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A LOCAL CLT FOR CONVOLUTION EQUATIONS WITH AN APPLICATION TO WEAKLY SELF-AVOIDING RANDOM WALKS

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We prove error bounds in a central limit theorem for solutions of certain convolution equations. The main motivation for investigating these equations stems from applications to lace expansions, in particular to weakly self-avoiding random walks in high dimensions. As an application we treat such self-avoiding walks in continuous space. The bounds obtained are sharper than those obtained by other methods.

1. Introduction.

1.1. *On some convolution equations.* Let ϕ be the standard normal density in $\mathbb{R}^d$, $\mathbf{B} = \{B_k\}_{k\geq 1}$ be a sequence of rotationally invariant integrable functions and $\lambda > 0$ a (small) parameter. Define recursively

$$C_0 = \delta_0, \qquad (1)$$

$$C_n = C_{n-1} * \phi + \lambda \sum_{k=1}^{n} c_k B_k * C_{n-k}, \qquad n \geq 1,$$

where

$$c_n \stackrel{\text{def}}{=} \int C_n(x)\, dx.$$

δ_0 denotes the Dirac "function."

As written above, the sequence $\mathbf{C} = \{C_n\}_{n\geq 0}$ is not quite recursively defined, as the right-hand side in (1) contains the summand $c_n B_n$. The sequence $\{c_n\}$ itself satisfies

$$c_0 = 1, \qquad (2)$$

$$c_n = c_{n-1} + \lambda \sum_{k=1}^{n} c_k b_k c_{n-k}, \qquad n \geq 1,$$

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where $b_k = \int B_k(x)\,dx$. Therefore, if $\lambda|b_n| < 1$ for all n, these equations define the sequence $\{c_n\}$ uniquely, and then also $\mathbf{C}$ is well defined. We will always assume that we are in this situation.

The main assumption is a decay property of the B_n for large n. We will also assume Gaussian decay properties in space which are natural for the applications to self-avoiding walks we have in mind. The method we present here can probably be adapted to treat situations with less severe decay assumptions in space, but we have not worked this out.

Our main interest is to prove a local central limit theorem for the signed density C_n/c_n under appropriate conditions on $\mathbf{B}$ and λ. Of course, the parameter λ can be incorporated into $\mathbf{B}$. However, the approach we follow is purely perturbative. We will give conditions on $\mathbf{B}$ and then state that if in addition λ is small enough, a CLT holds.

At the expense of a few complications, we could also investigate the case where the first summand in (1) is $C_{n-1} * S$ with a rotationally invariant density S. We, however, feel that this generalization would somehow obscure the main line of the argument. To step out from the rotationally invariant case leads, however, to new, complicated and interesting problems which will be presented elsewhere.

The main motivation for our investigation comes from weakly self-avoiding random walks (WSAW). Indeed, as we will show, by using the so-called *lace expansion*, WSAW satisfy an equation as in (1).

In the next section we state our main theorem on this type of convolution equation, Theorem 1.1. In Section 1.3, we introduce WSAW in continuous space and state a local CLT, Theorem 1.2, that will be deduced from Theorem 1.1. To conclude this introductory part, in Subsecton 1.4 we discuss how this work relates to the existent literature, and we describe the structure of the paper.

1.2. *Main result on convolution equations.* Before stating our general result on convolution equations as in (1), we first fix some notation and define the set of conditions we need for the B_k's in (1).

$\mathbb{N}$ is the set of natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}_0 \stackrel{\text{def}}{=} \mathbb{N} \cup \{0\}$. For $t > 0$, ϕ_t is the centered normal density in $\mathbb{R}^d$ with covariance matrix $t \times$ identity. We write ϕ for ϕ_1.

We write $\mathcal{C}_*(\mathbb{R}^d)$ for the set of continuous, integrable functions $f : \mathbb{R}^d \to \mathbb{R}$, vanishing at ∞, which are of the form $f(x) = f_0(|x|)$ for some continuous function $f_0 : [0, \infty) \to \mathbb{R}$. We also write $\mathcal{C}_*^+(\mathbb{R}^d)$ for the strictly positive ones.

Here are the conditions we need for $\mathbf{B}$:

CONDITION 1.1 (Decay assumptions on $\mathbf{B}$-sequence). Assume that the functions $B_m \in \mathcal{C}_*(\mathbb{R}^d)$ in (1) are dominated in absolute value by functions $\Gamma_m \in \mathcal{C}_*^+(\mathbb{R}^d)$ which satisfy the following conditions:

(B1) There exist numbers $\chi_n(s) > 0$, $1 \le s \le n$, satisfying $\chi_n(s) = \chi_n(n-s)$, and for some constant K_1

$$\sum_{s=1}^{n-1} \bigl(s \wedge (n-s)\bigr)\chi_n(s) \le K_1 \qquad \forall n, \tag{3}$$

such that

$$\Gamma_m * \Gamma_n \le \chi_{m+n}(m)\Gamma_{n+m}. \tag{4}$$

(B2) There exists a constant $K_2 > 0$ such that for $t \le s \le 2t$ one has

$$\Gamma_s \le K_2 \Gamma_{2t}. \tag{5}$$

(B3) There exists $K_3 > 0$ such that for $m \le t, m \in \mathbb{N}$, $t \in \mathbb{R}^+$, $k = 0, 1, 2$, one has

$$\int \phi_t(x-y)|y|^{2k}\Gamma_m(y)\,dy \le K_3 \gamma_m^{(k)} \phi_{t+m}(x), \tag{6}$$

where

$$\gamma_m^{(k)} \overset{\text{def}}{=} \int |y|^{2k}\Gamma_m(y)\,dy.$$

(B4) The three sequences $\{\gamma_n^{(i)}\}_{n\in\mathbb{N}}$, $i = 0, 1, 2$, are nonincreasing, and

$$\begin{aligned} &K_4 \overset{\text{def}}{=} \sum_n n\gamma_n^{(0)} < \infty, \qquad K_5 \overset{\text{def}}{=} \sum_n \gamma_n^{(1)} < \infty, \\ &K_6 \overset{\text{def}}{=} \sum_n n^{-1}\gamma_n^{(2)} < \infty. \end{aligned} \tag{7}$$

A simple example where conditions (B1)–(B4) are satisfied is $\Gamma_n = n^{-a}\phi_{n/2}$, $a > 2$, but the application to self-avoiding walks needs a slightly more complicated choice, as will be discussed later.

We will often write γ_m for $\gamma_m^{(0)}$.

We remark that under the above condition, one has for

$$b_n \overset{\text{def}}{=} \int B_n(x)\,dx \tag{8}$$

the estimate

$$|b_n| \le \gamma_n$$

with

$$\gamma_m\gamma_n \le \chi_{m+n}(m)\gamma_{n+m}. \tag{9}$$

Next, fix an arbitrary positive $\varepsilon > 0$, and write

$$\psi_n \overset{\text{def}}{=} \phi_{n\delta(1+\varepsilon)}, \tag{10}$$

with δ defined below in (30).

In the sequel, we will use L as a positive constant, not necessarily the same at different occurrences, which may depend on d, ε, K_1–K_6, but not on n, λ.

Let

$$\zeta_n^{(1)} \stackrel{\text{def}}{=} 1 + \sum_{i=0}^{2} \sum_{m=1}^{n} m^{2-i} \gamma_m^{(i)},$$

$$\zeta_n^{(2)} \stackrel{\text{def}}{=} \sum_{m=n}^{\infty} (\gamma_m^{(1)} + m\gamma_m), \tag{11}$$

$$\overline{\zeta}_n \stackrel{\text{def}}{=} n^{-2} \sum_{j=1}^{n} \zeta_j^{(1)} + n^{-1} \sum_{j=1}^{n} \zeta_j^{(2)}.$$

Because of (5) and (7) we have

$$\lim_{n\to\infty} \overline{\zeta}_n = 0, \qquad \sum_n n^{-1} \overline{\zeta}_n < \infty, \qquad \overline{\zeta}_m \le \overline{\zeta}_{2n} \qquad \text{for } n \le m \le 2n. \tag{12}$$

We remark that

$$\overline{\zeta}_n \ge \frac{1}{n^2} \sum_{j=1}^{n} \sum_{m=1}^{n} m^2 \gamma_m \ge \frac{n^2}{L} \gamma_n. \tag{13}$$

We can finally state our main theorem on convolution equations:

THEOREM 1.1 (Local CLT for convolution equations). *Assume Condition 1.1. Then if λ is small enough (depending on d, ε and K_1–K_6), the following estimate holds*:

$$\left|C_n(x)/c_n - \phi_{n\delta}(x)\right| \le L\lambda \left[\sum_{s=1}^{[n/2]} s(\psi_s * \Gamma_{n-s})(x) + \overline{\zeta}_n \psi_n(x) \right], \tag{14}$$

where $\delta = \delta(\mathbf{B}, \lambda) > 0$ is defined in (30) below.

In the example $\Gamma_n(x) = n^{-a}\phi_{n/2}(x)$, $2 < a < 3$, one has $\zeta_n^{(1)} = \text{const} \times n^{3-a}$, $\zeta_n^{(2)} = \text{const} \times n^{2-a}$, and therefore $\overline{\zeta}_n = \text{const} \times n^{2-a}$, and thus

$$\left|C_n(x)/c_n - \phi_{n\delta}(x)\right| \le L\lambda n^{2-a} \psi_n$$

giving a local CLT with a precise error estimate. For $a > 3$, we get

$$\left|C_n(x)/c_n - \phi_{n\delta}(x)\right| \le L\lambda n^{-1} \psi_n.$$

As we remarked above, this Γ_n cannot work for the application to self-avoiding walks, and in fact, a pure local CLT is not possible in this case.

