Transport coefficients and low energy excitations of a strongly interacting holographic fluid

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The cover shows the private and public transport phenomena of electricity, heat, humans and other pollutants. Incidentally, this photo was taken at Wongsawang junction (close to the author's house in Bangkok), infamous for its traffic congestion, a few seconds before a waterfall-like fluid (a.k.a. a tropical rain) starts to pour down and almost destroys the author's camera.

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On the other hand, one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition of quantum gauge theory in four dimensions. Will this change in the 21st century? We hope so!

A. Jaffe and E. Witten on Yang–Mills existence and mass gap [1]

1 Introduction

1.1 Preface : What is QFT?

Quantum field theory (QFT) has been an active area of research for several decades by now. By all means, it is one of the greatest accomplishments of theoretical physics from the last century, with applications ranging from high energy physics, such as the standard model, to low energy physics, such as our understanding of metals and semiconductors. We are able to predict the magnetic dipole moment of the electron which agrees with the experimental result to eight decimal places: an unprecedented accuracy in the entire history of science! We are able to predict the existence of anti-particles, $W^{\pm}$, Z bosons, gluons and quarks in the standard model of particle physics. We have a theory of electrons in solids, which enables us to build computers, smartphones, NMR machines, superconductors and many many more applications.

But perhaps we only see the tip of the iceberg. Our achievements from conventional QFT rely on single-particle classical physics intuition: we start from a microscopic Lagrangian description then treat the interaction and the quantum fluctuation as a perturbation. However, we seem to fail miserably when trying to compute physical quantities from the usual QFT formalism when the interactions are strong. One of the most glaring examples is the confinement phenomenon in quantum chromodynamics (QCD), which is responsible for most of the mass of the known matter in the universe. It is known that the interaction strength in QCD (describing physics of quarks and gluons in nuclear matter) is infinitely large at low energy and this dynamically generates a mass gap, resulting in bound states of quarks and gluons, such as protons and neutrons. We know that this is true thanks to the results from lattice simulations, but more evidently, because most of our mass came from protons and neutrons. Still[1], there is no analytic proof why they should form such bound states!

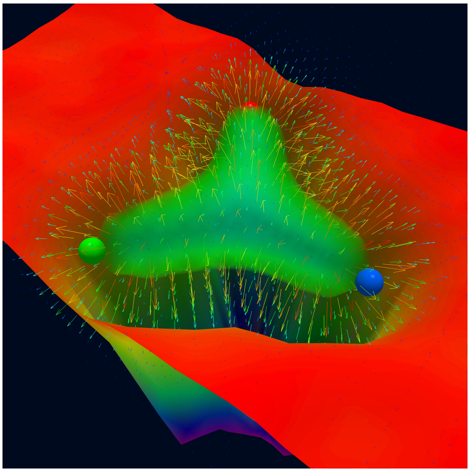

Figure 1.1. A visualisation of a proton in the “vacuum” filled with gluon field. The gluonic field in the proton interior is pushed away by quarks resulting in the flux tube structure confining the quarks together (figure from [2])

Strong interaction are also believed to play a crucial role in materials with high critical temperature superconductivity. There are many intriguing aspects to these materials but I will focus on the strange metallic phase at temperature

[1]See e.g. a Millennium Prize problem [1] by Clay Mathematics Institute involving the existence of this mass gap.

above the, unusually high, superconductor critical temperature[2]. As a name suggests, a strange metal has certain properties that are different from a typical metal. For example, the resistivity of a strange metal grows linearly in contrast to the quadratic growth found in a normal metal. This may not seems like a big deal, but the T^2 dependence is extremely robust. It was shown, from renormlisation group analysis, that any small perturbation around normal metal cannot change this property [5, 6] (see also a review [7]). Thus, one may argue that the "perturbation" has to be very strong in order to obtain the linear resistivity and, again, we enter the regime of strongly interacting quantum field theory. To make matters even worse, the brute-force numerical simulation is categorised as NP-hard in computational complexity [8], thus making the classical simulation of strongly coupled fermions in thermodynamic limits unachievable[3].

However, it an overstatement to say that we cannot compute anything when the interaction is strong. Numbers of non-perturbative techniques and many toy models where computations are doable even at strong interaction have been developed over the past few years. These non-perturbative results might make one doubt whether the Lagrangian-based formulation of QFT is the best way to understand quantum systems. For example, there are a number of non-perturbative methods which do not rely on microscopic Lagrangian such as conformal bootstrap [13, 14], integrability [15], modern analysis of scattering amplitude [16–18]. Instead they rely heavily on the symmetries of the systems. Many non-perturbative results are also obtained by utilising the property called *duality*. Duality is the phenomenon where by two or more Lagrangians which describe the same physics and, sometimes, enables us to map a difficult problem of strongly interacting QFT to a very simple one. Since this is the most important aspect of this thesis, let me be a little more precise. The strongest form of

[2]See [3] and chapter 2-3 of [4], for a brief history and introduction to high temperature superconductor.

[3]This problem is known as "the sign problem" which, sadly, is largely ignored by most introductions to quantum field theory. However, it is one of the biggest enemies when one tries to directly simulate strongly correlated fermion systems, both in condensed matter and lattice QCD (see e.g. [9, 10]). See also [11] for a discussion of the sign problem in a path integral formalism and the references therein. Solving this problem in polynomial time is equivalent to solving, yet another, Millennium Prize problem by Clay Mathematics Institute [12].

the duality is the following:

If a theory A is dual to a theory B, then for every physical quantity in A there are corresponding physical quantities in B. The theories A and B may have different microscopic constituents or gauge groups or even live in a different dimension.

If A and B describes the same physics, one may ask a philosophical question: which Lagrangian describe the "true nature" of quantum theory. Are they both physical? Is this just a mathematical trick? Or should we think of theory B as an emergent phenomenon in a certain limit of theory A? At this point, we simply do not know.

So what is QFT? The facts that a conventional Lagrangian formalism seems to hit a brick wall, most non-perturbative results rely so little on the microscopic Lagrangian and, sometimes, more than one Lagrangian describe the same physics, all seem to indicate that we should rethink what we know about it. How should QFT be reformulated? I am afraid I do not know. There seems to be a lot of things we need to understand before answering such a question. *The goal of this thesis* is to make a few small steps toward, hopefully, the answer by exploring a peculiar duality between strongly interacting quantum field theory and the theory of general relativity, known as the gauge/gravity duality.

1.2 Gauge/gravity duality and effective theory

The origin of this duality dates back to the study of black hole entropy. One very peculiar property of black holes is that their entropy is proportional to the area of a surface called black hole horizon, not the enclosed volume. This and many other pieces of evidence prompted 't Hooft and Susskind to conjecture that a black hole can be described by a quantum field theory in one lower dimension [19, 20]. In this picture, the statistical physics interpretation of black hole entropy is nothing but the number of microstates of the corresponding quantum field theory. This statement can be confirmed in the string theory setup by counting the microstates of extended objects called D-brane[21]. Shortly after, it was argued by Maldacena [22] that the large class of quantum field theories constructed in similar manner not only capture the entropy of the black hole but also the dynamics of the spacetime itself! The strongest statement of this duality is:

For a quantum field theory with a holographic dual, there is a theory of (quantum) gravity in higher dimensions in which every physical quantity in the field theory can be mapped into a physical quantity of the dual gravity.

The fact that a certain spacetime in a theory of (super)gravity seems to emerge from quantum field theory is, in itself, a very fascinating phenomenon. In addition, there is a very interesting usage of this duality. It turns out that the dual gravity theory becomes classical when the coupling of the QFT becomes infinitely large. Thus, with this duality, one might hope to use it to compute quantities in the, conventionally, unaccessible regime of quantum field theory.

Despite such a promising aspect, one has to be careful about the applicability of this duality. With this in mind, let me illustrate this caveat with some technical details. In the prototypical example of gauge/gravity duality, the aforementioned quantum field theoy is a well-known maximally supersymmetric quantum field theory in 3+1 spacetime dimensions, called $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $SU(N)$ gauge group. It consist of a non-Abelian $SU(N)$ gauge field, four species of fermions and six scalar fields. All of these transform under the adjoint representation of $SU(N)$. Roughly speaking, this theory is a matrix theory where all the fields are $N \times N$ matrices. The coupling between theses fields is extremely constrained by the supersymmetry, leaving only one free parameter in the Lagrangian: the Yang-Mills coupling constant g_{YM}.[4] At first glance, studying this theory or its deformations is very unlikely

[4]In terms of these parameters, the dual gravity theory only becomes tractable when $g_{\text{YM}}^2 N \gg 1$ and $N \gg 1$. Otherwise, one needs to include quantum gravity effects in string theory. This issue will be discussed again in chapter 2.

to give us any new insight on the microscopic parameters of a realistic theory (such as hopping parameters, spin orbit coupling etc).

However, it is certainly possible for theories with different microscopic structures to be macroscopically equivalent to one another. This concept, known as *universality*, where many theories with different microscopic constituents become macroscopically indistinguishable, is well-known in the study of critical phenomena. This is the situation we are hoping for: *the infrared physics of the system we are interested in is the same as that of a theory with holographic dual.* If this indeed happens, we should be asking for the macroscopic information rather than the microscopic.

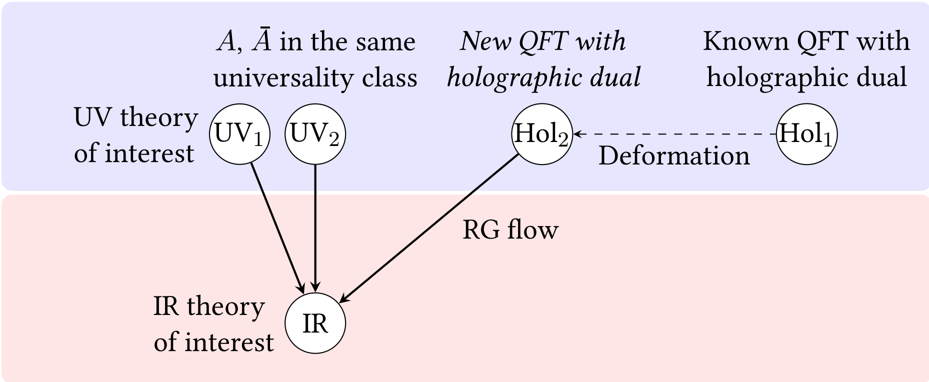

Figure 1.2. A schematic renormalisation group flow from small (UV) to large (IR) length scale where the QFT with holographic dual is in the same universality class as the system we are interested.

Focusing on the macroscopic quantities, such as thermodynamic variables, seems to be a very natural starting point. It is widely known that in strongly interacting quantum systems, thermodynamic quantities exhibit (non-mean field) scaling relations, which are very difficult to achieve using conventional methods. In the contrary, thermodynamic quantities of field theories with holographic dual are almost guaranteed to have such scaling relations. Interestingly, some of these relations are the same as those observed in strongly correlated systems.

Beyond equilibrium analysis, the gauge/gravity duality can be used to study transport phenomena and real-time dynamics of strongly interacting theories at finite temperature. Instead of having a quasi-particle description, a large class of theories with gravity dual exhibit fluid-like excitations. Intuitively, this is due to the fact that a quantum particle lose its individuality because the interaction with its neighbours is infinitely strong. This situation is well-summarised in Fig. 1.3. Such hydrodynamics features have also been observed in the study of the Nernst effect in quantum critical regions [23], graphene near the charge neutrality point [24, 25] and $PdCoO_2$ [26].

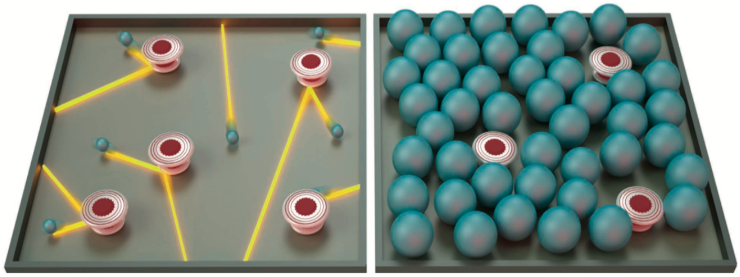

Figure 1.3. (LEFT) The ballistic transport where each quasi-particle barely interacted with each other. (RIGHT) The hydrodynamic transport where quantum particles collide with each other, losing their individuality and becoming a fluid-like entity (figure from [27]).

The fact that hydrodynamics is able to capture the dynamics of such systems may not be very surprising. After all, hydrodynamics is an almost universal effective theory that describes systems with excitation wavelength, λ_{excite} much larger than the mean free path, ℓ_{mfp}, of the constitute particles in the system (which is known to be vanishing as the interaction increases). What sets the strongly interacting quantum fluid apart from a classical fluid, however, is its viscosities. It was found that, in classes of strongly interacting quantum systems, the viscosity is governed by the ratio of Planck's constant and Boltzmann's constant $\hbar/k_B$ where the appearance of $\hbar$ signifies its quantum nature [28] (see also [4, 29] for more stories from the condensed matter viewpoints). In this aspect, gauge/gravity duality not only confirms this observation but also gives the value of the viscosity explicitly. By examining a large class of models with gravity duals, Kovtun, Son and Starinets (KSS) [30] found an interesting pattern and postulated that the ratio of the shear viscosity η and the entropy density s is bounded from below i.e.

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}, \tag{1.1}$$

where the inequality is saturated when the coupling is infinitely large. Remarkably, this ratio can also be extracted from experimental data in known strongly interacting quantum systems, see Fig. 1.4. Interestingly, the viscosity/entropy density ratio gets closer the value $\hbar/4\pi k_B$ as the interaction becomes stronger near the critical point and so far the bound has yet to be violated experimentally. This also strongly suggests that many strongly interacting quantum field theories may belong to the same universality class with the universal diffusion constant $D := \eta/sT = \hbar/4\pi k_B T$ at infinite coupling.

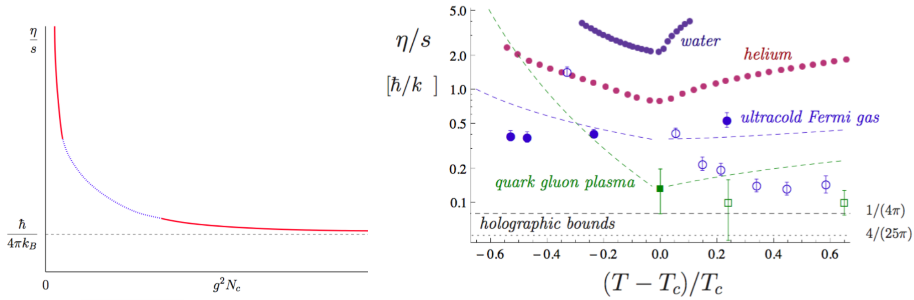

Figure 1.4. (LEFT) The interpolation between value of η/s in $\mathcal{N} = 4$ supersymmetric Yang-Mills between known results at weak coupling $g^2_{\text{YM}}N \ll 1$ and strong coupling $g^2_{\text{YM}}N \gg 1$. (RIGHT) The experimental data in various materials near the critical point compared to the KSS bound (1.1) (figures from [30, 31]).

I would like to emphasise that, although thinking about quantum theory from hydrodynamic point of view may not give us a straightforward knowledge of microscopic information, it nevertheless allows us to make a link between dissipative nature and quantum nature of many body quantum systems. From this perspective, gauge/gravity duality is a playground where computations involving these aspects can be performed explicitly. Moreover, the lower bound on shear viscosity/entropy density can, in principle, be experimentally tested.A result such as the KSS bound inspired searches for similar universal bounds e.g.[32–42] and hopefully these will help us better understand the nature of quantum field theory at strong coupling.

1.3 This thesis : A hunt for universality beyond standard hydrodynamics

One reason why hydrodynamics is such a universal effective theory is its underlying principle: global symmetry. From a more modern point of view, hydrodynamics is the low energy dynamics of the Noether currents whose time evolutions governed by conservation laws. In the long wavelength limit, the microscopic details are washed out and the effective theory is fully described by the equations of state and a few parameters in the gradient expansion (such as shear and bulk viscosity). For example the relativistic Navier-Stokes equation can be written as

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = (\mathcal{E}+\mathcal{P})u^\mu u^\nu + \mathcal{P}\eta^{\mu\nu} + \sum_{\text{All } \mathcal{O}(\partial^1) \text{ terms}} \alpha_i \, \mathbb{T}^{\mu\nu}_{(i)}, \quad (1.2)$$

where $T^{\mu\nu}$ is the stress-energy tensor, $\mathbb{T}_{(i)}$ are all possible symmetric traceless rank−2 tensors with one derivative. The conservation of $T^{\mu\nu}$ is linked to the spacetime translational symmetry of the system. The system of equations (1.2) can be extended to include dynamics of additional conserved currents if the system also possesses more global symmetries. The coefficients α_i are called *transport coefficients*. The behaviour of these transport coefficients in a strongly interacting theories is the main focus of this thesis, with an ultimate goal to explore universal relations analogous to (1.1).

Chapter 3,4,5 of this thesis are based on my original works with Piyabut Burikham [43] and Sašo Grozdanov [44, 45] where we study the interplay of additional (or broken) global symmetries, lower energy excitations and transport coefficients in strongly interacting quantum field theories with holographic dual. Chapter 2 contains a brief introduction and background material to relativistic hydrodynamics and implementations of gauge/gravity duality in this context. In the rest of this thesis, all the results are expressed in units where $\hbar = c = 1$. The outline of each chapter is the following.

In chapter 3, we construct a consistent hydrodynamic model where translational symmetry is broken. This is achieved by coupling the strongly interacting matter to the external spatial dependent scalar fields. We carefully define what is the shear viscosity in this system and study the validity of the KSS bound (1.1) in a hope to apply this bound to condensed matter systems where translational symmetry is broken by a lattice or disorder. We find that the bound is violated when we include the first order correction from the scalar fields. We are also able to identify how this violation occurs from the hydrodynamic point of view.

In chapter 4, the system now contains two additional global $U(1)$ symmetries and is subjected to a small external non-dynamical magnetic field. One Noether current is conserved but the other is broken by a quantum anomaly. This gives rise to a new transport channel called*anomalous transport*. Although quantum anomaly is famous for their non-renormalisability (or 1-loop exactness in the diagrammatic approach), it is not entirely clear why these new conductivities should inherit the same property. We find that the information regarding these conductivities is encoded on the black hole horizon in the dual gravity theory. With these horizon formulae, we show that, for all theories with a holographic dual at finite temperature, the form of these anomalous conductivities is fixed by quantum anomaly coefficients, unaffected by the value of the coupling constant and details of the renormalisation group flow.

In chapter 5, the matter no longer breaks any existing global symmetry but is now coupled to a *dynamical* external gauge field, a setup one usually finds in plasma physics. A standard treatment of this system is called magnetohydrodynamics (MHD), which is a very successful framework in the plasma physics community. The problem is that conventional MHD is constructed with an underlying assumption that the equation of state of the matter is independent of the magnetic field. This assumptions makes perfect sense in a normal earthly circumstances where the temperature is very high and the magnetic field is very small. However, there are examples in nature where this is not true. We use a novel concept of generalised global symmetry, put forth by Gaiotto, Kapustin, Seiberg and Willet [46], to identify the true global symmetry of this system. Together with the new formalism of hydrodynamics with generalised global symmetry, recently constructed by Grozdanov, Hofman and Iqbal [47], we implement this new concept in gauge/gravity duality. As a result, for the first time, we are able to investigate low energy dynamics of this system at arbitrary strength of magnetic field.

Lastly, in chapter 6, I summarise the results in this thesis, its contribution to our understanding of quantum field theory at strong coupling and possible future research directions.

You need a different way of looking at them than starting from single particle descriptions. You don't try to explain the ocean in terms of individual water molecules.

Sean Hartnoll, in Quanta Magazine

2 Lightning review of hydrodynamics and gauge/gravity duality

In this chapter, I briefly review some basic concepts of hydrodynamics and gauge/gravity duality. I will start by constructing the simplest relativistic hydrodynamics from global symmetry and gradually introduce more elements which are essential for chapters 3,4 and 5. The following section review more technical details on how to compute 2-point correlation function and extract transport coefficients from hydrodynamic equations. Then I present a very short introduction to the holographic duality with the minimum technical details possible. I will also restrict all of the discussion to relativistic hydrodynamics, ignoring non-relativistic model.

None of the material presented in this chapter is new. Hydrodynamics is, strictly speaking, a millennia-old subject. Many aspects of relativistic hydrodynamics can be found in a classic textbook by Landau and Lifshitz [48] or, for a more modern notation and applications, by Rezzolla and Zanotti [49]. The discussion in section 2.1-2.2 is inspired by [50, 51] complemented by the discussion about global symmetry in [46]. The procedures presented in section 2.2 are widely used in the holography community to compute correlation function and the review of the method itself (without reference to holography) is neatly summarised in [51]. The basic principle of gauge/gravity duality has been very well documented over the last few decades from both string theory [52, 53] and applications to condensed matter or quark-gluon plasma perspectives(see e.g. reviews, lecture notes [54–57]) and books [4, 58–60]). Therefore I will spare the reader the details and only cover the portion of the story that is most relevant to this thesis.

2.1 Global symmetry, conserved current and background fields

As mentioned in the previous chapter, one way to think about hydrodynamics as the effective theory describing the low-energy dynamics of many-body systems where the excitation wave length is much longer than the mean free path. Thess low-energy dynamics are governed only by the conservation laws of the system. Thus, one may say that hydrodynamics owes its existence to continuous global symmetries, which are related to conserved currents by the Noether theorem.

Usually, we derive the Noether current from a certain global symmetry transformation of fields in the microscopic theory. However, there is another way to look at it. A more Lagrangian-free way to access a global symmetry is to couple the system to a non-dynamical background field. For example, to introduce the chemical potential in the grand canonical ensemble, we deform the Hamiltonian in the following way

$$H \to H - \mu Q \,, \tag{2.1}$$

where Q is the number of particles. Now, we know that particle number is the

Noether charge of a conserved $U(1)$. In relativistic notation, this deformation can instead be written as

$$H \to H - \int d^{d-1}x \, J^\mu A_\mu, \qquad \text{where} \qquad \partial_\mu J^\mu = 0, \tag{2.2}$$

where the $U(1)$ charge is obtained by integrating the current J^μ over the spatial volume $Q = \int d^{d-1}S_\mu \, J^\mu$. The vector field A_μ plays the role of the non-dynamical *background field* we discussed earlier. This background field can also transform in the same way as the $U(1)$ gauge field: $A_\mu \to A_\mu + \partial_\mu \Lambda$. Thus we should think about this background field as a *non-dynamical background gauge field.*

Interestingly, the background gauge field with the gauge group $\mathfrak{G}$ can be used to singled out the Noether current associated to the global symmetry group $\mathfrak{G}$. This can be seen due to the fact that the minimum coupling between the matter and the background gauge field, $\int d^dx \, J^\mu A_\mu$, will not be invariant under the gauge transformation, unless J^μ is a conserved current.

The global symmetry also plays a crucial role in quantum physics. The state of a quantum system can be characterised by the eigenvalues of the conserved charge operators. In the study of the hydrodynamic limit of quantum systems, the quantity we are interested in is therefore the expectation of the operator $\hat{J}^\mu$ acting on the thermal state $\langle \hat{J}^\mu \rangle_{\text{thermal}}$. To access these quantities without directly specifying the Lagrangian, we put the theory on a curved manifold $\mathcal{M}$ with metric $g_{\mu\nu}$ and, if the theory possesses the $U(1)$ global symmetry, we couple it to the background $U(1)$ gauge field A_μ. The background metric is introduced to capture the stress-energy tensor $\langle T^{\mu\nu} \rangle_{\text{thermal}}$. With these ingredients, we can write down the partition function, which has the background fields as its arguments. For example, the partition function of the relativistic charged fluid in the presence of a background metric $g_{\mu\nu}$ and gauge field A_μ can be schematically written as

$$Z[g_{\mu\nu}, A_\mu] = \left\langle \exp \left[i \int d^dx \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} g_{\mu\nu} + J^\mu A_\mu \right) \right] \right\rangle \tag{2.3}$$

The background fields can now be used as a source for the conserved currents.

The variational derivative of the partition function with respect to the background fields gives

$$\langle T^{\mu\nu}\rangle = \frac{-2i}{\sqrt{-g}}\frac{\delta \log Z}{\delta g_{\mu\nu}}, \qquad \langle J^{\mu}\rangle = \frac{-i}{\sqrt{-g}}\frac{\delta \log Z}{\delta A_{\mu}} \tag{2.4}$$

The next step is to to express the conserved currents in the long wavelength limit. This can be done by gradient expanding the conserved currents and writing down all possible terms allowed by symmetry at a given order. This is not yet a physical hydrodynamics as the theory has to be augmented by additional physical requirements such as the extensivity condition and the positivity of the local entropy production near equilibrium.[1]

To make things more explicit, I first illustrate how this procedure works in a typical charge neutral fluid in section 2.1.1. The relevant global symmetries of this section are only spacetime translational symmetries (equivalently, $T^{\mu\nu}$ is the only conserved current). Then I will briefly review the hydrodynamic constructions relevant to the work of chapter 3-5, where some global symmetry groups are added or broken.

2.1.1 Charge neutral relativistic fluid

First of all, one needs to express $T^{\mu\nu}(x)$ in terms of the local macroscopic variables in order to solve the conservation equation

$$\nabla_{\nu} T^{\mu\nu} = 0 \tag{2.5}$$

which is a requirement that the system is invariant under diffeomorphism. To do this, we split the system into small pieces and assume that each piece, called fluid elements, which occupy an infinitesimal volume at position x^{μ}. This allows one to define local thermodynamics variables: the temperature $T(x)$, the entropy $s(x)$, the energy density $\varepsilon(x)$ and the pressure $p(x)$, which are small variations of their global equilibrium values. Each fluid element is also allowed

[1] Alternatively, one can also attempt to gradient expand the effective action. It turns out that this approach gives us more insight on the emergent symmetry of the system once dissipative effects are properly taken into account properly. We will come back to this approach in chapter 6.

to move around with relativistic velocity $u^\mu(x)$, which satisfied the condition $u^\mu u_\mu = -1$ (see Fig 2.1).

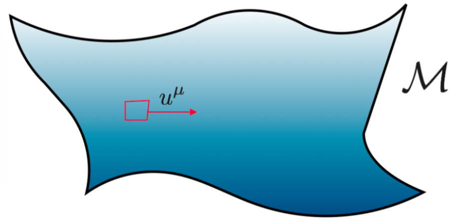

Figure 2.1. An illustration of a fluid living on the manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. The red box denotes a fluid element at point x^μ which moves with a velocity u^μ.

We are now ready to construct the stress-energy tensor. Since the stress-energy tensor is a rank-2 symmetric tensor, the possible 0^{th} order in derivative expansion is

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{O}(\partial^1), \qquad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{2.6}$$

where $\Delta^{\mu\nu}$ is acting as a projector to a $d-1$ dimensional plane perpendicular to u^μ i.e. $u_\mu \Delta^{\mu\nu} = 0$ and $\Delta^{\mu\nu} g_{\mu\nu} = d-1$. The reader might notice that this form of the stress-energy tensor is similar to the one describing an ideal fluid if $\mathcal{E} = \varepsilon$ and $\mathcal{P} = p$. However, at this level, $\mathcal{E}$ and $\mathcal{P}$ are just arbitrary functions of thermodynamics variables. The expression of the conserved currents in term of these variables is called *the constitutive relation.*

To rigorously give a physical meaning to these two coefficients, we will follow a beautiful analysis of the equilibrium partition function in the presence of a background fields, introduced in [61, 62]. In this setup, the finite temperature can be incorporated by putting the system in the cylinder $\tilde{\mathcal{M}}_{d-1} \times S^1$ (or equivalently by performing the Wick rotation which transforms the original manifold $\mathcal{M}$ to $\tilde{\mathcal{M}}_{d-1} \times S^1$), where we denote the radius of S^1 as L_0. The fluid velocity u^μ in equilibrium is proportional to the Killing vector ξ^μ of the system[2]. In this language, the temperature and fluid velocity can be expressed

[2]This is a more refined version of saying that if $\xi^\mu = (1, 0, ..., 0)$, then $\xi^\mu \partial_\mu(\text{Lagrangian}) = \partial_t(\text{Lagrangian}) = 0$

as

$$T = \frac{1}{L_0\sqrt{-\xi^\mu \xi_\mu}}, \qquad u^\mu = \frac{\xi^\mu}{\sqrt{-\xi^\lambda \xi_\lambda}}. \tag{2.7}$$

In the equilibrium configuration, the free energy can be expressible in terms of these two quantities. In the Ginzburg-Landau style, the only term at 0^{th} order in derivative expansion is

$$F = -\log Z = \int d^d x \sqrt{-g}\, p(T) \tag{2.8}$$

where F is the Gibbs free energy and $p(T)$ is the pressure. Now, varying the above partition function with respect to the metric $g_{\mu\nu}$ and substituting it in the definition of the stress-energy tensor in (2.4), we immediately find that

$$T^{\mu\nu}_{\text{equilbrium}} = \left(T\frac{\partial p}{\partial T} - p\right) u^\mu u^\nu + p\Delta^{\mu\nu} + \mathcal{O}(\partial^1). \tag{2.9}$$

The first term is nothing but the energy density obtained by the Euler relation $\varepsilon + p = sT$. With this, we obtain the stress-energy tensor for an ideal fluid as promised. The above method may seem like overkill but its true power will be manifest when the global symmetry structure is more complicated or when higher order derivative terms are involved.

Next, we move to add terms at first order in the derivative expansion. This gives the following stress-energy tensor

$$\begin{aligned} T^{\mu\nu} = \mathbb{E}\, u^\mu u^\nu + \mathbb{P}\Delta^{\mu\nu} + \sum_{\text{all vectors}} \alpha_{(i)}(T)\left(\mathbb{V}^\mu_{(i)} u^\nu + u^\mu \mathbb{V}^\nu_{(i)}\right) \\ + \sum_{\text{all tensors}} \beta_{(i)} \mathbb{T}^{\mu\nu} \end{aligned} \tag{2.10}$$

Here, we denote

$$\mathbb{E} = \varepsilon + \sum_{\text{all scalars}} \gamma_{(i)} \mathbb{S}_{(i)}, \qquad \mathbb{P} = p + \sum_{\text{all scalars}} \delta(i) \mathbb{S}_{(i)} \tag{2.11}$$

The structures $\mathbb{S}, \mathbb{V}, \mathbb{T}$ are respectively all possible independent scalar, vector, tensor constructed from u^μ and T at first order in the derivative expansions.

The list of all possible structures is

$$\begin{aligned} \text{scalar}: &\quad \nabla_\mu u^\mu, u^\mu \nabla_\mu T, \\ \text{vector}: &\quad \Delta^{\mu\nu}\nabla_\nu T, u^\nu \nabla_\nu u^\mu, \\ \text{tensor}: &\quad \sigma^{\mu\nu} := \Delta^{\mu\rho}\Delta^{\nu\sigma}\left(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho - \frac{2}{d-1} g_{\rho\sigma}\nabla_\lambda u^\lambda\right), \end{aligned} \tag{2.12}$$

and $\{\alpha_{(i)}, \beta_{(i)}, \gamma_{(i)}, \delta_{(i)}\}$ are unknown functions of thermodynamic variables, which are referred to as *transport coefficients*. However, a straightforward investigation will reveal that not all structures listed above are linearly independent. First of all, the variables $u^\mu(x), T(x)$ must satisfy the conservation law $\nabla_\nu T^{\mu\nu} = 0$ order by order. By substituting in the 0^{th} order terms in $T^{\mu\nu}$, one finds relations between the following quantities

$$u^\mu \nabla_\mu T \sim \nabla_\mu u^\mu, \qquad u^\nu \nabla_\nu u^\mu \sim \Delta^{\mu\nu}\nabla_\nu T. \tag{2.13}$$

Thus, one can see that $u^\mu \nabla_\mu T$ and $u^\nu \nabla_\nu u^\mu$ are not independent and can be remove from the list (2.12). The other type of redundancy is due to the fact that the temperature field $T(x)$ and fluid velocity u^μ have no microscopic specification out of equilibrium. This allows us to shift the velocity u^μ and temperature field $T(x)$ by terms subleading in the gradient expansion i.e.

$$u^\mu(x) \to u^\mu(x) + \mathcal{V}^\mu(\partial T, \partial u), \qquad T(x) \to T(x) + \mathcal{T}(\partial T, \partial u) \tag{2.14}$$

where the vector $\mathcal{V}^\mu$ and the scalar $\mathcal{T}$ only contain terms at first order in the gradient expansion. This redundancy is known in the literature as *frame choices*. For chapter 3 and 5, we will use the hydrodynamics construction in the *Landau frame* where $\mathcal{V}^\mu$ and $\mathcal{T}$ are chosen such that

$$u_\mu T^{\mu\nu} = -\varepsilon u^\nu, \tag{2.15}$$

which essentially set $\alpha_{(i)}$ and $\gamma_{(i)}$ to zero. Imposing these constrains, one finds that there are only two the remaining terms at 1^{st} order :

$$T^{\mu\nu}_{\text{first derivative}} = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}(\partial_\lambda u^\lambda), \tag{2.16}$$

where η and ζ are the shear and bulk viscosity respectively.

The last constraint to be imposed in this system is the positivity of local entropy production near equilibrium [48]. This condition is widely believed to be related by the second law of thermodynamics. This is, however, not entirely true since there is no such thing as the second law for local entropy (for example, refrigerators do exist). However, this condition gives a sensible constraint in most cases and, only recently, has finally been shown to emerge from consistency conditions of effective action for dissipative systems close to equilibrium [63].

This constraint demands the existence of a local entropy current J_S^μ such that $u^\mu J_S^\mu = s$ and $\nabla_\mu J_S^\mu \geq 0$. In practice, we can express J_S^μ in the gradient expansion as

$$J_S^\mu = s u^\mu + \sum_{\text{all vectors}} \tilde{\alpha}_{(i)}(T)\mathbb{V}_{(i)}^\mu + \sum_{\text{all scalars}} \tilde{\gamma}_{(i)}\mathbb{S}_{(i)} u^\mu \tag{2.17}$$

where $\tilde{\alpha}_{(i)}, \tilde{\gamma}_{(i)}$ are some unknown coefficients. Then, one can proceed by substituting the equation of motion $\nabla_\nu T^{\mu\nu} = 0$ in $\nabla_\mu J_S^\mu$. In many cases, the coefficients $\tilde{\alpha}, \tilde{\gamma}$ can be chosen to eliminate terms that are not positive definite[3], resulting in the desired structure

$$\nabla_\mu J_S^\mu = \sum F_j \left(\alpha_{(i)}, \beta_{(i)}, \gamma_{(i)}, \delta_{(i)}\right) (\text{positive definite combinations of } \partial u, \partial T)^2$$

where F_j are some linear combinations of the transport coefficients. The positivity of $\nabla_\mu J_S^\mu$ implies that $F_j \geq 0$. In this setup, fortunately, the entropy current is well-defined and one finds that

$$J_S^\mu = T^{-1}\left(p\, u^\mu - T^{\mu\nu} u_\nu\right), \qquad T\, \nabla_\mu J_S^\mu = \frac{1}{2}\eta\, \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\partial_\lambda u^\lambda)^2 \tag{2.18}$$

which implies that $\eta \geq 0$ and $\zeta \geq 0$. In this case, the positivity of the local entropy production does not put a lot of constraint in the system but as we crank up the complexity of the system, its consequences will be more dramatic.

[3] Unfortunately, this does not always happen, causing the definition of entropy current to be ambiguous as it depends on arbitrary constants $\{\tilde{\alpha}, ...\}$, see e.g. [64].

Again, this condition seems very simple but the constraints it impose will become more dramatic as we crank up the complexity of our system.

