Transport coefficients and low energy excitations of a strongly interacting holographic fluid

Poovuttikul, N.

Author: Poovuttikul, N.
Title: Transport coefficients and low energy excitations of a strongly interacting holographic fluid
Date: 2017-11-16

You need a different way of looking at them than starting from single particle descriptions. You don't try to explain the ocean in terms of individual water molecules.

Sean Hartnoll, in Quanta Magazine

2

Lightning review of hydrodynamics and gauge/gravity duality

In this chapter, I briefly review some basic concepts of hydrodynamics and gauge/gravity duality. I will start by constructing the simplest relativistic hydrodynamics from global symmetry and gradually introduce more elements which are essential for chapters 3,4 and 5. The following section review more technical details on how to compute 2-point correlation function and extract transport coefficients from hydrodynamic equations. Then I present a very short introduction to the holographic duality with the minimum technical details possible. I will also restrict all of the discussion to relativistic hydrodynamics, ignoring non-relativistic model.

None of the material presented in this chapter is new. Hydrodynamics is, strictly speaking, a millennia-old subject. Many aspects of relativistic hydrodynamics can be found in a classic textbook by Landau and Lifshitz [48] or, for a more modern notation and applications, by Rezzolla and Zanotti [49]. The discussion in section 2.1-2.2 is inspired by [50, 51] complemented by the discussion about global symmetry in [46]. The procedures presented in section 2.2 are widely used in the holography community to compute correlation function and the review of the method itself (without reference to holography) is neatly summarised in [51]. The basic principle of gauge/gravity duality has been very well documented over the last few decades from both string theory [52, 53] and applications to condensed matter or quark-gluon plasma perspectives(see e.g. reviews, lecture notes [54–57]) and books [4, 58–60]). Therefore I will spare the reader the details and only cover the portion of the story that is most relevant to this thesis.

2.1 Global symmetry, conserved current and background fields

As mentioned in the previous chapter, one way to think about hydrodynamics as the effective theory describing the low-energy dynamics of many-body systems where the excitation wave length is much longer than the mean free path. Thess low-energy dynamics are governed only by the conservation laws of the system. Thus, one may say that hydrodynamics owes its existence to continuous global symmetries, which are related to conserved currents by the Noether theorem.

Usually, we derive the Noether current from a certain global symmetry transformation of fields in the microscopic theory. However, there is another way to look at it. A more Lagrangian-free way to access a global symmetry is to couple the system to a non-dynamical background field. For example, to introduce the chemical potential in the grand canonical ensemble, we deform the Hamiltonian in the following way

$$H \to H - \mu Q\,, \tag{2.1}$$

where Q is the number of particles. Now, we know that particle number is the

Noether charge of a conserved $U(1)$. In relativistic notation, this deformation can instead be written as

$$H \to H - \int d^{d-1}x\, J^\mu A_\mu, \qquad \text{where} \qquad \partial_\mu J^\mu = 0, \tag{2.2}$$

where the $U(1)$ charge is obtained by integrating the current J^μ over the spatial volume $Q = \int d^{d-1}S_\mu\, J^\mu$. The vector field A_μ plays the role of the non-dynamical *background field* we discussed earlier. This background field can also transform in the same way as the $U(1)$ gauge field: $A_\mu \to A_\mu + \partial_\mu \Lambda$. Thus we should think about this background field as a *non-dynamical background gauge field.*

Interestingly, the background gauge field with the gauge group $\mathfrak{G}$ can be used to singled out the Noether current associated to the global symmetry group $\mathfrak{G}$. This can be seen due to the fact that the minimum coupling between the matter and the background gauge field, $\int d^dx\, J^\mu A_\mu$, will not be invariant under the gauge transformation, unless J^μ is a conserved current.

The global symmetry also plays a crucial role in quantum physics. The state of a quantum system can be characterised by the eigenvalues of the conserved charge operators. In the study of the hydrodynamic limit of quantum systems, the quantity we are interested in is therefore the expectation of the operator $\hat{J}^\mu$ acting on the thermal state $\langle \hat{J}^\mu \rangle_{\text{thermal}}$. To access these quantities without directly specifying the Lagrangian, we put the theory on a curved manifold $\mathcal{M}$ with metric $g_{\mu\nu}$ and, if the theory possesses the $U(1)$ global symmetry, we couple it to the background $U(1)$ gauge field A_μ. The background metric is introduced to capture the stress-energy tensor $\langle T^{\mu\nu} \rangle_{\text{thermal}}$. With these ingredients, we can write down the partition function, which has the background fields as its arguments. For example, the partition function of the relativistic charged fluid in the presence of a background metric $g_{\mu\nu}$ and gauge field A_μ can be schematically written as

$$Z[g_{\mu\nu}, A_\mu] = \left\langle \exp\left[i \int d^dx \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} g_{\mu\nu} + J^\mu A_\mu \right) \right] \right\rangle \tag{2.3}$$

The background fields can now be used as a source for the conserved currents.

The variational derivative of the partition function with respect to the background fields gives

$$\langle T^{\mu\nu}\rangle = \frac{-2i}{\sqrt{-g}}\frac{\delta \log Z}{\delta g_{\mu\nu}}, \qquad \langle J^{\mu}\rangle = \frac{-i}{\sqrt{-g}}\frac{\delta \log Z}{\delta A_{\mu}} \tag{2.4}$$

The next step is to to express the conserved currents in the long wavelength limit. This can be done by gradient expanding the conserved currents and writing down all possible terms allowed by symmetry at a given order. This is not yet a physical hydrodynamics as the theory has to be augmented by additional physical requirements such as the extensivity condition and the positivity of the local entropy production near equilibrium.[1]

To make things more explicit, I first illustrate how this procedure works in a typical charge neutral fluid in section 2.1.1. The relevant global symmetries of this section are only spacetime translational symmetries (equivalently, $T^{\mu\nu}$ is the only conserved current). Then I will briefly review the hydrodynamic constructions relevant to the work of chapter 3-5, where some global symmetry groups are added or broken.

2.1.1 Charge neutral relativistic fluid

First of all, one needs to express $T^{\mu\nu}(x)$ in terms of the local macroscopic variables in order to solve the conservation equation

$$\nabla_{\nu}T^{\mu\nu} = 0 \tag{2.5}$$

which is a requirement that the system is invariant under diffeomorphism. To do this, we split the system into small pieces and assume that each piece, called fluid elements, which occupy an infinitesimal volume at position x^{μ}. This allows one to define local thermodynamics variables: the temperature $T(x)$, the entropy $s(x)$, the energy density $\varepsilon(x)$ and the pressure $p(x)$, which are small variations of their global equilibrium values. Each fluid element is also allowed

[1] Alternatively, one can also attempt to gradient expand the effective action. It turns out that this approach gives us more insight on the emergent symmetry of the system once dissipative effects are properly taken into account properly. We will come back to this approach in chapter 6.

to move around with relativistic velocity $u^\mu(x)$, which satisfied the condition $u^\mu u_\mu = -1$ (see Fig 2.1).

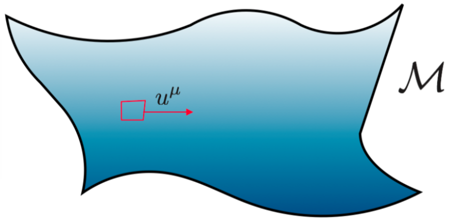

Figure 2.1. An illustration of a fluid living on the manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. The red box denotes a fluid element at point x^μ which moves with a velocity u^μ.