1.3. *WSAW on $\mathbb{R}^d$ and result.* The main motivation for our investigation of these types of convolution equations comes from WSAW, as was first investigated by Brydges and Spencer in the seminal paper [3]. Their results are for random walks on the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 5$. In contrast, we now introduce and investigate weakly self-avoiding random walks on $\mathbb{R}^d$ with standard normal increments. The model has two parameters, $\lambda, \rho > 0$, ρ being the range of the interaction and λ the strength. We set $\mathbb{I}_\rho(x) \stackrel{\text{def}}{=} 1_{\{|x| \leq \rho\}}$, and if $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, and $0 \leq i < j \leq n$, we set $U_{ij}^\rho(\mathbf{x}) \stackrel{\text{def}}{=} \mathbb{I}_\rho(x_j - x_i)$, where $x_0 = 0$. Then, for $0 \leq \lambda \leq 1$, define the probability measure $P_{n,\lambda,\rho}$ on $(\mathbb{R}^d)^n$ by its density with respect to Lebesgue measure

$$p_{n,\lambda,\rho}(\mathbf{x}) = \frac{1}{Z_{n,\lambda,\rho}} K_{\lambda,\rho}[0,n](\mathbf{x}) \Phi[0,n](\mathbf{x}), \tag{15}$$

where

$$K_{\lambda,\rho}[a,b](\mathbf{x}) \stackrel{\text{def}}{=} \prod_{a \leq i < j \leq b} \bigl(1 - \lambda U_{ij}^\rho(\mathbf{x})\bigr), \tag{16}$$

$$\Phi[a,b](\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=a+1}^{b} \phi(x_i - x_{i-1}). \tag{17}$$

$Z_{n,\lambda,\rho}$ is the usual partition function, that is, the norming factor which makes $p_{n,\beta,\rho}$ into a probability density. Our main interest is to prove a central limit theorem for this measure, in the simplest case for the last marginal measure. It is convenient to consider first the unnormalized kernel $C_n^{\text{SAW}}(x)$, $x \in \mathbb{R}^d$, which is defined to be the last marginal density of $Z_{n,\beta} p_{n,\beta,\rho}(\mathbf{x})$, that is,

$$C_n^{\text{SAW}}(x_n) = \int K_{\lambda,\rho}[0,n](\mathbf{x}) \Phi[0,n](\mathbf{x}) \prod_{i=1}^{n-1} dx_i. \tag{18}$$

By using the lace expansion (as we will show in Section 3.1), the C_n^{SAW} satisfy an equation of the form

$$C_n^{\text{SAW}} = C_{n-1}^{\text{SAW}} * \phi + \sum_{k=1}^{n} \Pi_k * C_{n-k}^{\text{SAW}}, \tag{19}$$

where the kernels Π_k describe the interactions through the weak self-avoidance. The Π_k are complicated functions and are hard to evaluate precisely. However, one crucial property is that the leading order decay is the same as that of the C_k^{SAW}. It therefore looks natural to write $\Pi_k = \lambda c_k^{\text{SAW}} B_k$, and one seeks for conditions on the B_k ensuring a CLT for solutions of (1). We can then apply Theorem 1.1, provided we can check Condition 1.1 on this $\mathbf{B}$ sequence. The theorem we obtain as a corollary of Theorem 1.1 is the following:

THEOREM 1.2 (Local CLT for WSAW). *For $d \geq 5$, $\rho \in (0,1]$ and $\varepsilon > 0$ there exists $\lambda_0(d,\varepsilon) > 0$ such that for all $\lambda \in (0,\lambda_0]$, there exist a parameter $\delta(d,\rho,\lambda) > 0$ and a constant $K(d,\varepsilon,\lambda) > 0$ such that for all $n \in \mathbb{N}$*

$$\left|\frac{C_n^{\mathrm{SAW}}(x)}{c_n^{\mathrm{SAW}}} - \phi_{n\delta}(x)\right| \leq K\left[r_n \phi_{n\delta(1+\varepsilon)}(x) + n^{-d/2}\sum_{j=1}^{\lceil n/2\rceil} j\phi_{j\delta(1+\varepsilon)}(x)\right], \tag{20}$$

with

$$r_n = \begin{cases} n^{-1/2}, & \text{for } d = 5, \\ n^{-1}\log n, & \text{for } d = 6, \\ n^{-1}, & \text{for } d \geq 7. \end{cases} \tag{21}$$

REMARK 1.1. (a) The bound leads to $\|C_n^{\mathrm{SAW}}/c_n^{\mathrm{SAW}} - \phi_{n\delta}\|_1 = O(r_n)$.

(b) The theorem does not give a local CLT as at $x = 0$ both $\phi_{n\delta}(0)$ and the bound are of order $n^{-d/2}$. A moment's reflection, however, reveals that there cannot be a local CLT, as the starting point continues to have a noticeable influence on $C_n^{\mathrm{SAW}}(x)/c_n^{\mathrm{SAW}}$ for points x at distance of order 1 from the origin. However, our bound proves

$$\lim_{r\to\infty}\limsup_{n\to\infty}\sup_{x:\,|x|\geq r} n^{d/2}\left|\frac{C_n^{\mathrm{SAW}}(x)}{c_n^{\mathrm{SAW}}} - \phi_{n\delta}(x)\right| = 0,$$

so the result comes as close as possible to a local CLT.

(c) The summation up to $\lceil n/2\rceil$ is somewhat arbitrary and can be replaced by $\lceil \alpha n\rceil$ for any $\alpha \in (0,1)$, adapting K. In fact, for $0 < \alpha < 1$, there exists a $K(\alpha)$ such that for all $x \in \mathbb{R}^d$,

$$n^{-d/2}\sum_{j=\lceil \alpha n\rceil}^{n} j\phi_{j\delta(1+\varepsilon)}(x) \leq K(\alpha) r_n \phi_{n\delta(1+\varepsilon)}(x).$$

We have chosen $\alpha = 1/2$ for convenience. The second summand on the right-hand side of (20) is important as it takes care of the failure of the local CLT for x near the origin.

(d) The choice of an $\varepsilon > 0$ on the right-hand side of (20) is essentially just for convenience, as it helps to swallow all kinds of polynomial factors in x with which we prefer not to be bothered. Note that if bound (20) is correct for a positive $\varepsilon > 0$, it is also true for any larger ε, with a changed constant K. It will be convenient to assume that ε is small, say $\varepsilon \leq 1/100$.

1.4. *Related literature and structure of the paper.* Self-avoiding random walks are models for polymer chains of relevance in statistical physics. Despite their simple definition, a mathematical rigorous analysis turns out to be a major challenge. We refer to [1] for a recent survey on this topic. Since the seminal paper by Brydges and Spencer [3], the analysis of these models in high dimensions ($d \geq 5$) has been carried out by using the so-called *lace expansion*. The latter is a diagrammatic type of expansion based on graphs (which we recall in Section 3.1) to deal with combinatorial objects of relevance in statistical mechanics, for example, self-avoiding walks, percolation models and lattice trees. For the interested reader, [5] represents the main reference on this type of expansion. While using the lace expansion for the analysis of high-dimensional WSAW or related models satisfying equation (1), the procedure is by now standard and can be roughly summarized via the following three steps:

(1) Show that the unnormalized densities C_n^{SAW} satisfy the convolution equation in (1).

(2) Estimate the B_k coefficents in (1).

(3) Deduce from the previous steps and equation (1) the growth of the normalized C_n^{SAW} and some detailed Gaussian behavior.

Step (3) is the most involved and technical, especially in [3]. A successful attempt to simplify this step has been obtained in [6, 7], where the authors introduced a new inductive approach. Both methods in [3, 6, 7] heavily rely on spatial Fourier transforms. In contrast, the method we use does not make use of Fourier analysis and is based on a fixed point iteration. This novel method is very different from the previous ones. It was originally developed in the thesis of Christine Ritzmann [2, 4], but it was never published. One of the main goals of this paper is to present this method with some improvements, generalizations and simplifications with respect to [2, 4]. The main new feature compared to [2, 4] is to use a more flexible and general way to define the operator whose fixed point characterizes the solution of the convolution equation. Also, [2, 4] was entirely tailored for the application to self-avoiding walks, whereas our main result on the convolution equations, Theorem 1.1, is much more general.

Our method gives error bounds in the local CLT that are better than those obtained with Fourier techniques. The second main novelty of this paper concerns the application to WSAW in continuous space. In fact, to our knowledge, all the previous works including [2, 4] focus on WSAW on $\mathbb{Z}^d$. One of the reasons to introduce this variant is that, to explain our approach based on fixed point iteration, continuous space is actually more convenient than the lattice. In other words, the emphasis here is to present an elementary and completely self-contained proof of a sharp CLT for solutions of (1), together with perhaps the simplest possible application. No knowledge of earlier versions of lace expansions or [4] are assumed.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of the local CLT for general convolution equation, Theorem 1.1. Section 3 focuses on the application to WSAW in continuous space. By performing the three steps sketched above we show how to derive the local CLT in Theorem 1.2 from Theorem 1.1.

2. Proof of the local CLT for convolution equations. In this section we prove Theorem 1.1. The proof is divided in three main steps which we perform in the following three sections. First, in Section 2.1 we analyze the normalizing sequence $\{c_n\}$. In the second step, Section 2.2, we prove Theorem 1.1 by assuming the technical Lemma 2.1 which we prove right after in Section 2.3.

2.1. *On the connectivity constants.* A first question we address is about the behavior of the sequence $\{c_n\}$.

PROPOSITION 2.1. *Assume Condition* 1.1, *and let* **c** *be the sequence defined by* (2). *Then if* λ *is small enough the following holds*:

(a) *There exists a unique* $\mu > 0$ *such that* $\alpha \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu^{-n} c_n$ *exists in* $(0, \infty)$.

(b) *Writing* $a_n \stackrel{\text{def}}{=} \mu^{-n} c_n$, *one has*

$$|a_{n+1} - a_n| < L\lambda\overline{\gamma}_n \stackrel{\text{def}}{=} L\lambda \sum_{j=n}^{\infty} \gamma_j. \tag{22}$$

(c)

$$\mu^{-1} = 1 - \lambda \sum_{k=1}^{\infty} a_k b_k. \tag{23}$$

REMARK 2.1. (a) Plugging expression (23) into (2), we see that $\mathbf{a} = \{a_n\}_{n\in\mathbb{N}_0}$ satisfies $a_0 = 1$, and

$$a_n = a_{n-1} - \lambda a_{n-1} \sum_{k=n+1}^{\infty} a_k b_k + \lambda \sum_{k=1}^{n} a_k b_k (a_{n-k} - a_{n-1}), \qquad n \geq 1. \tag{24}$$

(b) From (22) we get

$$|a_n - \alpha| \leq L\lambda \sum_{k=n}^{\infty} k\gamma_k. \tag{25}$$

The idea of the proof is simple: *assuming* that such a μ and a sequence $\{a_n\}$ exist, one gets from (2)

$$\mu^n a_n = \mu^{n-1} a_{n-1} + \lambda \mu^n \sum_{k=1}^{n} a_k b_k a_{n-k}.$$

Letting then $n \to \infty$, assuming that $\lim_{n\to\infty} a_n$ exists and is $\neq 0$, one sees that μ has to be given by (23) in terms of $\{a_n\}$. Plugging that back, one arrives at the conclusion, that the **a**-sequence has to satisfy (24). The idea therefore is first to prove by a fixed point argument that this equation has a nice solution, and then check that

$$d_n = \Big(1 - \lambda \sum_{k=1}^{\infty} a_k b_k\Big)^{-n} a_n$$

satisfies equation (2), and therefore $d_n = c_n$, completing the proof.