If the system also possesses a $U(1)$ global symmetry, we can also couple it to a background $U(1)$ gauge field A_μ and write down the constitutive relation. The stress energy tensor $T^{\mu\nu}$ and the conserved current J^μ can be expressed in terms of the same hydrodynamic quantities and background field. The caveat is that, since $T^{\mu\nu}$, J^μ are physical quantities and therefore cannot depend on the gauge choice, they can only depend on the field strength $F_{\mu\nu}$. In this case the constitutive relation for J^μ is

$$\begin{aligned} J^\mu = nu^\mu - \sigma\,\Delta^{\mu\nu}\left(F_{\nu\lambda}u^\lambda - T\nabla_\nu(\mu/T)\right) \\ + \chi_E\Delta^{\mu\nu}\left(F_{\nu\lambda}u^\lambda\right) + \chi_T\left(\Delta^{\mu\nu}\nabla_\nu T\right) \end{aligned} \tag{2.19}$$

The parameter n is the density of the $U(1)$ charge and $\{\sigma, \chi_E, \chi_T\}$ are transport coefficients. In this case, the coefficients χ_E and χ_T will generate a non-positive definite term in $\nabla_\mu J^\mu_S$ thus forcing them to be zero.

2.1.2 Breaking translational symmetry

Translational symmetry breaking is one of the defining properties of solid state physics. The lack of translational symmetry is caused by the presence of a lattice or disorder, which is responsible for the finite conductivity of the system. Typically, the lattice/disorder spacing ℓ_{dis} is very short compared to the mean free path ℓ_{mfp}, causing the hydrodynamic gradient expansions to breakdown as $\ell_{\text{dis}} \ll \ell_{\text{mfp}}$. However, we can still study hydrodynamic properties in the opposite limit, where the translation symmetry is only weakly broken such that the mean free path is still much shorter than the lattice/disorder spacing $\ell_{\text{dis}} \gg \ell_{\text{mfp}}$. Arguably, the quantum system will still behave like a fluid at a certain scale (see e.g. illustration in Fig. 1.3)

The simplest way to break translational symmetries in this system is to add, by hand, a term analogous to the Drude model. The Ward identity for $T^{\mu\nu}$ in flat space is modified to

$$\partial_\mu T^{\mu t} = 0, \qquad \partial_\mu T^{\mu i} = \frac{1}{\tau_{\text{imp}}} T^{ti} \tag{2.20}$$

where T^{ti} is the momentum in x^i direction and $1/\tau_{\text{imp}}$ is momentum relaxation rate [23, 65]. The second equation breaks the spatial translational symmetry in all directions. One can then proceed to compute hydrodynamic quantities by assuming that the stress-energy tensor still retain the original form in (2.6) and (2.16). This approach is a good approximation for conductivities, which are mostly governed by the zeroth order terms in $T^{\mu\nu}$ and continues to give interesting results [65–68]. However, first order hydrodynamics in (2.16) turns out to be inconsistent with the modified conservation equation (2.20).

This formalism can be made more systematic. Instead of adding the term $1/\tau_{\text{imp}}$ by hand, the translational symmetry can be incorporated by making the background metric or the background gauge field spatially dependent, similar to suspended graphene or an optical lattice in cold atom experiments. This program has been put forth [69–71] and found applications in a clean graphene experiment where hydrodynamic signatures have been found [24, 25], see e.g. figure 2.2.

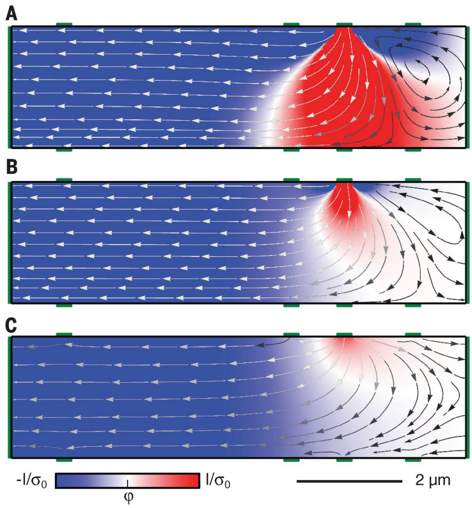

Figure 2.2. An illustration of a negative local resistance caused by viscous electron backflow in graphene experiment in [25]. The vortices, which are signatures of hydrodynamics are apparent in sub-panel A and B.

In chapter 3 of this thesis, we proceed with the same background field approach but instead of using the background metric to break translational symmetry, we introduce additional scalar fields to do the job. We couple the system to the background metric and spatially dependent scalar fields ϕ_i such that the new generating function is

$$Z[g_{\mu\nu}, \phi_i] = \left\langle \exp\left[i \int d^d x \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} g_{\mu\nu} + \mathcal{O}_i \phi_i \right) \right] \right\rangle \tag{2.21}$$

where $\langle \mathcal{O}_i \rangle$ is the operator that corresponds to the scalar field source ϕ_i. The stress-energy tensor of this theory obeys the following modified conservation law

$$\nabla_\nu T^{\mu\nu} = \langle \mathcal{O}_i \rangle \nabla^\nu \phi_i \tag{2.22}$$

The key advantage of this theory is that it has a much simpler holographic dual compared to the previous case, allowing us to compute transport coefficients explicitly. The constitutive relations can be derived systematically in terms of thermodynamic quantities, fluid velocity u^μ, background metric $g_{\mu\nu}$ and the scalar fields ϕ_i. We then explore this system and the fate of the KSS bound when the translational symmetry is broken in this particular way.

2.1.3 Introducing anomalous $U(1)$ current

Quantum anomalies are one of the most beautiful and genuine quantum effects. They are phenomena where the classical theory is invariant under a certain global symmetry but this symmetry does not survive when the theory transitions to the quantum regime. The most well-known anomaly is the chiral anomaly. An illustrative example is the massless fermion in even spacetime dimensions. For example, in 3+1 dimensions, the massless Dirac Lagrangian in the presence of a background gauge field is invariant under two $U(1)$ global symmetries called $U(1)_V$ and $U(1)_A$, which transform the fermion field as

$$\psi \to \psi \exp(i\theta), \qquad \text{and} \qquad \psi \to \psi \exp(i\gamma^5 \theta) \tag{2.23}$$

However, only one current is conserved. The conservation of current in this setup can be written in the following ways

$$\partial_\mu J^\mu_V = 0, \qquad \partial_\mu J^\mu_A = \kappa\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{2.24}$$

where the coefficient κ is called the*anomaly coefficient*. Similarly, one observes that putting the theory in curved space also has a similar effect but, this time, with an additional term associated to *the gravitational anomaly*

$$\nabla_\mu J^\mu_A = \epsilon^{\mu\nu\rho\sigma} \left(\kappa F_{\mu\nu} F_{\rho\sigma} + \lambda R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \right) \tag{2.25}$$

The quantum anomaly has been studied intensively over the last few decades since the study of pions in the pre-QCD era. Since then, it has been well understood and play an important role in our understanding of particle physics, string theory and condensed matter (see e.g. [72–75] for reviews). The quantum anomaly claim to fame came from its non-renormalisation nature: the diagrammatic computation of the anomaly coefficient is 1-loop exact [76–78], which originates from a topological quantity in the index theorem [79]. Despite all their theoretical success, these beautiful anomaly coefficients had never been measured in nature until the recent developments in the last few years.

The mentioned development materialised from the realisation that a quantum system with anomaly gives rise to a new kind of transport phenomena. We learned from high school physics that a charged particle will circulate around the magnetic field line as it is subjected to an external magnetic field. In a system with an anomaly, however, there exists an unusual current which flows along the magnetic field line and the magnitude of the current is proportional to the magnetic field itself! This phenomenon is dubbed *chiral magnetic effect* [80] and from known microscopic theories, the conductivity associated to this *anomalous current* is indeed fixed by the anomaly coefficients (see e.g. [80–86]). Subsequently, the strong interaction computation has also been done using gauge/gravity duality and the same relations between anomaly coefficients and anomalous conductivity are also found in simple models .

We are interested to see whether there exists a version of the non-renormalisation theorem for anomalous conductivities, especially at finite temperature and in the presence of the gravitational anomaly[4]. We investigate this possibility in a large class of strongly interacting quantum field theories with holographic duals and present the result in chapter 4. The theory we are interested in has two $U(1)$ global symmetries i.e. $U(1)_V \times U(1)_A$ and we couple them to two background gauge fields, V_μ and A_μ respectively. The partition function of this theory is

$$Z[g_{\mu\nu}, V_\mu, A_\nu] = \left\langle \exp\left[i \int d^4x\sqrt{-g}\left(\frac{1}{2}T^{\mu\nu}g_{\mu\nu} + J^\mu_V V_\mu + J^\mu_A A_\mu\right)\right]\right\rangle.$$

We construct the theory such that one of the global symmetries, namely $U(1)_A$ is anomalous. The Ward identities for the currents in this theory are

$$\begin{aligned} &\nabla_\mu J^\mu_V = 0, \\ &\nabla_\mu J^\mu_A = \epsilon^{\mu\nu\rho\sigma}\left(\kappa F_{A,\mu\nu}F_{A,\rho\sigma} + \gamma F_{V,\mu\nu}F_{V,\rho\sigma} + \lambda R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\rho\sigma}\right) \end{aligned} \tag{2.26}$$

where $F_{V,\mu\nu}, F_{A,\mu\nu}$ are the field strengths associated to the background field V_μ, A_μ respectively.

2.1.4 Generalised global symmetry

The global symmetries mentioned earlier are the one that people have been familiar with since the nineteenth or early twentieth centuries. As we mentioned in section 2.1, these symmetries (Poincaré symmetry and $U(1)$ global symmetry) are often associated with transformations of point-like objects (such as a particles) which leave the theory unchanged. Consequently, the conserved charges in d–dimensional systems are obtained by integrating the conserved currents over a spatial volume, e.g. for the $U(1)$ charge

$$Q = \int d\,(\text{volume})\, J^t = \int d^{d-1}S_\mu J^\mu. \tag{2.27}$$

where the spatial volume S_μ is a vector pointing in the time direction. We can also couple the system to the vector gauge field A_μ as we did in the previous

[4]See recents review about this issue in e.g. [44, 87–89]

section. In the language of differential geometry, the conserved current and the gauge field can be classified as a 1−form object (since it contains one index).

But of course, this is not the only global symmetry in quantum field theory, especially when they system consists of extended objects such as superfluid vortices, strings, domain walls, membranes etc. The study of such extended objects plays a crucial role in many areas of physics, particularly in string theory [90]. In these examples, there is also a notion of global symmetry and conserved charge. However, in contrast to the typical conserved charge (2.27) where we integrate the conserved current over the spatial volume, the conserved charge for extended objects is obtained by integrating over a surface (see figure 2.3).

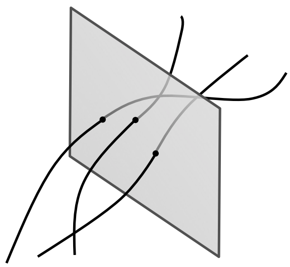

Figure 2.3. An illustration of a conserved charge for a system with string-like objects. In this case, the conserved charge is a flux obtained by integrating the number of a string/field line passing through a 2−dimensional surface (figure from [47]).

For an extended $q-1$ dimensional object (q=1 for a particle and q=2 for a string), the integration is done over a $p := d - q$ dimensional surface. Mathematically, the conserved charge for such an object is

$$Q_{\text{gen}} = \int dS_{\mu_1\mu_2\ldots\mu_p} J_{(p)}^{\mu_1\mu_2\ldots\mu_p}, \tag{2.28}$$

where $S_{\mu_1\mu_2\ldots\mu_p}$ is a totally antisymmetric tensor of rank p or a p−form object (and so is $J_{(p)}^{\mu_1\ldots\mu_p}$). Similar to the conventional conserved current (2.27), we can couple this p−form current $J_{(p)}$ to the background gauge field. However, instead of the 1-form gauge field, it will couple to the q−form gauge field $A_{\mu_1\mu_2\ldots\mu_q}$ which is a totally antisymmetric tensor of rank q. Recently, *higher–form symmtry* has been systematically categorised in [46], although it has been a recurring theme in various areas such as symmetry protected topological phase and topological order e.g. [46, 91, 92], phenomenological models for superfluid vortices [93–95], dislocation/disclination in liquid crystals [96, 97].

How does this have anything to do with plasma physics in 3+1 dimensions? Naively, one may argue that the plasma, which is charged matter coupled to an electromagnetic field, has a $U(1)$ symmetry associated to the conserved current j^μ in the Maxwell equation

$$\nabla_\mu F^{\mu\nu} = j^\nu, \qquad \nabla_\nu \left(\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\right) = 0. \tag{2.29}$$

Here, we can see from the first equation that $\nabla_\mu j^\mu = 0$ by definition. The second equation is usually thought of as a constraint and trivially vanishes when one expresses the field strength $F_{\mu\nu}$ in terms of the gauge field A_μ i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. However, this current j^μ is not associated with a global symmetry but a gauge symmetry. It is not a true symmetry but rather a redundancy of the description. The other subtle differences between global and gauge symmetry can be found in the table below

global symmetry	Gauge symmetry
Intrinsic property of the system	a redundancy rather than real symmetry
can be spontaneously broken	cannot be spontaneously broken
classifies physical states	all physical states are gauge invariant
can be anomalous	cannot have anomaly
can couple to background gauge fields	gauge field is dynamical

So, is there any global symmetry in a system described by (2.29) ? It was argued by [46] that the true global symmetry is encoded in the second set of Maxwell equation (2.29). It is the 2-form global symmetry with the current $J^{\mu\nu}$ defined as

$$J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{2.30}$$

The conserved charge $Q = \int dS_{\mu\nu} J^{\mu\nu}$ is nothing but the magnetic flux passing though the system as depicted in Fig. 2.3.

With the knowledge of global symmetries at our disposal, we can play the same game we did in the previous section. First we define a partition function of the theory coupled to the background metric $g_{\mu\nu}$ and the 2-form gauge field $b_{\mu\nu}$ which sources the 2-form current $J^{\mu\nu}$ in the following way :

$$Z[g_{\mu\nu}, b_{\mu\nu}] = \left\langle \exp\left[i \int d^4x \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} g_{\mu\nu} + J^{\mu\nu} b_{\mu\nu} \right) \right] \right\rangle .$$

The conserved currents can be obtained by varying the partition function

$$\langle T^{\mu\nu} \rangle = \frac{-2i}{\sqrt{-g}} \frac{\delta \log Z}{\delta g_{\mu\nu}}, \qquad \langle J^{\mu\nu} \rangle = \frac{-i}{\sqrt{-g}} \frac{\delta \log Z}{\delta b_{\mu\nu}}. \tag{2.31}$$

The low energy effective theory with these global symmetries can then be obtained by expressing $T^{\mu\nu}$ and $J^{\mu\nu}$ in terms of hydrodynamic variables. However, with this system, one has to take the direction of the magnetic field lines into account. This amounts to introducing a vector h^μ, pointing along the magnetic field lines, as a dynamical variable. After analysing the equilibrium partition function similar to those in section 2.1.1, the conserved currents at zeroth order in the gradient expansion are [47]

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} - \mu\rho\, h^\mu h^\nu + \mathcal{O}(\partial^1), \\ J^{\mu\nu} &= \rho\, (u^\mu h^\nu - u^\nu h^\mu) + \mathcal{O}(\partial^1). \end{aligned} \tag{2.32}$$

Here ρ denotes the magnetic flux density and μ is the chemical potential associated with the 2-form charge (or equivalently, the rate of change of the free energy with respect to the number of magnetic field line). The details of this derivation and the constitutive relations at first order in derivative expansion can be found in [47] and in chapter 5 of this thesis.

In the context of plasma physics, the situation where the matter (which obeys the Navier-Stoke equation) coupled to the dynamical electromagnetic field has been studied in the framework called *magnetohydrodynamics* (MHD). And while it is a successful self-consistent description, it has one unsatisfying underlying assumption, namely the equation of state is independent of the magnetic field. On the other hand, the framework of [47], discussed in this section, is obtained by utilising only the global symmetry of the system. Hence, the equation of state in this approach is free from the assumptions of the standard MHD formulation. This new framework allows us to explore a larger parameter space than that of the standard formulation. We apply this framework to a strongly interacting quantum field theory with holographic dual coupled to the dynamical $U(1)$ gauge field and report our findings in chapter 5.

2.2 2-point correlation functions and Kubo formulae

The focus of this section is the retarded 2-point correlation function, G^R, of conserved currents. The introduction of a background field allows us to compute this quantity in a very natural way. For example, the the stress-energy tensor in the almost flat metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small perturbation) can be written as[5]

$$\langle T^{\mu\nu}(x)\rangle_g = \langle T^{\mu\nu}(x)\rangle_{h=0} - \frac{1}{2}\int d^4y\, G^R_{T^{\mu\nu}T^{\rho\sigma}}(x,y)h_{\rho\sigma}(y) + \mathcal{O}(h^2) \quad (2.33)$$

where the retarded 2-point function is defined in the operator language as

$$G^R_{T^{\mu\nu}T^{\rho\sigma}}(x,y) := -\Theta(x^0 - y^0)\,\langle[T^{\mu\nu}(x), T^{\rho\sigma}(y)]\rangle \quad (2.34)$$

This method can be applied for all the conserved currents and background fields in previous sections.

The retarded 2-point function play a central role in this thesis for two reasons which are detailed below.

- Firstly, the pole of the 2-point function in Fourier space captures the dispersion relation of the low energy excitations. For hydrodynamics, there are two types of low energy excitations: diffusive mode and sound mode

$$G^R(\omega,k)^{-1} = 0 \quad \Rightarrow \begin{cases} \omega = -i\mathcal{D}_p k^2, & \text{diffusive mode} \\ \omega = \pm c_s k - i\mathcal{D}_l k^2 & \text{sound mode} \end{cases}$$

  where c_s is the speed of sound, $\mathcal{D}_p$ is the diffusion constant and $\mathcal{D}_l$ is the sound attenuation. The speed of sound only depends on the thermody-

[5] The computation for the real-time correlation function in the presence of the background field is best treated using Schwinger-Keldysh (or Closed-Time-Path) formalism. See, for example, [50, 98, 99] for the derivation.

namic quantities while the $\mathcal{D}_{p,l}$ are linear combinations of the first order transport coefficients and thermodynamics quantity.

- This bring us to the second point. The residue of some of the retarded functions $G^R(\omega, k)$ is proportional to the first-order transport coefficients. This allows us to express transport coefficients as linear combinations of correlation functions. Such expressions are referred to as *Kubo formulae.*

For the reader's convenience, I will use the charge neutral fluid as an example although the extension to a more complicated system is straightforward. The procedure to obtain (2.33) in hydrodynamic models is the following:

1) First, we perturb the background fields, such as $g_{\mu\nu}$, around their means values $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. For the metric, the mean value $g^{(0)}_{\mu\nu}$ is usually the flat space metric as in (2.33).

2) Then, we solve the conservation equation e.g. $\nabla_\mu T^{\mu\nu} = 0$ for dynamical variables (such as the temperature field $T(x)$ and fluid velocity u^μ). As a result, these dynamical variables will be expressed in terms of the background field perturbation $h_{\mu\nu}$.

3) Lastly, one can substitute the perturbed background fields and the solution of $\{T, u^\mu\}$, expressed in terms of $h_{\mu\nu}$, into the constitutive relations. The conserved currents can be expanded to linear order in $h_{\mu\nu}$ and by comparing this result to (2.33), we obtain the correlation function.

As an example, let me present a result for a few interesting correlation function in the charged neutral fluid. Firstly, the energy density correlation function is

$$G^R_{T^{tt}T^{tt}} = \frac{(\varepsilon + p)|k|^2}{\omega^2 - \left(\frac{\partial p}{\partial \varepsilon}\right)|k|^2 + i\frac{\omega|k|^2}{\varepsilon + p}\left(\zeta + \frac{2d-2}{d}\zeta\right)}. \tag{2.35}$$

The pole of this 2-point function indicates that the energy in the system is carried by the sound mode with the speed $c_s = (\partial p/\partial\varepsilon)^{1/2}$ and the sound attenuation $\mathcal{D}_l = \left(\zeta + (2d-2)\eta/d\right)/2(\varepsilon + p)$. On the other hand, the correlation function of momentum density $T^{t\perp}$, orthogonal to the momentum k_i, is governed by the diffusive mode

$$G^R_{T^{t\perp}T^{t\perp}} = \frac{\eta |k|^2}{i\omega - \frac{\eta}{\varepsilon+p}|k|^2} + \text{contact term} \tag{2.36}$$

with the diffusion constant $\mathcal{D}_p = \eta/(\varepsilon+p)$. The remaining correlation functions can be obtained by the Ward identities

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad k^\mu G^R_{T^{\mu\nu}T^{\rho\sigma}} = 0. \tag{2.37}$$

Finally, one can find linear combinations for these correlations functions and obtain Kubo formulae for the shear and bulk viscosity

$$\begin{aligned} \eta &= -\lim_{\omega\to 0} \operatorname{Im} \partial_\omega G^R_{T^{xy}T^{xy}}(\omega, k=0), \\ \zeta + \frac{2d-2}{d}\eta &= -\lim_{\omega\to 0} \operatorname{Im} \partial_\omega G^R_{T^{xx}T^{xx}}(\omega, k=0). \end{aligned} \tag{2.38}$$

It is important to notice that while these formulae do not depend on the microscopic information, they are direct consequences of the conservation law. Therefore they will be modified once we consider systems with different global symmetries e.g. those in section 2.1.1-2.1.3. Nevertheless, the procedure outlined here can still be applied, as we show explicitly in chapter 3-5.

2.3 Bottom-up approach to holographic duality

Obviously, there are a lot more details from string theory and supersymmetric gauge theory which play an important role in the fully fledged gauge/gravity duality. Nevertheless, the following discussion will be about a few essential aspects of the duality which allow us to show that the dual QFT exhibits hydrodynamic behaviour.

In the following sections, we present the gauge/gravity duality as a set of dictionary rules relating QFT physical quantities such as global symmetries, "source" background fields and conserved currents in section 2.1 to quantities in the gravity dual. We then continue by briefly review features in gravity which are relevant to chapters 3-5 of this thesis.

2.3.1 Capturing global symmetry

The precise form of the duality is obtained by stating that the partition function of a strongly interacting QFT discussed in section 2.1 is equal to the semi-classical partition function of a certain gravity theory with at least one dimension higher. Both partition function are also functions of the same arguments, namely the background metric and the other background fields, which act as sources. This is the celebrated Gubser-Klebanov-Polyakov-Witten (GKPW) relation [100, 101]

$$Z_{\text{QFT}}[g_{\mu\nu}, ...] = Z_{\text{gravity}}[g_{\mu\nu}, ...]. \tag{2.39}$$

But what is the generating function of the gravitational theory? How can it depend on the background fields in the QFT? In order to answer these question, let us analyse this dual gravity theory in more details to see how the global symmetry and background fields are manifested.

First of all, we need to recall that this gravity theory is, in fact, a certain low energy limit of string theory. It does not consist of only dynamical graviton but also all sort of scalar fields Φ_i and higher form gauge fields $A^{(p)}_{\mu_1...\mu_p}$. Schematically, the action of the "gravity dual" in $d+1$ dimensions is

$$\begin{aligned}\mathcal{L} = \mathcal{R} - \sum_{\text{all scalars}} Z_n(\Phi_i)(\partial\Phi_i)^2 - V(\Phi_i) + \sum_{\text{all gauge fields}} Y_m(\Phi_i)(F_{(p)})^2 \\ + \left(\text{even more complicated terms involving } \mathcal{R}, \Phi_i, A^{(p)}, F_{(p)}\right)\end{aligned} \tag{2.40}$$

where $\mathcal{R}$ is the Ricci scalar in $d+1$ dimensions. The functions $\{Z_m, Y_m\}$ can be either constant or non-trivial functions of scalar fields Φ_i. The detail of these functions and more complicated terms can be fixed by knowing their string theory origin. For example, the gravity dual to $d=4$, $SU(N_c)$ $\mathcal{N}=4$ supersymmetric Yang-Mills at strong coupling is governed by following action

$$S = \frac{N_c^2}{8\pi^2}\int d^5X\sqrt{-G}\,(\mathcal{R}+12) \tag{2.41}$$

Alternatively, one can treat $\{Z_m, Y_m\}$ as phenomenological parameters to "cook up" gravity duals with desirable properties. I will follow the second approach in this thesis.

Examples where we have evidence that the duality exists are when the solution of the action (2.40) has an asymptotic region which is described by the Anti-de Sitter (AdS) metric

$$ds^2 = G_{ab} dX^a dX^b = \frac{du^2}{4u^2} + \frac{L}{u}\left(g_{\mu\nu}(u) dx^\mu dx^\nu\right) \tag{2.42}$$

where X^a describes $d+1$ dimensional coordinate system (x^μ, u), x^μ describes the coordinates on a surface where u = constant and L is the characteristic length scale of the AdS space, called the AdS radius. The aforementioned asymptotic region is when $u \to 0$, referred to as the boundary of the AdS space. Here, the first dictionary rule can be introduced to relate the metric $g_{\mu\nu}(u)$ to the source metric $g_{\mu\nu}$ in the QFT.

- The background metric $g_{\mu\nu}$ coupled to the QFT stress-energy tensor $T^{\mu\nu}$ is the asymptotic value of the metric $g_{\mu\nu}(u)$ describing the geometry of a d dimensions surface at $u \to 0$.

$$g_{\mu\nu}|_{\text{QFT}} = \lim_{u\to 0} u\, G_{\mu\nu} = g_{\mu\nu}(u \to 0). \tag{2.43}$$

In fact, a similar procedure can be applied to other fields in the gravity dual. The small u expansion in the asymptotic region of all the fields $\mathcal{F} = \{g_{\mu\nu}, \Phi_i, ...\}$ in (2.40) can be written as

$$\begin{aligned}\mathcal{F}(u, x) = u^m \Big[& f^{(0)}(x) + f^{(1)}(x) u + ... + u^n f^{(n)}(x) + ... \\ & + \tilde{f}^{(n)}(x) u^n \log u \Big], \qquad \text{where } \tilde{f}^{(n)} = 0 \text{ for } d \text{ odd.}\end{aligned} \tag{2.44}$$

Solving this expansion using the equation of motion, one finds that all the functions $f^{(i)}$ can be written in terms of two independent functions $f^{(0)}(x)$ and

$f^{(n)}(x)$. A standard procedure [101, 102] indicates that the leading term in the expansion, $f^{(0)}$ should be interpreted as a source in the QFT picture. This leads us to a more general dictionary rule:

- The non-dynamical background field $f^{(0)}(x) = \{g_{\mu\nu}(x), A_\mu(x), ...\}$, which couple to the operator in QFT in d dimensions sets the asymptotic value of the classical dynamical fields $\mathcal{F}(u,x)$ propagating in the $d+1$ dimensional spacetime, i.e.

$$f^{(0)}(x)|_{\text{QFT}} = \lim_{u\to 0} u^{-m}\mathcal{F}(u,x). \tag{2.45}$$

Consequently, the global symmetry $\mathfrak{G}$ of the QFT is translated to a gauge symmetry $\mathfrak{G}$ on the dual gravity side. This is due to the fact that the gauge redundancy of the background fields $f^{(0)}$ is transferred to the dynamical fields $\mathcal{F}(u,x)$ via the expansion (2.44). As a result, the gravity action defined in (2.40) is gauge invariant under the gauge transformations of $\mathcal{F}$ if the QFT operator sourced by $f^{(0)}$ is a conserved current.

The other crucial point is how to incorporate finite temperature in the gravity dual. One can start by putting a QFT on a cylinder as in section 2.1.1. According to the dictionary rule (2.43), this introduces a time circle, set by the QFT temperature, at AdS boundary. One find that the thermal cycle smoothly shrinks to a point as we move further in the bulk of the AdS space, leaving the geometry in a cigar shape depicted in Fig. 2.4. Interestingly, this geometry is nothing but the Euclideanised black hole geometry. By undoing the Wick rotation, we find that the temperature effect can be obtained by introducing a black hole horizon with Hawking temperature equal to the temperature of the dual QFT.

The black hole also plays another important role in determining the unknown function coefficients $f^{(n)}(x)$ in the near boundary expansion (2.44). Essentially, the boundary condition at the black hole horizon (e.g. ingoing boundary condition, regularity condition) allows one to determine $f^{(n)}(x)$, thus the profile of $\mathcal{F}(u, x)$ is uniquely determined by $f^{(0)}$.

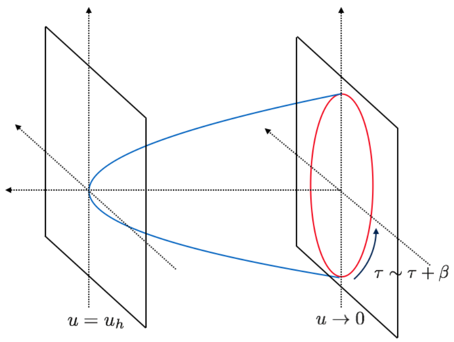

Figure 2.4. The evolution of the thermal cycle from the boundary $u \to 0$ into the bulk region $u > 0$. The circle shrinks smoothly to a point at distance $u = u_h$. The geometry is terminated at this point and forms a cigar-like structure.

By piecing together all this information, we now have the ingredients in the gravity dual that correspond to the physical quantities we are interested in, namely the global symmetries, background fields that source the conserved currents and finite temperature. With this, we will be able to construct a strongly interacting quantum field theory from a gravity dual that has a desirable global symmetry structure. To be more explicit, a few examples of bottom-up holographic models and their corresponding QFT properties are listed below.

- Conformal field theory in d dimensions with no additional global symmetries at finite temperature corresponds to the black hole geometry in the Einstein-Hilbert action

$$S = \frac{1}{2\kappa} \int d^{d+1}x \sqrt{-G} \left(\mathcal{R} + d(d-1) \right) . \tag{2.46}$$

  This can be seen from the fact that the only relevant background field and conserved charges are the background metric $g_{\mu\nu}$ and the stress-energy tensor $T^{\mu\nu}$. The cosmological constant originates from the string theory setup. For our purposes, we can view it as a term added to ensure that the geometry on the gravity dual side has an asymptotic AdS region.

- To add additional global symmetries, such as a global $U(1)$ symmetry on the QFT side, we can add a Maxwell term to the action:

$$S = \frac{1}{2\kappa}\int d^{d+1}x\sqrt{-G}\left(\mathcal{R} + d(d-1) - \frac{1}{4e^2}F_{ab}F^{ab}\right). \tag{2.47}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ where $A_\mu(u \to 0, x)$ is the QFT background gauge field that sources the $U(1)$ conserved current J^μ. If the global symmetry happens to be a non-Abelian group, such as $SU(2)$, we simply change the gauge field A_a in the gravity dual to a non-Ablelian gauge field A^i_a, where i runs over the group index.

2.3.2 Holographic thermal 1-point and 2-point function

We are now ready to use the GKPW to obtain expectation values and correlation functions.

In the semi-classical approximation, the partition function of the gravity theory is determined by the onshell action, i.e.

$$Z[g_{\mu\nu}, ...] = \exp\left[iS_{\text{gravity}}\right] = \exp\left[\frac{i}{2\kappa}\int d^{d+1}\sqrt{-G}\left(\mathcal{R} + ...\right)\right] \tag{2.48}$$

where fields $\mathcal{F}(u, x)$ in S_{gravity} are determined by the sources $f^{(0)}(x) = \{g_{\mu\nu}(x), ...\}$ and the boundary condition at the horizon. I would like to emphasise that, by keeping track of the background fields, we can conveniently use the same definition of the stress-energy tensor and the other conserved currents as in section 2.1, such as (2.4) and (2.31). Moreover, the retarded 2-point functions of these operators can also be obtained straightforwardly using the prescription in (2.33). For example, the stress-energy tensor 1-point and 2-point function are

$$\langle T^{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{gravity}}}{\delta g_{\mu\nu}}, \quad G^R_{T^{\mu\nu}T^{\rho\sigma}} = -2\frac{\delta}{\delta g_{\mu\nu}}\left(\sqrt{-g}\,\langle T^{\mu\nu}\rangle\right) \tag{2.49}$$

One can clearly see that this way of computing 1-point and 2-point function is a lot easier compared to the conventional QFT method beyond a few loops correction. Practically, one only needs to solve a differential equation for $\mathcal{F}(u, x)$ with the boundary condition determined by $f^{(0)}$, insert solutions in the gravity action S_{gravity} and apply the formulae (2.49). A similar procedure can be used to obtain correlation functions of the other conserved currents.

Historically, this procedure made contact with hydrodynamics in a series of heroic works by Policastro, Son and Starinets [103–105] for the gravity theory dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills. They found that the 2-point functions that agree with those predicted by hydrodynamics (2.35)-(2.36) with the equation of state

$$p = \frac{1}{3}\varepsilon \sim T^4, \quad \eta = \frac{s}{4\pi}, \quad \zeta = 0 \tag{2.50}$$

Soon after, this computation was extended to holographic system with different symmetries and equations of state (see e.g. [106–109] for early work on this construction). Later on, the way to extract the constitutive relation (in section 2.1.1) was developed [110–113], allowing one to link properties of the black hole dynamics and hydrodyhamics (this prescription is known under the name *fluid/gravity correspondence*). These works have been inspirations as well as the foundation for numerous works in the past few years, including those in this thesis.

2.3.3 Holographic RG flow

It is natural to ask what the physical meaning of the radial direction labelled by the coordinate u in the previous section is. This is a long-standing question which still remains unanswered. Earlier work [22, 114, 115] strongly suggested that it is related to the energy scale of the theory and that the gauge/gravity duality is the geometric realisation of the Wilsonian renormalisation group flow (see Fig. 2.5), where the region near the boundary $u = 0$ is associated to the UV fix point.

Although the precise relation between the radial direction and Wilsonian RG flow is still unclear, there are evidences to suggest that the above statement is not completely wrong. For example, the action of simple holographic models such as those in (2.46)–(2.47) diverges as $u \to 0$, reminiscent of a UV divergence in conventional QFT. Thus, for the holographic generating function to

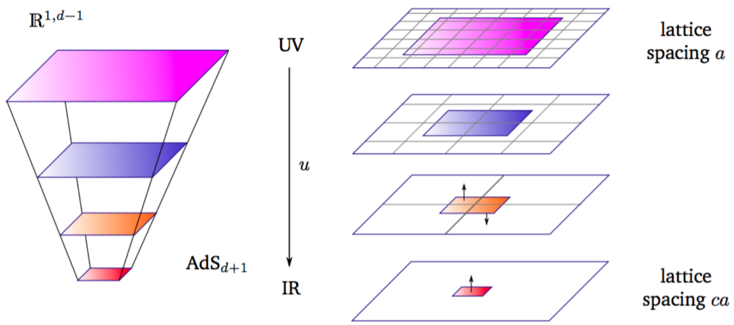

Figure 2.5. The comparison between the AdS space and series of block spin transformation along the Wilsonian RG flow.

be physical, one needs to regularise the gravity action by introducing a "cutoff surface" at $u = \epsilon \ll 1$ then add a local counter-term to S_{gravity} to subtract the divergence [101, 102]. One may also extract the Callan-Symanzik equation by integrating out the near boundary region $u < 1/\Lambda$, analogous to momentum shell renormalisation [116, 117] (see also [118] for a review of development before 2011 and references).