We are now ready to construct the stress-energy tensor. Since the stress-energy tensor is a rank-2 symmetric tensor, the possible 0^{th} order in derivative expansion is

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{O}(\partial^1), \qquad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{2.6}$$

where $\Delta^{\mu\nu}$ is acting as a projector to a $d-1$ dimensional plane perpendicular to u^μ i.e. $u_\mu \Delta^{\mu\nu} = 0$ and $\Delta^{\mu\nu} g_{\mu\nu} = d-1$. The reader might notice that this form of the stress-energy tensor is similar to the one describing an ideal fluid if $\mathcal{E} = \varepsilon$ and $\mathcal{P} = p$. However, at this level, $\mathcal{E}$ and $\mathcal{P}$ are just arbitrary functions of thermodynamics variables. The expression of the conserved currents in term of these variables is called *the constitutive relation.*

To rigorously give a physical meaning to these two coefficients, we will follow a beautiful analysis of the equilibrium partition function in the presence of a background fields, introduced in [61, 62]. In this setup, the finite temperature can be incorporated by putting the system in the cylinder $\tilde{\mathcal{M}}_{d-1} \times S^1$ (or equivalently by performing the Wick rotation which transforms the original manifold $\mathcal{M}$ to $\tilde{\mathcal{M}}_{d-1} \times S^1$), where we denote the radius of S^1 as L_0. The fluid velocity u^μ in equilibrium is proportional to the Killing vector ξ^μ of the system[2]. In this language, the temperature and fluid velocity can be expressed

[2]This is a more refined version of saying that if $\xi^\mu = (1, 0, ..., 0)$, then $\xi^\mu \partial_\mu(\text{Lagrangian}) = \partial_t(\text{Lagrangian}) = 0$

as

$$T = \frac{1}{L_0\sqrt{-\xi^\mu \xi_\mu}}, \qquad u^\mu = \frac{\xi^\mu}{\sqrt{-\xi^\lambda \xi_\lambda}}. \tag{2.7}$$

In the equilibrium configuration, the free energy can be expressible in terms of these two quantities. In the Ginzburg-Landau style, the only term at 0^{th} order in derivative expansion is

$$F = -\log Z = \int d^d x \sqrt{-g}\, p(T) \tag{2.8}$$

where F is the Gibbs free energy and $p(T)$ is the pressure. Now, varying the above partition function with respect to the metric $g_{\mu\nu}$ and substituting it in the definition of the stress-energy tensor in (2.4), we immediately find that

$$T^{\mu\nu}_{\text{equilibrium}} = \left(T\frac{\partial p}{\partial T} - p\right) u^\mu u^\nu + p\Delta^{\mu\nu} + \mathcal{O}(\partial^1). \tag{2.9}$$

The first term is nothing but the energy density obtained by the Euler relation $\varepsilon + p = sT$. With this, we obtain the stress-energy tensor for an ideal fluid as promised. The above method may seem like overkill but its true power will be manifest when the global symmetry structure is more complicated or when higher order derivative terms are involved.

Next, we move to add terms at first order in the derivative expansion. This gives the following stress-energy tensor

$$\begin{aligned} T^{\mu\nu} = \mathbb{E}\, u^\mu u^\nu + \mathbb{P}\Delta^{\mu\nu} + \sum_{\text{all vectors}} \alpha_{(i)}(T)\left(\mathbb{V}^\mu_{(i)} u^\nu + u^\mu \mathbb{V}^\nu_{(i)}\right) \\ + \sum_{\text{all tensors}} \beta_{(i)} \mathbb{T}^{\mu\nu} \end{aligned} \tag{2.10}$$

Here, we denote

$$\mathbb{E} = \varepsilon + \sum_{\text{all scalars}} \gamma_{(i)} \mathbb{S}_{(i)}, \qquad \mathbb{P} = p + \sum_{\text{all scalars}} \delta(i) \mathbb{S}_{(i)} \tag{2.11}$$

The structures $\mathbb{S}, \mathbb{V}, \mathbb{T}$ are respectively all possible independent scalar, vector, tensor constructed from u^μ and T at first order in the derivative expansions.

The list of all possible structures is

$$\begin{aligned} &\text{scalar :} && \nabla_\mu u^\mu, u^\mu \nabla_\mu T, \\ &\text{vector :} && \Delta^{\mu\nu}\nabla_\nu T, u^\nu \nabla_\nu u^\mu, \\ &\text{tensor :} && \sigma^{\mu\nu} := \Delta^{\mu\rho}\Delta^{\nu\sigma}\left(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho - \frac{2}{d-1} g_{\rho\sigma}\nabla_\lambda u^\lambda\right), \end{aligned} \tag{2.12}$$

and $\{\alpha_{(i)}, \beta_{(i)}, \gamma_{(i)}, \delta_{(i)}\}$ are unknown functions of thermodynamic variables, which are referred to as *transport coefficients*. However, a straightforward investigation will reveal that not all structures listed above are linearly independent. First of all, the variables $u^\mu(x), T(x)$ must satisfy the conservation law $\nabla_\nu T^{\mu\nu} = 0$ order by order. By substituting in the 0^{th} order terms in $T^{\mu\nu}$, one finds relations between the following quantities

$$u^\mu \nabla_\mu T \sim \nabla_\mu u^\mu, \qquad u^\nu \nabla_\nu u^\mu \sim \Delta^{\mu\nu}\nabla_\nu T. \tag{2.13}$$

Thus, one can see that $u^\mu \nabla_\mu T$ and $u^\nu \nabla_\nu u^\mu$ are not independent and can be remove from the list (2.12). The other type of redundancy is due to the fact that the temperature field $T(x)$ and fluid velocity u^μ have no microscopic specification out of equilibrium. This allows us to shift the velocity u^μ and temperature field $T(x)$ by terms subleading in the gradient expansion i.e.

$$u^\mu(x) \to u^\mu(x) + \mathcal{V}^\mu(\partial T, \partial u), \qquad T(x) \to T(x) + \mathcal{T}(\partial T, \partial u) \tag{2.14}$$

where the vector $\mathcal{V}^\mu$ and the scalar $\mathcal{T}$ only contain terms at first order in the gradient expansion. This redundancy is known in the literature as *frame choices.* For chapter 3 and 5, we will use the hydrodynamics construction in the *Landau frame* where $\mathcal{V}^\mu$ and $\mathcal{T}$ are chosen such that

$$u_\mu T^{\mu\nu} = -\varepsilon u^\nu, \tag{2.15}$$

which essentially set $\alpha_{(i)}$ and $\gamma_{(i)}$ to zero. Imposing these constrains, one finds that there are only two the remaining terms at 1^{st} order :

$$T^{\mu\nu}_{\text{first derivative}} = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}(\partial_\lambda u^\lambda), \tag{2.16}$$

where η and ζ are the shear and bulk viscosity respectively.

The last constraint to be imposed in this system is the positivity of local entropy production near equilibrium [48]. This condition is widely believed to be related by the second law of thermodynamics. This is, however, not entirely true since there is no such thing as the second law for local entropy (for example, refrigerators do exist). However, this condition gives a sensible constraint in most cases and, only recently, has finally been shown to emerge from consistency conditions of effective action for dissipative systems close to equilibrium [63].

