar to (2.8) and (2.9), we see that

$$\begin{aligned}\mathbb{E}_{y'}\left[\max_{0\leqslant s\leqslant\tau'(D_R)}\beta_1(s)-\min_{0\leqslant s\leqslant\tau'(D_R)}\beta_1(s)\right]^2&=\frac{64\log 2}{\pi^{1/2}}\int_{[0,\infty)}d\tau\int_{D_R}dz'\,p'_{D_R}(y',z';\tau)\\&=\frac{64\log 2}{\pi^{1/2}}\,v'_{D_R}(y')\leqslant\frac{16\log 2}{\pi^{1/2}}R^2.\end{aligned}$$

Together with (3.27), this yields

$$\mathbb{E}_y\left(|\hat{\mathfrak{B}}(C_R)|^2\right)\leqslant 16\pi^{3/2}(\log 2)R^6,$$

which gives us control over the first term in the right-hand side of (3.26). To estimate the second term in the right-hand side of (3.26), we note that the set $C_R-\hat{\mathfrak{B}}(C_R)$ consists of two semi-infinite cylinders. It is instructive to calculate this term explicitly. To simplify notation, we define $C_R^+=\{x\in\mathbb{R}^3\colon\ x_1>0,|x'|<R\}$, $Z_R=\{x\in\mathbb{R}^3\colon\ x_1=0,|x'|\leqslant R\}$, and $\vartheta(Z_R)=\inf\{s\geqslant 0\colon\,\bar{\beta}(s)\in Z_R\}$. Then, by separation of variables and integration by parts, we get

$$\begin{aligned}\bar{\mathbb{P}}_x\left(\vartheta(Z_R)\leqslant\bar{\tau}(C_R^+)\right)&=\int_{[0,\infty)}\bar{\mathbb{P}}_{x'}\left(\bar{\tau}'(D_R)\in d\tau\right)\bar{\mathbb{P}}_{x_1}\left(\vartheta(Z_R)\leqslant\tau\right)\\&=\int_{[0,\infty)}\bar{\mathbb{P}}_{x'}\left(\bar{\tau}'(D_R)\in d\tau\right)\,\frac{2}{\pi^{1/2}}\int_{[x_1/(2\tau^{1/2}),\infty)}d\xi\,e^{-\xi^2}\\&=\int_{[0,\infty)}d\tau\,\bar{\mathbb{P}}_{x'}\left(\bar{\tau}'(D_R)>\tau\right)\frac{2x_1}{\pi\tau^{3/2}}\,e^{-x_1^2/(4\tau)}.\end{aligned} \tag{3.28}$$

Integrating (3.28) with respect to $x_1\in\mathbb{R}^+$, we find that

$$\int_{\mathbb{R}^+}dx_1\bar{\mathbb{P}}_x\left(\vartheta(Z_R)\leqslant\bar{\tau}(C_R^+)\right)=\frac{4}{\pi^{1/2}}\int_{[0,\infty)}d\tau\,\tau^{-1/2}\,\bar{\mathbb{P}}_{x'}\left(\bar{\tau}'(D_R)>\tau\right). \tag{3.29}$$

Subsequently integrating both sides of (3.29) over $x'\in D_R$, we get

$$\int_{C_R^+}dx\,\bar{\mathbb{P}}_x\left(\vartheta(Z_R)\leqslant\bar{\tau}(C_R^+)\right)=\frac{4}{\pi^{1/2}}\int_{[0,\infty)}d\tau\,\tau^{-1/2}Q'_{D_R}(\tau).$$

It follows that

$$\left(\int_{C_R-\hat{\mathfrak{B}}(C_R)}dx\,\bar{\mathbb{P}}_x\left(\bar{\tau}(\hat{\mathfrak{B}}(C_R))\leqslant\bar{\tau}(C_R)\right)\right)^2=\frac{64}{\pi}\left(\int_{[0,\infty)}d\tau\,\tau^{-1/2}Q'_{D_R}(\tau)\right)^2. \tag{3.30}$$