If we adopt this notion of RG flow, one can drive the theory away from its UV fixed point by turning on a relevant deformation. The simplest way to do this is to add a scalar field ϕ, which sources a relevant operator $\langle \mathcal{O}_\phi \rangle$ with scaling dimensions $\Delta_{\mathcal{O}} < d$. In holography, this can be done by introducing a scalar field Φ, whose boundary condition set by the source ϕ, in the gravity action i.e.

$$S_{\text{gravity}} = \int d^{d+1}X\sqrt{-G}\left(\mathcal{R} - (\partial\Phi)^2 - V(\Phi)\right), \qquad (2.51)$$

where the potential $V(\Phi)$ triggers the deformation away from the AdS boundary. It can also be tuned to obtained a desirable geometry in the bulk (or IR QFT) [119, 120]. This procedure is also extendable to the holographic RG flow for a QFT with finite density [121, 122] by studying the action

$$S_{\text{gravity}} = \int d^{d+1}X\sqrt{-G}\left(\mathcal{R} - (\partial\Phi)^2 - V(\Phi) - \frac{Z(\Phi)}{4}F_{ab}F^{ab}\right), \qquad (2.52)$$

where the function $Z(\Phi)$ sets the relevance of the $U(1)$ charge at different stage along the RG flow. This ingredient of the holographic construction allows us to explore large classes of theories with the same global symmetry but different equations of state and thus allows us to check the validity of universal statements such as the KSS bound (1.1) and those in section 1.2. Interestingly, one can engineer holographic theories with the same scaling relations as those found in high temperature superconductor materials [123, 124].

2.3.4 The "membrane paradigm"

If we take seriously the AdS radius u as an energy scale in the Wilsonian renormalisation scheme, it is tempting to say that the low energy dynamics, such as DC conductivity and viscosities are captured by the deep interior of the AdS spacetime (see e.g. Fig. 2.5). In systems with finite temperature, the "IR cutoff" is generated by the black hole horizon, see Fig. 2.6. Since we know that the QFT dual to a black hole is governed by hydrodynamic principles, one might be tempted to say that the region near the black hole horizon (called the stretch horizon) can be thought of as some sort of fluid and might even be the same fluid in dual QFT. The idea that the black hole horizon behaves as a fictitious fluid is not new. In fact, it dates back to work on black hole in 70's-80's known as *the black hole membrane paradigm* [125].

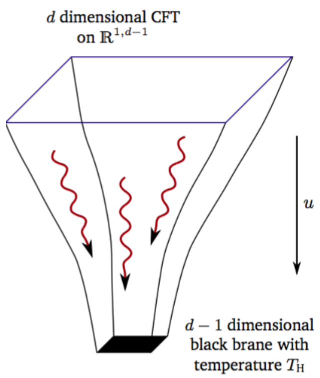

Figure 2.6. An illustration of AdS space with black hole. The horizon is located at $u = u_h > 0$ where the scale u_h is set by the temperature. This geometry can be obtained by Wick rotating the cigar geometry in Fig. 2.4. The excitations with low energy probe deeper in the interior near the horizon, suggesting that the low energy dynamics is governed by the physics near the black hole horizon.

However, a more careful analysis of the fluid/gravity correspondence indicates that the fictitious fluid at the horizon and the fluid described by the dual QFT are not the same in general [110, 116, 126–130]. This is due to the fact that the hydrodynamic data of the dual QFT contain not only a contribution from the stretched horizon but also from the bulk of AdS spacetime. Nevertheless, some hydrodynamic data of the fictitious fluid at the horizon can still be mapped to that of the dual QFT [126]. This is due to the fact that there exist "conserved currents" along the AdS radial direction (u–direction) which carry the hydrodynamic data from the stretched horizon over to the boundary, allowing one to map them to the dual QFT.

Let us briefly outline how this works for the computation of the shear viscosity. Typically, if one wants to compute $G^R_{T^{xy}T^{xy}}(\omega, 0)$ to feed into the Kubo formula (2.38), it is required to solve Einstein's equation, which is a second order differential equation. However, for the shear viscosity, one can arrange the equation of motion in such a way that it becomes a total derivative along the u–direction i.e.

$$\text{Second order DiffEqn}\Big|_{\text{low energy}} \quad \Rightarrow \quad \frac{d}{du}\Big[\mathcal{J}(u)\Big] = 0. \tag{2.53}$$

In this case, the radially conserved current $\mathcal{J}$ contains the information about the shear viscosity of the stretched horizon and the dual QFT when evaluate it at the horizon $u = u_h$ and at the AdS boundary $u \to 0$ respectively. Once the explicit form of $\mathcal{J}$ is known, one can use (2.53) to extract the shear viscosity of the dual QFT purely from the near-horizon data.

The existence of this radially conserved current not only simplifier the computation of the transport coefficients but is also a smoking gun for a universality relation. To be more precise, the physical quantities contained in the radial conserved currents are fully captured by the deep IR information (i.e. at the stretched horizon) and do not depend on the specific details of the " RG flow" along the u–direction. This concept plays a crucial role in identifying which quantities we should expect to display universal behaviour.

2.3.5 Higher derivative holography

From a string theory point of view, the fact that some QFTs have a simply gravity dual such as those dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills in (2.41) is not a coincidence. String theory is a quantum theory of gravity with two fundamental parameters: the string characteristic length ℓ_s and the string coupling g_s, which controls the quantum fluctuations in the dual gravity theory. For QFTs such as $SU(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills to have a simple gravity dual, both of these parameters have to be small. This limit corresponds to sending the rank N of the gauge group $SU(N)$ and the 't Hooft coupling $\lambda := g_{\mathrm{YM}}^2 N$ in the dual QFT to infinity. To be more precise, one can relate the string parameters ℓ_s, g_s to the QFT parameters λ, g_s, N as

$$\frac{L}{\ell_s} = \lambda \gg 1 \qquad \text{and} \qquad 4\pi g_s = g_{\mathrm{YM}}^2 = \frac{\lambda}{N} \ll 1. \tag{2.54}$$

Similar limits can also be found for the other supersymmetric theory with holographic dual, see e.g. [52].

While the large N limit is somewhat pathological, the large 't Hooft coupling limit represents the fact that the QFT we are interested in has infinitely strong coupling. The combination of the two limits greatly simplifies the computations in the gravity dual. However, this is nothing but one corner in the parameter space. To explore different regions of parameter space and make a universal statement such as the bound (1.1), one needs to find a way to extend our computation beyond large N, λ limit.

While it is very difficult to move away from the large N limit due to the more problematic nature of the string coupling g_s corrections[6], the large N limit with $1/\lambda$ correction is much more tame and very well-documented [135–141]. It turns out that, in order to include finite λ corrections in the gravity dual, one has to add higher-derivative terms. For example, in $\mathcal{N} = 4$ SYM, the first 't Hooft coupling correction comes from the following higher derivative

[6] See recent developments on $1/N$ corrections in the large 't Hooft coupling limit in e.g. [131–134].

term [141]

$$\Delta S = \int d^5X\sqrt{-G}\Big(\gamma W\Big) \tag{2.55}$$

where W is a linear combination of forth powers of the Weyl tensor C_{abcd} (c.f. [141]).

$$W = C^{abcd}C_{mbcn}C_{a}{}^{rsm}C^{n}{}_{rsc} + \frac{1}{2}C^{adbd}C_{mnbc}C_{a}{}^{rsm}C^{n}{}_{rsd}. \tag{2.56}$$

Note also that, in the string theory construction, the constant γ would also depends on scalar fields in the dual gravity side.

The above finite 't Hooft coupling correction is specific to $SU(N)$ $\mathcal{N}=4$ supersymmetric Yang-Mills. Given the vast landscape of string theory, it is reasonable to expect that generic higher derivative corrections could occur for the other QFTs with holographic duals. The higher derivative couplings e.g. γ could also depend on the scalar fields as in the case of $\mathcal{N}=4$ supersymmetric Yang-Mills but they are constrained by consistency conditions such as unitarity, causality, etc., as illustrated in [142–147].

3

Shear viscosity in holography and effective theory of transport without translational symmetry

3.1 Motivation

In recent years, numerous developments in relativistic strongly interacting quantum field theory at finite temperature have been made using the guage/gravity duality [22, 100, 101], which reduces the computations of 2-point functions to solving certain differential equations in the classical general relativity. In the IR limit, if the theory remains translational invariant, many theories of this type can be described using macroscopic variables governed by the conservation of energy-momentum : the hydrodynamic theory. Equipped with this description, the Green's functions obtained from gauge/gravity duality can be interpreted in terms of the language of relativistic hydrodynamics [103, 110] and allow us to predict universal bound for transport coefficients [30, 32–35, 148], defined by hydrodynamics constitutive relations. One of the most interesting bounds is the shear viscosity/entropy density, $\eta/s \geq 1/4\pi$ [30], which has been conjectured to be related to the minimum entropy production of the black hole in the dual gravity theory [149, 150].

Interesting applications of the gauge/gravity duality and relativistic hydrodynamics have also been found in the condensed matter systems [23, 31, 151, 152]. Despite the fact that the translational symmetry in such systems is broken due to lattice/disorder, the transport properties derived in holographic models [65, 69, 124, 153–174] fit surprisingly well with the hydrodynamic prescriptions. Moreover, the universal bounds, similar to those mentioned earlier, have been proposed [36] and some of them can also be demonstrated explicitly [38, 39]. Recently [129, 175, 176] also demonstrate that the DC transport coefficients can be extracted from the forced Navier-Stokes equations. Evidences from the work mentioned above hint that there should be a hydrodynamics-like description for the disordered theory.

If there is indeed a hydrodynamics-like description for theory without translational symmetry, one would naturally ask the following : how would such description differ from the standard relativistic hydrodynamics ? Which of the intuitions and universal results in the hydrodynamics are still applicable[1]? In this work, despite there are potentially interesting physics to be explored at strong disordered theory, we focus on the hydrodynamics-like theory when translational symmetry is weakly broken as it should be more closely related to the standard hydrodynamics. We also restrict ourselves to the type of models where translational symmetry breaking is the one in simple holographic models described below.

In ref [23], the effective theory motivated by hydrodynamics was proposed to describe the quantum critical transport where the translational symmetry is weakly broken. The dynamics of this theory is governed by the following

[1]Some aspect of this question has already been explored in [69]

equation of motion

$$\nabla_\mu T^{\mu 0} = 0, \qquad \nabla_\mu T^{\mu i} = -\Gamma\, T^{0i}, \tag{3.1}$$

where the index $i = 1, 2, d-1$ denotes the spatial dimensions. The dimensionful quantity Γ sets the scale for the broken translational symmetry and corresponds to the width of the Drude peak (see e.g. [65]). The stress-energy tensor is assumed to have the standard relativistic hydrodynamics form

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + p\, \Delta^{\mu\nu} - \eta\, \sigma^{\mu\nu}, \tag{3.2}$$

where the notation can be found in e.g. [51] and in the Appendix 3.5.1. The model successfully captures, in particular, thermo-electric conductivity and seems to be consistent with holographic computations mentioned above, see also [69] and references therein.

However, the theory described by (3.1)-(3.2) has a few draw back. As pointed out in [65, 177, 178], the above model's predictions do not agree with those from simple holographic model of [179, 180] beyond the leading order in the derivative expansion. Moreover, the correlation functions are not correctly related by the Ward identity derived from (3.1).

Alternatively, we use insight from holographic models [65, 158, 163, 168, 173, 180, 181]. In these models the translational symmetry is broken by the massive graviton or spatial dependent massless scalar fields in the dual gravity theory.[2] We following the terminology of [69] and refer to these models as theories with *mean field disorder*. From the dual theory point of view of the holographic theory with massless scalar fields, the source ϕ_i breaks the translational symmetry explicitly and the conservation of stress-energy tensor is modified to be

$$\nabla_\mu T^{\mu\nu} = \langle \mathcal{O}_i \rangle \nabla^\nu \phi_i \tag{3.3}$$

where $\langle \mathcal{O}_i \rangle$ is the expectation value of the operator sourced by ϕ_i. From the point of view of hydrodynamics, the above setup is equivalent to putting the

[2] Relations between classes of massive gravity and models with scalar fields are discussed in [173].

fluid in the manifold with background metric $g_{\mu\nu}$ and background source fields ϕ_i which breaks translational symmetry. At the equilibrium, the metric is set to be flat and the scalar sources have the profile $\phi_i = mx^i$. Taking the scalar field ϕ_i into account, the constitutive relation will also depend on the scalar fields, unlike (3.1). This coupling between fluid and spatial dependent scalar fields has already been explored earlier in [182] and more recently in [177, 178]. The modified constitutive relation for $T^{\mu\nu}$ generally has more terms than those in (3.2). The coefficients in front of independent structures in the modified constitutive relations in [177, 178, 182] are obtained by fluid/gravity method[110] for certain gravity dual theories. However, there should be general relations between the Green's function and the coefficients in the constitutive relations, which may differ from those in the standard hydrodynamics[3].

The purpose of this work is to find a systematic way of constructing the constitutive relations that also include the spatially dependent scalar fields and try to answer the questions mentioned earlier. We pay special attention to the shear viscosity and the viscosity/entropy density bound. One of our key result is that the shear viscosity η defined as coefficients of the shear tensor $\sigma^{\mu\nu}$, beyond the leading order in gradient expansion, differs from the value $\eta^\star$ extracted from standard definition $\eta^\star = -\lim_{\omega\to 0}(1/\omega)\text{Im}\, G^R_{T^{xy}T^{xy}}(\omega, k=0)$. This can be seen both from the constitutive relation, where we see that $\eta^\star$ is polluted by the additional terms due to the scalar fields, and from holographic computation, where η is extracted using fluid/gravity method [177, 178, 182] while $\eta^\star$ is obtained by directly computing the retarded Green's function.

The body of this work is consist of two main parts. In section 3.2, we focus on the constitutive relation of the effective hydrodynamics theory while the holographic computations are discussed in section 3.3. To be more precise, in section 3.2.1, we build up the constitutive relation of $T^{\mu\nu}$ and $\langle \mathcal{O}_i \rangle$ in terms of hydrodynamics variables and $\nabla\phi_i$, up to the second order in the derivative expansions. The gradient expansion in this work is organised using the anisotropic scaling of [177, 178]. This procedure is inspired by the construction of higher order hydrodynamics [50, 64, 110, 184]. In section 3.2.2, we outline

[3] The readers can find modern reviews of the subjects in e.g. [51, 183]

a consistent method to extract the retarded Green's function and show that $\eta^\star$ also include the other transport coefficients, not only the shear viscosity η. We then move on to the holographic computation, where the action and thermodynamics quantities are summarised in 3.3.1. We then compute η/s using the result from fluid/gravity [177, 178] and show that the KSS bound is violated in section 3.3.2. The computation of $\eta^\star/s$ at the leading order can be found in 3.3.3, which are differ from the expression of η/s in the previous section. The numerical profile of $\eta^\star/s$ and η/s at arbitrary value of disorder strength m/T are shown in 3.3.4. We discuss the results of this work and open questions in 4.5. An appendix contain structures in the constitutive relation.

Note added : Near the final stage of this work, we learned that [185] found the same result for $\eta^\star/s$. While the manuscript is in the preparation stage, [40] appears and has overlaps with our computations in section 3.3 but with different interpretation.

3.2 Effective theory for systems with broken translational symmetry

In this section, we first outline the procedure of how to construct the constitutive relation when the zero density fluid is coupled to the background metric $g_{\mu\nu}$ and the scalar field ϕ_i. Our expressions valid only in $2+1$ dimensions fluid but it would be straightforward to extend them to arbitrary dimensions. Our notation is closely related to those in [64] and are explained in Appendix 3.5.1. We make a small comment regarding how the role of shear viscosity, η, in the entropy production rate compared to the conformal fluid. Next, we describe the procedure to extract Green's function from the constitutive relation and the equation of motion. We show that $G^R_{T^{xy}T^{xy}}$ also contains higher derivative terms even at linear order in ω.

3.2.1 Constructing the constitutive relation

Just as in the construction of the standard hydrodynamics (those with translational symmetry), we expand $T^{\mu\nu}$, J^{μ}, $\langle\mathcal{O}_i\rangle$ in terms of the macroscopic variables $\{\mathcal{E}, u^{\mu}\}$ and background fields $\{g_{\mu\nu}, \phi_i\}$ order by order in the derivative expansion along x^{μ} direction. Since the scalar field, ϕ_i is explicitly proportional to x^i, Instead of the usual gradient expansion, we also set the momentum relaxation scale to be a small parameter as in [177, 178]. Let us call this small parameter δ, the magnitude of the gradient of the fluid variables $\{T, u^{\mu}, g_{\mu\nu}\}$ and the momentum relaxation scale m have the following scaling

$$\partial T \sim \delta, \qquad \partial u \sim \delta, \qquad \partial g \sim \delta, \qquad m \sim \delta^{1/2}. \tag{3.4}$$

This is done according to the previous study that the momentum relaxation rate $\Gamma \sim m^2$ e.g. [65]. Therefore, the frequency ω of the fluid is of the same scale as Γ.

To systematically construct the constitutive relation, it is convenient to decompose the stress energy tensor into the following form

$$T^{\mu\nu} = \mathcal{E}u^{\mu}u^{\nu} + \mathcal{P}\Delta^{\mu\nu} + t^{\mu\nu}, \tag{3.5}$$

where we choose to work with the Landau frame i.e. $u_{\mu}t^{\mu\nu} = 0$. Note that the above assumption might not be applicable for the theory without translational symmetry in general. In this work, we assume that the fluid remains translational invariant at equilibrium as this also happens in the holographic models with mean field disorder. Consequently, around the equilibrium, one can choose terms $\mathcal{E}, \mathcal{P}$ such that they contain no derivative in $\{u^{\mu}, \mathcal{E}\}$ and the scalar fields ϕ_i only enters $t^{\mu\nu}$ as $\nabla\phi_i$. Thus, the nontrivial task is reduced to building the transverse symmetric tensor out of the macroscopic variables $\{T(x), u^{\mu}, g_{\mu\nu}, \partial_{\mu}\phi_i\}$ and their derivatives upto order δ^2. Note that constitutive relation in (3.5) must also satisfy the equation of motion (3.3). In other words, the modified Ward identity (3.3) implies that the constitutive relations must satify one scalar and one vector equation

$$
\begin{aligned}
0 &= -D\mathcal{E} - (\mathcal{E}+\mathcal{P})\nabla_\mu u^\mu + u_\nu \nabla_\mu t^{\mu\nu} - \langle \mathcal{O}_i \rangle D\phi_i, \\
0 &= (\mathcal{E}+\mathcal{P})Du^\mu + \nabla_\perp^\mu \mathcal{P} + \Delta^\mu{}_\nu \nabla_\rho t^{\rho\nu} - \langle \mathcal{O}_i \rangle \nabla_\perp^\mu \phi_i.
\end{aligned} \tag{3.6}
$$

Here, we define the derivative $D \equiv u^\mu \nabla_\mu$ and $\nabla_\perp^\mu \equiv \Delta^{\mu\nu}\nabla_\nu$. The above equations put constraints on all scalars and vectors one can put into the constitutive relation. Using the first constraint, one may choose to write down a scalar in terms of the other scalars at the same order. The second constraint can be used in the same way to eliminate one vector. We follow the convention of [51] to eliminate $D\mathcal{E}$ and Du^μ so that the derivatives of $T(x)$ and u^μ only enter the constitutive relation as $\nabla_\perp^\mu T$ and $\nabla_\perp^\mu u^\nu$. The scalar fields, ϕ_i, however, contain both derivatives. Nevertheless, it is still convenient to decompose them into $D\phi_i$ and $\nabla_\perp^\mu \phi_i$ as the former vanishes at equilibrium $u^\mu = (1,0,0)$.

The procedure described so far is almost identical to the construction of the standard relativistic hydrodynamic constitutive relation. However, we would like to point out a few caveats in the above construction. First of all, despite the similarity of the notation, the parameters $\mathcal{E}$ is the energy density but $\mathcal{P}$ is not the pressure. Under our assumption, the energy density, $\epsilon \equiv T^{00} = \mathcal{E}$, as $t^{\mu\nu}$ is chosen in the Landau frame. At order δ^1, the spatial diagonal parts are $T^{xx} = T^{yy} = \mathcal{P}$. However, terms such as $\Delta^{\mu\nu}\nabla(\phi)^{2N}$ with $N = 1, 2, ..$ may also be part of $t^{\mu\nu}$ at higher order in δ due to the fact that they are not ruled out by the frame choice. Nevertheless, the correction terms to $\mathcal{P}$ will be vanishes in the traceless case $T^\mu{}_\mu = 0$. Regardless of the ambiguity, the spatial components T^{ii} of the stress-energy tensor is still not the pressure in the simple holographic theory [180]. There, the pressure, p, is obtained from the thermodynamics relation $\epsilon + p = sT$. Lastly, the scaling scheme (3.4), implies that the scalar expectation value $\mathcal{O}_i$ must be expanded up to order $\delta^{5/2}$ so that equation of motion (3.3) can be solved consistently order by order. We would also like to emphasize that it is not necessary to set the scaling such that $\omega \sim m^2$ as in (3.4)The constitutive relation for the fluid coupled to the scalar field with spatial dependence has already been considered in [182]. There, the consitutive relations are expanded with the scaling scheme $\partial u \sim \partial T \sim \partial g \sim \partial \phi$ upto the second order in the derivative expansion. The scaling scheme is indeed convenient to incorporate the effect of broken translational symmetry into the first order hydrodynamics. However, it should also be possible to take $\omega \sim m^N$ (with $N > 2$) to take into account the higher order effect of the translational symmetry breaking scale m. We will come back to comment on this point later in this section.

We list all possible independent scalars, vectors and transverse symmetric tensors, which we used to construct the consitutive relation up to order δ^1 in Appendix 3.5.1. The structures of higher order than δ^1 can be consistently built up but the number of independent terms grows very quickly. For the purpose of our work, we only list the tensors that would enter the stress-energy tensor.

The most general tensor $t^{\mu\nu}$ in (3.5), expanded up to order δ^2 can be written as

$$\begin{aligned} t^{\mu\nu} = & -\eta\sigma^{\mu\nu} - \eta_\phi \Phi^{\mu\nu} + t^{\mu\nu}_{(2)} \\ & - \Delta^{\mu\nu}\left(\zeta\nabla_\mu u^\mu + \zeta_1 D\phi_i D\phi_i + \zeta_2 \nabla_{\perp\mu}\phi_i \nabla^\mu_\perp \phi_i - P_{(2)}\right). \end{aligned} \tag{3.7}$$

The scalar, $P_{(2)}$, and orthogonal tensor, $t^{\mu\nu}_{(2)}$, of order δ^2 terms can be written explicitly as[4]

$$\begin{aligned} P_{(2)} = & \zeta\tau_\pi D(\nabla_\mu u^\mu) + \xi_1 \sigma^{\mu\nu}\sigma_{\mu\nu} + \xi_2 (\nabla_\mu u^\mu)^2 + \xi_3 \Omega^{\mu\nu}\Omega_{\mu\nu} + \tilde{\xi}_4 \nabla_{\perp\mu}\mathcal{E}\nabla^\mu_\perp \mathcal{E} \\ & + \xi_5 R + \xi_6 u^\mu u^\nu R_{\mu\nu} + \xi_7 (\nabla_{\perp\mu}\phi \cdot \nabla^\mu_\perp \phi)^2 + \xi_8 (D\phi \cdot D\phi)^2 \\ & + \xi_9 (\nabla_{\perp\mu}\phi \cdot \nabla^\mu_\perp \phi)(D\phi \cdot D\phi) + \xi_{10} (\nabla^\mu_\perp \phi \cdot D\phi)(\nabla_{\perp\mu}\phi \cdot D\phi) \\ & + \xi_{11} (\nabla_{\perp\mu}\phi \cdot D\phi)\nabla^\mu_\perp \mathcal{E} + \xi_{12} (\nabla_{\perp\mu}\phi \cdot \nabla^\mu_\perp \phi)(\nabla_\lambda u^\lambda) \\ & + \xi_{13} (D\phi \cdot D\phi)(\nabla_\lambda u^\lambda) + \xi_{14} \sigma^{\mu\nu} (\nabla_{\perp\mu}\phi \cdot \nabla^\mu_\perp \phi), \end{aligned} \tag{3.8}$$

[4]The notation of the first seven terms of $P_{(2)}$ and first eight terms of $t^{\mu\nu}_{(2)}$ are adopted from second order hydrodynamics constitutive relation of [50, 64, 184] where they write down the constitutive relation in terms of $\{u^\mu, \ln s\}$. We convert derivative of $\ln s$ into $\mathcal{E}$ using the thermodynamics relation, $d\mathcal{E} = Tds$. The coefficient $\tilde{a} \equiv a/(sT)^2$ where $a = \xi_4, \lambda_4$ in [64, 184]

and

$$
\begin{aligned}
t^{\mu\nu}_{(2)} =& \eta\tau_\pi \left[{}^{\langle} D\sigma^{\mu\nu\rangle} + \frac{1}{2}\sigma^{\mu\nu}\nabla_\lambda u^\lambda\right] + \kappa\left[R^{\langle\mu\nu\rangle} - u_\rho u_\sigma R^{\rho\langle\mu\nu\rangle\sigma}\right] \\
&+ \frac{1}{3}\eta\tau^\star_\pi\sigma^{\mu\nu}(\nabla_\lambda u^\lambda) + 2\kappa^\star u_\rho u_\sigma R^{\rho\langle\mu\nu\rangle\sigma} + \lambda_1\,\sigma^{\rho\langle\mu}\sigma^{\nu\rangle}{}_\rho + \lambda_2\,\sigma^{\rho\langle\mu}\Omega^{\nu\rangle}{}_\rho \\
&+ \lambda_3\,\Omega^{\rho\langle\mu}\Omega^{\nu\rangle}{}_\rho + \tilde{\lambda}_4\nabla^{\langle\mu}_\perp\mathcal{E}\nabla^{\nu\rangle}_\perp\mathcal{E} + \lambda_5\sigma^{\mu\nu}(D\phi\cdot D\phi) + \lambda_6\Phi^{\mu\nu}(D\phi\cdot D\phi) \\
&+ \lambda_7\,\sigma^{\mu\nu}(\nabla^\lambda_\perp\phi\cdot\nabla_{\perp\lambda}\phi) + \lambda_8\Phi^{\mu\nu}(\nabla_\lambda u^\lambda) + \lambda_9\,\Phi^{\mu\nu}_{ij}D\phi_i D\phi_j \\
&+ \lambda_{10}\Phi^{\mu\nu}(\nabla_{\perp\lambda}\phi\cdot\nabla^\mu_\perp\phi) + \lambda_{11}\Phi^{\mu\nu}_{ij}\nabla_{\perp\lambda}\phi_i\nabla^\lambda_\perp\phi_j.
\end{aligned}
\tag{3.9}
$$

Similarly, the scalar fields expectation value $\langle\mathcal{O}_i\rangle$ can be written in terms of linear combination of independent scalars with index i of the scalar fields, ϕ_i, namely

$$
\begin{aligned}
\langle\mathcal{O}_i\rangle =& c_0 D\phi_i + c_1(\nabla_\mu u^\mu)D\phi_i + c_2(\nabla^\mu_\perp\mathcal{E})\nabla_\mu\phi + c_3(D\phi\cdot D\phi)D\phi_i \\
&+ c_4(\nabla_{\perp\mu}\phi\cdot\nabla^\mu_\perp\phi)D\phi_i + c_5(D\phi\cdot\nabla_{\perp\mu}\phi)\nabla^\mu_\perp\phi_i + \mathcal{S}_i(\delta^{3/2},\delta^2,\delta^{5/2}).
\end{aligned}
\tag{3.10}
$$

where $\mathcal{S}_i$ is a linear combination of scalar of order $\delta^{3/2}$, δ^2, $\delta^{5/2}$ that transforms in the same way as $\mathcal{O}_i$. The explicit form of $\mathcal{S}_i$ is omitted as they are not relevant for the discussion in this work. In the holographic theory described by Einstein-Maxwell-scalar fields action in e.g.[180], the stress-energy tensor is traceless, $T^\mu{}_\mu = 0$. Such condition imposed on $t^{\mu\nu}$ implies that

$$
\zeta = 0, \qquad \zeta_1 = 0, \qquad \zeta_2 = 0, \qquad P_{(2)} = 0. \tag{3.11}
$$

Note that, even if $T^\mu{}_\mu = 0$ resembles the conformal field theory, this theory is not conformal due to the presence of nonzero expectation value $\langle\mathcal{O}_i\rangle$. Moreover, in the computation involving 2-point function, one can also perturb the fluid velocity as an additional small parameter. This allows one to ignore the term proportional to c_3 and terms with higher order of $D\phi$ in (3.7)-(3.10).

Before moving on, let us comments on the above form of $T^{\mu\nu}$ and $\mathcal{O}_i$, which are the result of the gradient expansions to the higher order while keeping the anisotropic scaling $\omega\sim m^2\sim\delta$. The main reason which cause these expressions to be so lenghtly is the fact that that the tensors and scalars structures built from ∂u and ∂g at higher order in δ. Keeping the same scaling and going beyond order δ^2 is simply overkill since most of the terms in the expressions similar to those in (3.8)-(3.10) are not even entering the 2-point functions' computations. It would be interesting to find the constitutive relation for theory with anisotropic scaling $\omega \sim m^N \sim \delta$ where N is a big number. This way, the constitutive relation will be able to capture more terms due to scalar fields.

We end this section by commenting on the entropy current. Demanding that the entropy production is positive locally implies that some of the coefficients in $t^{\mu\nu}$ and $\mathcal{O}_i$ are constrained [64, 186, 187]. In the case where the scalar field is not present, the entropy current is assumed to have the a canonical form [51]

$$TS^{\mu} = p\,u^{\mu} - T^{\mu\nu}u_{\nu} \tag{3.12}$$

which is reduced to the Smarr-like relation, $\epsilon + p = Ts$, when $u^{\mu} = \delta^{\mu t}$. Upon substituting the equation of motion and the constitutive relation for the conformal fluid at zero density, one will find that $\nabla_{\mu}S^{\mu} = \eta\,\sigma^{\mu\nu}\sigma_{\mu\nu} \geq 0$. Consequently, this inspired the origin of the bound on η to the minimum entropy production rate of the black hole [149, 150]. It turns out that the entropy production for the theory with broken translational symmetry is not as straightforward as in the standard conformal hydrodynamics. Let us demonstrate by consider the theory at order δ and assume that the entropy current take the canonical form(3.12), the entropy production rate contain three additional terms

$$T\nabla_{\mu}S^{\mu} = \left(sT - \mathcal{E} - \mathcal{P}\right)D\ln T - \langle \mathcal{O}_i \rangle D\phi_i + \eta_{\phi}\Phi^{\mu\nu}\sigma_{\mu\nu} + \eta\sigma^{\mu\nu}\sigma_{\mu\nu} \tag{3.13}$$

where we use the thermodynamics relation, $dp = sdT$ to eliminate $\nabla_{\mu}p$. The first three terms vanish in the absence of the scalar field but it is not so straightforward to eliminate or rearrange them to the positive definite structures. To be more precise, let us expand $\mathcal{O}_i$ at order $\delta^{3/2}$ (to make (3.3) consistent at order δ^3). One finds that

$$
\begin{aligned}
\langle \mathcal{O}_i \rangle D\phi_i = & c_0 (D\phi \cdot D\phi) + c_1 (D\phi \cdot D\phi) \nabla_\mu u^\mu + c_2 (\nabla_{\perp\mu} \cdot D\phi_i) \nabla_\perp^\mu \mathcal{E} \\
& + c_3 (D\phi \cdot D\phi)^2 + c_4 (\nabla_{\perp\mu}\phi \cdot \nabla_\perp^\mu \phi)(D\phi \cdot D\phi) \\
& + c_5 (D\phi \cdot \nabla_\perp^\mu \phi)(D\phi \cdot \nabla_{\perp\mu}\phi).
\end{aligned}
\tag{3.14}
$$

It is likely that one can add vectors that vanish at equilibrium to the canonical entropy current (3.12) to eliminate terms that contains $D\ln T, \nabla_\perp \mathcal{E}, \nabla_\mu u^\mu, \sigma^{\mu\nu}$. However, we can see that the term proportional to the coefficients of c_0, c_3, c_4, c_5 are already positive definite. Given a more complicated structure of the entropy current, it is possible that the entropy could also be produced by terms other than $\eta\,\sigma^{\mu\nu}\sigma_{\mu\nu}$. It would be very interesting to carefully analyse the entropy production in this type of models but we leave the complete analysis of the entropy current in the future work.

3.2.2 Kubo's formula for $\eta^\star$

In this section, we discuss the way to consistently extract the retarded Green's function. This method is slightly modified from variational method in [51] and is closely related to holographic computation. Extracting the Green's function in this way is also proven to be useful in deriving Kubo's formula for higher order hydrodynamics, see e.g. [99, 188]

The procedure for the variational method can be explained as the following. Firstly, one put the system in the manifold $\mathcal{M}$ with metric $g_{\mu\nu}$ and background scalar fields ϕ_i. We write down these background fields as their equilibrium value + small perturbations, namely

$$
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \phi_i = m x^i + \delta\phi_i \tag{3.15}
$$

where $\{h_{\mu\nu}, \delta\phi_i\}$ are small perturbations. At the same time, we perturb the energy density $\mathcal{E}$ and fluid velocity to linear order $\{\delta\mathcal{E}, \delta\rho, v^\mu\}$, which are also small perturbations. Then, we use the equation of motion (3.3) to solve for $\{\delta\mathcal{E}, \delta\rho, v^\mu\}$ in terms of $\{h_{\mu\nu}, a_\mu, \delta\phi_i\}$. After solving, substitute the solution for $\{\delta\mathcal{E}, \delta\rho, v^\mu\}$ into the constitutive relation (3.5).

We denoted the stress-energy tensor, where $\{\delta\mathcal{E}, \delta\rho, v^{\mu}\}$ are written in terms of $\{h_{\mu\nu}, a_{\mu}, \delta\phi_i\}$, as $\langle T^{\mu\nu}\rangle$. This is precisely the 1-point function from the field theory point of view. The retarded Green's function, G^R_{AB} of operator φ_A and φ_B where $\varphi_A = \{T^{\mu\nu}, J^{\mu}, \mathcal{O}_i\}, \varphi_B = \{h_{\mu\nu}, a_{\mu}, \delta\phi_i\}$ can be written as

$$G^R_{\mathcal{O}_i\mathcal{O}_j}(x) = -\frac{\delta\sqrt{-g}\langle\mathcal{O}_i(x)\rangle}{\delta\phi_j(0)}, \tag{3.16}$$

$$G^R_{\mathcal{O}_iT^{\mu\nu}}(x) = -2\frac{\delta\sqrt{-g}\langle\mathcal{O}_i(x)\rangle}{\delta h_{\mu\nu}(0)}, \tag{3.17}$$

$$G^R_{T^{\mu\nu}\mathcal{O}_i}(x) = -2\frac{\delta\sqrt{-g}\langle T^{\mu\nu}(x)\rangle}{\delta\phi_i(0)}, \tag{3.18}$$

$$G^R_{T^{\sigma\rho}T^{\mu\nu}}(x) = -2\frac{\delta\sqrt{-g}\langle T^{\sigma\rho}(x)\rangle}{\delta T_{\mu\nu}(0)}, \tag{3.19}$$

where all variations are performed with subsequent $\phi_i = h = 0$ insertion. Note that these 2-point functions are not entirely independent. They are related by the 2-point function's Ward's identity derived from (3.3).