This constraint demands the existence of a local entropy current J^μ_S such that $u^\mu J^\mu_S = s$ and $\nabla_\mu J^\mu_S \geq 0$. In practice, we can express J^μ_S in the gradient expansion as

$$J^\mu_S = s u^\mu + \sum_{\text{all vectors}} \tilde{\alpha}_{(i)}(T)\mathbb{V}^\mu_{(i)} + \sum_{\text{all scalars}} \tilde{\gamma}_{(i)}\mathbb{S}_{(i)} u^\mu \tag{2.17}$$

where $\tilde{\alpha}_{(i)}, \tilde{\gamma}_{(i)}$ are some unknown coefficients. Then, one can proceed by substituting the equation of motion $\nabla_\nu T^{\mu\nu} = 0$ in $\nabla_\mu J^\mu_S$. In many cases, the coefficients $\tilde{\alpha}, \tilde{\gamma}$ can be chosen to eliminate terms that are not positive definite[3], resulting in the desired structure

$$\nabla_\mu J^\mu_S = \sum F_j \left(\alpha_{(i)}, \beta_{(i)}, \gamma_{(i)}, \delta_{(i)}\right) (\text{positive definite combinations of } \partial u, \partial T)^2$$

where F_j are some linear combinations of the transport coefficients. The positivity of $\nabla_\mu J^\mu_S$ implies that $F_j \geq 0$. In this setup, fortunately, the entropy current is well-defined and one finds that

$$J^\mu_S = T^{-1}\left(p\, u^\mu - T^{\mu\nu} u_\nu\right), \qquad T\,\nabla_\mu J^\mu_S = \frac{1}{2}\eta\, \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\partial_\lambda u^\lambda)^2 \tag{2.18}$$

which implies that $\eta \geq 0$ and $\zeta \geq 0$. In this case, the positivity of the local entropy production does not put a lot of constraint in the system but as we crank up the complexity of the system, its consequences will be more dramatic.

[3] Unfortunately, this does not always happen, causing the definition of entropy current to be ambiguous as it depends on arbitrary constants $\{\tilde{\alpha}, \ldots\}$, see e.g. [64].

Again, this condition seems very simple but the constraints it impose will become more dramatic as we crank up the complexity of our system.

If the system also possesses a $U(1)$ global symmetry, we can also couple it to a background $U(1)$ gauge field A_μ and write down the constitutive relation. The stress energy tensor $T^{\mu\nu}$ and the conserved current J^μ can be expressed in terms of the same hydrodynamic quantities and background field. The caveat is that, since $T^{\mu\nu}$, J^μ are physical quantities and therefore cannot depend on the gauge choice, they can only depend on the field strength $F_{\mu\nu}$. In this case the constitutive relation for J^μ is

$$\begin{aligned} J^\mu = n u^\mu - \sigma\, \Delta^{\mu\nu} \left(F_{\nu\lambda} u^\lambda - T \nabla_\nu (\mu/T) \right) \\ + \chi_E \Delta^{\mu\nu} \left(F_{\nu\lambda} u^\lambda \right) + \chi_T \left(\Delta^{\mu\nu} \nabla_\nu T \right) \end{aligned} \tag{2.19}$$

The parameter n is the density of the $U(1)$ charge and $\{\sigma, \chi_E, \chi_T\}$ are transport coefficients. In this case, the coefficients χ_E and χ_T will generate a non-positive definite term in $\nabla_\mu J^\mu_S$ thus forcing them to be zero.

2.1.2 Breaking translational symmetry

Translational symmetry breaking is one of the defining properties of solid state physics. The lack of translational symmetry is caused by the presence of a lattice or disorder, which is responsible for the finite conductivity of the system. Typically, the lattice/disorder spacing ℓ_{dis} is very short compared to the mean free path ℓ_{mfp}, causing the hydrodynamic gradient expansions to breakdown as $\ell_{\text{dis}} \ll \ell_{\text{mfp}}$. However, we can still study hydrodynamic properties in the opposite limit, where the translation symmetry is only weakly broken such that the mean free path is still much shorter than the lattice/disorder spacing $\ell_{\text{dis}} \gg \ell_{\text{mfp}}$. Arguably, the quantum system will still behave like a fluid at a certain scale (see e.g. illustration in Fig. 1.3)

The simplest way to break translational symmetries in this system is to add, by hand, a term analogous to the Drude model. The Ward identity for $T^{\mu\nu}$ in flat space is modified to

$$\partial_\mu T^{\mu t} = 0, \qquad \partial_\mu T^{\mu i} = \frac{1}{\tau_{\text{imp}}} T^{ti} \tag{2.20}$$

where T^{ti} is the momentum in x^i direction and $1/\tau_{\text{imp}}$ is momentum relaxation rate [23, 65]. The second equation breaks the spatial translational symmetry in all directions. One can then proceed to compute hydrodynamic quantities by assuming that the stress-energy tensor still retain the original form in (2.6) and (2.16). This approach is a good approximation for conductivities, which are mostly governed by the zeroth order terms in $T^{\mu\nu}$ and continues to give interesting results [65–68]. However, first order hydrodynamics in (2.16) turns out to be inconsistent with the modified conservation equation (2.20).

This formalism can be made more systematic. Instead of adding the term $1/\tau_{\text{imp}}$ by hand, the translational symmetry can be incorporated by making the background metric or the background gauge field spatially dependent, similar to suspended graphene or an optical lattice in cold atom experiments. This program has been put forth [69–71] and found applications in a clean graphene experiment where hydrodynamic signatures have been found [24, 25], see e.g. figure 2.2.

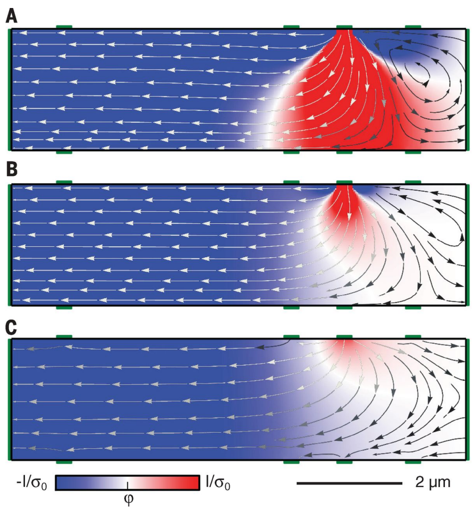

Figure 2.2. An illustration of a negative local resistance caused by viscous electron backflow in graphene experiment in [25]. The vortices, which are signatures of hydrodynamics are apparent in sub-panel A and B.

In chapter 3 of this thesis, we proceed with the same background field approach but instead of using the background metric to break translational symmetry, we introduce additional scalar fields to do the job. We couple the system to the background metric and spatially dependent scalar fields ϕ_i such that the new generating function is

$$Z[g_{\mu\nu},\phi_i] = \left\langle \exp\left[i\int d^dx\sqrt{-g}\left(\frac{1}{2}T^{\mu\nu}g_{\mu\nu} + \mathcal{O}_i\phi_i\right)\right]\right\rangle \tag{2.21}$$

where $\langle\mathcal{O}_i\rangle$ is the operator that corresponds to the scalar field source ϕ_i. The stress-energy tensor of this theory obeys the following modified conservation law

$$\nabla_\nu T^{\mu\nu} = \langle\mathcal{O}_i\rangle\nabla^\nu\phi_i \tag{2.22}$$

The key advantage of this theory is that it has a much simpler holographic dual compared to the previous case, allowing us to compute transport coefficients explicitly. The constitutive relations can be derived systematically in terms of thermodynamic quantities, fluid velocity u^μ, background metric $g_{\mu\nu}$ and the scalar fields ϕ_i. We then explore this system and the fate of the KSS bound when the translational symmetry is broken in this particular way.