The integral over τ in (3.30) is finite by (2.12). We conclude that, by (3.18),

$$\begin{aligned} A_3 &\leqslant \left(\mathbb{P}_y\left(\theta(\hat{L}) \leqslant \tau(C_R)\right)\right)^{1/2} \left(2\pi^{3/2}(\log 2)R^{10} + \frac{8}{\pi}R^4\left(\int_{[0,\infty)} d\tau\, \tau^{-1/2}Q'_{D_R}(\tau)\right)^2\right)^{1/2} \\ &= O\left(e^{-j_0 L^{1/2}/(4R^{1/2})}\right), \quad L \to \infty. \end{aligned} \tag{3.31}$$

(8) The integrand in (3.21) is independent of y_1. Since $\lim_{L\to\infty}(L - 2(RL)^{1/2})/L = 1$, we have by (3.21), (3.24), and (3.31) that

$$\liminf_{L\to\infty} B_3 \geqslant \frac{1}{\pi R^2}\int_{D_R} dy'\, \mathbb{E}_{(0,y')}\left[\int_{C_R} dx\, \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x)\right)\right]. \tag{3.32}$$

(9) It remains to obtain a lower bound on B_5 in (3.19) as $L \to \infty$. The integrand with respect to x is a non-negative harmonic function, which can be bounded from below by enlarging the set $\mathfrak{B}(C_R)$ to $\hat{C}_{R,L} := \{D_R \times [-\frac{L}{2} + \frac{1}{2}(RL)^{1/2}, \frac{L}{2} - \frac{1}{2}(RL)^{1/2}]\}$. Hence

$$\begin{aligned} B_5 &\geqslant \frac{1}{\pi R^2 L}\int_{\tilde{C}_{L,R}} dy\, \mathbb{E}_y\left[\mathbf{1}_{\{\mathfrak{B}(C_R)\cap\hat{L}=\emptyset\}}\int_{C_{L,R}} dx\, \left(v_{C_R-\hat{C}_{R,L}}(x) - v_{C_{L,R}-\hat{C}_{R,L}}(x)\right)\right] \\ &= \frac{1}{\pi R^2 L}\int_{\tilde{C}_{L,R}} dy\, \mathbb{P}_y\left(\mathfrak{B}(C_R)\cap\hat{L}=\emptyset\right)\int_{C_{L,R}-\hat{C}_{R,L}} dx\, \left(v_{C_R-\hat{C}_{R,L}}(x) - v_{C_{L,R}-\hat{C}_{R,L}}(x)\right). \end{aligned} \tag{3.33}$$

The set $C_{L,R} - \hat{C}_{R,L}$ consists of two cylinders with cross-section D_R and length $(RL)^{1/2}/2$ each. Hence, by Theorem 1.1(i), we have

$$\int_{C_{L,R}-\hat{C}_{R,L}} dx\, v_{C_{L,R}-\hat{C}_{R,L}}(x) = \mathcal{T}'(D_R)(RL)^{1/2} - \frac{8}{\pi^{1/2}}\int_{[0,\infty)} dt\, t^{1/2}Q'_{D_R}(t) + O(L^{-1/2}). \tag{3.34}$$

The set $C_R - \hat{C}_{R,L}$ consists of two semi-infinite cylinders, and we integrate the torsion function for that set over two cylinders of length $(RL)^{1/2}/2$, each near their base. Adopting previous notation, we get

$$\begin{aligned} &\int_{C_{L,R}-\hat{C}_{R,L}} dx\, v_{C_R-\hat{C}_{R,L}}(x) = 2\int_{[0,(RL)^{1/2}/2)} dx_1 \int_{D_R} dx'\, v_{C_R^+}(x) \\ &= 2\int_{[0,\infty)} dt \int_{[0,(RL)^{1/2}/2)} dx_1 \int_{D_R} dx' \int_{[0,\infty)} dx_1 \int_{D_R} dy' \int_{[0,\infty)} dy_1\, p'_{D_R}(x',y';t)p_{\mathbb{R}^+}(x_1,y_1;t) \\ &= 2\int_{[0,\infty)} dt \int_{[0,(RL)^{1/2}/2)} dx_1\, u_{\mathbb{R}^+}(x_1;t)Q'_{D_R}(t) \\ &= 2\int_{[0,\infty)} dt \int_{[0,(RL)^{1/2}/2)} dx_1 \left(1 - \frac{2}{\pi^{1/2}}\int_{[x_1/(4t)^{1/2},\infty)} d\xi\, e^{-\xi^2}\right)Q'_{D_R}(t) \\ &= \mathcal{T}'(D_R)(RL)^{1/2} - \frac{4}{\pi^{1/2}}\int_{[0,\infty)} dt\, Q'_{D_R}(t)\int_{[0,(RL)^{1/2}/2)} dx_1 \int_{[x_1/(4t)^{1/2},\infty)} d\xi\, e^{-\xi^2} \end{aligned}$$