To compute the shear viscosity, it is convenient to start from known result in translational invariant theory. In that case, the shear viscosity can be extracted from the retarded Greens' function of T^{xy} operator. Let us emphasize here again that, a priori, the relation between shear viscosity η and the 2-point functions is not necessary the same as in the usual hydrodynamics. For simplicity, we first study the perturbation that only depends on time. It turns out that one can bypass many steps in the above procedure as the stress-energy tensor δT^{xy} can be written in terms of the $\{h_{\mu\nu}, v^{\mu}, \delta\phi_i, \delta\mathcal{E}\}$ as

$$\delta T^{xy} = \frac{1}{2}\mathcal{P}h_{xy} + \frac{1}{2}\eta_{\phi}m^2 h_{xy} - \frac{1}{2}(\eta - m^2\lambda_7)\partial_t h_{xy} + \mathcal{O}(h^2) \tag{3.20}$$

where $\mathcal{O}(h^2)$ denotes the terms that are products of perturbations $\{h_{\mu\nu}, v^{\mu}, \delta\phi_i, \delta\mathcal{E}\}$. We can see that this component of the stress-energy tensor is independent of the primary variables i.e. $\{v^{\mu}, \delta\mathcal{E}\}$. Thus, by Fourier transform $h_{xy}(t) \sim$

$\int d\omega\, e^{i\omega t} h_{xy}(\omega)$, we immediately arrive at the 2-point function for $G^R_{T^{xy}T^{xy}}$,

$$\begin{aligned} G^R_{T^{xy}T^{xy}} &= \left(\mathcal{P} + \eta_\phi m^2\right) - i\omega\left(\eta - m^2\lambda_7\right) + \mathcal{O}(h^2), \\ \Rightarrow \quad \eta^\star &= \eta - \lambda_7 m^2 \end{aligned} \tag{3.21}$$

This implies that $-\omega^{-1}\mathrm{Im}\, G^R_{T^{xy}T^{xy}}$ are polluted by the terms proportional to m^2 and, unless one only consider $T^{\mu\nu}$ at order δ^1, the above Kubo formula is not the same as η in the constitutive relation. Note also that

$$\eta^\star = -\lim_{\omega\to 0}(1/\omega)\mathrm{Im}\, G^R_{T^{xy}T^{xy}} \tag{3.22}$$

is also bound from below at zero, for $\omega \geq 0$ because of the Hermitian property of T^{xy}. The relation between this lower bound of $\eta^\star$ and the entropy production is still unclear at this stage.

3.3 Holographic computation

If we use the effective "hydrodynamics" framework outlined in section 3.2 as a basis to define transport (or hydrodynamic) coefficients in arbitrary systems, it is then natural to expect that η and $\eta^\star$ are not identical evan at the leading order in δ expansion. However, from the hydrodynamics point of view, we do not known whether the quantites η/s and $\eta^\star/s$ violate the KSS bound or not. Moreover, as the coefficient λ_7 and possible higher order corrections are yet to be determined, we do not have an insight of how η and $\eta^\star$ are different before computing them explicitly.

To investigate these problems, we compute both η/s and $\eta^\star/s$ in a simple holographic model and shows that both of them violate the KSS bound. The ratio of η/s can be computed analytically using the results from fluid/gravity from [178]. The ratio $\eta^\star/s$ can also be computed analytically at small m and ω and are found be identical to η/s at the same order of m. Beyond the leading order, they start to deviate from each other.

To perform a holographic calculation of the shear viscosity and other thermodynamic quantities, we use a $3+1$ dimensional Einstein-Maxwell-Scalar action with a charged black brane solution ansatz. The scalar fields are assumed to have a fixed profile that explicitly breaks the translational symmetry. Thermodynamic quantities of the black hole are identified with those of the corresponding fluid. In Section 3.3.1, we specify the model and compute thermodynamic quantities. The fluid/gravity calculations are discussed in Section 3.3.2, demonstrating the violation of the KSS bound. Section 3.3.3 shows the perturbative calculation of the shear viscosity/entropy density ratio by the Kubo's formula method. The results of Section 3.3.2 and 3.3.3 shows that the η/s and $\eta^{\star}/s$ are not identical even at small m, as expected. Numerical calculations of $\eta^{\star}/s$ are in Section 3.3.4. Notably, Fig. 3.1 shows that the values of shear viscosity/entropy density ratio calculated by the two methods deviate more from one another as m increases.

3.3.1 Action and Thermodynamics

Let us start by specifying the action for the holographic model where the translational symmetry of the boundary theory is broken by the massless bulk scalar fields

$$S = \int_M d^{d+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\sum_{i=1}^{d-1}(\partial\phi_i)^2 - \frac{1}{4}F^2\right) + S_{\text{bnd}} \tag{3.23}$$

with appropriate boundary and counter terms S_{bnd}. This action exhibits a simple planar charged black hole solution where the translational symmetry of the boundary theory is broken explicitly by the scalar fields. For this solution, the background metric, gauge field and scalar fields can be written as the following [180]

$$
\begin{aligned}
ds^2 &= -r^2 f(r) dt^2 + r^2 dx_i dx^i + \frac{dr^2}{r^2 f(r)}, \quad A = A_t(r) dt, \quad \phi_i = m x^i, \\
f(r) &= 1 - \frac{m^2}{2(d-2)r^2} - \left(1 - \frac{m^2}{2(d-2)r_h^2} + \frac{(d-2)\mu^2}{2(d-1)r_h^2}\right)\left(\frac{r_h}{r}\right)^d \\
&\quad + \frac{(d-2)\mu^2}{2(d-1)r_h^2}\left(\frac{r_h}{r}\right)^{2(d-1)}, \\
A_t &= \mu\left(1 - \left(\frac{r_h}{r}\right)^{d-2}\right),
\end{aligned}
\tag{3.24}
$$

where $i = 1, 2, ..., d-1$. We denote the chemical potential by μ. For concreteness, we will focus on the theory with $d = 3$, which is an arena for many condensed matter systems. The temperature, entropy density, energy density and charge density can be written as

$$
\begin{aligned}
T &= \frac{r_h}{4\pi}\left(3 - \frac{m^2}{2r_h^2} - \frac{\mu^2}{4r_h^2}\right), \quad s = 4\pi r_h^2, \\
\epsilon &= 2r_h^3\left(1 - \frac{m^2}{2r_h^2} + \frac{\mu^2}{4r_h^2}\right), \quad \rho = \mu r_h.
\end{aligned}
\tag{3.25}
$$

Finally, the pressure can be computed using the renormalised Euclidean action [180].

$$
p = \langle T^{xx} \rangle + m^2 r_h = \frac{\epsilon}{2} + m^2 r_h = sT + \mu\rho - \epsilon. \tag{3.26}
$$

As mentioned earlier, the pressure here is not the same as the expectation value $\langle T^{ii} \rangle$.

In [65], the value of parameter m is restricted to be $0 < m < r_h\sqrt{6}$ so that the temperature remains non-negative for $\mu = 0$. Once the density of turned on, the allowed range of m becomes $0 < m < \sqrt{6r_h^2 - \mu^2/2}$.

3.3.2 Coherent regime and constitutive relation from fluid/gravity correspondence

The background parametrisation where we keep the entropy density fixed is suitable to find the numerical solution. However, it is more convenient to fix the energy density in order to compare with the result from fluid/gravity [177, 178] and the constitutive relation constructed in section 3.2.1.

We will work on zero density case for simplicity. It is also convenient to introduce a scale r_0 related to the energy density as $\epsilon = 2r_0^3$. In the absence of the scalar field, the position of the horizon in the gravity dual theory is precisely $r_h = r_0$. The relation between r_0 and r_h can be found by the following relation [178]

$$0 = 1 - \left(\frac{r_0}{r_h}\right)^3 - \frac{m^2}{2r_h^2}. \tag{3.27}$$

This relation can be found by equating the energy density where $m = 0, r = r_0$ and the case where m is nonzero given in Eqn. (3.25). The coefficients in the constitutive relation of $T^{\mu\nu}$ for theory with zero density were found using the fluid/gravity computation [178], where $T^{\mu\nu}$ is expanded up to order δ in the anisotropic scaling (3.4), to be

$$\mathcal{E} = 2r_0^3, \qquad \mathcal{P} = r_0^3, \qquad \eta = r_0^2, \qquad \eta_\phi = r_0. \tag{3.28}$$

Interestingly, if one fix the energy density and start to slightly break the translational symmetry, the shear viscosity remains unchanged. Now, the entropy density can be found, in terms of r_0, using (3.25) and (3.27) as

$$s = 4\pi r_h^2 = 4\pi \left(r_0^2 + \frac{m^2}{3} + \mathcal{O}(m^4)\right). \tag{3.29}$$

Note that the full expression of r_h is given by

$$r_h = \frac{\left(\sqrt{6}\sqrt{54r_0^6 - m^6} + 18r_0^3\right)^{2/3} + 6^{1/3}m^2}{6^{2/3}(\sqrt{6}\sqrt{54r_0^6 - m^6} + 18r_0^3)^{1/3}}. \tag{3.30}$$

This immediately implies the violation of the KSS bound [30] as

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - \frac{1}{3}\left(\frac{m}{r_0}\right)^2 + \mathcal{O}(m^4)\right), \qquad r_h = r_0 + \frac{m^2}{6r_0} + \mathcal{O}(m^4). \quad (3.31)$$

For completeness, we write down the coefficients c_i in the constitutive relation of $\langle \mathcal{O}_i \rangle$ obtained from fluid/gravity [178] i.e.

$$c_0 = -r_0^2, \quad c_1 = r_0(1-\lambda), \quad c_2 = -\frac{(1+\lambda)}{2r_0^3}, \quad c_4 = -\frac{1}{6}, \quad c_5 = \frac{2}{3}. \quad (3.32)$$

where λ can be found analytically for $\mu = 0$ to be

$$\lambda = -\frac{1}{2}\left(\frac{\pi}{3\sqrt{3}} - \log 3\right). \quad (3.33)$$

The coefficient c_3 is not specified as it depends on $(D\phi)^3$ and is subleading in the expansions $u^\mu = \delta^{0\mu} + v^\mu$ mentioned in section 3.2.1. It is interesting to observe that the value of $-2\lambda = \pi/3\sqrt{3} - \ln 3$ is identical to the coefficient of m^2 in Eqn. (3.48) of η^*/s calculated to $\omega m^2 \sim \delta^2$ order. Incidentally, λ appears in the two terms of order δ^2 in Eqn. (3.10) of $\langle \mathcal{O}_i \rangle$. It is possible that this is not a coincidence and the two quantities are actually the same.

We will not discuss the details of the transport coefficient at finite density, $\rho \neq 0$, but would like to mention that the relation between r_h and r_0 in that case can be found by solving

$$0 = 1 - \left(\frac{r_0}{r_h}\right)^3 - \frac{m^2}{2r_h^2} + \frac{\rho^2}{4r_h^4}. \quad (3.34)$$

The ratio between the entropies when $m = 0$ and nonzero value of m at the fixed energy density, in this case, at the leading order, is found to be

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - \frac{(2m/r_0)^2}{12 - \rho^2}\right) + \text{higher order terms}. \quad (3.35)$$

The above relation indicates that the shear viscosity/entropy density decreases more rapidly with the density.

3.3.3 Fluctuations and violation of the viscosity bound at leading order

Let us focus on the computation in the asymptotic AdS$_4$ space. We will choose the direction of the metric fluctuations to propagate in the x direction, i.e. $\vec{k}\cdot\hat{x} = k$ and consider the shear viscosity with respect to the perpendicular directions. In asymptotic AdS_4, the metric fluctuation can be split into those with odd and even parity under $y \leftrightarrow -y$. We are interested in odd parity modes namely $\{h_x^y, h_r^y, h_t^y\}$. In the presence of the two massless scalar fields, ϕ_1, ϕ_2, in AdS_4, only the fluctuation $\delta\phi_2$ couples to the odd parity channel. The full equations of motion of the relevant modes are

$$\begin{aligned}
\frac{d}{dr}\left[r^4 f(h_x^{y\prime} - ikh_r^y)\right] + \frac{\omega}{f}(\omega h_x^y + kh_t^y) - m^2 h_x^y + ikm\delta\phi_2 &= 0,\\
\frac{d}{dr}\left[r^4(h_t^{y\prime} + i\omega h_r^y)\right] - \frac{k}{f}(\omega h_x^y + kh_t^y) - \frac{m^2}{f}h_t^y - \frac{i\omega m}{f}\delta\phi_2 + r^2 a_y' A_t' &= 0,\\
\frac{d}{dr}\left[r^4 f(\delta\phi_2' - mh_r^y)\right] + \frac{1}{f}(\omega^2 - k^2 f)\delta\phi_2 - \frac{m}{f}(i\omega h_t^y + ikfh_x^y) &= 0,\\
i\omega h_t^{y\prime} + ikfh_x^{y\prime} - (\omega^2 - m^2 f - k^2 f)h_r^y - mf\delta\phi_2' + \frac{i\omega}{r^2}a_y A_t' &= 0.
\end{aligned}$$

The combination of the first and the third equations gives

$$\frac{d}{dr}\left(r^4 f\Psi'\right) + \frac{\omega^2 - (k^2+m^2)f}{f}\Psi = 0 \tag{3.36}$$

where $\Psi(r) = \Psi_y \equiv h_x^y - i(k/m)\delta\phi_2$. The scalar field generates mass term for the metric perturbation h_x^y proportional to its profile parameter m^2. It also breaks the translational invariance with respect to the infinitesimal shift in y direction.

To find the shear viscosity, we study the near boundary behaviour of $\Psi(r) = \Psi^{(0)} + r^{-3}\Psi^{(3)}$, which is equivalent to $h_x^y(r) = h_x^{y(0)} + r^{-3}h_x^{y(3)}$ in $k \to 0$ limit. Plugging this into the onshell action [65]

$$\begin{aligned}
S = &\int \frac{d\omega dk}{(2\pi)^2}\frac{3}{2(k^2+m^2-\omega^2)}\left[h_x^{y(0)}\left\{(m^2-\omega^2)h_x^{y(3)} - imk\delta\phi_2^{(3)}\right\}\right.\\
&\left. +\delta\phi_2^{(0)}\left\{imkh_x^{y(3)} + (k^2-\omega^2)\delta\phi_2^{(3)}\right\}\right]
\end{aligned}$$

and then apply the formula for the "shear viscosity" i.e.

$$\eta^* \equiv -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{T^{xy}T^{xy}}(\omega,k=0) = \frac{3}{\omega}\,\mathrm{Im}\left(\frac{\Psi^{(3)}}{\Psi^{(0)}}\right)\Bigg|_{\omega\to 0}. \tag{3.37}$$

The equation of motion (3.36) can be solved analytically for small ω, m limit. However, for the large m limit, one is required to solve it numerically. The numerical procedure to find η^* is straightforward as one only need to impose the ingoing boundary condition to in the region region close to the horizon, namely

$$\Psi_{\text{inner}} \;=\; \alpha_+ f(z)^{[-i\omega/(3-\frac{m^2}{2}-\frac{\mu^2}{4})]}\left(1+a(1-z)+b(1-z)^2+c(1-z)^3\right),$$

where we define the new coordinate to be $z = r_h/r$. We present the numerical results in Section 3.3.4.

Let us proceed by solving (3.36) analytically at the leading order in m^2. In the following calculation, the dimensionful parameters, ω, m, μ are rescaled by the horizon radius r_h to make them dimensionless. For simplicity, let us focus on the case where $\mu = 0, k = 0$. The gauge invariant field Ψ is assumed, consistently, to have the following expansion in m^2

$$\begin{aligned}\Psi &= f(z)^{i\omega/f'(1)}S(z),\\ S(z) &= A(z) + m^2 B(z) + \mathcal{O}(m^4),\end{aligned} \tag{3.38}$$

where at each m order we expand with respect to ω,

$$A(z) = A_0(z) + \omega A_1(z) + \omega^2 A_2(z) + \mathcal{O}(\omega^3), \tag{3.39}$$

$$B(z) = B_0(z) + \omega B_1(z) + \omega^2 B_2(z) + \mathcal{O}(\omega^3). \tag{3.40}$$

The equation of motion at $\mathcal{O}(m^0)$ order after substituting (3.38) into Eqn. (3.36) when $k \to 0$ is

$$0 \;=\; A''(z) - \frac{2+(1-2i\omega)z^3}{z(1-z^3)}A'(z) + \frac{\omega^2(1+z+z^2+z^3)}{(1-z)(1+z+z^2)^2}A(z).$$

This equation can be solved perturbatively by substituting (3.39) and solve order by order in ω. Once we obtain the solution satisfying the appropriate boundary condition, it can be used to solve for the solution at the higher order in m.

The equation of motion at $\mathcal{O}(m^2)$ order (the coefficient of m^2 in (3.36)) in $k \to 0$ limit is given by

$$\begin{aligned} 0 = & \frac{z\left(4i\omega + 2i\omega z^3 + 3z^2 + 3z - 6\right) A'(z)}{6(1-z)\left(z^2+z+1\right)^2} + \frac{g(z)A(z)}{3(1-z)\left(z^2+z+1\right)^3} \\ & + B''(z) - \frac{\left(2+(1-2i\omega)z^3\right) B'(z)}{z\left(1-z^3\right)} + \frac{\omega^2\left(z^3+z^2+z+1\right) B(z)}{(1-z)\left(z^2+z+1\right)^2}, \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} g(z) \equiv \Big(& -i\omega + \omega^2 z^5 + \left(\omega^2 - i\omega - 3\right) z^4 + \left(\omega^2 - 2i\omega - 6\right) z^3 \\ & + 3\left(\omega^2 - i\omega - 3\right) z^2 + (-6-2i\omega)z - 3 \Big). \end{aligned} \tag{3.42}$$

The boundary conditions of $A_0(z), A_1(z), A_2(z)$ are set as the following

$$A_0(0) = 1, |A_0(1)| < \infty;\; A_1(z=0,1) = A_2(z=0,1) = 0. \tag{3.43}$$

We can solve to obtain $A_0(z) = 1, A_1(z) = 0$ so that $A(z) = 1 + \omega^2 A_2(z)$. The full expression of $A_2(z)$ is lengthy but since we are interested in its behaviour near $z = 0$, we can Taylor expand $A(z)$ giving

$$A(z) \quad = \quad 1 + \omega^2 \left(\frac{z^2}{2} - \frac{z^3}{54}(18 + \sqrt{3}\pi - 9\ln 3) \right) + \mathcal{O}(z^4). \tag{3.44}$$

The function $B(z)$ can also be straightforwardly solved in a perturbative way by substituting $A(z)$ into (3.41) and solve order by order in ω. Requiring the boundary condition $B_0(0) = 0, |B_0(1)| < \infty$, the leading order solution is

$$B_0(z) = \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{1+2z}{\sqrt{3}} \right) - \frac{\pi}{6} \right] - \ln\left(\sqrt{\frac{3}{4} + (\frac{1}{2} + z)^2} \right). \tag{3.45}$$

The resulting functional form is a lengthy expression satisfying boundary conditionnext to leading order soluton, B_1, can be obtained in a similar way by requiring $B_1(z=0) = B_1(z=1) = 0$. Again, since we are interested in the behaviour of $B(z)$ near $z = 0$, we can Taylor expand to get

$$B(z) = -\frac{1}{6}(3+i\omega)z^2 + \frac{z^3}{3}\left(1 + \frac{i\omega}{9}(3+\sqrt{3}\pi - 9\ln 3)\right) + \mathcal{O}(z^4). \quad (3.46)$$

The perturbative solution is thus

$$\Psi(z) = 1 - \frac{z^2}{6}\left(3(m^2-\omega^2) + \frac{i\omega m^4}{m^2-6}\right) + z^3\left(\frac{i\omega(m^2-2)}{m^2-6} + \frac{m^2}{27}[9 + i\omega(3+\sqrt{3}\pi - 9\ln 3)] - \frac{\omega^2}{54}(18+\sqrt{3}\pi - 9\ln 3)\right) + \mathcal{O}(z^4). \quad (3.47)$$

Then the shear viscosity can be calculated by the usual relation

$$\eta^\star = \lim_{\omega\to 0}\frac{3}{\omega}\mathrm{Im}\left(\frac{\Psi^{(3)}(0)}{\Psi^{(0)}(0)}\right) \simeq 1 - m^2\left(\ln 3 - \frac{\pi}{3\sqrt{3}}\right), \quad (3.48)$$

where we expand $\Psi = \Psi^{(0)} + \Psi^{(1)}z + \Psi^{(2)}z^2 + \Psi^{(3)}z^3 + \ldots$.

Interestingly, the coefficient of $m^2, \pi/3\sqrt{3} - \ln 3$, is identical to the value of -2λ in (3.33) calculated from the fluid/gravity approach. We speculate that the two quantities could actually be related despite being at different order in the derivative expansion. [5]

3.3.4 Numerical results and beyond the leading order

In this section, we solve the equation for Ψ numerically with fixed $r_h = 1$, using the procedures outlined in the previous section. The purpose of these numerical computaion is two-fold. First of all, we would like to check the validity of the analytic computation and the prediction from fluid/gravity when the disorder strength is small. Secondly, it would be interesting to see the pattern of how the retarded correlation $G^R_{T^{xy}T^{xy}}$ behave at higher order. The main point of the latter part is to emphasize that, when the higher order in δ is included, the quantity $\eta^* = -\omega^{-1}\mathrm{Im}G^R_{\Psi\Psi}|_{\omega\to 0}$ is *not* the value of η in the constitutive relation. This is due to the fact that the 2-point function is polluted by the term

[5] **Note added :** We would like to mention that the expression for $\eta^\star/s$ here agrees with those presented in [40, 185].

of the form $(\text{scalars})\sigma^{\mu\nu}$ e.g. $\lambda_7\sigma^{\mu\nu}(\nabla_\perp\phi)^2$ in (3.8) and (3.8).

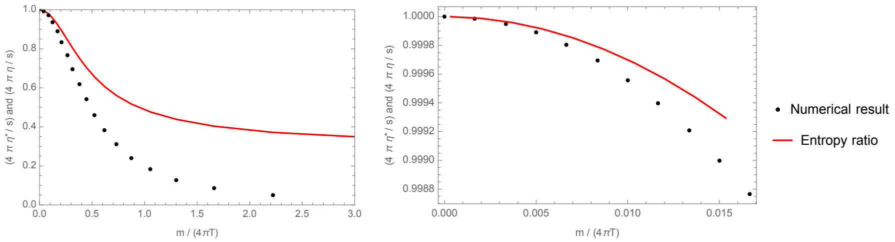

Figure 3.1. Numerical value of viscosity ratio $4\pi\eta^*/s$ at zero chemical potential compared with $4\pi\eta/s$ in the fluid/gravity calculation as a function of m/T. The dotted curve is the ratio $4\pi\eta^\star/s$ computed using Kubo's formula for $\eta^\star$ as described in section 3.3.3. The solid curve (fluid/gravity) is computed from η/s where $s = 4\pi r_h^2$ and r_h is given by the full expression in (3.30). We refer to this curve as entropy ratio since the value of η is proportional to the entropy density when $m = 0$ with the same energy density. It is clear that there is a large deviation between the numerical η^* and the fluid/gravity η.

In figure 3.1, we demonstrate that both η/s and $\eta^\star/s$ violate the KSS bound. The violation of KSS bound for η/s can be understood as η is only sensitive to r_0 as we pointed out in section 3.3.2. On the other hand, the violation of $\eta^\star/s$ comes from the change in entropy and the higher order terms in δ expansion.. Interestingly, our numerical result indicates that the differences $\eta - \eta^\star$ is monotonically increasing as m/T grows.

We can also consider what happens in the finite chemical potential case. In figure 3.2, we can see that the ratio $\eta^\star/s$ violate the bound for even small value of m. The numerical value of $\eta^\star/s$ decrease more rapidly as one increase the chemical potential. Although we don't have an analytic expression to see the explicit μ/r_h dependence, this feature can already be observed at a small value of m. In the regime where the difference between $\eta^\star$ and η is small, the above feature agrees with the prediction from (3.35).

A simple Mathematica code used to produced plots in this section is available upon request.

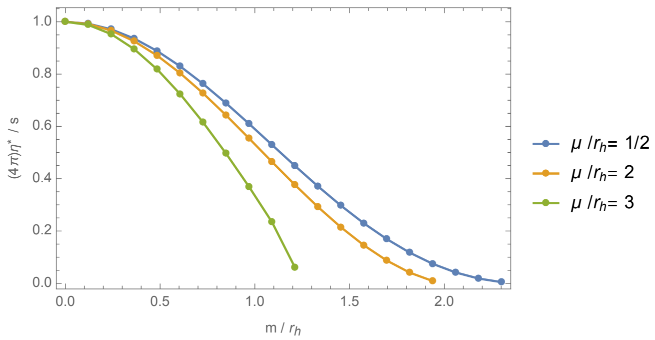

Figure 3.2. The numerical profile of $4\pi\eta^*/s$ with respect to the m/r_h at various μ/r_h, where $\eta^* = -\omega^{-1}\text{Im}G^R_{\Psi\Psi}|_{\omega\to 0}$ for different chemical potentials. Each curve truncates at zero temperature where $m/r_h = \sqrt{6 - \mu^2/2r_h^2}$.

3.4 Discussions and outlook

We follow up on the insight from [177, 178], which suggest that coupled the fluid to the background spatially dependent scalar fields ϕ_i is an accurate and consistent framework to study the hydrodynamics behaviour of the theory with broken translational symmetry. We construct the constitutive relation to order δ^2 and shows that the standard hydrodynamic formula we used to extract the usual shear viscosity, η, is no longer applicable when the scalar fields are included in the constitutive relation. With the modified constitutive relation, we speculate that the shear viscosity may not be the only channel to produce the entropy. However, the correct form of the entropy current has yet to be found. Thus, our constitutive relation should be considered as the worse case scenario, where no hydrodynamics coefficient is constrained by the positivity of local entropy production and we cannot make a clear statement on the minimum entropy production conjecture of [149, 150]. It would be very interesting to make the entropy production rate argument more precise in this class of theories and study the manifestation of the minimum entropy production conjecture in this class of theory, particularly, possible connection between the conjecture and the universal bound in disordered systems [36, 38, 39].

Regarding the holographic computation, we have analytically and numerically computed the "shear viscosity" per entropy density ratio, η^*/s, in the finite-density holographic models with translational symmetry breaking for an asymptotically AdS_4 spacetime. The analytic computation has been done using a perturbative method order by order in m^2 and ω. The ratio is found to violate the KSS bound $\eta/s = 1/4\pi$ for arbitrary translational symmetry breaking parameter m. In 4 ($d = 3$) dimensions for small m, the ratio is

$$\frac{4\pi\eta^*}{s} \simeq 1 - \frac{m^2}{r_h^2}\left(\log 3 - \frac{\pi}{3\sqrt{3}}\right) + \mathcal{O}(m^4).$$

At larger m, the deviation of η^*/s and η/s grows as we can see from Fig. 3.1. Incidentally, the difference $\eta - \eta^\star$ is monotonically increasing. As we saw that the difference is caused by the higher order terms e.g. λ_7, it would be interesting to understand whether the coefficient λ_7 and other terms participate in $\eta^\star$ are constrained by some underlying principles or not.

A simple explanation of the violation of KSS bound is the entropy contribution from the scalar fields. In the presence of the translational symmetry breaking scalar field profile, the entropy is increased as we can see from the enlarged horizon in Eqn. (3.29). On the other hand, the shear viscosity remains insensitive to m at the leading order. The η/s ratio thus becomes smaller than the KSS bound for any m. Remarkably, the violation persists even in the zero temperature limit the degree of violation depends on the chemical potential μ through dependency on m. Inspired by the viscosity bound violation, it is interesting to investigate other hydrodynamic bounds in the translational symmetry breaking axion-gravity model. First, let us consider the sound speed bound $c_s^2 \le 1/2$ [148]. From Eqn. (3.25), we might think that the sound speed c_s should be calculated from $p = m^2 r_0 + \epsilon/2$ by the quantity $(\partial p/\partial\epsilon)$. But if we choose to fix m, μ

$$\left.\frac{\partial p}{\partial \epsilon}\right|_{m,\mu} = \frac{1}{2} + m^2 \left.\frac{\partial r_0}{\partial \epsilon}\right|_{m,\mu} = \frac{1}{2} + \frac{2m^2}{\mu^2 + 2(6r_0^2 - m^2)} \ge \frac{1}{2}, \tag{3.49}$$

For $m = 0$, this quantity saturates the bound $(\partial p/\partial\epsilon) \le 1/2$. However, when m is turned on, the above definition of the speed of sound violates the sound-speed bound. A more consistent candidate for c_s^2 is the quantity $(\partial\mathcal{P}/\partial\mathcal{E})$ as the modified constitutive relation has the following sound pole

$$\omega^2 - \left(\frac{\partial \mathcal{P}}{\partial \mathcal{E}}\right)\bigg|_{\mu,\mu} k^2 + ... = 0, \tag{3.50}$$

instead of the physical pressure p in the standard hydrodynamics. Using (3.28), the speed of sound bound is trivially satisfied.

$$c_s^2 \equiv \frac{\partial \mathcal{P}}{\partial \mathcal{E}} = \frac{1}{2}, \tag{3.51}$$

saturating the sound-speed bound regardless of the translational symmetry breaking. The other interesting bound related to the sound speed is the bulk viscosity bound [32] for $d = 3$,

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{2} - c_s^2\right). \tag{3.52}$$

Since in our model the fluid is traceless so that the bulk viscosity $\zeta = 0$ [180], the bulk viscosity bound is trivially saturated.

One obvious next goal is also to find an effective hydrodynamic framework for a theory with strong disorder. As we also mentioned earlier, the main obstacle for the current framework is due to the complexity when one includes higher order terms in gradient expansions. It would be interesting to find a constituent way to incorporate terms higher order in $\nabla\phi_i$ without including higher order hydrodynamic terms containing ∂u and ∂g. In fact, the formalism to extract DC conductivities from forced Navier-Stokes equation has been recently developed in [129, 175, 176] without invoking the derivative expansions. The connection between this method and the one studied in this work has been discussed in [178]. It would be interesting to see how robust the connection between the two frameworks is when one includes higher order terms in $\nabla\phi$.

3.5 Appendices

3.5.1 Scalars, vectors and tensors from basic structures

The constitutive relation of the "hydrodynamics" effective theory in this work are constructed from the following local macroscopic variables $\mathcal{E}(x), u^\mu(x)$ and the background fields $g_{\mu\nu}(x), \phi_i(x)$ For simplicity, let us work on zero density. To find the structures that enter the consitutive relation, we organise the scalar, vector and tensor at each order in the expansion in δ.

- Structures of order δ^0 : For the system where the low energy limit is homogenous, as considered in this work, the zeroth order term cannot explicitly contain the scalar field $\phi_i = mx^i$. The objects at this order are

$$\begin{aligned} \text{Scalar}: &\quad \mathcal{E}(x) \\ \text{Vector}: &\quad u^\mu(x) \\ \text{Tensor}: &\quad u^\mu u^\nu, \Delta^{\mu\nu} \end{aligned} \tag{3.53}$$

  The projector, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is orthogonal to the 4-velocity i.e. $\Delta_{\mu\nu} u^\mu = 0$.

- Structures of order $\delta^{1/2}$: Terms at this order can only be linear in the derivative of ϕ_i as the expansion in δ is organised using anisotropic sclaing

$$\begin{aligned} \text{Scalar}: &\quad D\phi_i \\ \text{Vector}: &\quad u^\mu D\phi_i, \nabla_\perp^\mu \phi_i \end{aligned} \tag{3.54}$$

  where we introduce the notation for the directional derivative along the direction of the 4-velocity as $D = u^\mu \nabla_\mu$ and the derivative perpendicular to u^μ as $\nabla_\perp^\mu = \Delta^{\mu\nu}\nabla_\nu$

- Structures of order δ^1 : The basic structure at this order can be constructed from $\nabla\mathcal{E}$, ∇u and $(\nabla\phi_i)^2$. We only construct the tensors orthogonal to u^μ the Landau frame $u^\mu t_{\mu\nu}$ is chosen. Combining these objects together, we obtain

$$\begin{aligned} \text{Scalar}: &\quad \nabla_\mu u^\mu, (D\phi_i)(D\phi_j), \nabla^\perp_\mu \phi_i \nabla^\mu_\perp \phi_j \\ \text{Vector}: &\quad u^\mu D\phi_i D\phi_j, \nabla^\mu_\perp \mathcal{E}, \nabla^\mu_\perp \phi_i (D\phi_j), \\ \text{Tensor}: &\quad \sigma^{\mu\nu}, \Phi^{\mu\nu}_{ij} \end{aligned} \tag{3.55}$$

where $\sigma^{\mu\nu}$ and $\Phi^{\mu\nu}_{ij}$ are defined as

$$\begin{aligned} \sigma^{\mu\nu} &= 2\Delta^{\mu\alpha}\Delta^{\nu\beta}\nabla_{(\alpha}u_{\beta)} - \Delta^{\mu\nu}(\nabla_\lambda u^\lambda), \\ \Phi^{\mu\nu}_{ij} &= \nabla^\mu_\perp \phi_i \nabla^\nu_\perp \phi_j - \frac{1}{2}\Delta^{\mu\nu}(\nabla_{\perp\lambda}\phi_i \nabla^\lambda_\perp \phi_j) \end{aligned} \tag{3.56}$$

The trace of tensor $\Phi^{\mu\nu}_{ij}$ over the index i, j is denoted by $\Phi^{\mu\nu} = \sum_{i=1}^3 \Phi^{\mu\nu}_{ii}$. To avoid the cluttering of indices, we denote, $\phi \cdot \phi = \sum_i \phi_i \phi_i$ and $\Phi_{ij}\phi_i\phi_j = \sum_{i,j} \Phi_{ij}\phi_i\phi_j$. Note also that, the divergent of the fluid velocity $\nabla_\mu u^\mu$ is equivalent to $\nabla_{\perp\mu} u^\mu$ since $u_\mu D u^\mu = 0$.

- Structures of order $\delta^{3/2}$: Only relevant part in the constitutive relation that requires structure at this order is $\langle \mathcal{O}_i \rangle$. Thus, we need to construct scalar objects under spacetime transformation which contain the index i of the scalar fields ϕ_i. All possible combination of objects that satisfy the above requirements are listed below

$$\begin{aligned} \text{mixed term}: &\ (\nabla_\mu u^\mu) D\phi, \nabla_{\perp\mu}\phi_i \nabla^\mu_\perp \mathcal{E}, \\ \text{pure } \phi_i \text{ terms}: &\ D\phi_i(D\phi_j D\phi_j), (D\phi_i)(\Delta^{\mu\nu}\nabla_\mu\phi_j\nabla_\nu\phi_j), \\ &\ (D\phi_j)(\Delta^{\mu\nu}\nabla_\mu\phi_i\nabla_\nu\phi_j) \end{aligned} \tag{3.57}$$

4

Universality of anomalous conductivities in theories with higher-derivative holographic duals

4.1 More background materials and motivations

Anomalies

An anomaly is a quantum effect whereby a classically conserved current J^μ ceases to enjoy its conservation, $\nabla_\mu \langle J^\mu \rangle \neq 0$ [72–74, 189]. To date, a multitude of different anomalies have been discovered that can be classified into two main categories: local (gauge) and global anomalies. A gauge anomaly corresponds to a gauged symmetry (and current) and the consistency of a quantum field theory requires this anomaly to vanish. While global anomalies are permitted, their existence still imposes stringent conditions on the structure of quantum field theories due to the anomaly matching condition discovered by 't Hooft [190]. The condition states that a result of an anomaly calculation must be invariant under the renormalisation group flow and is thus independent of whether it is computed in the UV microscopic theory or an IR effective theory.