2.1.3 Introducing anomalous $U(1)$ current

Quantum anomalies are one of the most beautiful and genuine quantum effects. They are phenomena where the classical theory is invariant under a certain global symmetry but this symmetry does not survive when the theory transitions to the quantum regime. The most well-known anomaly is the chiral anomaly. An illustrative example is the massless fermion in even spacetime dimensions. For example, in 3+1 dimensions, the massless Dirac Lagrangian in the presence of a background gauge field is invariant under two $U(1)$ global symmetries called $U(1)_V$ and $U(1)_A$, which transform the fermion field as

$$\psi \to \psi\exp(i\theta), \qquad \text{and} \qquad \psi \to \psi\exp(i\gamma^5\theta) \tag{2.23}$$

However, only one current is conserved. The conservation of current in this setup can be written in the following ways

$$\partial_\mu J^\mu_V = 0, \qquad \partial_\mu J^\mu_A = \kappa\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{2.24}$$

where the coefficient κ is called the*anomaly coefficient*. Similarly, one observes that putting the theory in curved space also has a similar effect but, this time, with an additional term associated to *the gravitational anomaly*

$$\nabla_\mu J^\mu_A = \epsilon^{\mu\nu\rho\sigma} \left(\kappa F_{\mu\nu} F_{\rho\sigma} + \lambda R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \right) \tag{2.25}$$

The quantum anomaly has been studied intensively over the last few decades since the study of pions in the pre-QCD era. Since then, it has been well understood and play an important role in our understanding of particle physics, string theory and condensed matter (see e.g. [72–75] for reviews). The quantum anomaly claim to fame came from its non-renormalisation nature: the diagrammatic computation of the anomaly coefficient is 1-loop exact [76–78], which originates from a topological quantity in the index theorem [79]. Despite all their theoretical success, these beautiful anomaly coefficients had never been measured in nature until the recent developments in the last few years.

The mentioned development materialised from the realisation that a quantum system with anomaly gives rise to a new kind of transport phenomena. We learned from high school physics that a charged particle will circulate around the magnetic field line as it is subjected to an external magnetic field. In a system with an anomaly, however, there exists an unusual current which flows along the magnetic field line and the magnitude of the current is proportional to the magnetic field itself! This phenomenon is dubbed *chiral magnetic effect* [80] and from known microscopic theories, the conductivity associated to this *anomalous current* is indeed fixed by the anomaly coefficients (see e.g. [80–86]). Subsequently, the strong interaction computation has also been done using gauge/gravity duality and the same relations between anomaly coefficients and anomalous conductivity are also found in simple models .

We are interested to see whether there exists a version of the non-renormalisation theorem for anomalous conductivities, especially at finite temperature and in the presence of the gravitational anomaly[4]. We investigate this possibility in a large class of strongly interacting quantum field theories with holographic duals and present the result in chapter 4. The theory we are interested in has two $U(1)$ global symmetries i.e. $U(1)_V \times U(1)_A$ and we couple them to two background gauge fields, V_μ and A_μ respectively. The partition function of this theory is

$$Z[g_{\mu\nu}, V_\mu, A_\nu] = \left\langle \exp\left[i \int d^4x \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} g_{\mu\nu} + J^\mu_V V_\mu + J^\mu_A A_\mu \right) \right] \right\rangle .$$

We construct the theory such that one of the global symmetries, namely $U(1)_A$ is anomalous. The Ward identities for the currents in this theory are

$$\begin{aligned} \nabla_\mu J^\mu_V &= 0, \\ \nabla_\mu J^\mu_A &= \epsilon^{\mu\nu\rho\sigma} \left(\kappa F_{A,\mu\nu} F_{A,\rho\sigma} + \gamma F_{V,\mu\nu} F_{V,\rho\sigma} + \lambda R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\sigma} \right) \end{aligned} \tag{2.26}$$

where $F_{V,\mu\nu}, F_{A,\mu\nu}$ are the field strengths associated to the background field V_μ, A_μ respectively.

2.1.4 Generalised global symmetry

The global symmetries mentioned earlier are the one that people have been familiar with since the nineteenth or early twentieth centuries. As we mentioned in section 2.1, these symmetries (Poincaré symmetry and $U(1)$ global symmetry) are often associated with transformations of point-like objects (such as a particles) which leave the theory unchanged. Consequently, the conserved charges in d–dimensional systems are obtained by integrating the conserved currents over a spatial volume, e.g. for the $U(1)$ charge

$$Q = \int d\,(\text{volume})\, J^t = \int d^{d-1} S_\mu J^\mu . \tag{2.27}$$

where the spatial volume S_μ is a vector pointing in the time direction. We can also couple the system to the vector gauge field A_μ as we did in the previous

[4]See recents review about this issue in e.g. [44, 87–89]

section. In the language of differential geometry, the conserved current and the gauge field can be classified as a 1−form object (since it contains one index).

But of course, this is not the only global symmetry in quantum field theory, especially when they system consists of extended objects such as superfluid vortices, strings, domain walls, membranes etc. The study of such extended objects plays a crucial role in many areas of physics, particularly in string theory [90]. In these examples, there is also a notion of global symmetry and conserved charge. However, in contrast to the typical conserved charge (2.27) where we integrate the conserved current over the spatial volume, the conserved charge for extended objects is obtained by integrating over a surface (see figure 2.3).

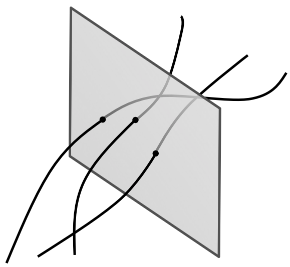

Figure 2.3. An illustration of a conserved charge for a system with string-like objects. In this case, the conserved charge is a flux obtained by integrating the number of a string/field line passing through a 2−dimensional surface (figure from [47]).

For an extended $q-1$ dimensional object (q=1 for a particle and q=2 for a string), the integration is done over a $p := d-q$ dimensional surface. Mathematically, the conserved charge for such an object is

$$Q_{\text{gen}} = \int dS_{\mu_1\mu_2\ldots\mu_p} J^{\mu_1\mu_2\ldots\mu_p}_{(p)}, \tag{2.28}$$

where $S_{\mu_1\mu_2\ldots\mu_p}$ is a totally antisymmetric tensor of rank p or a p−form object (and so is $J^{\mu_1\ldots\mu_p}_{(p)}$). Similar to the conventional conserved current (2.27), we can couple this p−form current $J_{(p)}$ to the background gauge field. However, instead of the 1-form gauge field, it will couple to the q−form gauge field $A_{\mu_1\mu_2\ldots\mu_q}$ which is a totally antisymmetric tensor of rank q. Recently, *higher–form symmtry* has been systematically categorised in [46], although it has been a recurring theme in various areas such as symmetry protected topological phase and topological order e.g. [46, 91, 92], phenomenological models for superfluid vortices [93–95], dislocation/disclination in liquid crystals [96, 97].

How does this have anything to do with plasma physics in 3+1 dimensions? Naively, one may argue that the plasma, which is charged matter coupled to an electromagnetic field, has a $U(1)$ symmetry associated to the conserved current j^μ in the Maxwell equation

$$\nabla_\mu F^{\mu\nu} = j^\nu, \qquad \nabla_\nu \left(\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\right) = 0. \tag{2.29}$$

Here, we can see from the first equation that $\nabla_\mu j^\mu = 0$ by definition. The second equation is usually thought of as a constraint and trivially vanishes when one expresses the field strength $F_{\mu\nu}$ in terms of the gauge field A_μ i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. However, this current j^μ is not associated with a global symmetry but a gauge symmetry. It is not a true symmetry but rather a redundancy of the description. The other subtle differences between global and gauge symmetry can be found in the table below

global symmetry	Gauge symmetry
Intrinsic property of the system	a redundancy rather than real symmetry
can be spontaneously broken	cannot be spontaneously broken
classifies physical states	all physical states are gauge invariant
can be anomalous	cannot have anomaly
can couple to background gauge fields	gauge field is dynamical

So, is there any global symmetry in a system described by (2.29) ? It was argued by [46] that the true global symmetry is encoded in the second set of Maxwell equation (2.29). It is the 2-form global symmetry with the current $J^{\mu\nu}$ defined as

$$J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{2.30}$$

The conserved charge $Q = \int dS_{\mu\nu} J^{\mu\nu}$ is nothing but the magnetic flux passing though the system as depicted in Fig. 2.3.