PROOF OF PROPOSITION 2.1. Let $l_1(\mathbb{N})$ be the Banach space of absolutely summable sequences $\mathbf{q} = \{q_n\}_{n\in\mathbb{N}}$, and $l_\gamma(\mathbb{N})$ be the set of sequences with $\|\mathbf{q}\|_\gamma \stackrel{\text{def}}{=} \sup_n \overline{\gamma}_n^{-1}|q_n| < \infty$. $(l_\gamma(\mathbb{N}), \|\cdot\|_\gamma)$ is a Banach space too, and by (7), $l_\gamma(\mathbb{N}) \subset l_1(\mathbb{N})$, and the embedding is continuous. The linear map $s : l_1(\mathbb{N}) \to l_\infty(\mathbb{N}_0)$ is defined by $s(\mathbf{q})_0 = 0$, and $s(\mathbf{q})_n \stackrel{\text{def}}{=} \sum_{j=1}^n q_j$, $n \geq 1$. Evidently, $\|s(\mathbf{q})\|_\infty \leq \|\mathbf{q}\|_1 \leq L\|\mathbf{q}\|_\gamma$. We also define the affine mapping $S : l_1(\mathbb{N}) \to l_\infty(\mathbb{N}_0)$ by $S(\mathbf{q}) \stackrel{\text{def}}{=} \mathbf{1} + s(\mathbf{q})$, where $\mathbf{1}$ is the sequence identical to 1. We define two mappings ψ_1, ψ_2 from $l_1(\mathbb{N})$ to the set of sequences with index set $\mathbb{N}$. We set

$$\psi_1(\mathbf{q})_n \stackrel{\text{def}}{=} S(\mathbf{q})_{n-1} \sum_{k=n+1}^{\infty} b_k S(\mathbf{q})_k,$$

$$\psi_2(\mathbf{q})_n \stackrel{\text{def}}{=} \sum_{k=2}^{n} S(\mathbf{q})_k b_k \big[s(\mathbf{q})_{n-k} - s(\mathbf{q})_{n-1}\big]$$

for $n \geq 1$. Finally we set $\psi \stackrel{\text{def}}{=} -\lambda\psi_1 + \lambda\psi_2$. Note first that

$$\psi(\mathbf{0})_n = \lambda\psi_1(\mathbf{0})_n = \lambda \sum_{k=n+1}^{\infty} b_k,$$

where $\mathbf{0}$ is the sequence identical to 0. We conclude that $\|\psi(\mathbf{0})\|_\gamma \leq L\lambda$, by (9).

$$\begin{aligned}\big|\psi_1(\mathbf{q})_n - \psi_1(\mathbf{p})_n\big| &\leq \big\|s(\mathbf{q}) - s(\mathbf{p})\big\|_\infty \Bigg[\sum_{k=n+1}^{\infty} \big|b_k S(\mathbf{q})_k\big| + \big|S(\mathbf{p})_{n-1}\big| \sum_{k=n+1}^{\infty} |b_k|\Bigg] \\ &\leq L\|\mathbf{q} - \mathbf{p}\|_\gamma \big[2 + L\|\mathbf{q}\|_\gamma + L\|\mathbf{p}\|_\gamma\big] \sum_{k=n+1}^{\infty} \gamma_k,\end{aligned}$$

$$\big\|\psi_1(\mathbf{q}) - \psi_1(\mathbf{p})\big\|_\gamma \leq L\|\mathbf{q} - \mathbf{p}\|_\gamma \big(1 + \|\mathbf{q}\|_\gamma + \|\mathbf{p}\|_\gamma\big).$$

Similarly, for $n \geq 2$, by resummation

$$\begin{aligned}\psi_2(\mathbf{q})_n - \psi_2(\mathbf{p})_n &= \sum_{j=1}^{n-1} q_j \sum_{k=n-j+1}^{n} \big(S(\mathbf{p})_k - S(\mathbf{q})_k\big) b_k \\ &\quad + \sum_{j=1}^{n-1} (p_j - q_j) \sum_{k=n-j+1}^{n} S(\mathbf{p})_k b_k.\end{aligned} \tag{26}$$

In the first summand, we estimate $|S(\mathbf{q})_k - S(\mathbf{p})_k|$ by $L\|\mathbf{q}-\mathbf{p}\|_\gamma$, so we get for this part an estimate

$$\leq L\|\mathbf{q}\|_\gamma \|\mathbf{q}-\mathbf{p}\|_\gamma \sum_{j=1}^{n-1}\sum_{t=j}^{\infty}\gamma_t \sum_{k=n-j+1}^{n} \gamma_k. \tag{27}$$

Further,

$$\begin{aligned}\sum_{j=1}^{n-1}\sum_{t=j}^{\infty}\gamma_t \sum_{k=n-j+1}^{n} \gamma_k &\leq \sum_{j=1}^{n-1}\sum_{t=j}^{\infty}\sum_{k=n-j+1}^{n} \chi_{t+k}(t)\gamma_{t+k} \\ &\leq \sum_{s=n+1}^{\infty}\gamma_s \sum_{t=1}^{s-1} N(s,t)\chi_s(t),\end{aligned} \tag{28}$$

where we have used (9), and where $N(s,t)$ is the number of indices j satisfying $1 \leq j \leq n-1, t \geq j, n-j+1 \leq s-t \leq n$, so that $N(s,t) \leq t \wedge (s-t)$, and using (3), from (27) and (28), we get for the first summand of (26) an estimate $\leq L\|\mathbf{q}\|_\gamma \|\mathbf{q}-\mathbf{p}\|_\gamma \overline{\gamma}_n$. In a similar way, we get for the second summand, an estimate $\leq L(1+\|\mathbf{p}\|_\gamma)\|\mathbf{q}-\mathbf{p}\|_\gamma \overline{\gamma}_n$, and therefore

$$\left\|\psi_2(\mathbf{q}) - \psi_2(\mathbf{p})\right\|_\gamma \leq L\|\mathbf{q}-\mathbf{p}\|_\gamma\left(1+\|\mathbf{q}\|_\gamma + \|\mathbf{p}\|_\gamma\right),$$

leading to

$$\left\|\psi(\mathbf{q}) - \psi(\mathbf{p})\right\|_\gamma \leq L\lambda\|\mathbf{q}-\mathbf{p}\|_\gamma\left(1+\|\mathbf{q}\|_\gamma + \|\mathbf{p}\|_\gamma\right).$$

From this and $\psi(\mathbf{0}) \in l_\gamma(\mathbb{N})$, it follows that ψ maps $l_\gamma(\mathbb{N})$ continuously into itself, and furthermore, if λ is small enough, the iterates $\psi^n(\mathbf{0})$ form a Cauchy sequence, and therefore converge in $l_\gamma(\mathbb{N})$ to an element $\boldsymbol{\xi}$ with $\|\boldsymbol{\xi}\|_\gamma \leq L\lambda$ which is a fixed point of ψ.

If we write

$$\boldsymbol{\eta} \stackrel{\text{def}}{=} S(\boldsymbol{\xi}), \qquad \varpi \stackrel{\text{def}}{=} \left(1-\lambda\sum_{k=1}^{\infty}\eta_k b_k\right)^{-1},$$

then it is evident, using the fact that $\boldsymbol{\xi}$ is a fixed point of ψ, that the sequence $\boldsymbol{\eta}$ satisfies (24), implying that the sequence $\{\eta_n \varpi^n\}$ satisfies (2), and therefore it *is* this sequence. So it follows that $\varpi = \mu$, and $\mu^{-n}c_n$ satisfies the properties listed in (a)–(c). □

2.2. *Proof of Theorem* 1.1. Before giving the proof, let us first start with a few observations.

As $B_m \in \mathcal{C}_*(\mathbb{R}^d)$, the "covariance" matrix satisfies

$$\int x^T x B_m(x)\,dx = \overline{b}_m I_d, \tag{29}$$

for some $\overline{b}_m \in \mathbb{R}$ (possibly negative), I_d being the $d \times d$ unit matrix. Evidently, $|\overline{b}_m| \leq \gamma_m^{(1)}$, and by Condition 1.1 (7), the following number is well defined (for small enough λ):

$$\delta \stackrel{\text{def}}{=} \frac{\mu^{-1} + \lambda \sum_{m=1}^{\infty} a_m \overline{b}_m}{\mu^{-1} + \lambda \sum_{m=1}^{\infty} m a_m b_m}, \tag{30}$$

where μ and b_m are given by (23) and (8), respectively. In particular, by choosing $\lambda > 0$ small enough, we can achieve that

$$|1 - \delta| \leq L\lambda, \qquad |1 - \mu| \leq L\lambda \tag{31}$$

and also

$$1/2 \leq a_n \leq 3/2 \qquad \forall n,$$

which we assume henceforward.

The idea of the proof of Theorem 1.1 is to consider an appropriate Banach space of sequences of functions with a norm that encodes the error we expect in the local CLT. We then prove that $\{\mu^{-n} C_n - \mu^{-n} c_n \phi_{n\delta}\}_{n \in \mathbb{N}}$ is an element of this Banach space by proving that it appears as a limit of a Cauchy sequence. This implies the desired result.

Let us start by describing the Banach space we need. Let $\mathbf{f} = \{f_n\}$ be a sequence of functions in $\mathcal{C}_*^+(\mathbb{R}^d)$ which satisfy $\lim_{n\to\infty} \sup_x f_n(x) = 0$. For any sequence $\mathbf{g} = \{g_n\}$, $g_n \in \mathcal{C}_*(\mathbb{R}^d)$ define

$$\|\mathbf{g}\|_{\mathbf{f}} \stackrel{\text{def}}{=} \sup_n \sup_{x \in \mathbb{R}^d} \frac{|g_n(x)|}{f_n(x)},$$

and write $\mathcal{B}_{\mathbf{f}} \stackrel{\text{def}}{=} \{\mathbf{g} : \|\mathbf{g}\|_{\mathbf{f}} < \infty\}$, which equipped with $\|\cdot\|_{\mathbf{f}}$ is a Banach space.

For our purposes, we consider the Banach space $(\mathcal{B}_{\mathbf{f}}, \|\cdot\|_{\mathbf{f}})$ with $\mathbf{f} = \{f_n\}$ defined by

$$f_n \stackrel{\text{def}}{=} \sum_{s=1}^{[n/2]} s\psi_s * \Gamma_{n-s} + \overline{\zeta}_n \psi_n, \tag{32}$$

where $\psi_n \stackrel{\text{def}}{=} \phi_{n\delta(1+\varepsilon)}$. (As we remarked before, the choice of $\varepsilon > 0$ is only of minor relevance, but it influences the notion of "small enough λ.") Note that the sequence $\{f_n\}$ is the same as the sequence of error terms on the right-hand side of (14).

Next, let $\mathbf{C}$ be the solution of (1), and put $A_n \stackrel{\text{def}}{=} C_n \mu^{-n}$. This sequence satisfies $A_0 = \delta_0$ and

$$A_n = \mu^{-1} A_{n-1} * \phi + \lambda \sum_{k=1}^{n} a_k B_k * A_{n-k}, \tag{33}$$

where $a_n = \int A_n(x)\, dx$, and $A_n/a_n = C_n/c_n$.

In particular, note that the statement of Theorem 1.1 is equivalent (given Proposition 2.1) to bounding $|A_n(x) - a_n\phi_{n\delta}(x)|$ in the same way, and this is what we will do.

We define the following operator Ψ on sequences of functions $\mathbf{G} = \{G_n\}_{n\geq 0}$, $G_n \in \mathcal{C}_*(\mathbb{R}^d)$, $\Psi(\mathbf{G})_0 \stackrel{\text{def}}{=} G_0$ and for $n \geq 1$:

$$\Psi(\mathbf{G})_n \stackrel{\text{def}}{=} a_n\phi_{n\delta} * G_0 - \sum_{j=1}^{n} G_{n-j} * \Delta_{j,j},$$

with

$$\Delta_{k,j} \stackrel{\text{def}}{=} a_j\phi_{k\delta} - \mu^{-1}a_{j-1}\phi_{(k-1)\delta+1} - \lambda\sum_{m=1}^{j} a_m a_{j-m} B_m * \phi_{(k-m)\delta} \tag{34}$$

for $k \geq j$. A resummation gives

$$\Psi(\mathbf{G})_n = G_n - \sum_{j=1}^{n} a_{n-j}\phi_{(n-j)\delta} * \left[G_j - \mu^{-1}\phi * G_{j-1} - \lambda\sum_{m=1}^{j} a_m B_m * G_{j-m}\right].$$

A crucial observation is that if $\mathbf{A}$ satisfies $A_0 = \delta_0$ and (33), then $\Psi(\mathbf{A}) = \mathbf{A}$, and vice versa: if $A_0 = \delta_0$, and $\mathbf{A}$ satisfies the fixed point equation, then (33) follows by induction on n.