Of particular importance to quantum field theory have been the chiral anomalies, which are present in theories with massless fermions. The values of the current divergences resulting from these anomalies are known to be one-loop exact. From the point of view of the topological structure of gauge theories, one can suspect that this should be true very generically due to the fact that the anomaly is related to the topologically protected index of the Dirac operator. Perturbatively, non-renormalisation of the one-loop anomalies was established in [76–78]. In a typical four dimensional chiral theory, there are two classically conserved currents: the axial J_5^μ (associated with the γ_5 Dirac matrix) and the vector current J^μ. By including quantum corrections, their Ward identities can be written as

$$\begin{aligned} \nabla_\mu \langle J_5^\mu \rangle &= \epsilon^{\mu\nu\rho\sigma} \left(\kappa F_{A,\mu\nu} F_{A,\rho\sigma} + \gamma F_{V,\mu\nu} F_{V,\rho\sigma} + \lambda R^{\alpha_1}{}_{\alpha_2\mu\nu} R^{\alpha_2}{}_{\alpha_1\rho\sigma} \right), \\ \nabla_\mu \langle J^\mu \rangle &= 0, \end{aligned} \tag{4.1}$$

where $F_{A,\mu\nu}$, $F_{V,\mu\nu}$ are the field strengths associated with the axial and the vector gauge fields. $R^\alpha{}_{\beta\mu\nu}$ is the Riemann curvature tensor of the curved manifold on which the four dimensional field theory is defined, and κ, γ and λ are the three Chern-Simons coupling constants. While the axial current conservation is violated by quantum effects, the vector current remains conserved. Among other works, various arguments in favour of non-renormalisation of one-loop anomalies have been presented in [62, 80, 83–85, 191–193]. The situation is much less clear when, as in [194], one considers the contributions of mixed, gauge-global anomalies. In such cases, it was shown in [194] that one should expect anomalous currents to receive radiative corrections at higher loops. The connection between this work and mixed, gauge-global anomalies will be elaborated upon below. A further set of open questions related to the non-renormalisation of anomalies enters the stage from the possibility of considering non-perturbative effects in QFT.

From a historically more unconventional point of view, anomalies have recently also been studied through the (macroscopic) hydrodynamic entropy current analysis [195, 196].[1] The effects of gravitational anomalies on the hydrodynamic gradient expansion were then studied by using the Euclidean partition function on a cone in [199]. Macroscopic transport properties associated with anomalous conservation laws have now been analysed in detail (at least theoretically) both at non-zero temperature and density. To date, the most prominent and well-understood anomaly-induced transport phenomena have been associated with the chiral magnetic effect [80, 84, 200] and the chiral vortical effect [195, 201].

Chiral conductivities in field theory

In the low-energy hydrodynamic limit, we expect that to leading order in the gradient expansion of relevant fields, the expectation values of these currents can be expressed in the form of Ohm's law. The corresponding conductivities can then be defined in the following way: If a chiral system is perturbed by a small external magnetic field $B^\mu = (1/2)\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$ and a spacetime vortex $\omega^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu \nabla_\rho u_\sigma$, where u^μ is the fluid velocity vector in the laboratory frame, then the expectation values of the two currents change by $\langle \delta J^\mu \rangle$ and $\langle \delta J_5^\mu \rangle$. Note that unlike in Eq. (4.1), both the axial and vector current conservation are now broken by the induced anomalies. To leading (dissipationless) order, the change can be expressed in terms of the conductivity matrix

$$\begin{pmatrix} \langle \delta J^\mu \rangle \\ \langle \delta J_5^\mu \rangle \end{pmatrix} = \begin{pmatrix} \sigma_{JB} & \sigma_{J\omega} \\ \sigma_{J_5 B} & \sigma_{J_5\omega} \end{pmatrix} \begin{pmatrix} B^\mu \\ \omega^\mu \end{pmatrix}, \tag{4.2}$$

where σ_{JB} is known as the *chiral magnetic conductivity*, $\sigma_{J\omega}$ as the *chiral vortical conductivity* and σ_{J_5B} as the *chiral separation conductivity*. The signature of anomalies can thus be traced all the way to the extreme IR physics and analysed by the linear response theory. This will be the subject studied in this work.

By following a set of rules postulated in [202] (see also [203]), a conve-

[1] For a recent discussion of anomalies from the point of view of UV divergences in classical physics and its connection to the breakdown of the time reversal symmetry, see [197, 198].

nient way to express the anomalous conductivities is in terms of the anomaly polynomials. We briefly review these rules in Appendix 4.6.1. They allow one to compute the anomalous conductivities from the structure of the anomaly polynomials in arbitrary (even) dimensions, independently of the value of the coupling constant [199, 202–204].

In the IR limit, we may assume that the stress-energy tensor and the charge current can be expressed in a hydrodynamic gradient expansion [50, 51, 64, 184]. The constitutive relations for a fluid with broken parity, in the Landau frame, are [112, 113, 195, 205]

$$\begin{aligned} T^{\mu\nu} &= \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla_\lambda u^\lambda + \mathcal{O}\left(\partial^2\right), \\ J^\mu_I &= n_I u^\mu + \sigma_I \Delta^{\mu\nu}\left(u^\rho F_{I,\rho\nu} - T\,\nabla_\nu\left(\frac{\mu_I}{T}\right)\right) + \xi_{I,B}B^\mu_I + \xi_{I,\omega}\omega^\mu + \mathcal{O}\left(\partial^2\right), \end{aligned} \tag{4.3}$$

where the index $I = \{A, V\}$ labels the axial and the vector currents ($J^\mu_5 = J^\mu_A$, $J^\mu = J^\mu_V$) and their respective transport coefficients. In the stress-energy tensor, ε, P, η and ζ are the energy density, pressure, shear viscosity and bulk viscosity. Furthermore, n, σ, T, μ and $F_{\mu\nu}$ are the charge density, charge conductivity, temperature, chemical potential and the gauge field strength tensor. The vector field u^μ is the velocity field of the fluid, the transverse projector (to the fluid flow) $\Delta^{\mu\nu}$ is defined as $\Delta^{\mu\nu} = u^\mu u^\nu + G^{\mu\nu}$, with $G^{\mu\nu}$ the metric tensor and $\sigma^{\mu\nu}$ the symmetric, transverse and traceless relativistic shear tensor composed of $\nabla_\mu u_\nu$. Plugging the above constitutive relations into the anomalous Ward identities, one can show that the anomalous conductivities are controlled by the transport coefficients ξ_B and ξ_ω (see e.g. [206]). It was shown in [195, 196] that by demanding the non-negativity of local entropy production (and similarly, by using a Euclidean effective action in [61, 62, 199])[2], the anomalous chiral separation conductivity σ_{J_5B} and the chiral magnetic conductivity σ_{JB} become fixed by the anomaly coefficient γ:

$$\sigma_{J_5B} = -2\gamma\mu, \qquad\qquad \sigma_{JB} = -2\gamma\mu_5. \tag{4.4}$$

[2]Note that the analysis in [61, 62, 195] only involves the axial gauge field. However, it is straightforward to generalise their results to the case with both the axial and the vector current.

On the other hand, the transport coefficient $\sigma_{J_5\omega}$ could not be completely determined by the anomaly and thermodynamic quantities. Its form contains an additional constant term,

$$\sigma_{J_5\omega} = \kappa\mu^2 + \tilde{c}T^2, \tag{4.5}$$

where $\tilde{c}$ is some yet-undetermined constant, which could run along the renormalisation group flow. By using perturbative field theory methods [86, 207] and simple holographic models [204, 206], it was then suggested that $\tilde{c}$ could be fixed by the gravitational anomaly coefficients, λ. However, the gravitational anomaly enters the equations of motion (4.1) with terms at fourth order in the derivative expansion while ξ_ω and ξ_B enter the equation of motion at second order. Thus, if one analysed the hydrodynamic expansion in terms of the naïve gradient expansion with all fluctuations of the same order, it would seem to be impossible to express $\tilde{c}$ in terms of the gravitational anomaly. The above paradox was resolved in [199]. There, the theory was placed on a product space of a cone and a two dimensional manifold. The deficit angle δ was defined along the thermal cycle, β, as $\beta \sim \beta + 2\pi(1+\delta)$. Demanding continuity of one-point functions in the vicinity of $\delta = 0$ then fixed the unknown coefficient $\tilde{c}$ in terms of the gravitational anomaly coefficient λ (the gradient expansion breaks down). The above construction can be extended to theories outside the hydrodynamic regime in arbitrary even dimensions and in the presence of other types of anomalies, so long as the theories only involve background gauge fields and a background metric [203].

In the presence of dynamical gauge fields, the anomalous transport coefficients do not seem to remain protected from radiative corrections. This is consistent with the fact that the chiral vortical conductivity $\sigma_{J\omega}$, given otherwise by the thermal field theory result

$$\sigma_{J\omega} = 2\gamma\mu_5\mu, \tag{4.6}$$

was also argued to get renormalised in theories with dynamical gauge fields by [208–210]. Furthermore, these various pieces of information regarding the renormalisation of the chiral conductivities are consistent with the findings of [194] (already noted above) and lattice results [211–214]: In theories with dynamical gauge fields and mixed, gauge-global anomalies, chiral conductivities renormalise.

Holography and universality of transport coefficients

Certain classes of strongly interacting theories at finite temperature and chemical potential can be formulated using gauge-gravity (holographic) duality. Thus, in comparison with the weakly coupled regime accessible to perturbative field theory calculations, holography can be seen as a convenient tool to investigate chiral transport properties at the opposite end of the coupling constant scale. Within holography, anomalous hydrodynamic transport was first studied in the context of fluid-gravity correspondence [110] by [112, 113] who added the Chern-Simons gauge field to the bulk. The two DC conductivities associated specifically with chiral magnetic and chiral vortical effects were then computed in the five-dimensional anti-de Sitter Reissner-Nördstrom black brane background in [206, 215, 216]. The results were extended to arbitrary dimensions in [204]. The work of [204] showed that these transport coefficients could be extracted from first-order differential equations (as opposed to the usual second-order wave equations in the bulk) due to the existence of a *conserved current* along the holographic radial direction. In a similar manner, this occurs in computations of the shear viscosity [30, 103] and other DC conductivities [126, 175]. We will refer to this situation as the case when the membrane paradigm is applicable (see Fig. 4.1). The existence of the membrane paradigm makes the calculation of chiral conductivities significantly simpler. Reassuringly, the holographic results for the chiral conductivities agree with the results obtained from conventional QFT methods described above and stated in Eqs. (4.4), (4.5) and (4.6) [86, 202, 207]. More recently, these calculations were generalised to cases of non-conformal holography (in which $T^{\mu}{}_{\mu} \neq 0$), giving the same results [88, 217]. A way to think of such holographic setups is as of geometric realisations of the renormalisation group flows.

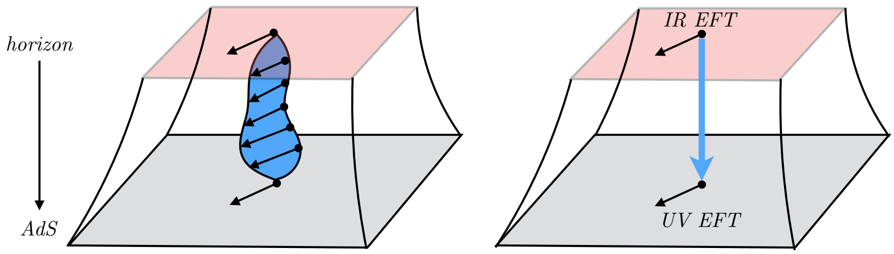

Figure 4.1. A schematic representation of the membrane paradigm: The image on the left-hand-side corresponds to a holographic calculation (without the membrane paradigm) in which one has to solve for the bulk fields all along the D dimensional bulk. On the right-hand-side (the membrane paradigm case), the field theory observable of interest can be read off from a conserved current (along the radial coordinate). Hence, we only need information about its dynamics at the horizon and the AdS boundary. The membrane paradigm enables us to consider independent *effective theories* at the two surfaces with $(D-1)$ dimensions. While the UV effective theory directly sources the dual field theory, it is the IR theory on the horizon that fixes the values of dual correlators in terms of the bulk black hole parameters. As in this chapter, such a structure may enable us to make much more general (universal) claims about field theory observables then if the calculation depended on the details of the full D−dimensional dynamics.

Universal holographic statements, most prominent among them being the ratio of shear viscosity to entropy density, $\eta/s = \hbar/(4\pi k_B)$ [30, 103, 126], can normally be reduced to an analysis of the dynamics of a minimally-coupled massless scalar mode and the existence of the membrane paradigm. The fact that the membrane paradigm exists in some theories for anomalous chiral conductivities thus naturally leads to the possibility of universality of these transport coefficients in holography. Motivated by this fact, in this work, we study whether and when non-renormalisation theorems for anomalous transport can be established in holography.

Recently, a work by Gürsoy and Tarrío [217] made the first step in this direction by proving the universality of chiral magnetic conductivity σ_{JB} in a two-derivative Einstein-Maxwell-dilaton theory with an arbitrary scalar field potential and anomaly-inducing Chern-Simons terms. The only necessary assumptions were that the bulk geometry is asymptotically anti-de Sitter (AdS) and that the Ricci scalar at the horizon must be regular. Because this statement is valid for two-derivative theories, it applies to duals at infinitely strong ('t Hooft) coupling λ and infinite number of adjoint colours, N. In this sense, it is applicable within the same class of theories as the statement of universality for η/s.

Higher-derivative corrections to supergravity actions arise when α' corrections are computed from string theory. Usually, this is done by either computing loop corrections to the β-functions of the sigma model or by computing string scattering amplitudes and guessing the effective supergravity action that could result in the same amplitudes (see e.g. [137, 218, 219]). Via the holographic dictionary, these higher-derivative corrections translate into (perturbative) coupling constant corrections in powers of the inverse coupling constant $(1/\lambda)$ expanded around $\lambda \to \infty$ [139]. The result of $\eta/s = 1/(4\pi)$ (having set $\hbar = k_B = 1$) is not protected from higher-derivative bulk corrections; it receives coupling constant corrections both in four-derivative theories (curvature-squared) [220–223] and in the presence of the leading-order top-down corrections to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with an infinite number of colours (these R^4 corrections are proportional to $\alpha'^3 \sim 1/\lambda^{3/2}$) [140]. An equivalent statement exists also in second-order hydrodynamics [50, 110]. There, a particular linear combination of three transport coefficients, $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2$, was shown to vanish for the same class of two-derivative theories as those that exhibit universality of η/s. It was then shown that the same linear combination of second-order transport coefficient vanishes to leading order in the coupling constant corrections even when curvature-squared terms [149, 224] and R^4 terms dual to the $\mathcal{N} = 4$ 't Hooft coupling corrections are included in the bulk action [149]. However, by using the non-perturbative results for these transport coefficients in Gauss-Bonnet theory [225], one finds that the universal relation is violated non-perturbatively (or at second order in the perturbative coupling constant expansion) [149].[3]

[3] The violation of universality in second-order hydrodynamics was later also verified in [226]

Our goal in this work is to study the universality of the four anomalous conductivities σ_{JB}, $\sigma_{J\omega}$, σ_{J_5B} and $\sigma_{J_5\omega}$ in general higher-derivative theories, thereby incorporating an infinite series of coupling constant corrections to results at infinite coupling (from two-derivative bulk theories). What we will show is that the expressions (4.4), (4.5) and (4.6) remain universal in any higher-derivative theory so long as the action (excluding the Chern-Simons terms) is gauge- and diffeomorphism-invariant.[4] All we will assume, in analogy with [217], is that the bulk theory is asymptotically AdS (it has a UV conformal fixed point) and that it permits a black brane solution with a regular, non-extremal horizon. In its essence, the proof will reduce to showing the validity of the membrane paradigm and then a study of a generic, higher-derivative effective theories (all possible terms present in the conserved current) at the horizon and the boundary (as depicted in Fig. 4.1). The condition of regularity of these constructions at the horizon will play a crucial role in the proof. By studying cases of theories for which the membrane paradigm fails, one can then find theories in which universality may be violated.

Our findings can be seen as a test of holography in reproducing the correct Ward identities for the anomalous currents. The fact that we find universality of chiral conductivities with an infinite series of coupling constant corrections (albeit expanded around infinite coupling) is an embodiment of the fact that when only global anomalies are present, anomalous transport is protected from radiative corrections. An example related to the presence of mixed, gauge-global anomalies, which will invalidate the membrane paradigm, will be studied in Section 4.4. Again, as expected from field theory arguments, a case like that will naturally be able to violate the universality (or non-renormalisation) of chiral conductivities.

The chapter is organised as follows: In Section 4.2, we describe the holographic theory at finite temperature and chemical potential that is studied in

by using fluid-gravity methods in Gauss-Bonnet theory.

[4] As we are mainly interested in theories in which the anomalous Ward identity retains the form of Eq. (4.1), the conditions of gauge- and diffeomorphism- invariance are imposed to avoid explicit violation of Eq. (4.1) by the bulk matter content (see Section 4.4.4 for a discussion of such an example that includes massive vector fields).

the main part of this work. We then turn to the proof of the universality of chiral conductivities in Section 4.3. First, in Section 4.3.1, we show how to compute anomalous conductivities by using the membrane paradigm and specify the conditions that must be obeyed in order for the membrane paradigm to be valid. In Section 4.3.2, we then prove that a gauge- and diffeomorphism- invariant action indeed satisfies those conditions and thus always gives the same anomalous conductivities. In Section 4.4, we study examples that obey and violate the conditions required for universality. In particular, those that violate the universality include either massive gauge fields or naked singularities in the bulk. We proceeds with a discussion of results and future directions in Section 4.5. Finally, Appendix 4.6.1 includes a discussion of anomaly polynomials and the replacement rule.

4.2 The holographic setup

In this work, we consider five dimensional bulk actions with a dynamical metric G_{ab}, two massless gauge fields A_a and V_a that are dual to the axial and the vector current in the boundary theory, respectively, and a set of scalar (dilaton) fields, ϕ_I:

$$S = \int d^5x\sqrt{-G}\ \{\mathcal{L}\left[A_a, V_a, G_{ab}, \phi_I\right] + \mathcal{L}_{CS}\left[A_a, V_a, G_{ab}\right]\}\,. \tag{4.7}$$

The Lagrangian density $\mathcal{L}$ should be thought of as a general, diffeomorphism- and gauge-invariant action that may include arbitrary higher-derivative terms of the fields. Since we are interested in anomalous transport, (4.7) must include the Chern-Simons terms, $\mathcal{L}_{CS}$, that source global chiral anomalies in the boundary theory. In holography, higher-than-second-derivative bulk terms correspond to the ('t Hooft) coupling corrections to otherwise infinitely strongly coupled states ($\lambda \to \infty$). Since $\mathcal{L}$ may include operators with arbitrary orders of derivatives (and corresponding bulk coupling constants), holographically computed quantities describing a hypothetical dual of (4.7) are able to incorporate an infinite series of coupling constant corrections to observables at infinite coupling.[5] However, one should still think of these corrections as perturbative in powers of $1/\lambda$ due to various potential problems that may arise in theories with higher derivatives, such as the Ostrogradsky instability [227, 228], see also [146] for a recent discussion of causality violation in theories with higher-derivative bulk actions, in particular with four-derivative, curvature-squared actions.

The second source of corrections are the quantum gravity corrections that need to be computed in order to find the $1/N$-corrections in field theory. If we consider S in Eq. (4.7) to be a *local* quantum effective action, expanded in a gradient expansion, we may also claim that our holographic results incorporate certain types of (perturbative) $1/N$ corrections, included in $\mathcal{L}$. What is important is the expectation (or the condition) that the anomalous Chern-Simons terms in $\mathcal{L}_{CS}$ do not renormalise under quantum bulk corrections.

It will prove convenient to write the action (4.7) as

$$\mathcal{L}\left[A_a, V_a, G_{ab}, \phi_I\right] \equiv \mathcal{L}_G + \mathcal{L}_\phi + \mathcal{L}_A + \mathcal{L}_V, \tag{4.8}$$

where $\mathcal{L}_G$ now contains the Einstein-Hilbert term (along with the cosmological constant) and higher-derivative terms of the metric, expressed in terms various contractions and derivatives of the Riemann curvature R_{abcd}. $\mathcal{L}_\phi$ contains kinetic and potential terms of a set of neutral scalar fields, ϕ_I. By $F_{A,ab}$ and $F_{V,ab}$, we denote the field strengths corresponding to A_a and V_a, respectively. Arbitrary derivatives of $F_{A,ab}$ and $F_{V,ab}$ may enter into $\mathcal{L}_A$ and $\mathcal{L}_V$, and along with the Chern-Simons terms,

$$\begin{aligned}
\mathcal{L}_A &= \mathcal{L}_A\left[F_{A,ab}, \nabla_a F_{A,bc}, \ldots, R_{abcd}, \nabla_a R_{bcde}, \ldots, \phi_I, \partial_a \phi_I, \ldots\right], \\
\mathcal{L}_V &= \mathcal{L}_V\left[F_{V,ab}, \nabla_a F_{V,bc}, \ldots, R_{abcd}, \nabla_a R_{bcde}, \ldots, \phi_I, \partial_a \phi_I, \ldots\right], \\
\mathcal{L}_{CS} &= \epsilon^{abcde} A_a \left(\frac{\kappa}{3} F_{A,bc} F_{A,de} + \gamma F_{V,bc} F_{V,de} + \lambda R^p{}_{qbc} R^q{}_{pde}\right).
\end{aligned} \tag{4.9}$$

The ellipses '...' stand for higher-derivative terms built from $F_{A,ab}$, $F_{V,ab}$, R, R_{ab}, R_{abcd} and ϕ_I.[6] Note also that we have chosen $\mathcal{L}_A$ and $\mathcal{L}_V$ so as not to

[5] In type IIB theory, higher-derivative bulk terms and corrections to infinitely coupled results in $\mathcal{N} = 4$ theory are proportional to powers of $\alpha' \propto 1/\lambda^{1/2}$. See e.g. [139] and numerous subsequent works.

[6] Latin letters $\{a, b, c, \ldots\}$ are used to label the spacetime indices in the five-dimensional bulk

mix the two gauge fields. If there were mixing terms like $F_{A,ab}F_V^{ab}$ in the Lagrangian, then the anomalous Ward identities would no longer be those from Eq. (4.1) and additional complications regarding operator mixing would have to be dealt with. We note that the normalisation of the Levi-Civita tensor is chosen to be $\epsilon_{trxyz} = \sqrt{-G}$.

Our goal is to study coupling constant corrections to the anomalous conductivities that arise from the Ward identity in Eq. (4.1). We therefore avoid any ingredients in the action (4.8) that would explicitly introduce additional terms into (4.1). Beyond imposing gauge- and diffeomorphism-invariance of (4.1), we will also restrict our attention to Lagrangians $\mathcal{L}_A$ and $\mathcal{L}_V$ that contain no Levi-Civita tensor. An explicit example with violated (bulk) gauge-invariance that can generate a mixed, gauge-global anomaly on the boundary (altering the Ward identity (4.1)) will be studied in Section 4.4.4.

Furthermore, we assume that the bulk theory admits a homogenous, translationally invariant and asymptotically anti-de Sitter black brane solution of the form

$$\begin{aligned} ds^2 &= r^2 f(r) d\bar{t}^2 + \frac{dr^2}{r^2 g(r)} + r^2 \left(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \right), \\ A &= A_t(r) d\bar{t}, \qquad V = V_t(r) d\bar{t}, \qquad \phi_I = \phi_I(r), \end{aligned} \tag{4.10}$$

with $f(r)$ and $g(r)$ two arbitrary functions of the radial coordinate r. At AdS infinity,

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} g(r) = 1. \tag{4.11}$$

The coordinates used in Eq. (4.10), $\{\bar{x}^\mu, r\}$, will be referred to as the un-boosted coordinates. Near the (outer) horizon, we assume that the metric can be written in a non-extremal, Rindler form

$$f(r) = f_1(r - r_h) + f_2(r - r_h)^2 + \mathcal{O}(r - r_h)^3, \tag{4.12}$$

$$g(r) = g_1(r - r_h) + g_2(r - r_h)^2 + \mathcal{O}(r - r_h)^3. \tag{4.13}$$

The Hawking temperature of this black brane background (and its dual) is given

theory while the spacetime indices in the dual boundary theory are denoted by the Greek letters $\{\mu, \nu, \rho, \ldots\}$. The indices $\{i, j, k, \ldots\}$ represent the spatial directions of the boundary theory.

by

$$T = \frac{r_h^2}{4\pi}\sqrt{f_1 g_1}. \tag{4.14}$$

The classical equations of motion describing this system can be obtained by varying the action (4.8). Firstly, the variations of the two gauge fields give[7]

$$d\star H_5 = 0, \qquad\qquad d\star H = 0, \tag{4.15}$$

where the two-forms H_5 and H are defined as

$$\begin{aligned} H_5 &= \frac{1}{2}\left(\frac{\delta\left(\mathcal{L}_A\right)}{\delta\left(\nabla_a A_b\right)} - \nabla_c \frac{\delta\left(\mathcal{L}_A\right)}{\delta\left(\nabla_c\nabla_a A_b\right)} + \dots\right) dx^a dx^b + H_{5,\text{CS}}, \\ H &= \frac{1}{2}\left(\frac{\delta\left(\mathcal{L}_V\right)}{\delta\left(\nabla_a V_b\right)} - \nabla_c \frac{\delta\left(\mathcal{L}_V\right)}{\delta\left(\nabla_c\nabla_a V_b\right)} + \dots\right) dx^a dx^b + H_{\text{CS}}, \end{aligned} \tag{4.16}$$

where

$$H_{5,\text{CS}} = \kappa\star\omega_A + \gamma\star\omega_V + \lambda\star\omega_\Gamma, \quad H_{\text{CS}} = \gamma\star(V\wedge dA), \tag{4.17}$$

The ellipses again denote expressions coming from the higher-derivative terms. The three abelian Chern-Simons three-forms are composed of the two gauge field one-forms $A = A_a dx^a$ and $V = V_a dx^a$, and the Levi-Civita connection one-form $\Gamma^a_{\ b} = \Gamma^a_{\ bc}\, dx^c$ as

$$\omega_X = \text{Tr}\left(X\wedge dX + \frac{2}{3}X\wedge X\wedge X\right), \tag{4.18}$$

where $X = \{A, V, \Gamma^a_{\ b}\}$.[8]

[7]In five spacetime dimensions, we define the Hodge dual of a p-form $\Omega = (p!)^{-1}\Omega_{a_1\dots a_p} dx^{a_1}\wedge\dots\wedge dx^{a_p}$ as

$$\star\Omega = \frac{1}{p!(5-p)!}\sqrt{-G}\,\Omega_{a_1\dots a_p}\epsilon^{a_1\dots a_p}{}_{a_{p+1}\dots a_5} dx^{a_{p+1}}\wedge\dots\wedge dx^{a_5}.$$

[8]In terms of the index notation, the Chern-Simons form built out of the Levi-Civita connection is given by

$$\omega_{abc} = \Gamma^{p_1}{}_{p_2 a}\partial_b\Gamma^{p_2}{}_{p_1 c} + (2/3)\Gamma^{p_1}{}_{p_2 a}\Gamma^{p_2}{}_{p_3 b}\Gamma^{p_3}{}_{p_1 c}.$$

Secondly, varying the metric gives the Einstein's equation

$$R_{ab} - \frac{1}{2}G_{ab}R + \ldots = T^M_{ab} + \frac{1}{2}\nabla_c \left(\Sigma_{ab}{}^c + \Sigma_{ba}{}^c\right), \tag{4.19}$$

where T^M_{ab} is the stress-energy tensor for the scalars and the gauge fields, excluding the Chern-Simons terms. The *spin current* $\Sigma_{ab}{}^c$ is defined as

$$\Sigma_{ab}{}^c = -\lambda\, \epsilon^{ad_1d_2d_3d_4} F_{d_1d_2} R_{d_3d_4b}{}^c. \tag{4.20}$$

We refer the reader to [204] for a more general definition of the spin current, its connection to the anomaly polynomial in Eq. (4.75) and expressions for $\Sigma_{ab}{}^c$ for different anomaly polynomials. We assume that the equations of motion coming from the variations of the scalar fields in (4.7) can also be solved, but we will make no further reference to that set of equations. As stated above, the full system of equations is assumed to result in a non-extremal, asymptotically AdS black brane solution and non-trivial, backreacted profiles for the gauge and the scalar fields.

To find the set of anomalous conductivities $\{\sigma_{J_5B}, \sigma_{JB}, \sigma_{J_5\omega}, \sigma_{J\omega}\}$ in all hypothetical duals of this holographic setup, it is convenient to consider the following perturbed metric in the boosted (fluid-gravity) frame [204]:

$$\begin{aligned} ds^2 = &-2\sqrt{\frac{f(r)}{g(r)}}u_\mu dr dx^\mu + r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu + r^2 \Delta_{\mu\nu} dx^\mu dx^\nu \\ &+ 2r^2 h(r) u_\mu \omega_\nu dx^\mu dx^\nu, \end{aligned} \tag{4.21}$$

where the projector $\Delta_{\mu\nu}$ is defined as $\Delta_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$, with $\eta_{\mu\nu}$ the four-dimensional Minkowski metric. Note that once we set the fluid to be stationary, i.e. $u^\mu_{eq} = \{-1, 0, 0, 0\}$, the metric (4.21) will return to the un-boosted form (4.10), but in the Eddington-Finkelstein coordinates, as is usual in the fluid-gravity correspondence [110, 183]. The perturbations are organised so that the fluid velocity u_μ depends only on the boundary coordinates x^μ and all of the r-dependence is encoded in $h(r)$. Since the vorticity is defined as $\omega^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma$, the last term in (4.21) corresponds to the metric perturbations at first order in the derivative expansion (in the x^μ coordinates). Similarly, the perturbed axial and vector gauge fields can be written as[9]

$$\begin{aligned} A &= -A_t(r)\, u_\mu dx^\mu + \tilde{a}(x^\mu) + a(r)\, \omega_\mu dx^\mu, \\ V &= -V_t(r)\, u_\mu dx^\mu + \tilde{v}(x^\mu) + v(r)\, \omega_\mu dx^\mu. \end{aligned} \tag{4.23}$$

One may use the one-forms $\tilde{a}$ and $\tilde{v}$ to define the magnetic field source $B^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \tilde{v}_\sigma$ and the (fictitious) axial magnetic field source $B_5^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \tilde{a}_\sigma$.

4.3 Proof of universality

In this section, we show that upon expanding the equations of motion (4.15) and (4.19) to first order in the (boundary) derivative expansion, the conserved currents can be expressed as a total radial derivative of some function. This type of a radially conserved quantity is necessary for the applicability of the membrane paradigm, used e.g. in [126] and many other holographic studies. To express all four anomalous conductivities purely in terms of the near-horizon data, our work will generalise the membrane paradigm result for the chiral magnetic conductivity of Gürsoy and Tarrío [217]. This will then enable us to establish the universality of the four transport coefficients in the presence of a general higher-derivative bulk theory specified in Section 4.2. Furthermore, the structure of the equations will single out the properties that holographic theories must violate in order for there to be a possibility that the dual conductivities

[9]Our choice of the metric and the gauge fields can be understood in the following way: If one considers the perturbed metric and the gauge fields with all possible terms at first order in gradient expansions, they have the form

$$\begin{aligned} ds^2 &= -2S(r)\, u_\mu dx^\mu dr + F(r)\, u_\mu u_\nu dx^\mu dx^\nu + G(r)\, \Delta_{\mu\nu} dx^\mu dx^\nu \\ &\quad + 2H_\mu^\perp(r,x)\, u_\nu dx^\mu dx^\nu + \Pi(r)\, \sigma_{\mu\nu} dx^\mu dx^\nu, \end{aligned} \tag{4.22}$$

$$A = C(r) u_\mu dx^\mu + a_\mu^\perp(r,x) dx^\mu, \qquad V = D(r) u_\mu dx^\mu + v_\mu^\perp(r,x) dx^\mu,$$

where $H_\mu^\perp$, $a_\mu^\perp$ and $v_\mu^\perp$ are vectors orthogonal to the fluid velocity u^μ. Using the equations of motion for $\{H_\mu^\perp, a_\mu^\perp, v_\mu^\perp\}$, one can show that they decouple from all other perturbations at the same order in the gradient expansion (see e.g. [112, 113]). Thus, to compute anomalous conductivities, one can consistently solve for only $\{H_\mu^\perp, a_\mu^\perp, v_\mu^\perp\}$, setting the remaining perturbations to zero. To first order, this gives our Eqs. (4.21) and (4.23).

may get renormalised.

Our proof can be divided into two steps: First (in Section 4.3.1), we expand the equations of motion for the gauge field (4.15) to first order in the (boundary coordinate) derivative expansions and arrange them into a total-derivative form of a conserved current along the radial direction. This radially conserved current can be written as a sum of the anomalous Chern-Simons terms and terms that come from the rest of the action. We identify the conditions that each of these terms has to satisfy in order for the anomalous conductivities to have a universal form fixed by the Chern-Simons action. Proving the validity of these conditions is then done in Section 4.3.2 by analysing the horizon and the boundary behaviour of the higher-derivative bulk effective action (all possible terms that can appear in the conserved current).

4.3.1 Anomalous conductivities and the membrane paradigm

Let us begin by considering the axial and the vector currents, $\langle \delta J^\mu_5 \rangle$ and $\langle \delta J^\mu \rangle$, sourced by a small magnetic field and a small vortex. As in [217], the membrane paradigm equations follow from the two Maxwell's equations in (4.15). For conciseness, we only show the details of the axial current computation, which involves H_5 from Eq. (4.16). A calculation for the vector current, involving H, proceeds along similar lines. In case of the vector current, we will only state the relevant results.

To first order in the gradient expansion along the boundary directions x^μ, both equations in (4.15) can be schematically written as

$$\partial_r \left(\sqrt{-G} H_5^{ra} \left(\partial^1 \right) \right) + \partial_\mu \left(\sqrt{-G} H_5^{\mu a} \left(\partial^0 \right) \right) = 0, \tag{4.24}$$

where $H_5^{ra}\left(\partial^0\right)$ and $H_5^{\mu a}\left(\partial^1\right)$ are the components of the conserved current two-form in Eq. (4.16) that contain zero- and one-derivative terms (derivatives are taken with respect to x^μ).

As our first goal is to rewrite the problem in terms of a radially conserved quantity, we need to consider the structure of second term in (4.24). We will set the index a to the four-dimensional index ν. It is easy to see that only the

Chern-Simons terms from $\mathcal{L}_{CS}$ can enter into this term at zeroth order in the (boundary) derivative expansion, i.e. $\partial_\mu \left(\sqrt{-G} H_5^{\mu\nu}\left(\partial^0\right)\right)|_{\kappa=g=\lambda=0} = 0$ (cf. Eq. (4.9)). This is because $H_5^{\mu\nu}$ can only be constructed out of the (axial) gauge field (4.23) and the metric tensor (4.21), containing no derivatives along x^μ. At zeroth-order in the derivative expansion, any two-tensor $X^{\mu\nu}$ can thus be decomposed as

$$X^{\mu\nu} = X_1\, u^\mu u^\nu + X_2\, \Delta^{\mu\nu} + X_3\, u^{(\mu} A^{\nu)} + X_4\, u^{[\mu} A^{\nu]}, \tag{4.25}$$

where X_i are scalar functions of the radial coordinate. For an anti-symmetric $X^{\mu\nu}$, as are $H_5^{\mu\nu}$ and $H^{\mu\nu}$, X_1, X_2 and X_3 must vanish and only X_4 can be non-zero. Since such a term can only come from $\mathcal{L}_{CS}$, $\mathcal{L}$ cannot contribute to the second term in (4.24). For $a = \nu$, the two terms in Eq. (4.24) are therefore given by

$$\begin{aligned} \partial_r \left[\sqrt{-G} H_5^{r\nu}\left(\partial^1\right)\right] &= \frac{\partial}{\partial r}\Bigg[\ldots + \kappa\left(A_t B_5^\nu + A_t^2 \omega^\nu\right) \\ &\qquad + \gamma\left(V_t B^\nu + V_t^2 \omega^\nu\right) + \lambda \frac{g (r^3 f')^2}{2 r^2 f}\omega^\nu\Bigg], \\ \partial_\mu \left[\sqrt{-G} H_5^{\mu\nu}\left(\partial^0\right)\right] &= \kappa\left(\partial_r A_t\right) B_5^\nu + \gamma\left(\partial_r V_t\right) B^\nu \\ &= \partial_r\left(\kappa A_t B_5^\nu + g V_t B^\nu\right). \end{aligned} \tag{4.26}$$

The ellipsis indicates the non-Chern-Simons terms. Hence, one can write the Maxwell's equation for the axial gauge field as a derivative of a conserved current along the r-direction:

$$\partial_r \mathcal{J}_5^\mu(r) = 0. \tag{4.27}$$

The axial bulk current is defined as

$$\mathcal{J}_5^\mu(r) = \mathcal{J}_{5,mb}^\mu(r) + \mathcal{J}_{5,r}^\mu(r) + \mathcal{J}_{5,CS}^\mu(r), \tag{4.28}$$

where the *membrane current* $\mathcal{J}_{5,mb}^\mu(r)$, the Chern-Simons current $\mathcal{J}_{5,CS}^\mu$ and

$\mathcal{J}^\mu_{5,r}$ are defined as

$$\begin{aligned}
\mathcal{J}^\mu_{5,mb} &= \sqrt{-G}\left(\frac{\partial \mathcal{L}_A}{\partial A'_\mu} - \partial_a \frac{\partial \mathcal{L}_A}{\partial(\partial_a A'_\mu)} + \dots\right)\Bigg|_{h(r)\to 0}, \\
\mathcal{J}^\mu_{5,r} &= \sqrt{-G}\left(\frac{\partial \mathcal{L}_A}{\partial A'_\mu} - \partial_a \frac{\partial \mathcal{L}_A}{\partial(\partial_a A'_\mu)} + \dots\right)\Bigg|_{a(r)\to 0}, \\
\mathcal{J}^\mu_{5,CS} &= 2\kappa A_t B_5^\mu + 2\gamma V_t B^\mu + \left(\kappa A_t^2 + \lambda \frac{g(r^2 f')^2}{2f}\right)\omega^\mu .
\end{aligned} \tag{4.29}$$

Note that the primes indicate derivatives with respect to the radial coordinate.