$$\geqslant \mathcal{T}'(D_R)(RL)^{1/2} - \frac{4}{\pi^{1/2}} \int_{[0,\infty)} dt\, Q'_{D_R}(t) \int_{[0,\infty)} dx_1 \int_{[x_1/(4t)^{1/2},\infty)} d\xi\, e^{-\xi^2}$$

$$= \mathcal{T}'(D_R)(RL)^{1/2} - \frac{4}{\pi^{1/2}} \int_{[0,\infty)} dt\, t^{1/2}\, Q'_{D_R}(t). \tag{3.35}$$

Combining (3.33), (3.34), and (3.35), we arrive at

$$B_5 \geqslant \frac{1}{\pi R^2 L} \int_{\tilde{C}_{L,R}} dy\, \mathbb{P}_y \left(\mathfrak{B}(C_R) \cap \hat{L} = \emptyset\right) \left(\frac{4}{\pi^{1/2}} \int_{[0,\infty)} dt\, t^{1/2}\, Q'_{D_R}(t) + O(L^{-1/2})\right).$$

We conclude that

$$\liminf_{L\to\infty} B_5 \geqslant \frac{4}{\pi^{1/2}} \int_{[0,\infty)} dt\, t^{1/2}\, Q'_{D_R}(t). \tag{3.36}$$

(10) From (3.20), (3.32), and (3.36), we get

$$\liminf_{L\to\infty} (B_3 + B_4 + B_5) \geqslant \frac{1}{\pi R^2} \int_{D_R} dy'\, \mathbb{E}_{(0,y')} \left[\int_{C_R} dx\, \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x)\right)\right].$$

Scaling each the space variables y' and x by a factor R, we gain a factor R^5 for the respective integrals with respect to y' and x. Furthermore, scaling the torsion functions v_{C_R} and $v_{C_R-\mathfrak{B}(C_R)}$, we gain a further factor R^2. Hence

$$\frac{1}{\pi R^2} \int_{D_R} dy'\, \mathbb{E}_{(0,y')} \left[\int_{C_R} dx\, \left(v_{C_R}(x) - v_{C_R - \mathfrak{B}(C_R)}(x)\right)\right]$$
$$= \frac{1}{\pi} R^5 \int_{D_1} dy'\, \mathbb{E}_{(0,y')} \left[\int_{C_1} dx\, \left(v_{C_1}(x) - v_{C_1 - \mathfrak{B}(C_1)}(x)\right)\right],$$

which is the required first formula in (3.1). □

The main modification for the proof for $\mathfrak{C}(C_{L,R})$ in the second formula of (3.1) is that no averaging takes place over the cross-section D_R as $y' = 0$ is fixed. Hence the absence of the factor $\frac{1}{\pi}$ and the integral with respect to y' over D_1 in the formula for c' in (3.2).

4. *Proofs of Theorems* 1.2 *and* 1.3

The proofs of Theorems 1.2 and 1.3 are given in Section 4.1 and 4.2, respectively, and rely on Proposition 3.1.

4.1. *Proof of Theorem* 1.2

To prove the upper bound we note that $\lambda_1'(D_R) = j_0^2/R^2$ and $\mathcal{T}'(D_R) = \pi R^4/8$ (see [6]). This gives the upper bound $\pi R^5/2j_0$ for the right-hand side of (1.15), which implies the upper bound for c in (1.16).