The main technical estimates are summarized in the following lemma which will be proved in the next section.

LEMMA 2.1. (a)

$$\sum_{j=1}^{n} |\Delta_{n,j}| \leq L\lambda f_n, \tag{35}$$

(b)

$$|\Delta_{n,n}| \leq L\lambda\kappa_n, \tag{36}$$

where

$$\kappa_n \stackrel{\text{def}}{=} \sum_{s=0}^{[n/2]} \psi_s * \Gamma_{n-s} + n^{-1}\overline{\zeta}_n\psi_n, \tag{37}$$

(c)

$$\sum_{j=1}^{n} \kappa_j * f_{n-j} \leq Lf_n. \tag{38}$$

We proceed with the proof of Theorem 1.1, assuming this lemma. Note that on the one hand, if $\mathbf{E}$ is the sequence $\{a_n\phi_{n\delta}\}$, then $\Psi(\mathbf{E})_n = E_n - \sum_{j=1}^n a_{n-j}\Delta_{n,j}$. By Lemma 2.1(a), we get that $\Psi(\mathbf{E}) - \mathbf{E} \in \mathcal{B}_{\mathbf{f}}$ with $\|\Psi(\mathbf{E}) - \mathbf{E}\|_{\mathbf{f}} \le L\lambda$. ($\mathbf{E}$ itself is of course not in $\mathcal{B}_{\mathbf{f}}$.)

On the other hand, if $\mathbf{G} \in \mathcal{B}_{\mathbf{f}}$, with $G_0 = 0$, then for $n \ge 1$,

$$\left|\Psi(\mathbf{G})_n(x)\right| \le \|\mathbf{G}\|_{\mathbf{f}} \sum_{j=1}^n \left|f_{n-j}(x)\Delta_{j,j}(x)\right|.$$

By applying Lemma 2.1(b) and (c), we obtain that

$$\left\|\Psi(\mathbf{G})\right\|_{\mathbf{f}} \le L\lambda\|\mathbf{G}\|_{\mathbf{f}}.$$

Thus, since $(\Psi(\mathbf{E}) - \mathbf{E})_0 = 0$, we conclude that for small enough $\lambda > 0$, $\{\Psi^n(\mathbf{E}) - \mathbf{E}\}$ is a Cauchy sequence in $\mathcal{B}_{\mathbf{f}}$, and therefore converges, say to $\mathbf{Y} \in \mathcal{B}_{\mathbf{f}}$, which satisfies $\|\mathbf{Y}\|_{\mathbf{f}} \le L\lambda$. Then

$$\begin{aligned}\mathbf{Y} + \mathbf{E} - \Psi(\mathbf{Y} + \mathbf{E}) = {} & \left[\mathbf{Y} + \mathbf{E} - \Psi^n(\mathbf{E})\right] \\ & + \left[\Psi^n(\mathbf{E}) - \Psi^{n+1}(\mathbf{E})\right] + \left[\Psi^{n+1}(\mathbf{E}) - \Psi(\mathbf{Y} + \mathbf{E})\right],\end{aligned}$$

and all three expressions in square brackets on the right-hand side converge to 0 in $\mathcal{B}_{\mathbf{f}}$. Therefore, $\mathbf{Y}+\mathbf{E}$ is a fixed point of Ψ, which we know has to be $\mathbf{A}$. Therefore $\|\mathbf{A} - \mathbf{E}\|_{\mathbf{f}} \le L\lambda$. So we have proved the theorem.

2.3. *Proof of Lemma* 2.1. We first recall some properties of the semigroup $\{\phi_t\}$. Of course, $\phi_t(x) = t^{-d/2}\phi(x/\sqrt{t})$. We often write $\dot{\phi}_t$ for the derivative in t, and we write $\partial_i\phi_t$ for the partial derivatives in x_i, and $\partial_{ij}^2\phi_t$ for the second partial derivatives, etc. We also write $\Delta\phi_t \stackrel{\text{def}}{=} \sum_{i=1}^d \partial_{ii}^2\phi_t$, as usual. The heat equation gives $\dot{\phi}_t = \frac{1}{2}\Delta\phi_t$. The partial derivatives in x of ϕ are of the form $p\phi$ for a polynomial p in x whose exact form is of no concern for us. Here are some elementary properties we will use:

- If $t \le s \le 2t$, then

$$\phi_t \le 2^{d/2}\phi_s. \tag{39}$$

- If p is any polynomial in x, then for any $\varepsilon > 0$, there exists $C_{\varepsilon,p} > 0$ such that

$$\left|p(x)\right|\phi(x) \le C_{\varepsilon,p}\phi_{1+\varepsilon}(x) \tag{40}$$

  implying

$$\left|p(x/\sqrt{t})\right|\phi_t(x) \le C_{\varepsilon,p}\phi_{t(1+\varepsilon)}(x). \tag{41}$$

From this, we see that for $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ with $|\mathbf{k}| = k_1 + \cdots + k_d$,

$$\left|\frac{\partial^{|\mathbf{k}|}\phi_t(x)}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}\right| \le C_{\varepsilon,k}t^{-|\mathbf{k}|/2}\phi_{t(1+\varepsilon)}(x), \tag{42}$$

and for $k \in \mathbb{N}$,

$$\left| \frac{\partial^k \phi_t(x)}{\partial t^k} \right| \le C_{\varepsilon,k} t^{-k} \phi_{t(1+\varepsilon)}(x). \tag{43}$$

Below, we use the convention $\sum_{m=a}^{b} = 0$ if $b < a$.

2.3.1. *Proof of* (35). Recall that $a_n = \mu^{-n} c_n$. Using (23) and (24), we can rewrite $\Delta_{k,j}$ as

$$\begin{aligned}\Delta_{k,j} &= \mu^{-1} a_{j-1}(\phi_{k\delta} - \phi_{(k-1)\delta+1}) - \lambda \sum_{m=1}^{j} a_m a_{j-m}(B_m * \phi_{(k-m)\delta} - b_m \phi_{k\delta}) \\ &= \Delta_{k,j}^{(1)} + \Delta_{k,j}^{(2)},\end{aligned}$$

where

$$\Delta_{k,j}^{(2)} \stackrel{\text{def}}{=} -\lambda \sum_{m=[k/2]+1}^{j} a_m a_{j-m}(B_m * \phi_{(k-m)\delta} - b_m \phi_{k\delta}).$$

Note that $\Delta^{(1)}$ and $\Delta^{(2)}$ deal with small and large m, respectively. As $\{a_m\}$ is bounded, we can estimate, using $|b_m| \le \gamma_m$, $|B_m| \le \Gamma_m$,

$$|\Delta_{k,j}^{(2)}| \le L\lambda \left[\sum_{s=k-j}^{[k/2]} \phi_{s\delta} * \Gamma_{k-s} + \phi_{k\delta} \sum_{m=[k/2]+1}^{j} \gamma_m \right], \tag{44}$$

where in the first summand on the RHS, we substituted $k - m = s$. From this we see that (35) [and also (36)] holds for $\Delta^{(2)}$ instead of Δ, and so it remains to check the inequalities for $\Delta^{(1)}$:

$$\begin{aligned}\Delta_{k,j}^{(1)} &= \alpha \left[\mu^{-1}(\phi_{k\delta} - \phi_{(k-1)\delta+1}) - \lambda \sum_{m=1}^{j\wedge[k/2]} a_m (B_m * \phi_{(k-m)\delta} - b_m \phi_{k\delta}) \right] \\ &\qquad + \mu^{-1}(a_{j-1} - \alpha)(\phi_{k\delta} - \phi_{(k-1)\delta+1}) \\ &\qquad - \lambda \sum_{m=1}^{j\wedge[k/2]} a_m (a_{j-m} - \alpha)(B_m * \phi_{(k-m)\delta} - b_m \phi_{k\delta}) \\ &= X_{k,j}^{(1)} + X_{k,j}^{(2)} - X_{k,j}^{(3)}, \qquad \text{say.}\end{aligned}$$

To estimate $X^{(1)}$, we use a Taylor approximation $\phi_t(x)$ in the x-variable up to fourth order. Note that in the expansion below, the odd contributions vanish due to the assumed symmetry of the B_m function, and in the second Taylor term, we replace $\frac{1}{2}\Delta\phi_t$ by $\dot{\phi}_t$. b_m and $\overline{b}_m$ are defined by (8) and (29).

$$(B_m * \phi_{(k-m)\delta})(x) = b_m\phi_{(k-m)\delta}(x) + \overline{b}_m\dot{\phi}_{(k-m)\delta}(x) + \frac{1}{24}E_\theta\left(\int \phi^{(4)}_{(k-m)\delta}(x-\theta y)[y^4]B_m(y)\,dy\right),$$

where E_θ refers to an expectation under the probability measure with density $4(1-\theta)^3$ on [0, 1]. $\phi^{(4)}(z)[y^4]$ is the fourth derivative of ϕ at z in the direction y. The third summand, we estimate by (6) and (42), using $m < k/2$,

$$\begin{aligned}
&\le Lk^{-2}E_\theta \int \phi_{(k-m)\delta(1+\varepsilon)}(x-\theta y)|y|^4\Gamma_m(y)\,dy \\
&= Lk^{-2}E_\theta\theta^{-d}\int \phi_{(k-m)\delta(1+\varepsilon)/\theta^2}\left(\frac{x}{\theta}-y\right)|y|^4\Gamma_m(y)\,dy \\
&\le Lk^{-2}\gamma_m^{(2)}E_\theta\theta^{-d}\phi_{(k-m)\delta(1+\varepsilon)/\theta^2+m}\left(\frac{x}{\theta}\right) \\
&= Lk^{-2}\gamma_m^{(2)}E_\theta\phi_{(k-m)\delta(1+\varepsilon)+m\theta^2}(x) \\
&\le Lk^{-2}\gamma_m^{(2)}\psi_k(x)
\end{aligned}$$

as $\theta^2 \le \delta(1+\varepsilon)$ if λ is small enough [by (31)]. Furthermore,

$$\begin{aligned}
\overline{b}_m\dot{\phi}_{(k-m)\delta} &= \overline{b}_m\dot{\phi}_{k\delta} + O(\gamma_m^{(1)}mk^{-2}\psi_k), \\
b_m\phi_{(k-m)\delta} &= b_m\phi_{k\delta} - b_m m\delta\dot{\phi}_{k\delta} + O(\gamma_m m^2k^{-2}\psi_k), \\
\phi_{(k-1)\delta+1} &= \phi_{k\delta} + (1-\delta)\dot{\phi}_{k\delta} + O(k^{-2}\lambda^2\psi_k).
\end{aligned}$$

So we get

$$X^{(1)}_{k,j} = \left[\mu^{-1}(1-\delta) - \lambda\sum_{m=1}^{j\wedge(k/2)} a_m(\overline{b}_m - b_m m\delta)\right]\dot{\phi}_{k\delta} + O(\lambda k^{-2}\zeta^{(1)}_{j\wedge[k/2]}\psi_k).$$