With the knowledge of global symmetries at our disposal, we can play the same game we did in the previous section. First we define a partition function of the theory coupled to the background metric $g_{\mu\nu}$ and the 2-form gauge field $b_{\mu\nu}$ which sources the 2-form current $J^{\mu\nu}$ in the following way :

$$Z[g_{\mu\nu}, b_{\mu\nu}] = \left\langle \exp\left[i \int d^4x \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} g_{\mu\nu} + J^{\mu\nu} b_{\mu\nu} \right) \right] \right\rangle .$$

The conserved currents can be obtained by varying the partition function

$$\langle T^{\mu\nu} \rangle = \frac{-2i}{\sqrt{-g}} \frac{\delta \log Z}{\delta g_{\mu\nu}}, \qquad \langle J^{\mu\nu} \rangle = \frac{-i}{\sqrt{-g}} \frac{\delta \log Z}{\delta b_{\mu\nu}}. \tag{2.31}$$

The low energy effective theory with these global symmetries can then be obtained by expressing $T^{\mu\nu}$ and $J^{\mu\nu}$ in terms of hydrodynamic variables. However, with this system, one has to take the direction of the magnetic field lines into account. This amounts to introducing a vector h^μ, pointing along the magnetic field lines, as a dynamical variable. After analysing the equilibrium partition function similar to those in section 2.1.1, the conserved currents at zeroth order in the gradient expansion are [47]

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} - \mu\rho\, h^\mu h^\nu + \mathcal{O}(\partial^1), \\ J^{\mu\nu} &= \rho \left(u^\mu h^\nu - u^\nu h^\mu \right) + \mathcal{O}(\partial^1). \end{aligned} \tag{2.32}$$

Here ρ denotes the magnetic flux density and μ is the chemical potential associated with the 2-form charge (or equivalently, the rate of change of the free energy with respect to the number of magnetic field line). The details of this derivation and the constitutive relations at first order in derivative expansion can be found in [47] and in chapter 5 of this thesis.

In the context of plasma physics, the situation where the matter (which obeys the Navier-Stoke equation) coupled to the dynamical electromagnetic field has been studied in the framework called *magnetohydrodynamics* (MHD). And while it is a successful self-consistent description, it has one unsatisfying underlying assumption, namely the equation of state is independent of the magnetic field. On the other hand, the framework of [47], discussed in this section, is obtained by utilising only the global symmetry of the system. Hence, the equation of state in this approach is free from the assumptions of the standard MHD formulation. This new framework allows us to explore a larger parameter space than that of the standard formulation. We apply this framework to a strongly interacting quantum field theory with holographic dual coupled to the dynamical $U(1)$ gauge field and report our findings in chapter 5.

2.2 2-point correlation functions and Kubo formulae

The focus of this section is the retarded 2-point correlation function, G^R, of conserved currents. The introduction of a background field allows us to compute this quantity in a very natural way. For example, the the stress-energy tensor in the almost flat metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small perturbation) can be written as[5]

$$\langle T^{\mu\nu}(x)\rangle_g = \langle T^{\mu\nu}(x)\rangle_{h=0} - \frac{1}{2}\int d^4y\, G^R_{T^{\mu\nu}T^{\rho\sigma}}(x,y)h_{\rho\sigma}(y) + \mathcal{O}(h^2) \quad (2.33)$$

where the retarded 2-point function is defined in the operator language as

$$G^R_{T^{\mu\nu}T^{\rho\sigma}}(x,y) := -\Theta(x^0 - y^0)\,\langle[T^{\mu\nu}(x), T^{\rho\sigma}(y)]\rangle \quad (2.34)$$

This method can be applied for all the conserved currents and background fields in previous sections.

The retarded 2-point function play a central role in this thesis for two reasons which are detailed below.

- Firstly, the pole of the 2-point function in Fourier space captures the dispersion relation of the low energy excitations. For hydrodynamics, there are two types of low energy excitations: diffusive mode and sound mode

$$G^R(\omega,k)^{-1} = 0 \quad \Rightarrow \begin{cases} \omega = -i\mathcal{D}_p k^2, & \text{diffusive mode} \\ \omega = \pm c_s k - i\mathcal{D}_l k^2 & \text{sound mode} \end{cases}$$

  where c_s is the speed of sound, $\mathcal{D}_p$ is the diffusion constant and $\mathcal{D}_l$ is the sound attenuation. The speed of sound only depends on the thermody-

[5]The computation for the real-time correlation function in the presence of the background field is best treated using Schwinger-Keldysh (or Closed-Time-Path) formalism. See, for example, [50, 98, 99] for the derivation.

namic quantities while the $\mathcal{D}_{p,l}$ are linear combinations of the first order transport coefficients and thermodynamics quantity.

- This bring us to the second point. The residue of some of the retarded functions $G^R(\omega, k)$ is proportional to the first-order transport coefficients. This allows us to express transport coefficients as linear combinations of correlation functions. Such expressions are referred to as *Kubo formulae.*

For the reader's convenience, I will use the charge neutral fluid as an example although the extension to a more complicated system is straightforward. The procedure to obtain (2.33) in hydrodynamic models is the following:

1) First, we perturb the background fields, such as $g_{\mu\nu}$, around their means values $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. For the metric, the mean value $g^{(0)}_{\mu\nu}$ is usually the flat space metric as in (2.33).

2) Then, we solve the conservation equation e.g. $\nabla_\mu T^{\mu\nu} = 0$ for dynamical variables (such as the temperature field $T(x)$ and fluid velocity u^μ). As a result, these dynamical variables will be expressed in terms of the background field perturbation $h_{\mu\nu}$.

3) Lastly, one can substitute the perturbed background fields and the solution of $\{T, u^\mu\}$, expressed in terms of $h_{\mu\nu}$, into the constitutive relations. The conserved currents can be expanded to linear order in $h_{\mu\nu}$ and by comparing this result to (2.33), we obtain the correlation function.

As an example, let me present a result for a few interesting correlation function in the charged neutral fluid. Firstly, the energy density correlation function is

$$G^R_{T^{tt}T^{tt}} = \frac{(\varepsilon + p)|k|^2}{\omega^2 - \left(\frac{\partial p}{\partial \varepsilon}\right)|k|^2 + i\frac{\omega|k|^2}{\varepsilon+p}\left(\zeta + \frac{2d-2}{d}\zeta\right)}. \tag{2.35}$$

The pole of this 2-point function indicates that the energy in the system is carried by the sound mode with the speed $c_s = (\partial p/\partial\varepsilon)^{1/2}$ and the sound attenuation $\mathcal{D}_l = \left(\zeta + (2d-2)\eta/d\right)/2(\varepsilon + p)$. On the other hand, the correlation function of momentum density $T^{t\perp}$, orthogonal to the momentum k_i, is governed by the diffusive mode

$$G^R_{T^{t\perp}T^{t\perp}} = \frac{\eta|k|^2}{i\omega - \frac{\eta}{\varepsilon+p}|k|^2} + \text{contact term} \tag{2.36}$$

with the diffusion constant $\mathcal{D}_p = \eta/(\varepsilon + p)$. The remaining correlation functions can be obtained by the Ward identities

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad k^\mu G^R_{T^{\mu\nu}T^{\rho\sigma}} = 0. \tag{2.37}$$

Finally, one can find linear combinations for these correlations functions and obtain Kubo formulae for the shear and bulk viscosity

$$\begin{aligned} \eta &= -\lim_{\omega\to 0} \operatorname{Im} \partial_\omega G^R_{T^{xy}T^{xy}}(\omega, k=0), \\ \zeta + \frac{2d-2}{d}\eta &= -\lim_{\omega\to 0} \operatorname{Im} \partial_\omega G^R_{T^{xx}T^{xx}}(\omega, k=0). \end{aligned} \tag{2.38}$$

It is important to notice that while these formulae do not depend on the microscopic information, they are direct consequences of the conservation law. Therefore they will be modified once we consider systems with different global symmetries e.g. those in section 2.1.1-2.1.3. Nevertheless, the procedure outlined here can still be applied, as we show explicitly in chapter 3-5.