To prove the lower bound we start from (3.2). Let $a \in (0, \frac{1}{4})$. We have the following estimate:

$$c = \frac{1}{\pi} \int_{D_1} dy'\, \mathbb{E}_{(0,y')} \left[\int_{C_1} dx\, \left(v_{C_1}(x) - v_{C_1 - \mathfrak{B}(C_1)}(x)\right)\right]$$
$$\geqslant \frac{1}{\pi} \int_{D_a} dy'\, \mathbb{E}_{(0,y')} \left[\int_{C_1} dx\, \left(v_{C_1}(x) - v_{C_1 - \mathfrak{B}(C_1)}(x)\right)\right]$$
$$\geqslant \frac{1}{\pi} \int_{D_a} dy'\, \mathbb{E}_{(0,y')} \left[\int_{\{x\in\mathbb{R}^3 :\ |x-\beta(0)|<a\}} dx\, \left(v_{C_1}(x) - v_{C_1 - \mathfrak{B}(B(\beta(0);a))}(x)\right)\right], \tag{4.1}$$

where we have used that $\mathfrak{B}(C_1) \supset \mathfrak{B}(B((0,y');a))$. To estimate the second integral, we consider a fixed compact set $K \subset B((0,y');a) \subset \mathbb{R}^3$ and derive a lower bound for $v_{C_1}(x) - v_{C_1-K}(x)$ uniformly in $|y'| \leqslant a$ and $|x-(0,y')| \leqslant a$.

First note that $x \mapsto v_{C_1}(x) - v_{C_1-K}(x)$ is harmonic on $C_1 - K$, equals 0 for $x \in \partial C_1$, and equals $\frac{1}{4}(1-|x'|^2)$ for $x \in K$. If $|y'| < a$, then $|x'| < 2a$, $x \in K$. Hence $v_{C_1}(x) - v_{C_1-K}(x) \geqslant \frac{1}{4}(1-4a^2)$ for $x \in K$. We therefore have

$$v_{C_1}(x) - v_{C_1-K}(x) \geqslant \frac{1-4a^2}{4}\, \bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \bar\tau(C_1)\right), \quad x \in C_1. \tag{4.2}$$

By the strong Markov property, we have

$$\begin{aligned}\bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \bar\tau(C_1)\right) &= \bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \infty\right) - \bar{\mathbb{P}}_x\left(\bar\tau(C_1) \leqslant \bar\tau_{\mathbb{R}^3-K} < \infty\right)\\ &\geqslant \inf_{\{|x-(0,y')|<a\}} \bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \infty\right) - \sup_{x\in\partial C_1} \bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \infty\right).\end{aligned} \tag{4.3}$$

Let μ_K denote the equilibrium measure for K. Then (see [**15**])

$$\bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \infty\right) = \int_K \mu_K(dz)\, \frac{1}{4\pi|x-z|}, \quad x \in K. \tag{4.4}$$

If $z \in K$ and $|x-(0,y')| \leqslant a$, then $|x-z| \leqslant 2a$. Hence (4.4) gives

$$\inf_{\{|x-(0,y')|<a\}} \bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \infty\right) \geqslant \frac{1}{8\pi a}\int_K \mu_K(dz) = \frac{1}{8\pi a}\,\mathrm{cap}(K). \tag{4.5}$$

Furthermore, if $x \in \partial C_1$, $z \in K$ and $|y'| \leqslant a$, then $|z-x| \geqslant 1-2a$. Hence (4.4) also gives

$$\sup_{x\in\partial C_1} \bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \infty\right) \leqslant \frac{1}{4\pi(1-2a)}\,\mathrm{cap}(K). \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\begin{aligned}&\bar{\mathbb{P}}_x\left(\bar\tau_{\mathbb{R}^3-K} < \bar\tau(C_1)\right) \geqslant \frac{1-4a}{8\pi a(1-2a)}\,\mathrm{cap}(K),\\ &K \subset B((0,y');a),\ |x-(0,y')| \leqslant a\ \ |y'| \leqslant a.\end{aligned} \tag{4.7}$$

Combining (4.1), (4.2), and (4.7), we arrive at

$$\begin{aligned}c &\geqslant \frac{1-4a^2}{4\pi}\ \frac{1-4a}{8\pi a(1-2a)} \int_{D_a} dy' \int_{\{x\in\mathbb{R}^3:\ |x-(0,y')|<a\}} dx\, \mathbb{E}_{(0,y')}\left[\mathrm{cap}\left(\mathfrak{B}(B(\beta(0);a))\right)\right]\\ &= \frac{(1-4a)(1+2a)a^4}{24}\, \mathbb{E}_0\left[\mathrm{cap}\left(\mathfrak{B}(B(0;a))\right)\right]\\ &= \frac{(1-4a)(1+2a)a^5}{24}\,\kappa,\end{aligned} \tag{4.8}$$

where we have used that $\mathcal{H}^2(D_a) = \pi a^2$, $|B(0;a)| = \frac{4\pi}{3}a^3$, and

$$\mathbb{E}_0\left[\mathrm{cap}\left(\mathfrak{B}(B(0;a))\right)\right] = a\,\mathbb{E}_0\left[\mathrm{cap}\left(\mathfrak{B}(B(0;1))\right)\right] = \kappa a.$$