The choice of δ was made such that the expression in square brackets is 0 if we extend the sum to ∞. Therefore, the expression in square brackets is in absolute value

$$\le L\lambda\sum_{m\ge j\wedge(k/2)}(|\overline{b}_m| + m|b_m|) \le L\lambda\sum_{m\ge j\wedge(k/2)}(\gamma_m^{(1)} + m\gamma_m) \le L\lambda\zeta^{(2)}_{j\wedge[k/2]},$$

and as $|\dot{\phi}_{k\delta}| \le Lk^{-1}\psi_k$, we get

$$|X^{(1)}_{k,j}| \le L\lambda\{k^{-2}\zeta^{(1)}_{j\wedge[k/2]} + k^{-1}\zeta^{(2)}_{j\wedge[k/2]}\}\psi_k. \tag{45}$$

For $X^{(2)}$, we simply use $\phi_{(k-1)\delta+1} = \phi_{k\delta} + O(\lambda k^{-1}\psi_k)$, and Proposition 2.1(c) to get

$$|X^{(2)}_{k,j}| \le L\lambda k^{-1}\zeta^{(2)}_j\psi_k, \tag{46}$$

and in a similar fashion, we get

$$|X_{k,j}^{(3)}| = L\lambda k^{-1}\zeta_{j\wedge[k/2]}^{(2)}\psi_k. \tag{47}$$

Using these estimates for $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, we get

$$\begin{aligned}\sum_{j=1}^{n}|\Delta_{n,j}^{(1)}| &\le L\lambda\left\{n^{-2}\sum_{j=1}^{n}\zeta_{j\wedge[n/2]}^{(1)} + n^{-1}\sum_{j=1}^{n}\zeta_{j\wedge[n/2]}^{(2)}\right\}\psi_n \\ &\le L\lambda\left\{n^{-2}\sum_{j=1}^{n}\zeta_j^{(1)} + n^{-1}\sum_{j=1}^{n}\zeta_j^{(2)}\right\}\psi_n = L\lambda\overline{\zeta}_n\psi_n,\end{aligned}$$

that is, estimate (35) for $\Delta^{(1)}$.

2.3.2. *Proof of* (36).

$$|\Delta_{j,j}^{(1)}| \le L\lambda\{j^{-2}\zeta_j^{(1)} + j^{-1}\zeta_j^{(2)}\}\psi_j \le \frac{L\lambda}{j}\overline{\zeta}_j\psi_j. \tag{48}$$

The first inequality is evident by (45)–(47). To see the second one, note first that $\zeta_j^{(2)}$ is decreasing in j, and therefore $\zeta_j^{(2)} \le \overline{\zeta}_j$ follows. It remains to prove $j^{-1}\zeta_j^{(1)} \le L\overline{\zeta}_j$ which is the same as to prove

$$1 + \sum_{i=0}^{2}\sum_{m=1}^{j} m^{2-i}\gamma_m^{(i)} \le L + Lj^{-1}\sum_{i=0}^{2}\sum_{m=1}^{j}(j-m+1)m^{2-i}\gamma_m^{(i)}. \tag{49}$$

If we restrict both sides to summations over $m \le 2j/3$, the inequality is evident. On the other hand, using the assumed monotonicity of the $\gamma_n^{(i)}$ sequences, we have

$$\begin{aligned}\sum_{m=2j/3}^{j} m^{2-i}\gamma_m^{(i)} &\le j^{2-i}\sum_{m=2j/3}^{j}\gamma_m^{(i)} \\ &\le j^{2-i}\sum_{m=j/3}^{2j/3}\gamma_m^{(i)} \\ &\le 27j^{-1}\sum_{m=j/3}^{2j/3}(j-m+1)m^{2-i}\gamma_m^{(i)}.\end{aligned}$$

As we had $|\Delta_{j,j}^{(2)}| \le \frac{L\lambda}{j}\overline{\zeta}_j\psi_j$ already by (44), the proof is complete.

2.3.3. *Proof of* (38). Recall (32), and write $f_n = f_n^{(1)} + f_n^{(2)}$ where $f_n^{(1)}$ is the first of the two summands, and $f_n^{(2)}$ the second. We similarly split $\kappa_n = \kappa_n^{(1)} + \kappa_n^{(2)}$.

Using (4), estimate

$$\sum_{j=1}^{n}\kappa_j^{(1)} * f_{n-j}^{(1)} = \sum_{j=1}^{n-1}\sum_{s=0}^{[j/2]}\sum_{t=1}^{[(n-j)/2]} t(\psi_s * \psi_t) * (\Gamma_{j-s} * \Gamma_{n-j-t})$$
$$\le L\sum_{j=1}^{n-1}\sum_{s=0}^{[j/2]}\sum_{t=1}^{[(n-j)/2]} t\chi_{n-s-t}(j-s)(\psi_{s+t} * \Gamma_{n-s-t})$$
$$\le L\sum_{r=1}^{[n/2]}\rho(r)(\psi_r * \Gamma_{n-r})$$

with

$$\rho(r) \stackrel{\text{def}}{=} \sum_{j=1}^{n-1}\sum_{s=0\vee(r-[(n-j)/2])}^{[j/2]\wedge(r-1)}(r-s)\chi_{n-r}(j-s) \le r\sum_{k=1}^{n-r-1}\alpha_{n,r}(k)\chi_{n-r}(k),$$

with

$$\alpha_{n,r}(k) \stackrel{\text{def}}{=} \#\{(j,s): 1 \le j \le n-1,\, j-s=k,$$
$$0 \vee (r-[(n-j)/2]) \le s \le [j/2] \wedge (r-1)\}.$$

It is elementary to check that $\alpha_{n,r}(k) \le \min(k, 2(n-r-k))$, which implies by (3) $\rho(r) \le 2K_1 r$, so we get

$$\sum_{j=1}^{n}\kappa_j^{(1)} * f_{n-j}^{(1)} \le Lf_n. \tag{50}$$

We next estimate

$$\sum_{j=1}^{n}\kappa_j^{(1)} * f_{n-j}^{(2)} = \sum_{j=1}^{n}\sum_{s=0}^{[j/2]}\overline{\zeta}_{n-j}\Gamma_{j-s} * \psi_{n-j+s}. \tag{51}$$

For the summands with $j-s \le [n/2]$ we have by (6) $\Gamma_{j-s} * \psi_{n-j+s} \le L\gamma_{j-s} * \psi_n$ and by (12), as $n-j \ge n/4$, we have $\overline{\zeta}_{n-j} \le L\overline{\zeta}_n$. So we get for this part of the sum on the RHS,

$$\le L\overline{\zeta}_n\psi_n\sum_{j=1}^{n}\sum_{s:\, s\le[j/2],\, j-s\le[n/2]}\gamma_{j-s} \le L\overline{\zeta}_n\psi_n.$$

For the summands on the RHS of (51) with $j-s > [n/2]$, we get, by substituting k for $n-j+s$, that it is $\le \sum_{k=1}^{[n/2]}[\sum_{s\le(n-k)/2}]\overline{\zeta}_{k-s}\psi_k\Gamma_{n-k} \le \sum_{k=1}^{[n/2]}k\psi_k\Gamma_{n-k}$, so that we have proved

$$\sum_{j=1}^{n}\kappa_j^{(1)} * f_{n-j}^{(2)} \le Lf_n. \tag{52}$$

We next prove

$$\sum_{j=1}^{n} \kappa_j^{(2)} * f_{n-j}^{(1)} \le L f_n, \tag{53}$$

$$\sum_{j=1}^{n} \kappa_j^{(2)} * f_{n-j}^{(1)} = \sum_{j=1}^{n-1} \frac{\overline{\zeta}_j}{j} \sum_{s=1}^{[(n-j)/2]} s[\psi_{j+s} * \Gamma_{n-j-s}].$$

We split $Q \stackrel{\text{def}}{=} \{(j,s): 1 \le j \le n-1, 1 \le s \le [(n-j)/2]\}$ into the part Q_1 with $j+s \le n/2$, the part Q_2 with $n/2 < j+s \le 3n/4$ and the part Q_3 with $j+s > 3n/4$. On $Q_2 \cup Q_3$ we again use (6) and estimate $\psi_{j+s} * \Gamma_{n-j-s} \le L\gamma_{n-j-s}\psi_n$. On Q_3, we must have $j \ge n/4$, and therefore

$$\sum_{Q_3} \frac{\overline{\zeta}_j}{j} s\gamma_{n-j-s} \le L\frac{\overline{\zeta}_n}{n} \sum_{Q_3} s\gamma_{n-j-s} \le L\overline{\zeta}_n,$$

$$\begin{aligned}\sum_{Q_2} \frac{\overline{\zeta}_j}{j} s\gamma_{n-j-s} &\le L\gamma_n \sum_{Q_2} \frac{\overline{\zeta}_j}{j} \le Ln\gamma_n \sum_{j=1}^{\infty} \frac{\overline{\zeta}_j}{j} \\ &\le Ln\gamma_n \le L\overline{\zeta}_n,\end{aligned}$$

the last inequality by (13). Finally,

$$\begin{aligned}\sum_{Q_1} \frac{\overline{\zeta}_j}{j} s[\psi_{j+s} * \Gamma_{n-j-s}] &= \sum_{k=1}^{[n/2]} \psi_k * \Gamma_{n-k} \sum_{Q_1 \cap \{(j,s):\, j+s=k\}} \frac{\overline{\zeta}_j}{j} s \\ &\le L \sum_{k=1}^{[n/2]} k(\psi_k * \Gamma_{n-k}).\end{aligned}$$

Therefore, we have proved (53).

Finally, it remains to investigate

$$\sum_{j=1}^{n} \kappa_j^{(2)} * f_{n-j}^{(2)} = \psi_n \sum_{j=1}^{n} \frac{\overline{\zeta}_j}{j} \overline{\zeta}_{n-j}.$$

The summation over $j \le n/2$ is $\le L\overline{\zeta}_n \sum_j \overline{\zeta}_j/j \le L\overline{\zeta}_n$ by (12), and the summation over $j > n/2$ is $\le (\overline{\zeta}_n/n)\sum_{j \le n} \overline{\zeta}_j \le \overline{\zeta}_n \sum_j (\overline{\zeta}_j/j) \le L\overline{\zeta}_n$. Therefore

$$\sum_{j=1}^{n} \kappa_j^{(2)} * f_{n-j}^{(2)} \le L f_n. \tag{54}$$

Combining (50), (52), (53) and (54) proves the claim.

3. Application to weakly self-avoiding walks: Proof of Theorem 1.2. We choose an ε with $0 < \varepsilon \leq 1/100$ which will be fixed through the rest of this section.

We derive Theorem 1.2 by applying the main Theorem 1.1 with

$$\Gamma_n \stackrel{\text{def}}{=} K n^{-d/2} \sum_{k=1}^{n} k^{1-d/2} \phi_{2k/5}, \tag{55}$$

with

$$K \stackrel{\text{def}}{=} 8\mathrm{e}^{5/4}\left(1 + \tfrac{3}{2}\left(1 + \tfrac{1}{100}\right)^{d/2}\right). \tag{56}$$

Let us first show that this Γ_n satisfies (B1)–(B4) in Condition 1.1:

LEMMA 3.1. *If $d \geq 5$, then the sequence $\{\Gamma_n\}$ defined in* (55) *satisfies* (B1)–(B4) *from Condition* 1.1.