2.3 Bottom-up approach to holographic duality

Obviously, there are a lot more details from string theory and supersymmetric gauge theory which play an important role in the fully fledged gauge/gravity duality. Nevertheless, the following discussion will be about a few essential aspects of the duality which allow us to show that the dual QFT exhibits hydrodynamic behaviour.

In the following sections, we present the gauge/gravity duality as a set of dictionary rules relating QFT physical quantities such as global symmetries, “source” background fields and conserved currents in section 2.1 to quantities in the gravity dual. We then continue by briefly review features in gravity which are relevant to chapters 3-5 of this thesis.

2.3.1 Capturing global symmetry

The precise form of the duality is obtained by stating that the partition function of a strongly interacting QFT discussed in section 2.1 is equal to the semi-classical partition function of a certain gravity theory with at least one dimension higher. Both partition function are also functions of the same arguments, namely the background metric and the other background fields, which act as sources. This is the celebrated Gubser-Klebanov-Polyakov-Witten (GKPW) relation [100, 101]

$$Z_{\text{QFT}}[g_{\mu\nu}, ...] = Z_{\text{gravity}}[g_{\mu\nu}, ...]. \tag{2.39}$$

But what is the generating function of the gravitational theory? How can it depend on the background fields in the QFT? In order to answer these question, let us analyse this dual gravity theory in more details to see how the global symmetry and background fields are manifested.

First of all, we need to recall that this gravity theory is, in fact, a certain low energy limit of string theory. It does not consist of only dynamical graviton but also all sort of scalar fields Φ_i and higher form gauge fields $A^{(p)}_{\mu_1...\mu_p}$. Schematically, the action of the "gravity dual" in $d+1$ dimensions is

$$\begin{aligned}\mathcal{L} = \mathcal{R} - \sum_{\text{all scalars}} Z_n(\Phi_i)(\partial\Phi_i)^2 - V(\Phi_i) + \sum_{\text{all gauge fields}} Y_m(\Phi_i)(F_{(p)})^2 \\ + \left(\text{even more complicated terms involving } \mathcal{R}, \Phi_i, A^{(p)}, F_{(p)}\right)\end{aligned} \tag{2.40}$$

where $\mathcal{R}$ is the Ricci scalar in $d+1$ dimensions. The functions $\{Z_m, Y_m\}$ can be either constant or non-trivial functions of scalar fields Φ_i. The detail of these functions and more complicated terms can be fixed by knowing their string theory origin. For example, the gravity dual to $d = 4$, $SU(N_c)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills at strong coupling is governed by following action

$$S = \frac{N_c^2}{8\pi^2}\int d^5X\sqrt{-G}\,(\mathcal{R} + 12) \tag{2.41}$$

Alternatively, one can treat $\{Z_m, Y_m\}$ as phenomenological parameters to "cook up" gravity duals with desirable properties. I will follow the second approach in this thesis.

Examples where we have evidence that the duality exists are when the solution of the action (2.40) has an asymptotic region which is described by the Anti-de Sitter (AdS) metric

$$ds^2 = G_{ab} dX^a dX^b = \frac{du^2}{4u^2} + \frac{L}{u} \left(g_{\mu\nu}(u) dx^\mu dx^\nu \right) \tag{2.42}$$

where X^a describes $d+1$ dimensional coordinate system (x^μ, u), x^μ describes the coordinates on a surface where $u =$ constant and L is the characteristic length scale of the AdS space, called the AdS radius. The aforementioned asymptotic region is when $u \to 0$, referred to as the boundary of the AdS space. Here, the first dictionary rule can be introduced to relate the metric $g_{\mu\nu}(u)$ to the source metric $g_{\mu\nu}$ in the QFT.

- The background metric $g_{\mu\nu}$ coupled to the QFT stress-energy tensor $T^{\mu\nu}$ is the asymptotic value of the metric $g_{\mu\nu}(u)$ describing the geometry of a d dimensions surface at $u \to 0$.

$$g_{\mu\nu}|_{\text{QFT}} = \lim_{u \to 0} u \, G_{\mu\nu} = g_{\mu\nu}(u \to 0). \tag{2.43}$$

In fact, a similar procedure can be applied to other fields in the gravity dual. The small u expansion in the asymptotic region of all the fields $\mathcal{F} = \{g_{\mu\nu}, \Phi_i, ...\}$ in (2.40) can be written as

$$\begin{aligned} \mathcal{F}(u,x) = u^m \Big[& f^{(0)}(x) + f^{(1)}(x) u + ... + u^n f^{(n)}(x) + ... \\ & + \tilde{f}^{(n)}(x) u^n \log u \Big], \qquad \text{where } \tilde{f}^{(n)} = 0 \text{ for } d \text{ odd.} \end{aligned} \tag{2.44}$$

Solving this expansion using the equation of motion, one finds that all the functions $f^{(i)}$ can be written in terms of two independent functions $f^{(0)}(x)$ and

$f^{(n)}(x)$. A standard procedure [101, 102] indicates that the leading term in the expansion, $f^{(0)}$ should be interpreted as a source in the QFT picture. This leads us to a more general dictionary rule:

- The non-dynamical background field $f^{(0)}(x) = \{g_{\mu\nu}(x), A_\mu(x), ...\}$, which couple to the operator in QFT in d dimensions sets the asymptotic value of the classical dynamical fields $\mathcal{F}(u,x)$ propagating in the $d+1$ dimensional spacetime, i.e.

$$f^{(0)}(x)|_{\text{QFT}} = \lim_{u\to 0} u^{-m}\mathcal{F}(u,x). \tag{2.45}$$

Consequently, the global symmetry $\mathfrak{G}$ of the QFT is translated to a gauge symmetry $\mathfrak{G}$ on the dual gravity side. This is due to the fact that the gauge redundancy of the background fields $f^{(0)}$ is transferred to the dynamical fields $\mathcal{F}(u,x)$ via the expansion (2.44). As a result, the gravity action defined in (2.40) is gauge invariant under the gauge transformations of $\mathcal{F}$ if the QFT operator sourced by $f^{(0)}$ is a conserved current.

The other crucial point is how to incorporate finite temperature in the gravity dual. One can start by putting a QFT on a cylinder as in section 2.1.1. According to the dictionary rule (2.43), this introduces a time circle, set by the QFT temperature, at AdS boundary. One find that the thermal cycle smoothly shrinks to a point as we move further in the bulk of the AdS space, leaving the geometry in a cigar shape depicted in Fig. 2.4. Interestingly, this geometry is nothing but the Euclideanised black hole geometry. By undoing the Wick rotation, we find that the temperature effect can be obtained by introducing a black hole horizon with Hawking temperature equal to the temperature of the dual QFT.

The black hole also plays another important role in determining the unknown function coefficients $f^{(n)}(x)$ in the near boundary expansion (2.44). Essentially, the boundary condition at the black hole horizon (e.g. ingoing boundary condition, regularity condition) allows one to determine $f^{(n)}(x)$, thus the profile of $\mathcal{F}(u,x)$ is uniquely determined by $f^{(0)}$.