The expectation value of the external boundary current $\langle \delta J_5^\mu \rangle$ that we turned on to excite anomalous transport (cf. Eq. (4.2)) is obtained by varying the perturbed on-shell action (4.8) with respect to the bulk axial gauge field fluctuation at the boundary. We find that it is the membrane current $\mathcal{J}^\mu_{5,mb}$ evaluated at the boundary ($r \to \infty$) that can be interpreted as its expectation value:

$$\langle \delta J_5^\mu \rangle = \lim_{r\to\infty} \mathcal{J}^\mu_{5,mb}(r). \tag{4.30}$$

This result is of central importance to the existence of the membrane paradigm in our discussion.

Let us now study how $\mathcal{J}^\mu_{5,mb}$ can be related to the full conserved current $\mathcal{J}^\mu$ from Eq. (4.28). What will prove very convenient is the gauge choice for A and V whereby (see e.g. [215])

$$\lim_{r\to\infty} A_t(r) = 0, \qquad \lim_{r\to\infty} V_t(r) = 0. \tag{4.31}$$

Such a choice results in[10]

$$\lim_{r\to\infty} \mathcal{J}_{5,CS}(r) = 0, \tag{4.32}$$

which together with the conservation equation (4.27) and Eq. (4.30) implies that

$$\langle \delta J_5^\mu \rangle = \mathcal{J}^\mu_{5,mb}(r_h) + \mathcal{J}^\mu_{5,r}(r_h) - \mathcal{J}^\mu_{5,r}(\infty) + \mathcal{J}^\mu_{5,CS}(r_h). \tag{4.33}$$

What we will prove in the next section (Sec. 4.3.2) will be the statement that

[10] For an alternative gauge choice, see e.g. formalism B from Ref. [229].

for any theory specified by the action in (4.7),

$$\mathcal{J}^{\mu}_{5,mb}(r_h) + \mathcal{J}^{\mu}_{5,r}(r_h) - \mathcal{J}^{\mu}_{5,r}(\infty) = 0, \tag{4.34}$$

implying that the current $\langle \delta J^{\mu}_5 \rangle$ can be completely determined by only the Chern-Simons current evaluated at the horizon,

$$\langle \delta J^{\mu}_5 \rangle = \mathcal{J}^{\mu}_{5,CS}(r_h). \tag{4.35}$$

The same reasoning and equations (4.30)–(4.35) apply also to the case of the vector current, up to the appropriate replacements of A_a by V_a, $\mathcal{L}_A$ by $\mathcal{L}_V$ and the axial Chern-Simons current by

$$\mathcal{J}^{\mu}_{CS} = 2\gamma\left(A_t B^{\mu} + V_t B^{\mu}_5\right) + 2\gamma A_t V_t\,\omega^{\mu}. \tag{4.36}$$

Let us for now assume that the condition (4.34) is satisfied and proceed to compute the anomalous conductivities. In our gauge choice, the gauge fields at the horizon are related to the two chemical potentials via

$$A_t(r_h) = -\mu_5, \qquad\qquad V_t(r_h) = -\mu. \tag{4.37}$$

By using the near-horizon expansions (4.12) and (4.13), the last term in $\mathcal{J}^{\mu}_{5,CS}$ from (4.29) can be related to the temperature

$$\frac{g\left(r^2 f'\right)^2}{f} = r^4 f_1 g_1 = 4\left(2\pi T\right)^2. \tag{4.38}$$

Furthermore, using the horizon values of the gauge fields from Eq. (4.37) along with the definitions of the anomalous conductivities from (4.2), we find

$$\begin{aligned} \sigma_{J_5 B} &= -2\gamma\mu, & \sigma_{JB} &= -2\gamma\mu_5, \\ \sigma_{J_5\omega} &= \kappa\mu_5^2 + \gamma\mu^2 + 2\lambda(2\pi T)^2, & \sigma_{J\omega} &= 2\gamma\mu_5\mu. \end{aligned} \tag{4.39}$$

Hence, so long as the condition (4.34) is satisfied, the bulk theory (4.7) gives precisely the non-renormalised, universal conductivities stated in Eqs. (4.4), (4.5) and (4.6).

4.3.2 Universality

We will now show that the condition (4.34) always holds in gauge- and diffeomorphism - invariant theories, thus establishing the universality of the anomaly-induced conductivities $\sigma_{J_5 B}$, σ_{JB}, $\sigma_{J_5\omega}$ and $\sigma_{J\omega}$ from Eq. (4.39) in theories with arbitrary higher-derivative actions, dual to an infinite series of coupling constant corrections expanded around infinite coupling. The condition (4.34) requires us to understand how $\mathcal{J}^{\mu}_{5,mb}$ and $\mathcal{J}^{\mu}_{5,r}$ behave at the two ends of the five-dimensional geometry (boundary and horizon). To make general statements about that, we construct an effective field theory (or the effective current) in terms of the metric, gauge fields and dilatons with first-order perturbations to quadratic order in the amplitude expansion. The two conditions that we impose on the effective theory and the resulting currents are the following:

(1) The theory must be regular at the non-extremal horizon, by which we mean that any Lorentz scalar present in the action (or a current) must be regular (non-singular) when evaluated at the horizon.

(2) The bulk spacetime is asymptotically anti-de Sitter.

For conciseness, we again only analyse the axial gauge field, A_a. A completely equivalent procedure can be applied to the case of the vector gauge field, V_a.

From the definitions of $\mathcal{J}^{\mu}_{5,mb}$ and $\mathcal{J}^{\mu}_{5,r}$ in Eq. (4.29), it is clear that the only relevant part of the action (4.8) for this analysis is $\mathcal{L}_A$. Because the two currents are independent of the Chern-Simons terms, they only depend on the terms encoded in $H_5^{ra}\left(\partial^1\right)$ (see discussion below Eq. (4.24)). The possible terms in $H_5^{ra}\left(\partial^1\right)$ that correspond to $\mathcal{J}^{\mu}_{5,mb}$ and $\mathcal{J}^{\mu}_{5,r}$ can be written (schematically, up to correct tensor structures of $\mathcal{C}_{A,n}$ and $\mathcal{C}_{G,n}$) as

$$H_5^{r\mu}\left(\partial^1\right) = \sum_{n=1}^{\infty}\left[\mathcal{C}_{A,n}\partial_r^n a(r) + \mathcal{C}_{G,n}\partial_r^n h(r)\right]\omega^{\mu} + H_{5,CS}^{r\mu}\left(\partial^1\right), \qquad (4.40)$$

where $H_{5,CS}^{r\mu}$ is the irrelevant Chern-Simons part of $H_5^{r\mu}$, stated explicitly in Eq. (4.26). Since the action $\mathcal{L}_A$ does not contain any Levi-Civita tensors, the terms in $\{\mathcal{C}_{A,n}, \mathcal{C}_{G,n}\}$ can only depend on $a(r)$ and $h(r)$. This implies that

$\mathcal{C}_{A,n} = \mathcal{C}_{G,n} = 0$ when $a(r) = h(r) = 0$, to first order in the boundary-coordinate derivative expansion. Hence, the problem reduces to the question of finding all possible structure of the tensorial coefficients $\{\mathcal{C}_{A,n}, \mathcal{C}_{G,n}\}$ at the horizon and at the boundary.

It is now convenient to return to the un-boosted coordinates, $\{r, \bar{x}^\mu\}$, used in Eq. (4.10). In these coordinates, the perturbed metric and the axial gauge field are (in analogy with (4.21) and (4.23))

$$ds^2 = -r^2 f(r) d\bar{t}^2 + \frac{dr^2}{r^2 g(r)} + r^2(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) + 2h_{\bar{t}i}(r, \bar{x}^i) d\bar{t} d\bar{x}^i, \tag{4.41}$$

$$A = A_t dt + a_i(r, \bar{x}^i) d\bar{x}^i, \tag{4.42}$$

where the perturbations are now denoted by $h_{\bar{t}i}$, a_i and v_i with $i = \{x, y, z\}$. One can relate $\{h_{\bar{t}i}, a_i\}$ to $\{h(r), a(r)\}$ by using the appropriate coordinate transformations, which give

$$\begin{aligned} h_{\bar{t}i} &= \ldots + r^2 h(r)\, u_\mu \omega_\nu \frac{\partial x^\mu}{\partial \bar{t}} \frac{\partial x^\nu}{\partial \bar{x}^i} + \mathcal{O}\left(\partial^2\right), \\ a_i &= \ldots + a(r)\, \omega_\mu \frac{\partial x^\mu}{\partial \bar{x}^i} + \mathcal{O}\left(\partial^2\right). \end{aligned} \tag{4.43}$$

Here, the ellipses denote the zeroth-order terms in the derivative expansion. It is convenient to consider $u^\mu - u^\mu_{eq}$ to be small, which gives

$$u_\mu dx^\mu = dt + \delta u_i dx^i, \quad dt = d\bar{t} + \frac{1}{r^2}\sqrt{\frac{1}{f(r)g(r)}} dr, \quad dx^i = d\bar{x}^i. \tag{4.44}$$

This choice of the fluid velocity further gives $\omega^t = B^t = 0$. Thus, in the remainder in this section, we will only write down the tensors $\{H_5^{r\mu}, \mathcal{J}_5^\mu, \mathcal{J}^\mu_{5,CS}\}$ with spatial components of $\mu = \{i, j, k, \ldots\}$. It immediately follows that $H_5^{ri}(r, x^\mu)$ in the boosted coordinates and $H_5^{ri}(r, \bar{x}^\mu)$ in the un-boosted coordinates have identical expressions. In analogy with (4.40), expanding H_5^{ri} in the un-boosted coordinates to first order in amplitudes of a_i and $h_{\bar{t}i}$,

$$\begin{aligned} H_5^{ri}\left[a_i, h_{ti}\right] = &\left(\mathcal{I}_{A,1}^{rirj}\partial_r a_j + \mathcal{I}_{A,2}^{rirrj}\partial_r^2 a_j + \dots\right) \\ &+ \left(\mathcal{I}_{G,0}^{ri\bar{t}j} h_{\bar{t}j} + \mathcal{I}_{G,1}^{ri\bar{t}rj}\partial_r h_{\bar{t}j} + \mathcal{I}_{G,2}^{ri\bar{t}rrj}\partial_r^2 h_{\bar{t}j} + \dots\right) \\ &+ \left(\text{terms with derivatives along } x^i\right). \end{aligned} \tag{4.45}$$

Note that $\mathcal{I}_{A,0}^{rij} = 0$ because gauge-invariance of $\mathcal{L}_A$ excludes the possibility of any explicit dependence on a_i (only derivatives of a_i may appear). The ellipses represent terms with higher derivatives in r and $\{\mathcal{I}_{A,n}, \mathcal{I}_{G,n}\}$ are tensors contracted with $\partial_r^n a_i$ and $\partial_r^n h_{\bar{t}i}$. To verify (4.45), we can use the coordinate transformations (4.43), which show that all relevant terms from (4.40) are indeed contained in (4.45). Thus, one can determine the coefficients $\{\mathcal{C}_{A,n}, \mathcal{C}_{G,n}\}$ by applying (4.44) to (4.45) and matching the coefficients of $\partial_r^n a(r)\,\omega^i$ and $\partial_r^n h(r)\,\omega^i$.

The structure of the $\{\mathcal{I}_{G,n}, \mathcal{I}_{A,n}\}$ tensors near the horizon and the AdS-boundary can be understood in the following way: In the un-boosted frame, we define five mutually orthogonal unit-vectors or vielbeins, $e_{\hat{p}a} = \delta_{\hat{p}a}$, where the hatted indices $\{\hat{p}, \hat{q}, ..\} = \{\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}\}$ are used as (local flat space) book-keeping indices. The full set of the five-dimensional vectors with upper Lorentz indices can now be written as $e^a{}_{\hat{p}} = \left[\sqrt{G}\right]^{ab}\delta_{\hat{p}b}$:

$$\begin{aligned} e_{\hat{0}} &= \left(\left(r^2 f\right)^{-1/2}, 0, 0, 0, 0\right), \\ e_{\hat{1}} &= \left(0, 1/r, 0, 0, 0\right), \\ e_{\hat{2}} &= \left(0, 0, 1/r, 0, 0\right), \\ e_{\hat{3}} &= \left(0, 0, 0, 1/r, 0\right), \\ e_{\hat{4}} &= \left(0, 0, 0, 0, \left(r^2 g\right)^{1/2}\right). \end{aligned} \tag{4.46}$$

These normal vectors allow us to write the tensors $\{\mathcal{I}_{G,n}, \mathcal{I}_{A,n}\}$ as

$$\begin{aligned} \mathcal{I}_{A,n}^{a_1 a_2 \dots a_m} &= \sum_{\hat{p}_1,\dots,\hat{p}_m} \mathcal{S}_{A,n}^{\hat{p}_1\dots\hat{p}_m} e^{a_1}_{\ \hat{p}_1} \dots e^{a_m}_{\ \hat{p}_m}, \\ \mathcal{I}_{G,n}^{a_1 a_2 \dots a_m} &= \sum_{\hat{p}_1,\dots,\hat{p}_m} \mathcal{S}_{G,n}^{\hat{p}_1\dots\hat{p}_m} e^{a_1}_{\ \hat{p}_1} \dots e^{a_m}_{\ \hat{p}_m}, \end{aligned} \tag{4.47}$$

where $\{\mathcal{S}_{A,n}, \mathcal{S}_{G,n}\}$ are (spacetime) Lorentz-scalars. The regularity condition imposed at the horizon demands that these scalar have to be non-singular at $r = r_h$. The question of whether $\mathcal{I}_{G,n}$ and $\mathcal{I}_{A,n}$ vanish at the horizon is therefore completely determined by the values the projectors $e^{a_1}_{\ \hat{p}_1} \dots e^{a_m}_{\ \hat{p}_m}$ take when evaluated at the horizon. To demonstrate this fact more clearly, let us write down the first few relevant components of the tensors $\mathcal{I}_{G,n}$ and $\mathcal{I}_{A,n}$ explicitly:

$$\begin{aligned} \mathcal{I}_{G,0}^{ri\bar{t}j} &= \left(r^{-2}\sqrt{g/f}\right) \mathcal{S}_{G,0}^{4\hat{i}0\hat{j}}\,, & \mathcal{I}_{A,0}^{rij} &= 0\,, \\ \mathcal{I}_{G,1}^{ri\bar{t}rj} &= \left(r^{-1}\sqrt{g^2/f}\right) \mathcal{S}_{G,2}^{4\hat{i}04\hat{j}}\,, & \mathcal{I}_{A,1}^{rirj} &= g\,\mathcal{S}_{A,1}^{4\hat{i}4\hat{j}}\,, \\ \mathcal{I}_{G,2}^{ri\bar{t}rrj} &= \left(\sqrt{g^3/f}\right) \mathcal{S}_{G,2}^{4\hat{i}044\hat{j}}\,, & \mathcal{I}_{A,2}^{rirrj} &= \left(r g^{3/2}\right) \mathcal{S}_{A,2}^{4\hat{i}44\hat{j}}\,, \\ \mathcal{I}_{G,3}^{ri\bar{t}rrrj} &= \left(r\sqrt{g^4/f}\right) \mathcal{S}_{G,3}^{4\hat{i}0444\hat{j}}\,, & \mathcal{I}_{A,3}^{rirrrj} &= \left(r^2 g^2\right) \mathcal{S}_{A,3}^{4\hat{i}444\hat{j}}\,, \end{aligned}$$

with $r = r_h$. As before, the tensor $\mathcal{I}_{A,0}^{rij} = 0$ because of the gauge-invariance of $\mathcal{L}_A$.

With this decomposition, the problem of determining the non-zero terms in H_5^{ri} has been reduced to simple power-counting. Namely, a tensor $\mathcal{I}^{a_1 a_2 \dots}$ can only be non-zero at the horizon if the number of $e^{\bar{t}}_{\ \hat{0}}$ in its decomposition is greater or equal to the number of $e^{r}_{\ \hat{4}}$. The regularity of the scalars $\mathcal{S}_{A,n}$ and $\mathcal{S}_{G,n}$ at the horizon plays a crucial role here. Hence, one can see that the only non-zero tensor from the set of $\{\mathcal{I}_{A,n}, \mathcal{I}_{G,n}\}$ is $\mathcal{I}_{G,0}^{ri\bar{t}j}$. The conserved current evaluated at the horizon thus becomes

$$\mathcal{J}_5^i = \sqrt{-G}\left(\sqrt{\frac{g}{f}}\,\mathcal{S}_{G,0}^{4\hat{j}0\hat{i}}\right) h(r_h)\, u_\mu \omega_\nu \frac{\partial x^\mu}{\partial \bar{t}} \frac{\partial x^\nu}{\partial \bar{x}^j} + \mathcal{J}_{5,CS}^i(r_h). \tag{4.48}$$

To see why the first term in (4.48) has to vanish, recall that as other scalars, the

Ricci scalar also has to be regular at the horizon. As pointed out in [217], this condition implies that $h_{ti} \sim (r - r_h)$ at the horizon. Therefore, the conserved current at the horizon is indeed fully determined by the anomalous Chern-Simons term:

$$\mathcal{J}_5^i = \mathcal{J}_{5,CS}^i(r_h). \tag{4.49}$$

With $H_5^{rt} = 0$, Eq. (4.49) implies the first two terms from the condition (4.34) vanish:

$$\mathcal{J}_{5,mb}^\mu(r_h) + \mathcal{J}_{5,r}^\mu(r_h) = 0. \tag{4.50}$$

Similarly, we can determine the value of the current $\mathcal{J}_{5,r}^\mu$ at the boundary. Since $\mathcal{J}_{5,r}^\mu$ includes only terms linear in $h(r)$, it is enough to consider

$$H_5^{ri} = \left(\mathcal{I}_{G,0}^{ri\bar{t}j} h_{\bar{t}j} + \mathcal{I}_{G,1}^{ri\bar{t}rj} \partial_r h_{\bar{t}j} + \mathcal{I}_{G,2}^{ri\bar{t}rrj} \partial_r^2 h_{\bar{t}j} + \dots\right) + \dots . \tag{4.51}$$

Now, because the boundary is asymptotically AdS and higher-derivative terms considered here do not change the scaling behaviour near the boundary, we can use the near-AdS solution for $h(r)$ [204]:

$$h(r) = \frac{\mathcal{H}}{r^4} + \mathcal{O}\left(r^{-5}\right). \tag{4.52}$$

Substituting the expansion for $h(r)$ into (4.51), it immediately follows that the third term in the condition (4.34) vanishes as well when it is evaluated at the boundary (note again that $H_5^{rt} = 0$):

$$\mathcal{J}_{5,r}^\mu(\infty) = 0. \tag{4.53}$$

Together, Eqs. (4.50) and (4.53) imply the validity of the condition stated in Eq. (4.34), which completes our proof. The analysis of the vector current $\mathcal{J}^\mu$ and a proof of a condition analogous to (4.34) follow through along exactly the same lines. This implies that all four anomalous conductivities take the universal form of (4.39) for all holographic theories specified in (4.7) so long as the (effective) theory is regular at the non-extremal horizon and the bulk is asymptotically anti-de Sitter.

4.4 Examples and counter-examples

In this section, we turn our attention to explicit examples of theories that obey and violate the conditions used in our proof in Section 4.3 and thus result in universal and renormalised anomalous conductivities, respectively. We will first demonstrate their universality in two- and four-derivative theories with a non-extremal horizon and then move on to describing two holographic models, which violate the assumptions in the proof of Eq. (4.34). More precisely, in Section 4.4.1, we compute the conductivities in the two-derivative Einstein-Maxwell-Dilaton theory. In Section 4.4.2, we then show explicitly how our proof works in the case of the most general four-derivative action with Maxwell fields and dynamical gravity. In both of those case, the conductivities are universal and the current at the horizon only depends on the metric fluctuation, as established by our effective theory method in (4.48).

In Section 4.4.3, we comment on the validity of our proof in gravity duals without a horizon. We use the examples of the confining soft/hard-wall models and charged dilatonic black holes at zero temperature. The membrane paradigm computation goes through as before in the case of confining geometry. However, the conductivities no longer have any temperature dependence, which would require us to augment the replacement rule discussed in Appendix 4.6.1. As for the latter example, the family of theories considered suffers from naked singularities in the bulk. Lastly, in Section 4.4.4, we point out how the bulk terms corresponding to field theories with a gauge-global anomaly violate the assumptions in our proof. This is consistent with the known fact that anomalous conductivities in systems with mixed anomalies receive corrections along the renormalisation group flow. We will not review the details behind the holographic constructions of such systems but rather focus on the reasons for why these models may violate the universality from the point of view of Section 4.3.2.

4.4.1 Einstein-Maxwell-dilaton theory at finite temperature

As for our first example, we consider the two-derivative Einstein-Mawell-dilaton theory with a non-trivial dilaton profile:

$$\mathcal{L}_G = R - 2\Lambda, \qquad \mathcal{L}_\phi = -(\partial\phi)^2 - V(\phi), \tag{4.54}$$

$$\mathcal{L}_A = -\frac{1}{4} Z_A(\phi) F_{A,ab} F_A{}^{ab}, \qquad \mathcal{L}_V = -\frac{1}{4} Z_V(\phi) F_{V,ab} F_V{}^{ab}, \tag{4.55}$$

having used the notation of the action in Eq. (4.8). This is an extension of the case studied in [217], which includes the gravitational anomaly and anomalous conductivities that follow from a response to a small vortex.

The theory has two charges that are conserved along the radial direction at zeroth-order in the boundary-derivative expansion. The expressions follow from the $a = \mu$ component of the Maxwell's equations:

$$Q_5 = r^3 \sqrt{\frac{g}{f}} Z_A \partial_r A_t, \tag{4.56}$$

$$Q = r^3 \sqrt{\frac{g}{f}} Z_V \partial_r V_t. \tag{4.57}$$

At first order in derivatives, the two conserved currents $\mathcal{J}_5^\mu$ and $\mathcal{J}^\mu$ are given by

$$\begin{aligned} \mathcal{J}_5^\mu &= \left[Q_5 h + r^3 \sqrt{fg} Z_A \partial_r a \right] \omega^\mu + \mathcal{J}_{5,CS}^\mu, \\ \mathcal{J}^\mu &= \left[Q h + r^3 \sqrt{fg} Z_V \partial_r v \right] \omega^\mu + \mathcal{J}_{CS}^\mu. \end{aligned} \tag{4.58}$$

Thus, we can immediately read off the membrane currents:

$$\delta J_{5,mb}^\mu = r^3 \sqrt{\frac{g}{f}} Z_A \partial_r a, \tag{4.59}$$

$$\delta J_{mb}^\mu = r^3 \sqrt{\frac{g}{f}} Z_V \partial_r v. \tag{4.60}$$

Moreover, the regularity of the black hole horizon implies that that metric fluctuation has to vanish at the horizon [217], i.e. $h(r_h) = 0$. At the horizon, the two currents $\mathcal{J}_5^\mu(r_h)$ and $\mathcal{J}^\mu(r_h)$ are therefore completely determined by the anomalous terms $\mathcal{J}_{5,CS}^\mu(r_h)$ and $\mathcal{J}_{CS}^\mu(r_h)$.

Next, we investigate the behaviour of $\mathcal{J}_5^\mu$ and $\mathcal{J}^\mu$ at the boundary. Substituting the near-boundary solutions (4.52) into (4.58), one can see that $Q_5 h$ and Qh are sub-leading, which implies that $\mathcal{J}_5^\mu$ and $\mathcal{J}^\mu$ at $r \to \infty$ become determined by the membrane currents evaluated at the boundary.

4.4.2 Four-derivative Einstein-Maxwell theory

In this section, we consider the most general four-derivative theory of massless gravitons and gauge fields. The action $\mathcal{L}_A$ can be written as (see [219, 223, 230–232]):

$$\begin{aligned}\mathcal{L}_A = & -\frac{1}{4}F_{ab}F^{ab} + \alpha_4 R F_{ab}F^{ab} + \alpha_5 R^{ab}F_{ac}F_b{}^c + \alpha_6 R^{abcd}F_{ab}F_{cd} \\ & + \alpha_7 (F_{ab}F^{ab})^2 + \alpha_8 \nabla_a F_{bc}\nabla^a F^{bc} + \alpha_9 \nabla_a F_{bc}\nabla^b F^{ac} \\ & + \alpha_{10}\nabla_a F^{ab}\nabla^c F_{cb} + \alpha_{11} F^{ab}F_{bc}F^{cd}F_{da},\end{aligned} \tag{4.61}$$

and similarly $\mathcal{L}_V$. Note that in Eq. (4.61), all indices A denoting that F_{ab} is the axial field strength have been suppressed. The conserved current two-form, H_5^{ab}, in this theory is

$$\begin{aligned}H_5^{ab} = & - F^{ab} + 4\alpha_4 R F^{ab} + 2\alpha_5 (R^{ac}F_c{}^b - R^{bc}F_c{}^a) + 4\alpha_6 R^{cdab}F_{cd} \\ & + 8\alpha_7 F_{cd}F^{cd}F^{ab} - 4\alpha_8 \Box F^{ab} - 2\alpha_9 \nabla_c (\nabla^a F^{cb} - \nabla^b F^{ca}) \\ & + 2\alpha_{10}(\nabla^b\nabla_c F^{ca} - \nabla^a \nabla_c F^{cb}) + 8\alpha_{11}F^{bc}F_{cd}F^{da}.\end{aligned} \tag{4.62}$$

The current $\mathcal{J}_5^i$ is then

$$\mathcal{J}_5^\mu = \mathcal{J}_{5,\text{Maxwell}}^\mu + \sum_{n=4}^{11} \alpha_n \mathcal{J}_{5,(n)}^\mu + \mathcal{J}_{CS}^\mu, \tag{4.63}$$

where $\mathcal{J}_{5,\text{Maxwell}}^\mu$ is the axial current that follows from the two-derivative Maxwell action analysed in Section 4.4.1. The remaining terms, $\mathcal{J}_{5,(n)}^\mu$, all have the schematic form

$$\mathcal{J}_{5,(n)}^\mu = \left[C_{n,1}h + C_{n,2}\partial_r h + C_{n,3}\partial_r^2 h + D_{n,1}\partial_r a + D_{n,2}\partial_r^2 a + D_{n,3}\partial_r^3 a\right]\omega^\mu, \tag{4.64}$$

where the coefficients $C_{n,i}$ and $D_{n,i}$ depend on the background and parameters of the action. The full expressions for these coefficients are lengthy and will not be presented here.

Near the non-extremal horizon (assumed to exist), the metric must behave as in Eqs. (4.12) and (4.13). What we find is that when evaluated at the horizon, all coefficients except $C_{n,1}$ vanish. This result therefore precisely agrees with the structure of $\mathcal{J}_5^\mu$ predicted in (4.48), which followed from our general treatment of $H_5^{r\mu}$ in Section 4.3.2. At the horizon, the full set of $\mathcal{J}^\mu_{5,(n)}$ is given by

$$\begin{aligned}
\mathcal{J}^\mu_{5,(4)}(r_h) &= -\frac{2r_h^2\sqrt{g_1}A_t'}{f_1^{3/2}}\left(20f_1g_1+3f_2g_1r_h+f_1g_2r_h\right)h(r_h)\,\omega^\mu,\\
\mathcal{J}^\mu_{5,(5)}(r_h) &= -\frac{r_h\sqrt{g_1}A_t'}{f_1^{3/2}}\left(14r_hf_1g_1+2r_h^2g_1f_2+r_h^2f_1g_2\right)h(r_h)\,\omega^\mu,\\
\mathcal{J}^\mu_{5,(6)}(r_h) &= -\frac{2r_h^2\sqrt{g_1}A_t'}{f_1^{3/2}}\left(8f_1g_1+3r_hg_1f_2+r_hf_1g_2\right)h(r_h)\,\omega^\mu,\\
\mathcal{J}^\mu_{5,(7)}(r_h) &= -\frac{16r_hg_1^{3/2}(A_t')^3}{f_1^{3/2}}h(r_h)\,\omega^\mu,\\
\mathcal{J}^\mu_{5,(8)}(r_h) &= -\frac{28r_h^3\sqrt{g_1}}{f_1^{3/2}}\left(-g_1f_2+f_1g_2+2f_1g_1A_t''/A_t'\right)h(r_h)\,\omega^\mu,\\
\mathcal{J}^\mu_{5,(9)}(r_h) &= \frac{1}{2}\mathcal{J}^\mu_{5,(8)},\\
\mathcal{J}^\mu_{5,(10)}(r_h) &= \frac{r_h^2\sqrt{g_1}}{f_1^{3/2}}\left(6f_1g_1-r_hg_1f_2+r_hf_1g_2+2r_hf_1g_1A_t''/A_t'\right)h(r_h)\,\omega^\mu,\\
\mathcal{J}^\mu_{5,(11)}(r_h) &= -\frac{1}{2}\mathcal{J}^\mu_{5,(7)}.
\end{aligned} \tag{4.65}$$

Finally, imposing the horizon Ricci scalar regularity condition (see the discussion after Eq. (4.48)), $h(r_h)=0$, we find that all $J^\mu_{5,(n)}(r_h)=0$.

At the AdS boundary ($r\to\infty$), we further find that all coefficients $C_{n,i}\sim r^{-m}$, where $m>0$. With this explicit verification, our results imply that the most general gauge- and diffeomorphism - invariant four-derivative theory (4.61) satisfies the condition (4.34) and that the anomalous conductivities in its dual all have the universal form of Eq. (4.39).

4.4.3 Theories without horizons and theories with scaling geometries at zero temperature

In this section, we consider two classes of backgrounds, each one a possible solution of the Einstein-Maxwell-dilaton theory of Section 4.4.1. The first one belongs to the family of soft/hard wall model that are dual to a field theory with a mass gap [233–236]. The second example is the scaling geometry that can arise as a solution of the Einstein-Maxwell-dilaton theory at zero temperature (see e.g. [121]). What we show is that the criterion for the universality of anomalous conductivities, i.e. Eq. (4.34), is still satisfied in the gapped system. However, the conductivities can no longer computed by using the replacement rule in the form stated in Eq. (4.77). For the scaling geometries, the universality may be violated due to the presence of naked singularities. A way to retain a holographic theory at zero temperature in which the condition (4.34) is satisfied is to put very strong constraints on the geometry that avoid the naked singularity. These constraints restrict the allowed range of value of the hyperscaling violation exponent, θ, and the dynamical critical exponent, z.

Let us start with an example of the soft/hard wall geometry at zero density. In an un-boosted frame, the metric for these models can be written as

$$ds^2 = e^{-(M/u)^\nu} \left(-u^2 d\bar{t}^2 + \frac{du^2}{u^2} + u^2 (d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) \right), \tag{4.66}$$

where the parameter M sets the scale of the mass gap. The nature of the spectrum is also controlled by the parameter ν: while the gapped spectrum is continuous above the gap when $\nu = 1$, it is discrete when $\nu > 1$. The hard wall model in which the AdS radius is capped off at $u \ll M$ corresponds to the limiting value of $\nu \to \infty$ [235, 237].

One can change coordinates of the above metric to bring them to the form of (4.10) by redefining the radial coordinate as $r = e^{-\frac{1}{2}(M/u)^\nu}$. In the deep IR region, $u \ll M$, the functions $f(r)$ and $g(r)$ can be written as

$$f_{IR}(r) = 1, \qquad g_{IR}(r) = g\,(u \ll M) = \nu^2 \left(\frac{M}{u}\right)^\nu e^{M/u}. \tag{4.67}$$

Despite there being no horizon, the dual of the above geometry can still have non-zero temperature; it can be interpreted as a thermal state before undergoing a phase transition to the black hole phase at high temperature, analogously to the Hawking-Page transition [238].

The two currents, $\mathcal{J}_5^\mu$ and $\mathcal{J}^\mu$, must now be evaluated at $r = 0$ and at the boundary ($r = \infty$). Because the geometry is still asymptotically AdS, their near-boundary behaviour is the same as in all the cases studied before. The fact that $g(r)$ exponentially diverges in the IR appears problematic at first. However, the volume form, which is proportional to $\sqrt{-G}$, is exponentially suppressed. Evaluating $\mathcal{J}_5^i$ at $u = 0$, one finds that $\mathcal{J}^\mu_{5,mb}(0) + \mathcal{J}^\mu_{5,r}(0) = 0$ as in 4.3.2. Thus, the universality condition (4.34) is still satisfied.

On the other hand, the Chern-Simons current $\mathcal{J}^\mu_{5,CS}$ no longer behaves the same way. Although the profiles of the gauge fields A_t, and V_t can be assumed to asymptote to a constant value at $r = 0$, the derivative of f' can no longer be interpreted as the temperature of the dual theory (substituting (4.67) into (4.29), we see that $\mathcal{J}^\mu_{5,CS}$ has no temperature dependence). Therefore, in the confining phase, the replacement rules discussed in the Appendix 4.6.1 are no longer applicable even if the condition (4.34) is satisfied. The above statements also apply to the AdS soliton-like geometries.

Next, we explore the scaling geometries at zero temperature. In the unboosted frame, the metric can now be written as

$$ds^2 = r^2\left(-r^{n_0}dt^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2\right) + \frac{dr^2}{r^{n_1}}, \tag{4.68}$$

or in terms of θ and z,

$$n_0 = 2 + \frac{6(z-1)}{3-\theta}, \qquad n_1 = 2 + \frac{2\theta}{3-\theta}. \tag{4.69}$$

As mentioned in [121], many of these geometries contain a naked singularity. As a result, the scalars $\{\mathcal{S}_{A,n}, \mathcal{S}_{G,n}\}$ used in Eq. (4.47) no longer have to be finite. Such systems can therefore easily violate the universality condition (4.34). Thus, the universality of the anomalous conductivities is no longer guaranteed in the presence of a naked singularity. In this work, we do not study in detail what happens to anomalous conductivities in such cases and whether they nevertheless remain universal for some geometries.