PROOF. (B2) and (B4) are readily checked.

(B1)

$$\begin{aligned}\Gamma_n * \Gamma_m &= K^2 (nm)^{-d/2} \sum_{k \leq n} \sum_{l \leq m} (kl)^{1-d/2} \phi_{2(k+l)/5} \\ &= K^2 \left(\frac{n+m}{nm}\right)^{d/2} (n+m)^{-d/2} \sum_{t=2}^{n+m} \left(\sum_{k=1}^{t-1} \big(k(t-k)\big)^{1-d/2} \right) \phi_{2t/5} \\ &\leq C(d) \left(\frac{n+m}{nm}\right)^{d/2} \Gamma_{n+m},\end{aligned}$$

for some constant $C(d) > 0$ depending only on d, which proves (B1). Note that the last inequality holds only when $d \geq 5$.

(B3) We use the fact that $|y|^{2k} \phi_j \leq L j^k \phi_{3j/2}$ for $j \in \mathbb{N}$ and $k = 0, 1, 2$. Therefore, we have for $m \leq t$,

$$\begin{aligned}\int \phi_t(\cdot - y) |y|^{2k} \Gamma_m(y)\, dy &\leq C(d) m^{-d/2} \sum_{j=1}^{m} j^{1-d/2+k} \phi_{t+3j/5} \\ &\leq C(d) \phi_{t+m} m^{-d/2} \sum_{j=1}^{m} j^{1-d/2+k} \\ &\leq L \gamma_m^{(k)} \phi_{t+m}.\end{aligned}$$

□

We keep our convention of the last section concerning the constant L. However, as we have chosen ε fixed, and a concrete Γ which specifies K_1–K_6, depending only on the dimension $d \geq 5$, L now depends only on the dimension d.

With this choice of Γ, we have $\overline{\zeta}_n = O(r_n)$, where r_n is defined in (21), and therefore the bound in Theorem 1.1 is

$$
\begin{aligned}
&L\Bigg[\sum_{s=1}^{[n/2]} s\Bigg(\phi_{s\delta(1+\varepsilon)} * (n-s)^{-d/2}\sum_{k=1}^{n-s} k^{1-d/2}\phi_{2k/5}\Bigg)(x) + r_n\phi_{n\delta(1+\varepsilon)}(x)\Bigg] \\
&\quad \le L\Bigg[n^{-d/2}\sum_{s=1}^{[n/2]} s\Bigg(\phi_{s\delta(1+\varepsilon)} * \sum_{k=1}^{n-s} k^{1-d/2}\phi_{2k/5}\Bigg)(x) + r_n\phi_{n\delta(1+\varepsilon)}(x)\Bigg] \qquad (57)\\
&\quad \le L\Bigg[n^{-d/2}\sum_{s=1}^{[n/2]} s\phi_{s\delta(1+\varepsilon)}(x) + r_n\phi_{n\delta(1+\varepsilon)}(x)\Bigg],
\end{aligned}
$$

the last inequality provided

$$\delta(1+\varepsilon) \ge 4/5, \qquad (58)$$

which is achieved by choosing λ small enough. To see the second inequality in (57), we sum $sk^{1-d/2}\phi_{s\delta(1+\varepsilon)+2k/5}$ over s, k satisfying $s\delta(1+\varepsilon) + 2k/5 \in (s'-1, s']\delta(1+\varepsilon)$, estimate $\phi_{s\delta(1+\varepsilon)+2k/5}$ by $L\phi_{s'\delta(1+\varepsilon)}$ and finally sum over s'. This leads to

$$L\sum_{s'} s'\phi_{s'\delta(1+\varepsilon)}(x),$$

but the summation extends beyond $[n/2]$. However, the sum over $s' > [n/2]$ can be estimated by $Ln^{d/2}r_n\phi_{n\delta(1+\varepsilon)}(x)$ provided all the s' are $\le n\delta(1+\varepsilon)$ which is guaranteed by (58).

In order to prove Theorem 1.2 we have to show that the connectivity function in (18) satisfies the recursion in (19). This is done in Section 3.1. Finally, we have to show that the B_n's, defined through $\Pi_n = \lambda c_n^{\mathrm{SAW}} B_n$, are bounded from above by the Γ_n sequence in (55). This is the content of Section 3.2.

There is nothing mysterious in our choice of $\{\Gamma_n\}$: simply *assume* that a (near) local CLT is correct. Then estimating the B_n for WSAW from the lace expansion, immediately leads to an estimate $|B_n| \le \Gamma_n$. On the other hand, $|B_n| \le \Gamma_n$ implies a (near) local CLT. There is sufficient "contraction" in this circle to make it work.

3.1. *Definition of the lace functions and recursion for WSAW.* This section contains standard material on the lace expansion adapted to the model in continuous space.

Given an interval $I = [a, b] \subset \mathbb{Z}$ of integers with $0 \le a \le b$, we refer to a pair $\{s, t\}$ $(s < t)$ of elements of I as an *edge*. To abbreviate the notation, we write st for $\{s, t\}$. A set of edges is called a *graph*. A graph Γ on $[a, b]$ is said to be *connected* if both a and b are endpoints of edges in Γ, and if, in addition, for any $c \in [a, b]$, there is an edge $st \in \Gamma$ such that $s < c < t$. Note that this is *not* in agreement with the usual notion of connectedness in graph theory. The set of all graphs on $[a, b]$

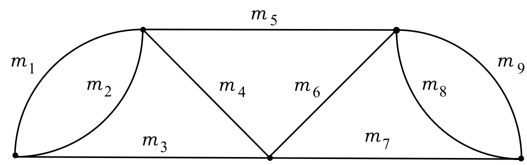

FIG. 1.

is denoted by $\mathcal{B}[a,b]$, and the subset consisting of all connected graphs is denoted by $\mathcal{G}[a,b]$. A *lace* is a minimally connected graph, that is, a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a,b]$ is denoted by $\mathcal{L}[a,b]$, and the set of laces on $[a,b]$ consisting of exactly N edges is denoted by $\mathcal{L}^{(N)}[a,b]$.

A lace $\ell = \{s_1t_1, \dots, s_Nt_N\}$ on $[0,n]$, with $s_1 = 0, t_N = n$, satisfies $s_i < t_{i-1}$, $i = 2, \dots, N$, and $t_i \le s_{i+2}$, $i = 1, \dots, N-2$. We can describe the lace by the interdistances $m_1, \dots, m_{2N-1}$ between the points s_i, t_i ordered increasingly, $s_1 = 0 < s_2 < t_1 \le s_3 < t_2 < \cdots$, that is, $m_1 = s_2, m_2 = t_1 - s_2$, etc. Then of course $\sum_{i=1}^{2N-1} m_i = n$. We switch freely between the s_i–t_i-representation of the lace and the representation by the m_i, without special notice. The restrictions on the m_i are $m_i > 0$ for i even and $m_i \ge 0$ for i odd, with the additional restriction at the boundary $m_1 > 0$ and $m_{2N-1} > 0$. (For $N = 2$, all the m_i are positive.) It is customary to visualize the laces as graphs by identifying the vertices connected by a bond. Figure 1 illustrates the example of a lace with $N = 4$.

The "basic" N-lace is the graph

$$\ell_N^0 \stackrel{\text{def}}{=} \{(0,2),(1,4),(3,6),\dots,(2N-5,2N-2),(2N-3,2N-1)\} \tag{59}$$

on $\{0,\dots,2N-1\}$. We will write $b_i = (\underline{i}, \overline{i})$ for the ith bond in this graph, that is, $\underline{1} = 0$, and $\underline{i} = 2i-3$ for $i = 1,\dots,N$, $\overline{i} = 2i$ for $i \le N-1$ and $\overline{N} = 2N-1$. Conversely, for $i = 0,\dots,2N-1$, we write $\beta(i) \in \{1,\dots,N\}$ for the unique element with $i \in \{\underline{\beta(i)}, \overline{\beta(i)}\}$, that is, $\beta(0) = 1, \beta(1) = 2$, etc. With this notation, we have for a lace in $\mathcal{L}^{(N)}[a,b]$,

$$s_i = \sum_{j=1}^{\underline{i}} m_j, \qquad t_i = \sum_{j=1}^{\overline{i}} m_j.$$

If $G = \{G_t\}_{t>0}$ is any family of functions in $\mathcal{C}_*^+(\mathbb{R}^d)$, augmented by $G_0 = \delta_0$, and $\ell \in \mathcal{L}^{(N)}[0,n]$, we write with $x_0 = 0, x_{2N-1} = x$,

$$\Xi_\ell(G,\rho)(x) \stackrel{\text{def}}{=} \int dx_1 \cdots dx_{2N-2} \prod_{i=1}^{2N-1} G_{m_i}(x_i - x_{i-1}) \prod_{i=1}^{n} \mathbb{I}_\rho(x_{\overline{i}} - x_{\underline{i}}). \tag{60}$$

For the moment, we need G only for integer m, but the more general situation is needed below.

Given a connected graph Γ on $[a,b]$, the following prescription associates to Γ a unique lace ℓ_Γ. The lace consists of edges $s_1t_1, s_2t_2, \dots,$ with $t_1, s_1, t_2, s_2, \dots$ determined (in that order) by

$$t_1 = \max\{t : at \in \Gamma\}, \qquad s_1 = a,$$
$$t_{i+1} = \max\{t : \exists s < t_i \text{ such that } st \in \Gamma\}, \qquad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}.$$

Given a lace ℓ, the set of all edges $st \notin \ell$ such that $\ell_{\ell\cup\{st\}} = \ell$ is denoted by $\mathcal{C}(\ell)$. Edges in $\mathcal{C}(\ell)$ are said to be *compatible* with ℓ. With this formalism, we can expand the product in (16), obtaining

$$K_{\lambda,\rho}[a,b](\mathbf{x}) = \sum_{\Gamma\in\mathcal{B}[a,b]} \prod_{st\in\Gamma} \bigl(-\lambda U^\rho_{st}(\mathbf{x})\bigr). \tag{61}$$

We also define an analogous quantity, in which the sum over graphs is restricted to connected graphs, namely,

$$J[a,b](\mathbf{x}) \stackrel{\text{def}}{=} \sum_{\Gamma\in\mathcal{G}[a,b]} \prod_{st\in\Gamma} \bigl(-\lambda U^\rho_{st}(\mathbf{x})\bigr). \tag{62}$$

Recalling (17), this allows us to define the *lace functions*, which are the key quantities in the lace expansion

$$\Pi_n(x_n) \stackrel{\text{def}}{=} \int J[0,n](\mathbf{x})\Phi[0,n](\mathbf{x}) \prod_{i=1}^{n-1} dx_i \tag{63}$$

for any $n \ge 1$ and $x_n \in \mathbb{R}^d$. Identity (19) is shown in the following lemma.