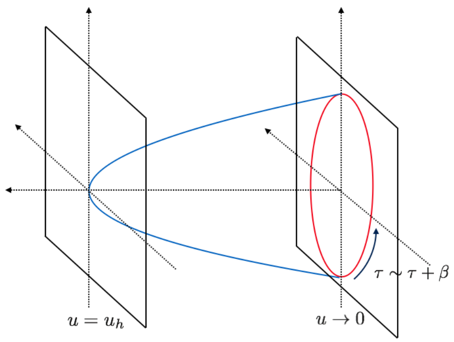

Figure 2.4. The evolution of the thermal cycle from the boundary $u \to 0$ into the bulk region $u > 0$. The circle shrinks smoothly to a point at distance $u = u_h$. The geometry is terminated at this point and forms a cigar-like structure.

By piecing together all this information, we now have the ingredients in the gravity dual that correspond to the physical quantities we are interested in, namely the global symmetries, background fields that source the conserved currents and finite temperature. With this, we will be able to construct a strongly interacting quantum field theory from a gravity dual that has a desirable global symmetry structure. To be more explicit, a few examples of bottom-up holographic models and their corresponding QFT properties are listed below.

- Conformal field theory in d dimensions with no additional global symmetries at finite temperature corresponds to the black hole geometry in the Einstein-Hilbert action

$$S = \frac{1}{2\kappa}\int d^{d+1}x\sqrt{-G}\,(\mathcal{R} + d(d-1))\,. \tag{2.46}$$

  This can be seen from the fact that the only relevant background field and conserved charges are the background metric $g_{\mu\nu}$ and the stress-energy tensor $T^{\mu\nu}$. The cosmological constant originates from the string theory setup. For our purposes, we can view it as a term added to ensure that the geometry on the gravity dual side has an asymptotic AdS region.

- To add additional global symmetries, such as a global $U(1)$ symmetry on the QFT side, we can add a Maxwell term to the action:

$$S = \frac{1}{2\kappa} \int d^{d+1}x\sqrt{-G} \left(\mathcal{R} + d(d-1) - \frac{1}{4e^2} F_{ab}F^{ab} \right). \tag{2.47}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ where $A_\mu(u \to 0, x)$ is the QFT background gauge field that sources the $U(1)$ conserved current J^μ. If the global symmetry happens to be a non-Abelian group, such as $SU(2)$, we simply change the gauge field A_a in the gravity dual to a non-Ablelian gauge field A^i_a, where i runs over the group index.

2.3.2 Holographic thermal 1-point and 2-point function

We are now ready to use the GKPW to obtain expectation values and correlation functions.

In the semi-classical approximation, the partition function of the gravity theory is determined by the onshell action, i.e.

$$Z[g_{\mu\nu}, ...] = \exp\left[iS_{\text{gravity}}\right] = \exp\left[\frac{i}{2\kappa}\int d^{d+1}\sqrt{-G}\left(\mathcal{R} + ...\right)\right] \tag{2.48}$$

where fields $\mathcal{F}(u, x)$ in S_{gravity} are determined by the sources $f^{(0)}(x) = \{g_{\mu\nu}(x), ...\}$ and the boundary condition at the horizon. I would like to emphasise that, by keeping track of the background fields, we can conveniently use the same definition of the stress-energy tensor and the other conserved currents as in section 2.1, such as (2.4) and (2.31). Moreover, the retarded 2-point functions of these operators can also be obtained straightforwardly using the prescription in (2.33). For example, the stress-energy tensor 1-point and 2-point function are

$$\langle T^{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{gravity}}}{\delta g_{\mu\nu}}, \quad G^R_{T^{\mu\nu}T^{\rho\sigma}} = -2\frac{\delta}{\delta g_{\mu\nu}}\left(\sqrt{-g}\,\langle T^{\mu\nu}\rangle\right) \tag{2.49}$$

One can clearly see that this way of computing 1-point and 2-point function is a lot easier compared to the conventional QFT method beyond a few loops correction. Practically, one only needs to solve a differential equation for $\mathcal{F}(u, x)$ with the boundary condition determined by $f^{(0)}$, insert solutions in the gravity action S_{gravity} and apply the formulae (2.49). A similar procedure can be used to obtain correlation functions of the other conserved currents.

Historically, this procedure made contact with hydrodynamics in a series of heroic works by Policastro, Son and Starinets [103–105] for the gravity theory dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills. They found that the 2-point functions that agree with those predicted by hydrodynamics (2.35)-(2.36) with the equation of state

$$p = \frac{1}{3}\varepsilon \sim T^4, \quad \eta = \frac{s}{4\pi}, \quad \zeta = 0 \tag{2.50}$$

Soon after, this computation was extended to holographic system with different symmetries and equations of state (see e.g. [106–109] for early work on this construction). Later on, the way to extract the constitutive relation (in section 2.1.1) was developed [110–113], allowing one to link properties of the black hole dynamics and hydrodyhamics (this prescription is known under the name *fluid/gravity correspondence*). These works have been inspirations as well as the foundation for numerous works in the past few years, including those in this thesis.

2.3.3 Holographic RG flow

It is natural to ask what the physical meaning of the radial direction labelled by the coordinate u in the previous section is. This is a long-standing question which still remains unanswered. Earlier work [22, 114, 115] strongly suggested that it is related to the energy scale of the theory and that the gauge/gravity duality is the geometric realisation of the Wilsonian renormalisation group flow (see Fig. 2.5), where the region near the boundary $u = 0$ is associated to the UV fix point.

Although the precise relation between the radial direction and Wilsonian RG flow is still unclear, there are evidences to suggest that the above statement is not completely wrong. For example, the action of simple holographic models such as those in (2.46)–(2.47) diverges as $u \to 0$, reminiscent of a UV divergence in conventional QFT. Thus, for the holographic generating function to

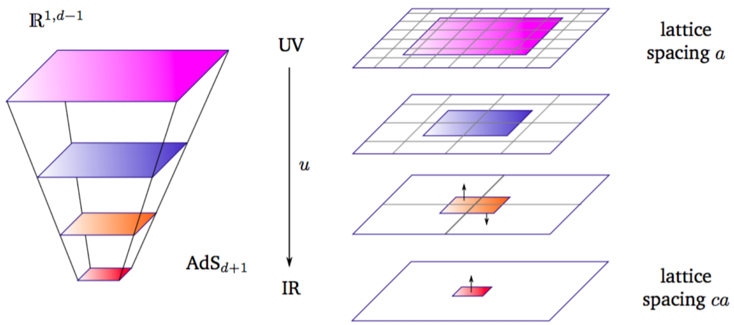

Figure 2.5. The comparison between the AdS space and series of block spin transformation along the Wilsonian RG flow.

be physical, one needs to regularise the gravity action by introducing a "cutoff surface" at $u = \epsilon \ll 1$ then add a local counter-term to S_{gravity} to subtract the divergence [101, 102]. One may also extract the Callan-Symanzik equation by integrating out the near boundary region $u < 1/\Lambda$, analogous to momentum shell renormalisation [116, 117] (see also [118] for a review of development before 2011 and references).