Are there special values of z and θ for which it is easy to see that the condition (4.34) remains satisfied? In other words, what are the ranges of $\{z, \theta\}$ for which the theory has no naked singularity? This problem was addressed in [239], where it was found that the geometries that satisfy either one of the following two conditions,

$$n_0 = n_1 = 2, \qquad n_0 = n_1 \geq 4, \tag{4.70}$$

have no naked singularities. The authors assumed that the matter content has to satisfy the null energy condition, which, for this geometry, is equivalent to imposing the following two inequalities:

$$n_0 \geq n_1, \qquad (n_0 - 2)(n_0 + n_1 + 4) \geq 0. \tag{4.71}$$

The first solution in (4.70) is simply the empty AdS solution with $z = 1$ and $\theta = 0$. The second solution (or a family of solutions) is more involved and requires non-trivial matter to support such geometries.

Of particular interest are charged dilatonic black holes with $z \to \infty$, $\theta \to -\infty$ and a fixed ratio $-\theta/z = \eta$, dual to strongly interacting theories with finite density (see e.g. [240, 241]). While such systems still satisfy the null energy condition, the geometries nevertheless exhibit a naked singularity at zero temperature. This means that unless there is a way to resolve the singularity, the universal structure of anomalous conductivities, although not necessarily, may be violated at zero temperature for all values of η. One way to resolve this issue, as mentioned in [240] for $\eta = 1$, is to lift the black hole solution to a ten- or eleven- dimensional solution of string or M-theory [242]. To study such solutions, one also needs to find the ten- and eleven-dimensional analogues of the Chern-Simons terms ($\mathcal{L}_{CS}$ in (4.9)). In case of a supergravity setup, this was studied in [243] and many subsequent works. An explicit computation of chiral magnetic conductivity, σ_{JB}, in a top-down setup of probe flavour branes can be found in [244]. More generally, it is plausible that the problems of IR singularities can be avoided when they are of the “good type" [245].[11] We defer a more detailed study of these issues and of top-down constructions to future works.

4.4.4 Bulk theories with massive vector fields

In this section, we comment on the universality of anomalous conductivities in field theories with mixed, gauge-global anomalies. Such theories exhibit the following anomalous Ward identity:

$$\partial_\mu \langle J_5^\mu \rangle = \beta \epsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma} + (\text{global anomaly terms}) \, , \tag{4.72}$$

where $\mathcal{F}_{\mu\nu}$ is the field strength of the gluon fields (e.g. in QCD). The global anomaly terms were stated in Eq. (4.1). As shown by perturbative quantum field theory calculations [194, 196, 208, 209], the anomalous conductivities in such theories are known to be renormalised, i.e. they receive quantum corrections.

Holographic models dual to theories with the anomalous Ward identity of the form of Eq. (4.72) were proposed and studied in [88, 243, 246, 247]. In this work, we focus on the bottom-up construction of [247], where the following terms are added to the bulk action (4.8):

$$\Delta S = \int d^5x \sqrt{-G} \left(-\frac{m^2}{2} (A_a - \partial_a \theta)(A^a - \partial^a \theta) - \frac{\kappa}{3} \epsilon^{abcde} (\partial_a \theta) F_{bc} F_{de} \right) \tag{4.73}$$

We have set the vector and the gravitational Chern-Simons terms to zero, i.e. $\gamma = \lambda = 0$ (see Eq. (4.1)). The scalar field θ is the Stückelberg axion.

A holographic theory with ΔS in the action can clearly evade the arguments of the proof of universality from Section 4.3. The reason is that the equation of motion for a massive vector field cannot be written in the form of Eq. (4.15). The right-hand-side of (4.15) now contains terms which explicitly depend on A_a and one cannot reduce the equations into a total derivative form, $\partial_r \mathcal{J}^\mu = 0$. Hence, in models with massive vector fields, dual to field theories with mixed, gauge-global anomalies, anomalous conductivities can be renormalised. This is

[11] We thank Umut Gürsoy for discussions on this point.

consistent with field theory calculations mentioned above.

4.5 Discussion

In this work, we studied the coupling constant dependence of the universality of chiral conductivities associated with the anomalous axial and vector currents in holographic models with arbitrary higher-derivative actions of the metric, gauge fields and scalars. We showed that so long as the action was gauge- and diffeomorphism- invariant, the membrane paradigm construction for the chiral conductivities remained valid, resulting in universal chiral conductivities (see Eq. (4.39)). The proof assumed the existence of a regular, non-extremal black brane with an asymptotically AdS geometry. This result is valid for an infinite-order expansion of coupling constant corrections to holographic results at infinite coupling. Hence, it is complementary to perturbative field theory proofs (expanded around zero coupling) of the non-renormalisation of chiral conductivities in systems with global anomalies and therefore of the anomalous Ward identities with the form of Eq. (4.1). Furthermore, this chapter also explored cases which may violate universality, in particular, in cases with naked singularities and massive vector fields that explicitly violate Eq. (4.1) through mixed, gauge-global anomalies.

This work provides a consistency test of holography in its ability to reproduce the expected non-renormalisation of global Ward identities at the level of (non-zero temperature and density) transport in very general bulk constructions that include arbitrary higher-derivative actions. Furthermore, we believe that the methods presented in this work can be of wider use to other holographic statements of universality that employ the membrane paradigm.

We end this chapter by listing some problems that are left to future works. Most importantly, there exists another anomalous conductivity in the stress-energy tensor, which can be sourced by a small vortex, $\delta T^{\mu\nu} = \sigma^{\epsilon} u^{(\mu}\omega^{\nu)}$. The analysis of this conductivity was not performed in this work. In the fluid-gravity framework, σ^{ϵ} was studied in the Einstein-Maxwell theory by [204]. Forming a conserved bulk current for computing components of the stress-energy tensor tends to be significantly more complicated than for those of a boundary current. However, it may be possible to achieve this by using the Hamiltonian methods recently employed for the calculations of the thermo-electric DC conductivities [129, 175, 248] in two-derivative theories, which should be extended to computations of anomalous transport in higher-derivative theories.

One may also wonder what happens to anomalous transport in inhomogeneous and anisotropic systems. In standard non-anomalous transport, it is known that universal relations can be violated, e.g. in η/s [40, 43, 249, 250]. While analysing such systems is in general significantly more difficult, the existence of the membrane paradigm, as e.g. in case of the DC thermo-electric conductivities [129, 175, 176], may still enable one to prove general statements about the behaviour of conductivities in disordered systems [38, 39]. These methods remain to be explored in the context of anomalous transport.

In even-dimensional theories, anomalous conductivities are directly related to the parity-odd hydrodynamic constitutive relation of [85, 195, 206]. These parity-odd terms are related to global anomalies. In odd dimension, one can still construct hydrodynamics with parity-odd terms, as e.g. in [251]. A well-known parity-odd transport coefficients is the Hall viscosity [252, 253]. This quantity has relations to topological states of matter, such as fractional quantum Hall systems (see e.g. [254] and references therein). A holographic theory with non-zero Hall viscosity can be obtained by adding a topological term similar to the dimensionally-reduced gravitational Chern-Simons term [255]. Recently, in [150], the constitutive relation term associated with the Hall viscosity was generalised to a class of hydrodynamic terms that resemble the Berry curvature. Despite these similarities, there is no known non-renormalisation theorem for parity-odd transport coefficients in odd dimensions.

Lastly, we point out that many recent works have found novel structures in entanglement entropy of theories with anomalies [256–258, 258–260]. As a result of non-renormalisation, one may expect there to exist strong constraints on the structure of extremal bulk surfaces associated with entanglement entropy. It would be interesting to better understand the connection between geometric constraints on holographic entanglement entropy and non-renormalisation theorems for anomalies.

4.6 Appendices

4.6.1 Anomaly polynomials and the replacement rule

As noted in the Introduction, the full set of chiral conductivities (4.2) can be encoded in the anomaly polynomial defined in terms of the Chern-Simons action [73, 74, 202, 203]:

$$\mathcal{P}(F,R) = dS_{CS}\left[A,\Gamma\right]. \tag{4.74}$$

If we restrict ourselves only to global anomalies in four spacetime dimensions, then the anomaly polynomial can be written as

$$\mathcal{P} = \frac{\kappa}{3}\left(F_A \wedge F_A \wedge F_A\right) + \gamma\left(F_A \wedge F_V \wedge F_V\right) + \lambda\left(F_A \wedge R^{\mu}{}_{\nu} \wedge R^{\nu}{}_{\mu}\right). \tag{4.75}$$

The replacement rule states that, for an anomaly polynomial $\mathcal{P}$, one can define the generating function $\mathcal{G}[\mu_5, \mu, T]$:

$$\mathcal{G}\left[\mu_5,\mu,T\right] = \mathcal{P}\left[F_A \to \mu_5,\; F_V \to \mu,\; \mathrm{Tr}\, R^2 \to 2(2\pi T)^2\right], \tag{4.76}$$

where T is the temperature and μ_5, and μ are chemical potentials associated with the axial and the vector currents J_5^{μ} and J^{μ}. The anomalous conductivities can then be computed by using

$$\sigma_{J_5 B} = -\frac{\partial^2 \mathcal{G}}{\partial\mu_5 \partial\mu}, \qquad \sigma_{JB} = -\frac{\partial^2 \mathcal{G}}{\partial\mu\partial\mu},$$
$$\sigma_{J_5\omega} = \frac{\partial\mathcal{G}}{\partial\mu_5}, \qquad \sigma_{J\omega} = \frac{\partial\mathcal{G}}{\partial\mu}. \tag{4.77}$$

For the anomaly polynomial in (4.75), the anomalous conductivities are precisely those stated in Eq. (4.39).

In the work of [204], the replacement rule (4.76) with (4.77) was derived for a field theory dual to the AdS Reissner-Nördstrom background. Our work can be seen a check of the validity of this replacement rule prescription for more general, higher-derivative holographic theories.

5

Magnetohydrodynamic waves in a strongly interacting holographic plasma

5.1 Introduction

Magnetohydrodynamics (MHD) is a hydrodynamic theory of long-range excitations in plasmas (ionised gases) (see e.g. [261, 262]), which has been applied to systems ranging from the physics of fusion reactors to astrophysical objects. In the modern language of hydrodynamics formulated as an effective field theory [63, 263–276], MHD should describe the dynamics of infrared (IR) charge-neutral states in terms of massless effective degrees of freedom. These plasma ground states are characterised by an equation of state with a finite magnetic field. The electric field is suppressed due to the screening of electromagnetic interactions and is only induced on shorter length scales than the (thermodynamic) size of the system. In their standard form, the equations of motion that describe the evolution of plasmas are formulated as a combination of macroscopic fluid equations (continuity equation and the non-dissipative Euler, or dissipative Navier-Stokes equation), coupled to the microscopic electromagnetic Maxwell's equations. In ideal, non-dissipative form, the set of dynamical equations is

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \, \vec{v}) = 0 \,, \tag{5.1}$$

$$\rho \left(\partial_t + \vec{v} \cdot \nabla \right) \vec{v} = -\vec{\nabla} \, p + \vec{J} \times \vec{B} \,, \tag{5.2}$$

$$\partial_t \vec{B} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) , \tag{5.3}$$

$$\left(\partial_t + \vec{v} \cdot \nabla \right) \left(\frac{p}{\rho^\gamma} \right) = 0 \,. \tag{5.4}$$

The magnetic field is constrained by

$$\vec{\nabla} \cdot \vec{B} = 0 \,. \tag{5.5}$$

Eq. (5.1) is the continuity equation and Eq. (5.2) the Euler equation in the presence of the Lorentz force $\vec{J} \times \vec{B}$, with $\vec{J}$ given by the low-frequency limit of the Ampere's law ($\partial_t \vec{E} \to 0$)

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} \,. \tag{5.6}$$

Eq. (5.3) is the Faraday's induction law with the electric field fixed by the assumption of the ideal Ohm's law

$$\vec{E} + \vec{v} \times \vec{B} = 0 \,, \tag{5.7}$$

which is derived by taking the conductivity in the (Lorentz transformed) Ohm's law $\vec{J}/\sigma = \vec{E} + \vec{v} \times \vec{B}$ to infinity, i.e. $\sigma \to \infty$. The constraint equation (5.5) is the magnetic Gauss's law. Since the ideal Ohm's law completely fixes $\vec{E}$, the electric Gauss's law plays no role in the equations of MHD. Eq. (5.4) is the adiabatic equation of state relating density and pressure. Usually, one takes $\gamma = 5/3$. Altogether, Eqs. (5.1)–(5.4) give eight dynamical equations for eight unknown functions ρ, p, $\vec{v}$ and $\vec{B}$, subject to the magnetic field constraint (5.5).

While the above equations are closed, solvable and have been successfully applied to a variety of phenomena in plasma physics, they are only applicable within the specific assumptions used to construct them. This means that they are only valid for electromagnetism controlled by Maxwell's equations in the limit of ideal Ohm's law (no possibility of strong-field pair production, etc.) and for the specific equation of state in Eq. (5.4). This equation of state encodes a separation between the fluid and the charge carrying sectors, for which the justification, beyond assuming weakly coupled Maxwell electromagnetism, also assumes very weak interactions between the fluid degrees of freedom and electromagnetism inside the plasma. Concretely, the latter statement is reflected in the equation of state permitting no dependence on the magnetic properties controlled by the charged sector. Furthermore, because of a lack of a symmetry principle behind the construction of ideal MHD, these equations are difficult to extend unambiguously to the most general, higher-order, dissipative theory in the gradient expansion (the Knudsen number expansion) [50, 64, 110, 184].[1] As such, the traditional formulation of MHD lacks generality and cannot be compatible with a variety of IR effective theories of plasmas that could (in principle) be derived from quantum field theory, in particular, in the presence of a strong magnetic field.

These issues were addressed in a recent work [47], where MHD was formulated by following the effective field theory philosophy behind the construction of relativistic hydrodynamics (see e.g. [51, 184]). Namely, MHD was formulated by only considering global conserved operators and writing them in terms of the most general hydrodynamic gradient expansion of the IR hydrodynamic fields [47].[2] With such an expansion in hand, conservation equations then completely determine the temporal dynamics of a plasma with any equation of state. As in hydrodynamics, all of the details of the equation of state and transport coefficients are left to be determined by the microscopics of the underlying the-

[1]We note that in standard MHD, as formulated in Eqs. (5.1)–(5.4), only the fluid sector has a well-defined and finite Knudsen number.

[2]See also [277] and Ref. [278], which includes a valuable comparison of various related past works, such as [279–281]. For a new treatment of charged fluids in an external electromagnetic field, see [278, 282]. Of further interest is also a recently proposed field theory description of polarised fluids [283].

ory.

The two relevant global symmetries describing the long-range dynamics of a plasma were argued to give the stress-energy tensor $T^{\mu\nu}$ and a conserved anti-symmetric two-form current $J^{\mu\nu}$ [47]:

$$\nabla_\mu T^{\mu\nu} = H^\nu{}_{\mu\sigma} J^{\mu\sigma} \,, \tag{5.8}$$

$$\nabla_\mu J^{\mu\nu} = 0 \,. \tag{5.9}$$

While $T^{\mu\nu}$ corresponds to conserved energy-momentum, $J^{\mu\nu}$ is the manifestation of a generalised global $U(1)$ symmetry, which can be sourced (and gauged) by a two-form gauge field $b_{\mu\nu}$ [46]. $H^\nu{}_{\mu\sigma}$ is a three-form field strength that can be turned on by an external two-form gauge field, $H = db_{ext}$. This generalised global symmetry is a consequence of the absence of magnetic monopoles and directly corresponds to the conserved number of magnetic flux lines crossing a co-dimension two surface (in a four dimensional plasma). Normally, it is expressed in terms of the (topological) Bianchi identity

$$dF = 0 \,, \tag{5.10}$$

where $F = dA$ and A is the abelian electromagnetic field. In the language of a two-form current used in Eq. (5.9),

$$J^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \,. \tag{5.11}$$

Eqs. (5.8) and (5.9) give seven dynamical equations of motion (and one constraint). To solve the equations, we introduce the following hydrodynamical fields: a velocity field u^μ, a temperature field T, a chemical potential μ that corresponds to the density of magnetic flux lines and a vector h^μ, which can be though as a hydrodynamical realisation of a fluctuating magnetic field. The vectors are normalised as $u_\mu u^\mu = -1$, $h_\mu h^\mu = 1$, $u_\mu h^\mu = 0$, together resulting in $10 - 3 = 7$ degrees of freedom. The velocity flow of the plasma breaks the Lorentz symmetry from $SO(3,1)$ to $SO(3)$, which is further broken by the additional vector (magnetic field) to $SO(2)$.[3] The projector transverse to both

[3]Note that at zero temperature, in a plasma with a non-fluctuating temperature field, the

u^μ and h^μ is defined as $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu - h^\mu h^\nu$ and has a trace $\Delta^\mu_{\ \mu} = 2$.

The constitutive relations for the conserved tensors with a positive local entropy production [63] and charge conjugation symmetry can now be expanded to first order in derivatives as [47]

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu} - \mu\rho\, h^\mu h^\nu + \delta f\, \Delta^{\mu\nu} + \delta\tau\, h^\mu h^\nu + 2\,\ell^{(\mu} h^{\nu)} + t^{\mu\nu}\,, \tag{5.12}$$

$$J^{\mu\nu} = 2\rho\, u^{[\mu} h^{\nu]} + s^{\mu\nu}\,, \tag{5.13}$$

where

$$\delta f = -\zeta_\perp \Delta^{\mu\nu}\nabla_\mu u_\nu - \zeta_\times^{(1)} h^\mu h^\nu \nabla_\mu u_\nu\,, \tag{5.14}$$

$$\delta\tau = -\zeta_\times^{(2)} \Delta^{\mu\nu}\nabla_\mu u_\nu - \zeta_\parallel h^\mu h^\nu \nabla_\mu u_\nu\,, \tag{5.15}$$

$$\ell^\mu = -2\eta_\parallel \Delta^{\mu\sigma} h^\nu \nabla_{(\sigma} u_{\nu)}\,, \tag{5.16}$$

$$t^{\mu\nu} = -2\eta_\perp \left(\Delta^{\mu\rho}\Delta^{\nu\sigma} - \frac{1}{2}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)\nabla_{(\rho} u_{\sigma)}\,, \tag{5.17}$$

$$m^\mu = -2r_\perp T \Delta^{\mu\beta} h^\nu \nabla_{[\beta}\left(\frac{h_{\nu]}\mu}{T}\right)\,, \tag{5.18}$$

$$s^{\mu\nu} = -2r_\parallel \mu \Delta^{\mu\rho}\Delta^{\nu\sigma}\nabla_{[\rho} h_{\sigma]}\,. \tag{5.19}$$

The thermodynamic relations between ε, p and ρ, which need to be obeyed by the equation of state $p(T,\mu)$ are

$$\varepsilon + p = sT + \mu\rho\,, \tag{5.20}$$

$$dp = s\, dT + \rho\, d\mu\,. \tag{5.21}$$

Furthermore, for the theory to be invariant under time-reversal, the Onsager relation implies that $\zeta_\times^{(1)} = \zeta_\times^{(2)} \equiv \zeta_\times$. Thus, first-order dissipative corrections to ideal MHD are controlled by seven transport coefficients: $\eta_\perp$, $\eta_\parallel$, $\zeta_\perp$, $\zeta_\parallel$, $\zeta_\times$, $r_\perp$ and $r_\parallel$. Each one can be computed from a set of Kubo formulae presented in [47, 278] and reviewed in Appendix 5.6.1. The transport coefficients should obey the following positive entropy production constraints: $\eta_\perp \geq 0$, $\eta_\parallel \geq 0$,

symmetry is enhanced to $SO(1,1) \times SO(2)$ [47].

$r_\perp \geq 0$, $r_\parallel \geq 0$, $\zeta_\perp \geq 0$ and $\zeta_\perp\zeta_\parallel \geq \zeta_\times^2$. In absence of charge conjugation symmetry, the theory has four additional transport coefficients, resulting in total in eleven transport coefficients [278]. The precise connection between the above formalism of MHD using the concept of generalised global symmetries and MHD expressed in terms of electromagnetic fields, which match in the limit of a small magnetic field, was established in Ref. [278].

Since the effective theory [47] makes no assumption regarding the microscopic details of the plasma, then, should such details somehow be computable from quantum field theory, or otherwise, the effective MHD can be used in solar plasma physics, fusion reactor physics, astrophysical plasma physics and even QCD quark-gluon plasma resulting from nuclear collisions. Of course, computing the microscopic properties of such systems is extremely difficult. In this work, we will resort to using holographic duality. By using standard holographic methods applicable to hydrodynamics [103–105], our analysis will provide us with the required microscopic data of a strongly interacting toy model plasma needed to describe the phenomenology of MHD waves.

The paper is structure as follows: first, in Section 5.2 we review important aspects of gauge theories with a sector coupled to dynamical $U(1)$, which can describe a plasma in the IR limit. In particular, we focus on the discussion of how to couple a strongly interacting field theory with a holographic dual to dynamical electromagnetism, all within a holographic setup. Then, in Section 5.3, we explore this holographic setup in detail, develop the holographic dictionary and use it to compute the microscopic properties of the dual plasma, i.e. the equation of state and first-order transport coefficients. In Section 5.4, we then use this data to analyse the phenomenology of propagating MHD modes—Alfvén and magnetosonic waves. Finally, we conclude with a discussion and a summary of the most important findings in Section 5.5. Three appendices are devoted to a derivation of the relevant Kubo formulae (Appendix 5.6.1), details regarding the derivation of horizon formulae for the transport coefficients (Appendix 5.7) and a derivation of the magnetosonic dispersion relations (Appendix 5.8).

5.2 Matter coupled to electromagnetic interactions

A microscopic theory from which an effective description of a plasma can arise comprises of a matter sector that interacts through an electromagnetic $U(1)$ gauge field. The simplest example of such a theory is quantum electrodynamics. In other theories, the matter sector may itself exhibit complicated physics with additional gauge interactions, such as in QCD. In this work, the theory that we will study contains an infinitely strongly coupled holographic matter sector (closely related to $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills) with infinite N_c. Because of the coupling between matter and dynamical electromagnetism, the holographic setup and the interpretation of results is somewhat subtle. For this reason, we begin our discussion by reviewing some relevant aspects of quantum field theory in a line of arguments similar to [284].

5.2.1 Quantum electrodynamics

The simplest example of a theory coupling matter to electromagnetism is quantum electrodynamics (QED). QED is a $U(1)$ gauge theory that contains a (massive) Dirac fermion ψ (describing electrons and positrons) and a massless photon field A_μ:[4]

$$S_{QED} = -\int d^4x \left[i\bar{\psi}\gamma^\mu D_\mu\psi + m\bar{\psi}\psi + \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right]. \tag{5.22}$$

D_μ is the gauge covariant derivative that couples A_μ to the fermion current (with the coupling e scaled out from the interaction). For a detailed discussion of various properties of QED, see e.g. [189, 285, 286].

The stress-energy tensor of the theory is

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}i\left(\gamma^\mu D^\nu + \gamma^\nu D^\mu\right)\psi - \eta^{\mu\nu}\bar{\psi}\left(i\gamma^\mu D_\mu + m\right)\psi \tag{5.23}$$

$$+ \frac{1}{e^2}\left[F^{\mu\lambda}F^\nu_{\ \lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right]. \tag{5.24}$$

In the massless limit ($m = 0$), the theory is classically scale invariant, which is

[4] We use the mostly positive convention for the metric tensor, so that $\eta_{\mu\nu} = \{-1, +1, +1, +1\}$.

reflected in the vanishing trace of the stress-energy tensor, $T^{\mu}_{\ \mu} = 0$. Quantum mechanically, the theory does not remain scale invariant. The trace receives a correction proportional to the beta function of the electromagnetic coupling

$$T^{\mu}_{\ \mu} = -\frac{\beta(e)}{2e^3} F_{\mu\nu} F^{\mu\nu} \,. \tag{5.25}$$

This is the anomalous breaking of scale invariance—the so-called trace anomaly. The running electromagnetic coupling $e(\mu)$ depends on the renormalisation group scale μ. To first order in perturbation theory, the beta function is

$$\beta(e) = \mu \frac{de}{d\mu} = \frac{e^3}{12\pi^2} \,, \tag{5.26}$$

which, integrated on the interval $\mu \in [M, \Lambda]$, gives the running coupling

$$\frac{1}{e(\Lambda)^2} = \frac{1}{e(M)^2} - \frac{\ln(\Lambda/M)}{6\pi^2} \,. \tag{5.27}$$

Here, M is some IR RG scale at which the electric charge takes the renormalised physical value, $e_r = e(M)$, and Λ is the UV cut-off. Note that at the Landau pole, $\Lambda = \Lambda_{EM}$, the left-hand-side of (5.27) vanishes. On the other hand, the expectation value of the stress-energy tensor is a physical quantity and therefore cannot depend on μ. This statement is encoded in the following identity, which leads to the Callan-Symanzik equation:

$$\mu \frac{d}{d\mu} \langle T^{\mu\nu} \rangle = 0 \,. \tag{5.28}$$

Since we are interested in neutral IR plasma states in QED that can be described by an effective theory of MHD, we can consider the expectation value of the photon field to produce a non-zero magnetic field and zero electric field,

$$\langle A_\mu \rangle = \frac{1}{2} \mathcal{B} \left(x^1 \delta^2_{\ \mu} - x^2 \delta^1_{\ \mu} \right) . \tag{5.29}$$

$\mathcal{B}$ is the magnitude of the “background" magnetic field in the $x^3 = z$ direction. The IR spectrum of the theory has a gapped-out photon, i.e. long-range charge neutrality, which allows us to neglect quantum fluctuations of A_μ. For such a plasma state, Eq. (5.25) yields

$$\langle T^{\mu}_{\nu} \rangle = -\frac{\beta(e)}{e^3}\mathcal{B}^2 = -\frac{1}{12\pi^2}\mathcal{B}^2 + \mathcal{O}\left(e^2\right). \tag{5.30}$$

Furthermore, the expectation value of the stress-energy tensor can be conveniently split into the matter (containing matter-light interactions) and the purely electromagnetic parts

$$\begin{aligned} \langle T^{\mu\nu} \rangle &= \langle T^{\mu\nu}_{matter}(\mu) \rangle + \frac{1}{e(\mu)^2}\left[F^{\mu\lambda}F^{\nu}_{\ \lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right] \\ &= \langle T^{\mu\nu}_{matter}(\Lambda/M) \rangle + \left(\frac{1}{e_r^2} - \frac{\ln(\Lambda/M)}{6\pi^2}\right)\frac{\mathcal{B}^2}{2} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \end{aligned} \tag{5.31}$$

where in the second line, we chose to evaluate the expectation value at the UV cut-off $\mu = \Lambda$. Note that because $\langle T^{\mu\nu} \rangle$ is μ-independent (cf. (5.28)), this choice does not influence the final value of $\langle T^{\mu\nu} \rangle$.

5.2.2 Strongly interacting holographic matter coupled to dynamical electromagnetism

We now turn our attention to the holographic strongly interacting theory that will be investigated in the remainder of this paper. Throughout our discussion, it will prove useful to think of the matter sector as that of the best understood holographic example—the conformal $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with an infinite number of colours N_c and an infinite 't Hooft coupling λ. However, as will become clear below, the theory dual to our holographic setup will not be precisely the $\mathcal{N} = 4$ SYM coupled to a $U(1)$ gauge field, but rather its deformation, of which the microscopic definition will not be investigated in detail. Instead, the model studied here should be considered as a bottom-up construction—the simplest dual of a strongly coupled plasma, which can be described with magnetohydrodynamics in the infrared limit.

The field content of $\mathcal{N} = 4$ SYM are four Weyl fermions, three complex scalars and a vector field, all transforming under the adjoint representation of $SU(N_c)$. The theory also has an $SU(4)_R$ R-symmetry owing to its extended supersymmetry. The adjoint fields together represent the matter content of a hypothetical plasma, which further requires the fields to be (minimally) coupled to an electromagnetic $U(1)$ gauge group (with e the electromagnetic coupling). In $\mathcal{N} = 4$ SYM, this can be achieved by gauging the $U(1)_R$ subgroup of $SU(4)_R$. Under $U(1)_R$, the Weyl fermions transform with the charges $\{+3, -1, -1, -1\}/\sqrt{3}$ and the complex scalars all have charge $+2/\sqrt{3}$ (for details regarding the choice of the normalisation, see [284]). Such a system can be considered as a strongly coupled toy model for a QCD plasma in which the quarks interact with photons as well as with the $SU(3)$ vector gluons.

A crucial fact about $\mathcal{N} = 4$ SYM is that the R-current of $\mathcal{N} = 4$ becomes anomalous in the presence of electromagnetism. For this reason, the $U(1)_R$, which is gauged, is also anomalous and thus the theory has to be deformed in some way to reestablish its self-consistency. As pointed out in [284], one way to do this is by adding a set of spectator fermions that only interact electromagnetically and "absorb" the anomaly. We will assume that the gauge anomaly can be cancelled by some way of deforming the theory, so that the quantum expectation value of the $U(1)_R$ R-current J^μ_R remains conserved, $\nabla_\mu \langle J^\mu_R \rangle = 0$. We can then write the total bare action of the $SU(N_c) \times U(1)$ gauge theory as

$$S_{plasma} = S_{matter} + \int d^4x\, A_\mu J^\mu_R - \frac{1}{4e^2} \int d^4x\, F_{\mu\nu} F^{\mu\nu} \,, \tag{5.32}$$

where A_μ is the dynamical electromagnetic gauge field and $F = dA$. The expectation value of the conserved operator J^μ_R contains a trace over the colour index of the adjoint matter field and therefore scales as N_c^2. Since it is coupled to a single photon, the Maxwell part of the total plasma action S_{plasma} contains no powers of N_c.

As in the QED plasma, we will consider the photons to be gapped out from the IR spectrum so that A_μ will only produce a (classical) magnetic field

$$\langle A_\mu \rangle = \frac{1}{2} \mathcal{B} \left(x^1 \delta^2{}_\mu - x^2 \delta^1{}_\mu \right) . \tag{5.33}$$

In order to maintain the neutrality of the plasma, we will set the electric $U(1)_R$

chemical potential to zero, $\mu_R = \langle A_0 \rangle = 0$.[5] For this reason, the electric one-form (or vector) conserved $U(1)_R$ R-current will play no role in the hydrodynamic IR limit of the theory, so $\langle J^\mu_R \rangle = 0$.

The plasma has a conserved stress-energy tensor to which both the matter (along with its interaction with the electromagnetic field) and the purely electromagnetic sectors contribute,

$$\langle T^{\mu\nu} \rangle = \langle T^{\mu\nu}_{matter}(\Lambda/M) \rangle + \frac{1}{e(\Lambda/M)^2} \left[\langle F^{\mu\lambda} F^{\nu}_{\ \lambda} \rangle - \frac{1}{4} \eta^{\mu\nu} \langle F^{\rho\sigma} F_{\rho\sigma} \rangle \right] . \tag{5.34}$$

The trace of the superconformal theory again experiences an anomaly proportional to the beta function of the electromagnetic coupling (cf. Eq. (5.30)), which in $\mathcal{N} = 4$ theory turns out to be one-loop exact in the presence of a background electromagnetic field and follows from a special case of the NSVZ beta function (see Refs. [284, 289, 290]),[6]

$$\langle T^\mu_{\ \mu} \rangle = -\frac{\beta(e)}{e^3} \mathcal{B}^2 = -\frac{N_c^2}{4\pi^2} \mathcal{B}^2 . \tag{5.35}$$

The beta function for the inverse electromagnetic coupling is then

$$\beta\left(1/e^2\right) = \mu \frac{de^{-2}}{d\mu} = -\frac{N_c^2}{2\pi^2} \left[\frac{1}{6} \sum_{\alpha=1}^{4} \left(q_{\rm f}^\alpha\right)^2 + \frac{1}{12} \sum_{a=1}^{3} \left(q_{\rm s}^a\right)^2 \right] = -\frac{N_c^2}{2\pi^2} , \tag{5.36}$$

with the fermionic and the scalar R-charges being $q_{\rm f}^\alpha = \{+3, -1, -1, -1\}/\sqrt{3}$ and $q_{\rm s}^\alpha = \{2, 2, 2\}/\sqrt{3}$, respectively. In analogy with Eq. (5.27) in QED, by integrating the beta function equation, we find

$$\frac{1}{e^2(\Lambda)} = \frac{1}{e^2(M)} - \frac{N_c^2}{2\pi^2} \ln\left(\Lambda/M\right) . \tag{5.37}$$

It is essential to stress that even though our holographic theory will not be exactly dual to the $\mathcal{N} = 4$ SYM theory, it will give us the same trace anomaly

[5] For a discussion of supersymmetric gauge theories with non-zero R-charge densities, see e.g. [287, 288]

[6] Note that as $N_c \to \infty$, $N_c^2 - 1 \approx N_c^2$.

and thus the same electromagnetic beta function. Since the NSVZ beta function (5.36) is only sensitive to the matter content, this match can be interpreted as our working with a theory with the $U(1)$-gauged matter content and R-charges of $\mathcal{N}=4$ but with a deformed Lagrangian and possibly additional matter that is ungauged under the $U(1)$.

Beyond the stress-energy tensor of the theory discussed thus far, the only other (generalised) global symmetry of interest to describing a plasma state is the higher-form $U(1)$ that corresponds to the conserved number of magnetic flux lines crossing a two-surface. The symmetry results in a conserved two-form current $\langle J^{\mu\nu}\rangle \neq 0$ and was discussed in Section 5.1. The generating function of the field theory that can be used to study MHD of a magnetised plasma in which the two globally conserved operators are $T^{\mu\nu}$ and $J^{\mu\nu}$ is

$$W\left[g_{\mu\nu}, b_{\mu\nu}\right] = \left\langle \exp\left[i\int d^4x\sqrt{-g}\left(\frac{1}{2}T^{\mu\nu}g_{\mu\nu} + J^{\mu\nu}b_{\mu\nu}\right)\right]\right\rangle . \tag{5.38}$$

The simplest holographic dual of such a state is one that contains a five-dimensional bulk with a dynamical graviton (metric tensor G_{ab}) described by the Einstein-Hilbert action, a negative cosmological constant and a two-form bulk gauge field B_{ab}:[7]

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-G}\left(R + \frac{12}{L^2} - \frac{1}{3e_H^2}H_{abc}H^{abc}\right). \tag{5.39}$$

In standard (Dirichlet) quantisation, the two fields asymptote to $g_{\mu\nu}$ and $b_{\mu\nu}$ at the boundary and source $T^{\mu\nu}$ and $J^{\mu\nu}$. Furthermore, H is the three-form defined as $H = dB$. In component notation, $B = \frac{1}{2}B_{ab}\,dx^a \wedge dx^b$ and $H = \frac{1}{6}H_{abc}\,dx^a \wedge dx^b \wedge dx^c$. The two-form gauge field action is the bulk Maxwell Lagrangian $F \wedge \star F$ written in terms of the five-dimensional Hodge dual three-form $H = \star F$, giving the Lagrangian term $H \wedge \star H$. In most of our work, we will set $e_H = L = 1$. Because the two bulk theories are related by dualisation, the background solution to the equations of motion derived from (5.39) give rise to the same magnetised black brane solution known from the Einstein-Maxwell

[7] Throughout this paper, we use Greek and Latin letters to denote the boundary and bulk theory indices, respectively.

theory [291].

In the absence of the two-form term, the action (5.39) arises from a consistent truncation of type IIB string theory on S^5 and is upon identification of the Newton's constant $\kappa_5 = 2\pi/N_c$ dual to pure $\mathcal{N} = 4$ SYM at infinite N_c and infinite 't Hooft coupling λ. For reasons discussed above, the full dual of the action (5.39) is unknown and we are not aware of the mechanism for deriving this action from a consistent truncation of ten-dimensional type IIB supergravity. Nevertheless, for purposes of comparing the sizes of matter and electromagnetic contributions to total operator expectation values, it will prove useful to keep the definition of κ_5 in terms of the number of colours N_c of the hypothetical dual deformed $\mathcal{N} = 4$ SYM coupled to dynamical electromagnetism.