LEMMA 3.2 (Convolution equation for WSAW). *For $n \ge 1$,*

$$C_n^{\text{SAW}} = C_{n-1}^{\text{SAW}} * \phi + \sum_{k=1}^{n} \Pi_k * C_{n-k}^{\text{SAW}}.$$

PROOF. It suffices to show that for each path $\mathbf{x}$, we have (suppressing $\mathbf{x}$ in the formulas)

$$K[0,n] = K[1,n] + \sum_{m=1}^{n} J[0,m]K[m,n]. \tag{64}$$

Then (19) is obtained after the insertion of (64) into (18) followed by factorization of the integral over $\mathbf{x}$. To prove (64), we note from (61) that the contribution to $K[0,n]$ from all graphs Γ for which 0 is not in an edge is exactly $K[1,n]$. To resum the contribution from the remaining graphs, we proceed as follows. When Γ does contain an edge ending at 0, we let $m[\Gamma]$ denote the largest value of m

such that the set of edges in Γ with at least one end in the interval $[0,m]$ forms a connected graph on $[0,m]$. Then resummation over graphs on $[m,n]$ gives

$$K[0,n]=K[1,n]+\sum_{m=1}^{n}\sum_{\Gamma\in\mathcal{G}[0,m]}\prod_{st\in\Gamma}(-\lambda U_{st})K[m,n]. \tag{65}$$

With (62) this proves (64). □

We next rewrite (63) in a form that can be used to obtain good bounds on $\Pi_n(x)$. First, splitting the sum over $\Gamma\in\mathcal{G}[a,b]$ according to the number of bonds in ℓ_Γ, we get

$$\begin{aligned} J[a,b] &= \sum_{N\geq 1} J_N[a,b],\\ J_N[a,b] &\stackrel{\text{def}}{=} \sum_{\ell\in\mathcal{L}^{(N)}[a,b]}\sum_{\Gamma:\ell_\Gamma=\ell}\prod_{st\in\ell}(-\lambda U_{st})\prod_{s't'\in\Gamma\setminus\ell}(-\lambda U_{s't'})\\ &= (-\lambda)^N\sum_{\ell\in\mathcal{L}^{(N)}[a,b]}\prod_{st\in\ell}U_{st}\prod_{s't'\in\mathcal{C}(\ell)}(1-\lambda U_{s't'})\\ &= (-\lambda)^N J^{(N)}[a,b], \qquad \text{say}. \end{aligned} \tag{66}$$

Implementing into (63), we get a splitting

$$\Pi_n=\sum_{N\geq 1}(-\lambda)^N\Pi_n^{(N)},$$

where $\Pi_n^{(N)}$ is obtained by replacing $J[0,n]$ in (63) by $J^{(N)}[0,n]$. Note that the sum over N is restricted to $N<n$.

An important point is that we obtain an upper bound for $\Pi_n^{(N)}$ by dropping into (66) the factors $(1-\lambda U_{s't'})$ for all $s't'$ which cross an endpoint of any st bond of the lace ℓ. This gives the upper bound

$$\Pi_n^{(N)}(x)\leq\sum_{\ell\in\mathcal{L}^{(N)}[0,n]}\Xi_\ell(C,\rho)(x) \tag{67}$$

for $N\geq 2$, where $C=\{C_n\}$. For $N=1$, there is the slight modification from "restoring" the $0n$ bond, $\Pi_n^{(1)}(x)=\Xi_{0n}(C,\rho)(x)/(1-\lambda)$.

3.2. *Bounds on the lace function.* We need below a slight generalization of the notion in (60). Given G_t, defined for real $t>0$, we define for an additional sequence $\mathbf{t}=(t_1,\ldots,t_{2N-1})$, $\Xi_\ell(G,\rho,\mathbf{t})(x)$ by replacing m_i on the right-hand side of (60) by m_i+t_i. Also, given an arbitrary sequence $\mathbf{r}=(r_1,\ldots,r_{2N-1})$ of elements in $\mathbb{N}_0$, we write

$$\xi_n^{(N)}(G,\rho,\mathbf{t},\mathbf{r})(x)\stackrel{\text{def}}{=}\sum_{\mathbf{m}\in\mathcal{L}^{(N)}[0,n],m_i\geq r_i}\Xi_\ell(G,\rho,\mathbf{t})(x).$$

Of course, finally we are interested only in the case where the r_i are the "natural" ones from the restriction of the laces, that is, $r_1 = r_2 = 1, r_3 = 0$ (if $N \geq 3$), etc. We write $\mathbf{r}^{(0)}$ for this starting sequence. If $\mathbf{t}$ is the sequence of 0's, and $\mathbf{r} = \mathbf{r}^{(0)}$, we drop these arguments in the notation. We will need the more general ones in an induction argument.

We first state a simple lemma regarding normal densities.

LEMMA 3.3. *If* $u, v, s, t > 0, x, y \in \mathbb{R}^d$, *then*

$$\int \phi_u(z)\phi_v(x-z)\phi_s(z)\phi_t(y-z)\,dz \leq L\left[\frac{u+v}{uv}\right]^{d/4}\left[\frac{s+t}{st}\right]^{d/4}\phi_{u+v}(x)\phi_{s+t}(y). \tag{68}$$

PROOF. By Cauchy–Schwarz, the left-hand side is

$$\leq \sqrt{\int \phi_u^2(z)\phi_v^2(x-z)\,dz}\sqrt{\int \phi_s^2(z)\phi_t^2(y-z)\,dz},$$

which equals the RHS of (68) by an elementary computation. □

Let us fix some more notation. We saw that an N-lace is nothing but a sequence $\mathbf{m} = (m_1, \ldots, m_{2N-1})$ with $\sum_i m_i = n$, and satisfying some restrictions, like $m_1 \geq 1, m_2 \geq 1, m_3 \geq 0$. We write $\mathbf{r}^{(0)} = (1, 1, 0, 1, 0, \ldots)$ for this sequence of restrictions. For an arbitrary sequence $\mathbf{r} \in \mathbb{N}_0^{2N-1}$ with $\sum_i r_i \leq n$, we write $\mathcal{L}_{\mathbf{r}}^{(N)}[0,n]$ for the set of $\mathbf{m}$ satisfying $m_i \geq r_i, \forall i$, and $\sum_i m_i = n$. The r_i need not satisfy $r_i \geq r_i^{(0)}$.

LEMMA 3.4. *For* $v > 0$, $\mathbf{m} \in \mathbb{N}_0^{2N-1}$, $t_i \geq 0$, $\mathbf{x} = (x_1, \ldots, x_{N-1}) \in (\mathbb{R}^d)^{N-1}$, *let*

$$\Phi_{N,\mathbf{m},\mathbf{t}}^{(v)}(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^{2N-1} \phi_{vm_i+t_i}(x_{\beta(i)-1} - x_{\beta(i-1)-1}),$$

with $x_0 = 0$. *If for any* i, *either* $r_i \geq 1$ *or* $t_i \geq c$, *then for* $d \geq 5$ *and* $N \geq 3$,

$$\sum_{\mathbf{m}\in\mathcal{L}_{\mathbf{r}}^{(N)}[0,n]} \int dx_1 \Phi_{N,\mathbf{m},\mathbf{t}}^{(v)}(\mathbf{x}) \leq L(c) \sum_{\mathbf{m}'\in\mathcal{L}_{\mathbf{r}'}^{(N-1)}[0,n]} \Phi_{N-1,\mathbf{m}',\mathbf{t}'}^{(v)}(x_2, \ldots, x_{N-1}),$$

where $\mathbf{r}' \stackrel{\text{def}}{=} (r_3, r_1 + r_4, r_2 + r_5, r_6, \ldots, r_{2N-1})$, $\mathbf{t}' \stackrel{\text{def}}{=} (t_3, t_1 + t_4, t_2 + t_5, t_6, \ldots, t_{2N-1})$ *which both have* $2N-3$ *components.*

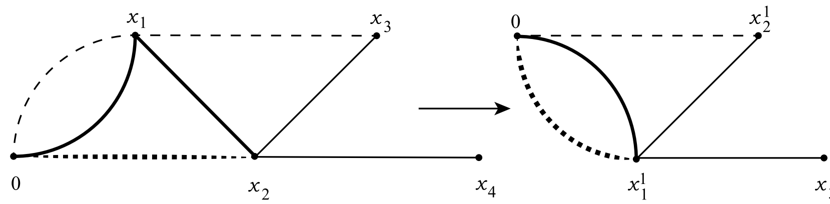

FIG. 2.

PROOF. The part of $\Phi^{(\nu)}_{N,\mathbf{m},\mathbf{t}}(\mathbf{x})$ which contains x_1 is

$$\phi_{m_1\nu+t_1}(x_1)\phi_{m_2\nu+t_2}(x_1)\phi_{m_4\nu+t_4}(x_2-x_1)\phi_{m_5\nu+t_5}(x_3-x_1).$$

In case $N=3$, we have $x_3=x_2$. Using the previous lemma for the integration over x_1, and summing over m_1, m_2, m_4, m_5, keeping $m_1+m_4=m_2', m_2+m_5=m_3'$ fixed, we get for the x_1-integration and this restricted summation of the above expression, a bound

$$\le L(c)\phi_{m_2'\nu+t_1+t_4}(x_2)\phi_{m_3'\nu+t_2+t_5}(x_3).$$

We write $\mathbf{m}'\in\mathbb{N}_0^{2N-3}$ with $m_1'=m_3, m_2'=m_1+m_4, m_3'=m_2+m_5$ and $m_i'=m_{i+2}$ otherwise. The restrictions on the m_i' are evidently given by $m_i'\ge r_i'$. Summing over $\mathbf{m}'$ gives the desired bound. $\square$

In Figure 2 is the illustration of the "collapsing mechanism."

LEMMA 3.5. *Assume $d\ge 5$. If for some $\nu\in[\frac{19}{20},\frac{21}{20}]$ and $m\in\mathbb{N}$, $m\ge 3$, one has*

$$G_n(x)\le\phi_{n\nu}(x), \tag{69}$$

for all $n<m$, then for $N\ge 2$, $0<\rho\le 1$, we have with $L=L(d)$, not depending on m, N

$$\xi_m^{(N)}(G,\rho)\le L^N\rho^{Nd}\Gamma_m,$$

where Γ_m is defined in (55).

PROOF. We choose $\nu'\stackrel{\text{def}}{=}20\nu/19$. Note that $\nu''\stackrel{\text{def}}{=}\nu'+1/100<6/5$, and therefore $2\nu''/3<4/5$.

Assumption (69) implies

$$\xi_n^{(N)}(G,\rho)\le\xi_n^{(N)}(\phi^{(\nu)},\rho), \tag{70}$$

where $\phi^{(\nu)}=\{\phi_{\nu t}\}$.