If we adopt this notion of RG flow, one can drive the theory away from its UV fixed point by turning on a relevant deformation. The simplest way to do this is to add a scalar field ϕ, which sources a relevant operator $\langle \mathcal{O}_\phi \rangle$ with scaling dimensions $\Delta_{\mathcal{O}} < d$. In holography, this can be done by introducing a scalar field Φ, whose boundary condition set by the source ϕ, in the gravity action i.e.

$$S_{\text{gravity}} = \int d^{d+1}X \sqrt{-G} \left(\mathcal{R} - (\partial\Phi)^2 - V(\Phi) \right), \tag{2.51}$$

where the potential $V(\Phi)$ triggers the deformation away from the AdS boundary. It can also be tuned to obtained a desirable geometry in the bulk (or IR QFT) [119, 120]. This procedure is also extendable to the holographic RG flow for a QFT with finite density [121, 122] by studying the action

$$S_{\text{gravity}} = \int d^{d+1}X \sqrt{-G} \left(\mathcal{R} - (\partial\Phi)^2 - V(\Phi) - \frac{Z(\Phi)}{4} F_{ab}F^{ab} \right), \tag{2.52}$$

where the function $Z(\Phi)$ sets the relevance of the $U(1)$ charge at different stage along the RG flow. This ingredient of the holographic construction allows us to explore large classes of theories with the same global symmetry but different equations of state and thus allows us to check the validity of universal statements such as the KSS bound (1.1) and those in section 1.2. Interestingly, one can engineer holographic theories with the same scaling relations as those found in high temperature superconductor materials [123, 124].

2.3.4 The "membrane paradigm"

If we take seriously the AdS radius u as an energy scale in the Wilsonian renormalisation scheme, it is tempting to say that the low energy dynamics, such as DC conductivity and viscosities are captured by the deep interior of the AdS spacetime (see e.g. Fig. 2.5). In systems with finite temperature, the "IR cutoff" is generated by the black hole horizon, see Fig. 2.6. Since we know that the QFT dual to a black hole is governed by hydrodynamic principles, one might be tempted to say that the region near the black hole horizon (called the stretch horizon) can be thought of as some sort of fluid and might even be the same fluid in dual QFT. The idea that the black hole horizon behaves as a fictitious fluid is not new. In fact, it dates back to work on black hole in 70's-80's known as *the black hole membrane paradigm* [125].

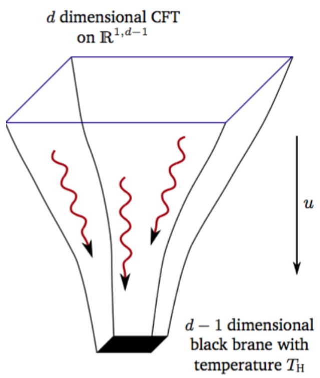

Figure 2.6. An illustration of AdS space with black hole. The horizon is located at $u = u_h > 0$ where the scale u_h is set by the temperature. This geometry can be obtained by Wick rotating the cigar geometry in Fig. 2.4. The excitations with low energy probe deeper in the interior near the horizon, suggesting that the low energy dynamics is governed by the physics near the black hole horizon.

However, a more careful analysis of the fluid/gravity correspondence indicates that the fictitious fluid at the horizon and the fluid described by the dual QFT are not the same in general [110, 116, 126–130]. This is due to the fact that the hydrodynamic data of the dual QFT contain not only a contribution from the stretched horizon but also from the bulk of AdS spacetime. Nevertheless, some hydrodynamic data of the fictitious fluid at the horizon can still be mapped to that of the dual QFT [126]. This is due to the fact that there exist "conserved currents" along the AdS radial direction (u–direction) which carry the hydrodynamic data from the stretched horizon over to the boundary, allowing one to map them to the dual QFT.

Let us briefly outline how this works for the computation of the shear viscosity. Typically, if one wants to compute $G^R_{T^{xy}T^{xy}}(\omega, 0)$ to feed into the Kubo formula (2.38), it is required to solve Einstein's equation, which is a second order differential equation. However, for the shear viscosity, one can arrange the equation of motion in such a way that it becomes a total derivative along the u–direction i.e.

$$\text{Second order DiffEqn}\Big|_{\text{low energy}} \quad \Rightarrow \quad \frac{d}{du}\Big[\mathcal{J}(u)\Big] = 0. \tag{2.53}$$

In this case, the radially conserved current $\mathcal{J}$ contains the information about the shear viscosity of the stretched horizon and the dual QFT when evaluate it at the horizon $u = u_h$ and at the AdS boundary $u \to 0$ respectively. Once the explicit form of $\mathcal{J}$ is known, one can use (2.53) to extract the shear viscosity of the dual QFT purely from the near-horizon data.

The existence of this radially conserved current not only simplifier the computation of the transport coefficients but is also a smoking gun for a universality relation. To be more precise, the physical quantities contained in the radial conserved currents are fully captured by the deep IR information (i.e. at the stretched horizon) and do not depend on the specific details of the " RG flow" along the u–direction. This concept plays a crucial role in identifying which quantities we should expect to display universal behaviour.

2.3.5 Higher derivative holography

From a string theory point of view, the fact that some QFTs have a simply gravity dual such as those dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills in (2.41) is not a coincidence. String theory is a quantum theory of gravity with two fundamental parameters: the string characteristic length ℓ_s and the string coupling g_s, which controls the quantum fluctuations in the dual gravity theory. For QFTs such as $SU(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills to have a simple gravity dual, both of these parameters have to be small. This limit corresponds to sending the rank N of the gauge group $SU(N)$ and the 't Hooft coupling $\lambda := g_{\text{YM}}^2 N$ in the dual QFT to infinity. To be more precise, one can relate the string parameters ℓ_s, g_s to the QFT parameters λ, g_s, N as

$$\frac{L}{\ell_s} = \lambda \gg 1 \qquad \text{and} \qquad 4\pi g_s = g_{\text{YM}}^2 = \frac{\lambda}{N} \ll 1. \tag{2.54}$$

Similar limits can also be found for the other supersymmetric theory with holographic dual, see e.g. [52].

While the large N limit is somewhat pathological, the large 't Hooft coupling limit represents the fact that the QFT we are interested in has infinitely strong coupling. The combination of the two limits greatly simplifies the computations in the gravity dual. However, this is nothing but one corner in the parameter space. To explore different regions of parameter space and make a universal statement such as the bound (1.1), one needs to find a way to extend our computation beyond large N, λ limit.

While it is very difficult to move away from the large N limit due to the more problematic nature of the string coupling g_s corrections[6], the large N limit with $1/\lambda$ correction is much more tame and very well-documented [135–141]. It turns out that, in order to include finite λ corrections in the gravity dual, one has to add higher-derivative terms. For example, in $\mathcal{N} = 4$ SYM, the first 't Hooft coupling correction comes from the following higher derivative

[6] See recent developments on $1/N$ corrections in the large 't Hooft coupling limit in e.g. [131–134].

term [141]

$$\Delta S = \int d^5X\sqrt{-G}\Big(\gamma W\Big) \tag{2.55}$$

where W is a linear combination of forth powers of the Weyl tensor C_{abcd} (c.f. [141]).

$$W = C^{abcd}C_{mbcn}C_{a}{}^{rsm}C^{n}{}_{rsc} + \frac{1}{2}C^{adbd}C_{mnbc}C_{a}{}^{rsm}C^{n}{}_{rsd}. \tag{2.56}$$

Note also that, in the string theory construction, the constant γ would also depends on scalar fields in the dual gravity side.

The above finite 't Hooft coupling correction is specific to $SU(N)$ $\mathcal{N}=4$ supersymmetric Yang-Mills. Given the vast landscape of string theory, it is reasonable to expect that generic higher derivative corrections could occur for the other QFTs with holographic duals. The higher derivative couplings e.g. γ could also depend on the scalar fields as in the case of $\mathcal{N}=4$ supersymmetric Yang-Mills but they are constrained by consistency conditions such as unitarity, causality, etc., as illustrated in [142–147].