The right-hand side of (4.8) is maximal when

$$a = \frac{\sqrt{79}-3}{28}.$$

This choice of a yields the left-hand side of (1.16). $\square$

4.2. *Proof of Theorem 1.3*

We first prove the upper bound. By (2.8),

$$\mathbb{E}_0\left[\tau'(D_R)^{1/2}\right] = \int_{[0,\infty)} d\tau\, \tau^{1/2}\, \mathbb{P}_{y'}\left(\tau'(D_R) \in d\tau\right) = \frac{1}{2}\int_{[0,\infty)} d\tau\, \tau^{-1/2} \int_{D_R} dz'\, p'_{D_R}(0,z';\tau). \tag{4.9}$$

By the monotonicity of the Dirichlet heat kernel,

$$p'_{D_R}(0,z';\tau) \leqslant p'_{\mathbb{R}^2}(0,z';\tau) = (4\pi\tau)^{-1} e^{-|z'|^2/(4\tau)}. \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\mathbb{E}_0\left[\tau'(D_R)^{1/2}\right] \leqslant \frac{1}{2}\int_{[0,\infty)} d\tau\, \tau^{-1/2} \int_{D_R} dz'\, (4\pi\tau)^{-1}\, e^{-|z'|^2/(4\tau)} = \tfrac{1}{2}\pi^{1/2} R. \tag{4.11}$$

Combining (2.6), (2.7), and (4.11), we obtain

$$\mathbb{E}_0\left[\mathcal{T}(C_{L,R} - \mathfrak{B}(C_{L,R}))\right] \geqslant (L-2R)\mathcal{T}'(D_R) - \frac{8}{\pi^{1/2}}\int_{[0,\infty)} dt\; t^{1/2} Q'_{D_R}(t). \tag{4.12}$$

From (1.12) we have

$$\mathcal{T}(C_{L,R}) \leqslant \mathcal{T}'(D_R)L - \frac{4}{\pi^{1/2}}\int_{[0,\infty)} dt\; t^{1/2} Q'_{D_R}(t) + \frac{8}{L\lambda'_1(D_R)}\mathcal{T}'(D_R). \tag{4.13}$$

Combining (1.17), (4.12), and (4.13), we get

$$\mathfrak{C}(C_{L,R}) \leqslant 2R\mathcal{T}'(D_R) + \frac{4}{\pi^{1/2}}\int_{[0,\infty)} dt\; t^{1/2} Q'_{D_R}(t) + \frac{8}{L\lambda'_1(D_R)}\mathcal{T}'(D_R).$$

Since $\mathcal{T}'(D_R) = \frac{\pi}{8}R^4$, we conclude by (2.13) with $\Omega = D_R$, that

$$\limsup_{L\to\infty} \mathfrak{C}(C_{L,R}) \leqslant \frac{\pi}{4}\left(1 + \frac{1}{j_0}\right) R^5.$$

To prove the lower bound we start from (3.2). Let $a \in (0, \frac{1}{3})$. We have the following estimate:

$$c' = \mathbb{E}_0\left[\int_{C_1} dx\, \left(v_{C_1}(x) - v_{C_1 - \mathfrak{B}(C_1)}(x)\right)\right] \geqslant \mathbb{E}_0\left[\int_{D_a} dx\, \left(v_{C_1}(x) - v_{C_1 - \mathfrak{B}(B(\beta(0);a))}(x)\right)\right].$$

Fix a compact set $K \subset B(B(0);a) \subset \mathbb{R}^3$. Note that $x \mapsto v_{C_1}(x) - v_{C_1-K}(x)$ is harmonic on $C_1 - K$, equals 0 for $x \in \partial C_1$, and equals $\frac{1}{4}(1-|x'|^2)$ for $x \in K$. If $|x| < a$, then $|x'| < a$, $x \in K$. Hence $v_{C_1}(x) - v_{C_1-K}(x) \geqslant \frac{1}{4}(1-a^2)$ for $x \in K$. We therefore have

$$v_{C_1}(x) - v_{C_1-K}(x) \geqslant \frac{1-a^2}{4}\, \bar{\mathbb{P}}_x\left(\bar{\tau}_{\mathbb{R}^3-K} < \bar{\tau}(C_1)\right), \quad x \in C_1.$$