We first want to get rid of the $\mathbb{I}_\rho$. In $\Xi_\ell(\phi^{(\nu)},\rho)(x)$, if all the m_i are ≥ 1, we can simply use $\phi_{m\nu}(x)\le L\phi_{m\nu'}(x')$ for $|x-x'|\le\rho\le 1$ from which we easily get

$$\Xi_\ell(\phi^{(\nu)},\rho)(x)\le L^N\rho^{Nd}\Xi_\ell(\phi^{(\nu')},0)(x).$$

There is, however, a complication due to the possibility of having $m_i = 0$ in the summation. Such i have to be odd, and the possibility is not present for m_1 and m_{2N-1}. Using the fact that if $m_i = 0$, then $m_{i-1}, m_{i+1} \geq 1$, we get

$$\Xi_\ell(\phi^{(\nu)}, \rho)(x) \leq L^N \rho^{Nd} \Xi_\ell(\phi^{(\nu')}, 0, \mathbf{t}^{(0)})(x) \tag{71}$$

for all $\ell \in \mathcal{L}^{(N)}[0,n]$ where $t_i^{(0)} = 0$ for i even and $i = 1, 2N-1$ and $t_i^{(0)} = 1/200$ for the other i odd. Actually, the adding of the constant $1/200$ would be necessary only if m_i in fact equals 0, but there is no harm adding it always with those i for which m_i *can* be 0. It remains to estimate

$$\xi_m^{(N)}(\phi^{(\nu')}, 0, \mathbf{t}^{(0)}) = \sum_{\mathbf{m}\in\mathcal{L}^{(N)}[0,n]} \int dx_1 \cdots dx_{N-2} \Phi^{(\nu')}_{N,\mathbf{m},\mathbf{t}^{(0)}}(\mathbf{x})$$

with $x = x_{N-1}$.

For $N = 2$, there is $t_i^{(0)} = 0$ for all $i = 1, 2, 3$ and no integration,

$$\begin{aligned} \xi_m^{(2)}(\phi^{(\nu')}, 0) &\leq 6 \sum_{1\leq k\leq l\leq j, k+l+j=m} \phi_{k\nu'}\phi_{l\nu'}\phi_{j\nu'} \\ &\leq Lm^{-d/2} \sum_{k=1}^{\lceil m/3\rceil} k^{-d/2+1}\phi_{k\nu'} \\ &\leq Lm^{-d/2} \sum_{k=1}^{[m/2]} k^{-d/2+1}\phi_{4k/5} \leq L\Gamma_m. \end{aligned} \tag{72}$$

For $N \geq 3$, we apply Lemma 3.4. Starting with $\mathbf{r}^{(0)}$ and $\mathbf{t}^{(0)}$, we recursively define $\mathbf{r}^{(k+1)} \stackrel{\text{def}}{=} \mathbf{r}^{(k)\prime}, \mathbf{t}^{(k+1)} \stackrel{\text{def}}{=} \mathbf{t}^{(k)\prime}$. Applying the lemma $N-2$ times we arrive at

$$\xi_m^{(N)}(\phi^{(\nu')}, 0, \mathbf{t}^{(0)})(x) \leq L^{N-2} \sum_{\mathbf{m}\in\mathcal{L}^{(2)}_{\mathbf{r}^{(N-2)}}[0,m]} \Phi^{(\nu')}_{2,\mathbf{m},\mathbf{t}^{(N-2)}}(x).$$

(There is no integration left when $N = 2$.) The $\hat{\boldsymbol{\tau}}^{(N)} \stackrel{\text{def}}{=} 200\mathbf{t}^{(N-2)}, \hat{\mathbf{r}}^{(N)} \stackrel{\text{def}}{=} \mathbf{r}^{(N-2)}$ can easily be computed in the following way: $\hat{\mathbf{r}}^{(2)} = (1,1,1), \hat{\mathbf{r}}^{(3)} = (0,2,2), \hat{\mathbf{r}}^{(4)} = (1,1,3), \hat{\boldsymbol{\tau}}^{(2)} = (0,0,0), \hat{\boldsymbol{\tau}}^{(3)} = (1,0,0), \hat{\boldsymbol{\tau}}^{(4)} = (1,1,0)$ and $\hat{\mathbf{r}}^{(k+3)} = \hat{\mathbf{r}}^{(k)} + (1,1,1), \hat{\boldsymbol{\tau}}^{(k+3)} = \hat{\boldsymbol{\tau}}^{(k)} + (1,1,1)$. Therefore, the only case where an $\hat{r}_i$ can be 0 is $N = 3$. Here one estimates by a similar expression as that on the right-hand side of (72) with the only difference being that summation over k starts at 0, but instead of $\phi_{k\nu'}$, one has $\phi_{k\nu'+1/200}$. However, for $k = 0$, one estimates $\phi_{1/200} \leq L\phi_{\nu'}$, giving an estimate similar to (72) with a different L. If $N > 3$, all the $\hat{r}_i^{(N)}$ are ≥ 1, and it is easily checked that $2\hat{r}_i^{(N)} \geq \hat{\tau}_i^{(N)}$. Using this, one estimates

$$\phi_{k\nu'+\hat{t}_i^{(N-2)}} \leq L\phi_{k\nu''}$$

for $k \geq \hat{r}_i^{(N-2)}$, so one gets the same estimate as in (72) replacing ν' by ν''. As $\nu'' < 6/5$, the argument is the same, leading to the desired estimate. □

3.3. *Checking Condition* 1.1 *and proof of Theorem* 1.2. We prove that given $\varepsilon \leq 1/100$, there exists $\lambda_0(d,\varepsilon)$ such that for $0 < \lambda \leq \lambda_0(d,\varepsilon)$, one has $|B_m| \leq \Gamma_m$ for all m, where $B_m \stackrel{\text{def}}{=} \Pi_m/\lambda c_m$, and Γ_m is given by (55). This is proved by induction on m. Below, we use the phrase "for small enough λ," in the sense that "small enough" may depend on ε and d, but on nothing else.

For $m = 1$, $\Pi_1(x) = -\lambda\phi(x)\mathbb{I}_\rho(x)$ and as $c_1 = 1 - \lambda\int_{|x|\leq\rho}\phi(x)\,dx \geq 1-\lambda$, we have, provided $\lambda_0(d,\varepsilon) \leq 1/2$,

$$|B_1| \leq 2\mathrm{e}^{3/4}\phi_{2/5} \leq 5\phi_{2/5} \leq \Gamma_1. \tag{73}$$

So the base of the induction is proved.

Assume now that $|B_k| \leq \Gamma_k$ for $k < m$, and define the truncated sequence $\overline{B}_k$ by B_k for $k < m$, and 0 for $k \geq m$. This sequence defines $\{\overline{C}_n\}$ via (1), and then $\overline{\mu}$ given by (23), and $\overline{A}_n = \overline{\mu}^{-n}\overline{C}_n$. Furthermore $\overline{\delta}$ is defined by (30). As $|\overline{\delta}-1| \leq L\lambda$, with L depending only on d,ε, we have

$$\left|\overline{\delta}(1+\varepsilon) - 1\right| \leq \tfrac{1}{20} \tag{74}$$

if λ is small enough. We can apply Theorem 1.1 leading to

$$|\overline{A}_n - \overline{a}_n\phi_{n\overline{\delta}}| \leq L\lambda\left[r_n\phi_{n\overline{\delta}(1+\varepsilon)} + n^{-d/2}\sum_{j=1}^{[n/2]} j\phi_{j\overline{\delta}(1+\varepsilon)}\right]. \tag{75}$$

As $\sup_n|\overline{a}_n - 1| \leq L\lambda$, we have for small enough λ that $\overline{a}_n\phi_{n\overline{\delta}} \leq (3/2)(1 + 1/100)^{d/2}\phi_{n\overline{\delta}(1+\varepsilon)}$, and that the right-hand side of (75) is $\leq\phi_{n\overline{\delta}(1+\varepsilon)}$, if λ is small enough, so that $\overline{A}_n \leq K_1(d)\phi_{n\overline{\delta}(1+\varepsilon)}$, where $K_1(d) \stackrel{\text{def}}{=} 1 + (3/2)(1+1/100)^{d/2}$, and therefore

$$\overline{C}_n \leq K_1(d)\overline{\mu}^n\phi_{n\overline{\delta}(1+\varepsilon)}. \tag{76}$$

As $\overline{B}_k = B_k$ for $k < m$, we have $\overline{C}_n = C_n$ for $n < m$.

With estimate (76), we can bound Π_m:

$$\Pi_m^{(1)}(x) = \mathbb{I}_\rho(x)\int\prod_{\substack{0\leq s<t\leq m,\\ st\neq 0m}}\bigl(1-\lambda U_{st}(\mathbf{x})\bigr)\Phi[0,n](\mathbf{x})\prod_{i=1}^{n-1}dx_i.$$

We bound the product inside the integral from above by dropping all bonds with $t = m$ leading to

$$\begin{aligned}\Pi_m^{(1)}(x) &\leq \mathbb{I}_\rho(x)(\phi * C_{m-1})(x)\\ &\leq K_1(d)\mathbb{I}_\rho(x)\overline{\mu}^{m-1}\phi_{(m-1)\overline{\delta}(1+\varepsilon)+1}(x)\\ &\leq K_1(d)\overline{\mu}^{m-1}\mathbb{I}_\rho(x)\phi_{(m-1)\overline{\delta}(1+\varepsilon)+1}(0).\end{aligned}$$

As $(m-1)\overline{\delta}(1+\varepsilon)+1 \geq m/2$, by (74), $\mathbb{I}_\rho(x) \leq (4\pi/5)^{d/2} \mathrm{e}^{5/4} \phi_{2/5}(x)$, by $\rho \leq 1$ and $\overline{\mu} \geq 1/2$, by (31), if λ is small enough, we get

$$\Pi_m^{(1)}(x) \leq K_2(d)\overline{\mu}^m m^{-d/2} \phi_{2/5}(x) \leq \tfrac{1}{4}\overline{\mu}^m \Gamma_m(x),$$

with $K_2(d) \stackrel{\text{def}}{=} 2\mathrm{e}^{5/4} K_1(d)$, the second inequality, chosen similarly to the way K is chosen in (56).

For $\Pi_m^{(N)}$ with $N \geq 2$, we use (67), (76) and Lemma 3.5 and obtain $\Pi_m^{(N)} \leq K_1(d)^N \overline{\mu}^m \Gamma_m$, and therefore

$$\begin{aligned} |\Pi_m| &\leq \left[\frac{\lambda}{4} + \sum_{N=2}^{\infty} (K_1(d)\lambda)^N\right] \overline{\mu}^m \Gamma_m \\ &\leq \frac{\lambda}{2} \overline{\mu}^m \Gamma_m, \end{aligned}$$

if λ is small enough, implying

$$|B_m| \leq \frac{\overline{\mu}^m}{2c_m} \Gamma_m. \tag{77}$$

It remains to bound $\overline{\mu}^m/c_m$. Note that by (2), $\overline{b}_m = 0$ and $\overline{b}_k = b_k$ for $k < m$, we get

$$\begin{aligned} \overline{c}_m &= c_{m-1} + \lambda \sum_{k=1}^{m-1} c_k b_k c_{m-k} \\ &= c_{m-1} + \lambda \sum_{k=1}^{m} c_k b_k c_{m-k} - \lambda c_m b_m = c_m(1 - \lambda b_m). \end{aligned}$$

However, $|\overline{\mu}^m/\overline{c}_m - 1| \leq L\lambda$, and from (77), we have $|b_m| \leq L m^{-d/2} \overline{\mu}^m / c_m$. Using this, we get $|b_m| \leq L$, and from that $|\overline{\mu}^m/c_m - 1| \leq L\lambda$, so we have $\overline{\mu}^m/c_m \leq 2$ for λ small enough. This shows that

$$|B_m| \leq \Gamma_m. \tag{78}$$

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