It is straightforward to check that (4.5) holds for $y' = 0$. Furthermore, if $x \in \partial C_1$ and $z \in K$, then $|z - x| \geqslant 1 - a$. Hence, by (4.4),

$$\sup_{x\in\partial C_1} \bar{\mathbb{P}}_x\left(\tau_{\mathbb{R}^3-K} < \infty\right) \leqslant \frac{1}{4\pi(1-a)}\,\mathrm{cap}(K).$$

Combining (4.5) and (4.6), we get

$$\bar{\mathbb{P}}_x\left(\bar{\tau}_{\mathbb{R}^3-K} < \bar{\tau}(C_1)\right) \geqslant \frac{1-3a}{8\pi a(1-a)}\,\mathrm{cap}(K), \quad K \subset B(\beta(0);a),\ |x| \leqslant a.$$

Combining (4.2), (4.5), and (4.7), we arrive at

$$\begin{aligned}
&\mathbb{E}_0\left[\int_{C_1} dx\left(v_{C_1}(x)-v_{C_1-\mathfrak{B}(C_1)}(x)\right)\right]\\
&\quad\geqslant \frac{1-a^2}{4}\,\frac{1-3a}{8\pi a(1-a)}\int_{\{x\in\mathbb{R}^3:\,|x|<a\}} dx\,\mathbb{E}_0\left[\mathrm{cap}\left(\mathfrak{B}(B(0;a))\right)\right]\\
&\quad=\frac{(1-3a)(1+a)a^3}{24}\,\kappa. \qquad (4.14)
\end{aligned}$$

The right-hand side of (4.14) is maximal when

$$a=\frac{\sqrt{61}-4}{15}.$$

This choice of a yields the left-hand side of (1.19). □

Appendix

The following estimate was used in Step 1 of the proof of Proposition 3.1.

LEMMA A.1. *Let $\Omega_1\subset\Omega_2$ be non-empty open sets in $\mathbb{R}^m$ and K a compact set in $\mathbb{R}^m$. Let the torsion functions for $\Omega_1,\Omega_2,\Omega_1-K,\Omega_2-K$ be denoted by $v_{\Omega_1},v_{\Omega_2},v_{\Omega_1-K},v_{\Omega_2-K}$, respectively. Suppose that inf[spec$(-\Delta_{\Omega_2})$] >0. Then*

$$v_{\Omega_2}(x)-v_{\Omega_2-K}(x)\geqslant v_{\Omega_1}(x)-v_{\Omega_1-K}(x),\quad x\in\Omega_1-K,$$

and

$$v_{\Omega_2}(x)-v_{\Omega_2-K}(x)\leqslant\frac{1}{8}(m+cm^{1/2}+8)\lambda(\Omega_2)^{-1},\quad x\in\Omega_1-K, \qquad (A.1)$$

with

$$c=\sqrt{5(4+\log 2)}.$$

Proof. We extend the torsion functions v_{Ω_2-K} and v_{Ω_1-K} to all of Ω_1 by putting them equal to 0 on $K\cup(\mathbb{R}^m-\Omega_1)$. Define $h(x)=(v_{\Omega_2}(x)-v_{\Omega_2-K}(x))-(v_{\Omega_1}(x)-v_{\Omega_1-K}(x))$, $x\in\Omega_1-K$. Then h is harmonic on Ω_1-K, and $h(x)=v_{\Omega_2}(x)-v_{\Omega_1}(x)\geqslant 0$, $x\in K$, by the domain monotonicity of the torsion function. Furthermore, $h(x)=v_{\Omega_2}(x)-v_{\Omega_2-K}(x)\geqslant 0$, $x\in\partial\Omega_1$, by the domain monotonicity, and $h(x)\geqslant 0$, $x\in\Omega_1-K$, by the maximum principle of harmonic functions. The estimate in (A.1) follows from the non-negativity of the torsion function, together with the estimate in [**18**]. □

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