To show further evidence that the action (5.39) is a sensible dual of a strongly coupled MHD plasma, it is useful to elucidate the connection between Eq. (5.39) and the Einstein-Maxwell theory. To put an uncharged holographic theory in an external magnetic field, one normally adds the Maxwell action F^2 with $F = dA$ to the Einstein-Hilbert bulk action. If one imposes the Dirichlet boundary conditions on the bulk one-form A_a, then A_a sources the R-current J^μ_R at the boundary, $\int d^4x\, J^\mu_R \delta A_\mu$, and thus the electromagnetic field A_μ is external and non-dynamical. The investigation of the physics of such a setup was initiated in [291] and studied in numerous subsequent works, including recent [280, 281, 284, 292, 293]. Instead, one can work in alternative quantisation and impose Neumann boundary conditions on A_a. Such a choice exchanges the interpretation of the normalisable and the non-normalisable mode in A_a. From the dual field theory point of view, this can be interpreted as the Legendre transform of the boundary coupling, leading to the variation $\int d^4x\, A_\mu \delta J^\mu_R$. Physically, this means that in alternative quantisation, an external current sources a dynamical (boundary) vector field (see e.g. [294, 295]). The two boundary theories, one with Dirichlet and one with Neumann boundary conditions, are related by a double-trace deformed RG flow.[8] Since J^μ_R is conserved, one can express it through an anti-symmetric $b_{\mu\nu}$ as $\epsilon^{\mu\nu\rho\sigma}\partial_\nu b_{\rho\sigma}$, which, upon integration by parts, yields a dualised $\int d^4x\, J^{\mu\nu}\delta b_{\mu\nu}$, where $J^{\mu\nu}$ is the anti-symmetric

[8] See Refs. [116–118, 296] and references therein.

current from Eq. (5.11).

From the point of view of the bulk, as in a lower-dimensional theory [297], the Einstein-Maxwell bulk (quantum) path integral runs over the metric and the Maxwell field A_a. Alternatively, one can write the path integral over the fields strength F_{ab}, but at the expense of ensuring the Bianchi identity $dF = 0$ by introducing a Lagrange multiplier B_{ab}:

$$Z \supset \int \mathcal{D}F_{ab}\,\mathcal{D}B_{ab}\,\exp\left\{i\,\frac{N_c^2}{8\pi^2}\int d^5x\sqrt{-G}\,\left(F_{ab}F^{ab} + e_H^{-1}B_{ab}\epsilon^{abcde}\nabla_c F_{de}\right)\right\}. \tag{5.40}$$

Since the second (Bianchi identity) term vanishes for any classical field solution, it has no influence on the saddle point of the path integral. However, it does generate a non-zero contribution to the boundary action, which is precisely the source term $\int d^4x\, J^{\mu\nu}b_{\mu\nu}$ once we identify $B_{\mu\nu} \sim b_{\mu\nu}$ and $J^{\mu\nu} \sim \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. The precise dictionary between the bulk and boundary quantities will be discussed in Section 5.3.2. By varying the action with respect to F_{ab}, one obtains the equation of motion

$$F^{ab} = e_H^{-1}\epsilon^{abcde}\nabla_c B_{de}\,. \tag{5.41}$$

Then, the field strength F_{ab} can be integrated out in the saddle point approximation which gives the two-form gauge field Lagrangian term from Eq. (5.39). Furthermore, in the language of the Einstein-Maxwell theory, by using Eq. (5.41), one finds the relation between the one-form R-current J_R^μ and B_{ab} field:

$$\langle J_R^\mu\rangle = -\frac{N_c^2}{2\pi^2}\lim_{u\to 0}F^{u\mu} = -\frac{N_c^2}{2\pi^2 e_H}\lim_{u\to 0}\epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}\,, \tag{5.42}$$

where u is the radial coordinate and $u \to 0$ the boundary of the bulk spacetime. Thus, imposing the Dirichlet boundary condition on B_{ab} corresponds to treating J_R^μ as a source, which is the same as performing alternative quantisation discussed above. This consistent with our interpretation that the dual field theory of (5.39) contains a dynamical photons. Furthermore, as we will see from a detailed holographic renormalisation in Section 5.3.2, the boundary counter-terms, which are required to keep the on-shell action finite will give us precisely the Maxwell theory for A_μ (dual of $b_{\mu\nu}$) on the boundary, including a renormalised electromagnetic coupling e_r, as in QED.[9] All further details of this holographic setup, in particular, the renormalisation of our strongly coupled theory, will be presented in Section 5.3.

5.3 Holographic analysis: equation of state and transport coefficients

In this section, we study the relevant details of the simplest holographic theory with Einstein gravity coupled to a two-form bulk field, cf. (5.39), which can source a two-form current associated with the $U(1)$ generalised global symmetry in the boundary theory. As our goal is to study the phenomenology of MHD waves in a strongly coupled plasma using the dispersion relations of [47], we will use holography only to provide us with the necessary microscopic data: the equation of state and the transport coefficients.

In Section 5.3.1, we will begin by discussing details of the magnetic brane solution [291, 299] supported by the bulk action introduced in Section 5.2.2. In Section 5.3.2, we will consider holographic renormalisation of the theory in question and show how the bulk gives rise to a dual theory coupled to dynamical electromagnetism (as in Section 5.2). In particular, we will derive the expectation values of the stress-energy tensor $\langle T^{\mu\nu}\rangle$ and the two-form $\langle J^{\mu\nu}\rangle$ and show that they satisfy the Ward identities (5.8) and (5.9). We will also match and reproduce all of the expected renormalisation group properties, such as the beta function of the electromagnetic coupling, from the point of view of the bulk calculation. In Section 5.3.3, we will then compute and analyse thermal and magnetic properties of the equation of state of the dual plasma. Finally, in Section 5.3.4, we will derive the membrane paradigm formulae for the seven

[9] We note that the way the Maxwell Lagrangian arises on the boundary is equivalent to the way holographic matter can be coupled to dynamical gravity on a cut-off brane [298]. There too, a holographic counter-term gives rise to the Einstein-Hilbert action at the cut-off brane (the boundary) of a more intricately foliated bulk. As shown by Gubser in [298], such a theory can result in a radiation (CFT)-dominated FRW universe at the boundary with the stress-energy tensor of the $\mathcal{N} = 4$ SYM driving the expansion.

transport coefficients required to describe first-order dissipative MHD [47] and compute them.[10] Further details regarding the horizon formulae for the transport coefficients can be found in Appendix 5.7.

5.3.1 Holographic action and the magnetic brane

A holographic action dual to a plasma state with a low-energy limit that can be described by MHD was stated in Eq. (5.39). Including the boundary Gibbons-Hawking term and the holographic counter-term, the full action is

$$\begin{aligned} S = \frac{N_c^2}{8\pi^2} \Bigg[\int d^5X\sqrt{-G} \left(R + 12 - \frac{1}{3e_H^2} H_{abc}H^{abc} \right) \\ + \int_{\partial M} d^4x \sqrt{-\gamma} \left(2\,\mathrm{Tr}K - 6 + \frac{1}{e_H^2} \mathcal{H}_{\mu\nu}\mathcal{H}^{\mu\nu} \ln \mathcal{C} \right) \Bigg], \end{aligned} \tag{5.43}$$

where $\mathrm{Tr}K$ is the trace of the extrinsic curvature of the boundary (∂M) defined by an outward normal vector n^a. For convenience, we now set $e_H = 1$. The two-form $\mathcal{H}_{\mu\nu}$ is defined as a projection of the three-form field strength in the direction normal the boundary, $\mathcal{H}_{\mu\nu} = n^a H_{a\mu\nu}$. $\mathcal{C}$ is a dimensionless number that needs to be adjusted to fix the renormalisation condition, of which the details will be discussed below. The equations of motion that follow are

$$R_{ab} + 4G_{ab} - \left(H_{acd}H_b{}^{cd} - \frac{2}{9} G_{ab} H_{cde}H^{cde} \right) = 0\,, \tag{5.44}$$

$$\frac{1}{\sqrt{-G}} \partial_a \left(\sqrt{-G} H^{abc} \right) = 0\,. \tag{5.45}$$

Since the theory (5.43) is S-dual to the Einstein-Maxwell theory, we can express the magnetised black brane solution of [291] by dualising the Maxwell

[10] These horizon formulae are analogous to the expression for shear viscosity in $\mathcal{N} = 4$ theory [126]. For more recent discussions of other transport coefficients that can be computed directly from horizon data, see e.g. [44, 129, 217, 248, 255, 300, 301].

terms and writting

$$\begin{aligned} ds^2 &= G_{ab}dx^a dx^b \\ &= r_h^2\left(-F(u)dt^2 + \frac{e^{2\mathcal{V}(u)}}{v}(dx^2+dy^2) + \frac{e^{2\mathcal{W}(u)}}{w}dz^2\right) + \frac{du^2}{4u^3F(u)}\,, \\ H &= \frac{Br_h^2 e^{-2\mathcal{V}+\mathcal{W}}}{2u^{3/2}\sqrt{w}}\,dt\wedge dz\wedge du\,. \end{aligned} \tag{5.46}$$

The equations of motion (5.44) for this ansatz reduce to three second-order ordinary differential equations (ODE's) for $\{F,\,\mathcal{V},\,\mathcal{W}\}$ and one additional first-order constraint. The equation of motion derived from the variation of the two-form (5.45) is automatically satisfied. The equations are equivalent to those derived from the Einstein-Maxwell theory [291] upon identification of the Maxwell bulk two-form F with $F=\mathcal{B}\,dx\wedge dy$, where $\mathcal{B}=Br_h^2/v$.[11] The undetermined functions F, $\mathcal{V}$ and $\mathcal{W}$ are can be obtained numerically by using the shooting method. We first expand the background fields near the horizon as

$$\begin{aligned} F &= f_1^h(1-u) + f_2^h(1-u)^2 + \mathcal{O}(1-u)^3\,, \\ \mathcal{V} &= v_0^h + v_1^h(1-u) + \mathcal{O}(1-u)^2\,, \\ \mathcal{W} &= w_0^h + w_1^h(1-u) + \mathcal{O}(1-u)^2\,, \end{aligned} \tag{5.47}$$

where the constants $\{f_i^h,\,v_i^h,\,w_i^h\}$ can be written in terms of $\{f_1^h,\,v_0^h,\,w_0^h\}$ after solving the equations of motion order-by-order near the horizon. The scaling symmetry of our background ansatz then allows us to rescale dx and dy so that $v_0^h=w_0^h=0$. Next, we match the numerical solutions generated by shooting from the horizon towards the boundary, where the analytical near-boundary

[11]The metric ansatz is chosen to have the form used in [292]. It can be obtained from the ansatz $ds^2 = -U(r)dt^2 + e^{2V(r)}(dx^2+dy^2) + e^{2W(r)}dz^2 + dr^2/U(r)$ used in [291] by performing a coordinate transformation $r = r_h/\sqrt{u}$ and shifting V and W by constant $-\ln v$ and $-\ln w$, respectively, which are chosen so that the near-boundary expansion has the form $ds^2 = (1/u)\,\eta_{\mu\nu}dx^\mu dx^\nu + du^2/(4u^2)$.

expansions of the metric functions are

$$
\begin{aligned}
u\,F &= 1 + f_1^b\sqrt{u} + \frac{f_1^b}{4}u + f_4^b u^2 + \left(\frac{\mathcal{B}^2}{3} +\right) u^2 \ln u + \mathcal{O}(u^{3/2}),\\
u\,e^{2\mathcal{V}} &= v + v f_1^b\sqrt{u} + \frac{v(f_1^b)^2}{4}u + v_4^b u^2 - \left(\frac{\mathcal{B}^2}{6}\right) u^2 \ln u + \mathcal{O}(u3/2),\\
u\,e^{2\mathcal{W}} &= w + w f_1^b\sqrt{u} + \frac{w(f_1^b)^2}{4}u - \frac{2wv_4^b}{v}u^2 - \left(\frac{w\mathcal{B}^2}{3}\right) u^2 \ln u + \mathcal{O}(u^{3/2}).
\end{aligned}
\tag{5.48}
$$

As before, one can solve for the coefficients $\{f_i^b,\ v_i^b,\ w_i^b\}$ in terms of $\{f_1^b,\ f_4^b,\ v_4^b\}$. Furthermore, f_1^b can be removed by residual diffeomorphism freedom of the metric ansatz [292]. For a given value of $B = v\mathcal{B}/r_h^2$, we can therefore generate a numerical background by shooting from the initial conditions of the functions set by the near-horizon expansion with $\{f_1^h, v_0^h, w_0^h\} = \{\hat{f}, 0, 0\}$. The numerical value of $\hat{f}$ is chosen so that the near-boundary expansion has $f_1^b = 0$. The near-boundary behaviour of this function then determines the properties of the dual field theory. Note that the theory is governed by a one-parameter family of such numerical solutions characterised by the dimensionless ratio $T/\sqrt{\mathcal{B}}$. In practice, this ratio can be tuned by changing the parameter B of the background ansatz. The numerical solver encounters stiffness problems when $B \approx \sqrt{3}$, i.e. where the temperature is close to zero. All of our numerical results will therefore stop near $T/\sqrt{\mathcal{B}} = 0$. In this work, we do not attempt an independent analysis of the theory at $T = 0$.

5.3.2 Holographic renormalisation and the bulk/boundary dictionary

The next step in analysing the dual of (5.43) is a systematic holographic renormalisation. In this section, we derive the one-point functions of the stress-energy tensor $\langle T_{\mu\nu}\rangle$ and the two-form current $\langle J_{\mu\nu}\rangle$, and show that they satisfy the Ward identities of magnetohydrodynamics (5.8) and (5.9) [47], which in terms of operator expectation values take the form

$$\nabla_\nu \langle T^{\mu\nu} \rangle = \tilde{H}^{\mu}_{\ \lambda\sigma} \langle J^{\lambda\sigma} \rangle \,, \qquad \nabla_\mu \langle J^{\mu\nu} \rangle = 0 \,, \tag{5.49}$$

where $\tilde{H} = db$ is the field strength of the background gauge field b in field theory. The precise definition of these quantities will become clear below. Since we are only interested in the expansion of MHD to first order in the gradient expansion around a flat (boundary) background, it will be sufficient to only work with terms that contain no more than two derivatives along the boundary directions. The procedure for obtaining holographic renormalisation will closely follow Refs. [102, 302].[12]

We begin by writing the bulk metric in the Fefferman-Graham coordinates [102]

$$ds^2_{\rm FG} = G_{ab}\, dx^a dx^b = \frac{d\rho^2}{4\rho^2} + \gamma_{\mu\nu}(\rho, x) dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(\rho, x) dx^\mu dx^\nu \,, \tag{5.50}$$

so that near the boundary, $\rho \approx 0$, the metric $g_{\mu\nu}$ can be expanded as

$$g_{\mu\nu}(\rho, x) = g^{(0)}_{\mu\nu}(x) + \rho g^{(1)}_{\mu\nu}(x) + \rho^2 \left(g^{(2)}_{\mu\nu}(x) + \tilde{h}_{\mu\nu}(x) \ln \rho \right) + \mathcal{O}(\rho^3) \,. \tag{5.51}$$

Note that Greek (boundary) indices in a tensor $A^{\mu\nu}$ are raised with the metric $g^{\mu\nu}_{(0)}$, which satisfies $g^{(0)}_{\mu\nu} g^{\mu\nu}_{(0)} = 4$. There are two types of covariant derivative that we will use: $\nabla^{(g)}_\mu$ and ∇_μ. Firstly, $\nabla^{(g)}_\mu$ and $\nabla^\mu_{(g)} \equiv g^{\mu\nu} \nabla^{(g)}_\mu$ are defined with respect to the metric $g_{\mu\nu}(\rho, x)$, while ∇_μ and $\nabla^\mu \equiv g^{\mu\nu}_{(0)} \nabla_\mu$ are defined through the metric $g^{(0)}_{\mu\nu}(x)$. The Ricci tensors of $g_{\mu\nu}$ and $g^{(0)}_{\mu\nu}$ are denoted by $R^{(g)}_{\mu\nu}$ and $R^{(0)}_{\mu\nu}$, respectively.

The components of bulk two-form gauge field B_{ab} in the boundary field theory directions can similarly be expanded near the boundary as

$$B_{\mu\nu}(\rho, x) = B^{(0)}_{\mu\nu}(x) + B^{(1)}_{\mu\nu}(x) \ln \rho + \mathcal{O}(\rho) \,. \tag{5.52}$$

In the boundary directions, the three-form field strength is defined as $H_{\mu\nu\sigma} = \partial_\mu B_{\nu\sigma} + \partial_\nu B_{\sigma\mu} + \partial_\sigma B_{\mu\nu}$, with the near-boundary expansion $H_{\mu\nu\sigma}(\rho, x) =$

[12] This part of the calculation was performed by using the Mathematica package xAct [303].

$H^{(0)}_{\mu\nu\sigma}(x) + H^{(1)}_{\mu\nu\sigma}(x)\ln\rho + \mathcal{O}(\rho)$. Each $H^{(n)}$ is defined in terms of $B^{(n)}$, i.e. in the same way at each order. $B^{(0)}$ can now be related to the two-form gauge field source of the boundary theory, $\int d^4x\sqrt{-g}\,J^{\mu\nu}\delta b_{\mu\nu}$. In the bulk, the variation of the on-shell contribution from the H^2 term is

$$\delta S_{on-shell} = \frac{N_c^2}{2\pi^2}\int d^4x\sqrt{-g}\,\mathcal{H}^{\mu\nu}\delta B^{(0)}_{\mu\nu} + \dots\,. \tag{5.53}$$

The expectation value of the operator sourced by $B^{(0)}_{\mu\nu}$ thus depends on a factor of N_c^2. However, since, by definition, $J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$, of which the expectation value contains no colour trace, we need to identify the bulk $B^{(0)}_{\mu\nu}$ and the field theory source $b_{\mu\nu}$ as

$$B^{(0)}_{\mu\nu} = \frac{2\pi^2}{N_c^2}b_{\mu\nu}\,. \tag{5.54}$$

The expectation value of $J^{\mu\nu}$ can then be obtained by taking a variational derivative of the on-shell action with respect to the source $b_{\mu\nu}$.

The Ward identities (5.49) can be obtained by solving the equations of motion (5.44) and (5.45) [302]. In Fefferman-Graham coordinates (5.50), these equations (together with the trace of (5.44)) become

$$\begin{aligned}
0 &= \frac{1}{2}\mathrm{Tr}\left[g^{-1}g''\right] - \frac{1}{4}\mathrm{Tr}\left[g^{-1}g'g^{-1}g'\right] + \frac{1}{3}\rho^2\mathrm{Tr}\left[g^{-1}B'g^{-1}B'\right] - \frac{1}{18}\rho\,\mathrm{Tr}[g^{-1}H^2]\,,\\
0 &= \frac{1}{2}\left(\nabla^{(g)}_\mu\mathrm{Tr}g' - \nabla^\nu_{(g)}g'_{\mu\nu}\right) - 2\rho^2 H_{\mu\alpha\beta}\left(g^{-1}B'g^{-1}\right)^{\alpha\beta}\,,\\
0 &= \rho\left(2g''_{\mu\nu} - 2(g'g^{-1}g')_{\mu\nu} + g'_{\mu\nu}\mathrm{Tr}[g^{-1}g']\right) + R^{(g)}_{\mu\nu} - 2g'_{\mu\nu} - g_{\mu\nu}\mathrm{Tr}[g^{-1}g']\\
&\quad + 8\rho^3\left((B'g^{-1}B')_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\mathrm{Tr}\left[g^{-1}B'g^{-1}B'\right]\right)\\
&\quad + \rho^2\left(H^2_{\mu\nu} - \frac{2}{9}g_{\mu\nu}\mathrm{Tr}[g^{-1}H^2]\right)\,,\\
0 &= \frac{d}{d\rho}\left(2\rho\left(g^{-1}B'g^{-1}\right)^{\mu\nu}\right) + \frac{1}{2}\nabla^\lambda_{(g)}\left(g^{\mu\alpha}g^{\nu\beta}H_{\lambda\alpha\beta}\right)\,,\\
0 &= \nabla_\nu\left(g^{-1}B'g^{-1}\right)^{\mu\nu}\,,
\end{aligned}$$

where g^{-1} denotes the matrix inverse of g (in components, this is $g^{\mu\nu}$) and where

$$\begin{aligned} \mathrm{Tr}[g^{-1}B'g^{-1}B'] &= -B'_{\mu_1\mu_2}B'_{\nu_1\nu_2}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\,, \\ H^2_{\mu\nu} &= H_{\mu\lambda_1\lambda_2}H_{\nu\sigma_1\sigma_2}g^{\lambda_1\sigma_1}g^{\lambda_2\sigma_2}\,. \end{aligned} \tag{5.55}$$

Expanding equations (5.55) around small ρ, we find that

$$g^{(1)}_{\mu\nu} = \frac{1}{2}\left(R^{(0)}_{\mu\nu} - \frac{1}{6}g^{(0)}_{\mu\nu}R^{(0)}\right), \qquad (g^{(1)})^{\mu}_{\ \mu} = \frac{1}{6}R\,. \tag{5.56}$$

Since $g^{(1)}_{\mu\nu}$ is proportional to second derivatives of the boundary metric, and we are only keeping track of terms up to second orders in boundary derivaties, we can ignore terms with $g^{(1)}_{\mu\nu}$. The remaining equations of motion can thus be written as

$$(g^{(2)})^{\mu}_{\ \mu} + \frac{1}{3}B^{(1)}_{\mu\nu}B^{(1)\mu\nu} = 0\,, \qquad \tilde{h}^{\mu}_{\ \mu} = 0\,, \qquad \nabla_{\nu}B^{(1)\mu\nu} = 0\,, \tag{5.57}$$

$$-2H^{(0)}_{\mu\nu\lambda}B^{(1)\nu\lambda} + \nabla^{\nu}_{(0)}\left(g^{(0)}_{\mu\nu}(g^{(2)})^{\lambda}_{\ \lambda} - g^{(2)}_{\mu\nu} - \frac{1}{2}\tilde{h}_{\mu\nu}\right) = 0\,, \tag{5.58}$$

$$\tilde{h}_{\mu\nu} + \frac{1}{2}\left(4B^{(1)}_{\mu\lambda}(B^{(1)})_{\nu}^{\ \lambda} - g^{(0)}_{\mu\nu}B^{(1)}_{\lambda\sigma}B^{(1)\lambda\sigma}\right) = 0\,. \tag{5.59}$$

The expectation values of the stress-energy tensor and the two-form current follow from the generating functional (5.38):

$$\langle T^{\mu\nu}\rangle = -\frac{2i}{\sqrt{-g^{(0)}}}\frac{\delta \ln W}{\delta g^{(0)}_{\mu\nu}}\,, \qquad \langle J^{\mu\nu}\rangle = -\frac{i}{\sqrt{-g^{(0)}}}\frac{\delta \ln W}{\delta b_{\mu\nu}}\,. \tag{5.60}$$

In holography, W is computed from the (on-shell) action (5.43), giving us[13]

$$\begin{aligned} \langle T_{\mu\nu}\rangle = &-\frac{N_c^2}{4\pi^2}\lim_{\epsilon\to 0}\frac{r_h^2}{\epsilon}\left(K_{\mu\nu} - \gamma_{\mu\nu}\mathrm{Tr}K - 3\gamma_{\mu\nu} + \frac{1}{2}R[\gamma]_{\mu\nu}\right. \\ &\left.\left. -\frac{1}{4}\gamma_{\mu\nu}R[\gamma] - \left(\mathcal{H}_{\mu\lambda}\mathcal{H}_{\nu}^{\ \lambda} - \frac{1}{4}\gamma_{\mu\nu}\mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta}\right)\ln(\mathcal{C}\rho)\right)\right|_{\rho=\epsilon}\,, \end{aligned} \tag{5.61}$$

$$\langle J_{\mu\nu}\rangle = -\lim_{\epsilon\to 0}\,\mathcal{H}_{\mu\nu}\Big|_{\rho=\epsilon}\,. \tag{5.62}$$

Note that by taking $\kappa_5 \sim N_c$, while the expectation value of $T^{\mu\nu}$ scales as N_c^2, the expectation value of $J^{\mu\nu}$ is of order $\mathcal{O}(1)$.

[13]Note that in order to raise indices of the boundary theory expectation values, one needs to use the induced metric $\gamma_{\mu\nu}$.

By using Eq. (5.57) and the fact that $\mathcal{H}_{\mu\nu} = n^\rho H_{\rho\mu\nu} = -2B^{(1)}_{\mu\nu} + \mathcal{O}(\rho)$, we find that the boundary two-form current is conserved:

$$\nabla^\mu_{(0)} \langle J_{\mu\nu} \rangle = -2\nabla^\mu B^{(1)}_{\mu\nu} = 0\,. \tag{5.63}$$

Using the definition (5.11), which gives $\langle J_{\mu\nu}\rangle = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\langle F^{\rho\sigma}\rangle$ and connects Eq. (5.63) with the Bianchi identity, we find that $\star B^{(1)}$ sets the expectation value of the Maxwell field strength $\langle F \rangle$. Furthermore, the (regularised) stress-energy tensor (5.61) becomes

$$\langle T_{\mu\nu} \rangle = \lim_{\epsilon\to 1/\Lambda^2} \frac{N_c^2}{2\pi^2}\left(-g^{(2)}_{\mu\nu} + g^{(0)}_{\mu\nu}(g^{(2)})^\lambda_\lambda - \frac{1}{2}\tilde{h}_{\mu\nu} - \frac{1}{2}\tilde{h}_{\mu\nu}\ln\left(\mathcal{C}_M\epsilon\right)\right) \tag{5.64}$$

where Λ is the UV cutoff of the theory. As discussed in Section 5.2, the choice of the dimensionful constant $\mathcal{C}_M$ (including $\mathcal{C}$ and e_H from Eq. (5.61)) must now be made in order to fix the renormalisation condition, which will render the renormalised expectation value $\langle T_{\mu\nu}\rangle$ physical.

To see how the constant $\mathcal{C}_M$ in Eq. (5.64) is related to our discussion in Section 5.2, we write the last term by introducing a mass scale M:

$$\frac{N_c^2}{2\pi^2}\tilde{h}_{\mu\nu}\ln(\Lambda/\mathcal{C}) = \frac{N_c^2}{2\pi^2}\tilde{h}_{\mu\nu}\ln(\Lambda/M) + \tilde{h}_{\mu\nu}\left(\frac{1}{e_r^2} - \frac{N_c^2}{2\pi^2}\ln(\Lambda/M)\right). \tag{5.65}$$

What can be seen from Eq. (5.65) is that this splitting precisely reproduces the way the logarithmic divergence enter into the stress-energy tensor from two different pieces of the Lagrangian: the matter content (with its coupling to the photons) and the electromagnetic (Maxwell) part:

$$\langle T_{\mu\nu}\rangle = \langle T^{matter}_{\mu\nu}\rangle + \langle T^{EM}_{\mu\nu}\rangle\,, \tag{5.66}$$

with the two terms being

$$\langle T^{matter}_{\mu\nu}\rangle = \frac{N_c^2}{2\pi^2}\left(-g^{(2)}_{\mu\nu} + g^{(0)}_{\mu\nu}(g^{(2)})^\lambda_\lambda - \frac{1}{2}\tilde{h}_{\mu\nu}\right) + \frac{N_c^2}{2\pi^2}\tilde{h}_{\mu\nu}\ln(\Lambda/M)\,, \tag{5.67}$$

$$\tag{5.68}$$

and

$$\langle T^{EM}_{\mu\nu} \rangle = \frac{1}{e_r^2}\tilde{h}_{\mu\nu} - \frac{N_c^2}{2\pi^2}\tilde{h}_{\mu\nu} \ln\left(\Lambda/M\right) . \tag{5.69}$$

Finally, we note that the electromagnetic $\langle T^{EM}_{\mu\nu} \rangle$ would follow precisely from the Maxwell boundary action

$$S_{EM} = -\frac{1}{4e(\Lambda/M)^2} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} , \tag{5.70}$$

upon using Eq. (5.59) and the fact that the bulk $\star B^{(1)}$ determines $\langle F_{\mu\nu} \rangle$:

$$\begin{aligned} \langle T^{EM}_{\mu\nu} \rangle &= \frac{1}{e(\Lambda/M)^2} \left(\langle F_{\mu\alpha} F_{\nu}{}^{\alpha} \rangle - \frac{1}{4}\eta_{\mu\nu} \langle F_{\alpha\beta} F^{\alpha\beta} \rangle \right) \\ &= \frac{1}{e(\Lambda/M)^2} \left(\langle F_{\mu\alpha} \rangle \langle F_{\nu}{}^{\alpha} \rangle - \frac{1}{4}\eta_{\mu\nu} \langle F_{\alpha\beta} \rangle \langle F^{\alpha\beta} \rangle \right) , \end{aligned} \tag{5.71}$$

where the last equality follows from the fact that quantum fluctuations of the photon field are suppressed in the boundary QFT. Our holographic calculation thus fully reproduces Eq. (5.34), which followed from the field theory discussion in Section 5.2.2. Furthermore, the running electromagnetic coupling constant matches the one found from field theory (cf. Eq. (5.37)) [284]. Hence, our holographic setup contains the a $U(1)$-gauged matter content of the $\mathcal{N} = 4$ SYM theory. In terms of bulk quantities, the renormalised stress-energy tensor and the two-form current are

$$\langle T_{\mu\nu} \rangle = \frac{N_c^2}{2\pi^2} \left(-g^{(2)}_{\mu\nu} + g^{(0)}_{\mu\nu} (g^{(2)})^{\lambda}_{\ \lambda} - \frac{1}{2}\tilde{h}_{\mu\nu} \right) + \frac{1}{e_r^2}\tilde{h}_{\mu\nu} , \tag{5.72}$$

$$\langle J_{\mu\nu} \rangle = 2B^{(1)}_{\mu\nu} , \tag{5.73}$$

where, as in Section 5.2, e_r is the renormalised coupling which needs to be set by experimental input—the renormalisation condition. In practice, this constant is fixed by choosing the value of $\mathcal{C}$ in (5.61). For the same reasons as in QFT, there is therefore an inherent ambiguity in holographic results, which has to be fixed by external physically-motivated input.

We conclude this section by noting that since $\nabla^{\mu} \langle T^{EM}_{\mu\nu} \rangle = 0$ follows from Maxwell's equations, using the relation (5.58) implies that the Ward identity for the stress-energy tensor satisfies Eq. (5.8), or in terms of our holographic notation, $\nabla_\nu \langle T^{\mu\nu}\rangle = \tilde{H}^{\mu}_{\ \lambda\sigma}\langle J^{\lambda\sigma}\rangle$ in Eq. (5.49).

5.3.3 Equation of state

To find the equation of state of our theory, we can use the renormalised stress-energy tensor (5.72) and the two-form current (5.73) computed in the previous section. Expressed in terms of the near-boundary expansions (5.48), which can be read off from the numerical background, we find

$$\left\langle T^{tt}\right\rangle = \frac{N_c^2}{2\pi^2}\left[-\frac{3}{4}f_4^b r_h^4 + \frac{\mathcal{B}^2}{8\pi\bar{\alpha}}\right], \tag{5.74}$$

$$\langle T^{xx}\rangle = \langle T^{yy}\rangle = \frac{N_c^2}{2\pi^2}\left[\left(-\frac{1}{4}f_4^b + \frac{v_4^b}{v}\right)r_h^4 - \frac{\mathcal{B}^2}{4} + \frac{\mathcal{B}^2}{8\pi\bar{\alpha}}\right], \tag{5.75}$$

$$\langle T^{zz}\rangle = \frac{N_c^2}{2\pi^2}\left[\left(-\frac{1}{4}f_4^b - 2\frac{v_4^b}{v}\right)r_h^4 - \frac{\mathcal{B}^2}{8\pi\bar{\alpha}}\right], \tag{5.76}$$

where we have used the (renormalised) fine-structure constant $\alpha = e_r^2/4\pi$ of the electromagnetic coupling in the plasma, which, as e_r, is fixed by the choice of $\mathcal{C}$ in (5.61), and rescaled it by $N_c^2/2\pi^2$ (or $|\beta(1/e^2)|$) as

$$\bar{\alpha} = \frac{N_c^2}{2\pi^2}\alpha\,. \tag{5.77}$$

The coupling $\bar{\alpha}$ has to be fixed by experimental observations as in any other quantum field theory, which is not easy in an unrealistic toy model.

To studying strongly coupled MHD, it is phenomenologically relevant to not only consider the matter and light-matter interactions, but also include large electromagnetic self-interactions encoded in the Maxwell action. However, since we are working with a holographic large-N_c matter sector and a single photon, it is rather unnatural to expect a Maxwell term of the same order, even though N_c controls the running of the electromagnetic coupling. The choice that we make here is to set the rescaled constant $\bar{\alpha}$ to the physically motivated $\bar{\alpha} = 1/137$. There are several ways to think about this choice: one is imagining that our plasma contains magnetic properties, which have non-trivial scalings with N_c, while another interpretation may assume that the bulk studied here could remain a valid dual of a theory with a reasonably small N_c. Of course, by considering only a classical bulk theory, we are restricting the strict validity of any computed observable to the limit of $N_c \to \infty$. As soon as one moves towards finite N_c, it becomes crucial to estimate the size of subleading $1/N_c^2$ corrections (topological expansion in the string coupling g_s)—an endeavour in holography (and string theory) which to date has been largely neglected and will continue being neglected in this work.[14] A less problematic limit is that of the infinite 't Hooft coupling, which is also implied by the choice of our action.[15] Perhaps the best interpretation is one of an "agnostic choice" led by our having to fix a free parameter to some value. We will return to a more careful investigation of the dependence of our results on this choice in Section 5.4.3.

The expectation values of the stress-energy tensor expressed in (5.74)–(5.76) are related to the MHD stress-energy tensor in Eq. (5.12) by

$$\langle T^{tt} \rangle = \varepsilon \,, \qquad \langle T^{xx} \rangle = p \,, \qquad \langle T^{zz} \rangle = p - \mu\rho \,. \tag{5.78}$$

We note that, as required in a conformal field theory with a trace anomaly induced by electromagnetic interactions, the trace of stress-energy tensor is non-zero. The holographic two-form current,

$$\langle J^{tz} \rangle = \mathcal{B} = \frac{B r_h^2}{v} \,, \tag{5.79}$$

is related to the equilibrium magnetic flux line density appearing in the MHD equation (5.13) as $\langle J^{tz} \rangle = \rho$. Temperature and entropy can be expressed in terms of the background geometry as

$$T = \frac{1}{2\pi} f_1^h r_h \,, \qquad s = \frac{N_c^2}{2\pi^2} \left(\frac{\pi r_h^3}{v\sqrt{w}} \right) , \tag{5.80}$$

[14] For some discussions of $1/N_c^2$ corrections to the thermodynamic free energy (the equilibrium partition function) and hydrodynamic long-time tails, see [131–133].

[15] For recent discussions of coupling-dependent holography, see [232, 304–307] and references therein.

and are therefore independent of the renormalised electromagnetic charge. The chemical potential, which corresponds to the density of magnetic flux lines, can be computed by using the thermodynamic identity $\varepsilon + p = sT + \mu\rho$ (cf. (5.20)):

$$\mu = \frac{\langle T^{xx}\rangle - \langle T^{zz}\rangle}{\langle J^{tz}\rangle} = \frac{N_c^2}{2\pi^2}\left(\frac{3v_4^b}{B} - \frac{B}{v} + \frac{B}{4\pi v\bar{\alpha}}\right) r_h^2 \,. \tag{5.81}$$

Note that with our choice of the bulk theory scalings, $\rho \sim \mathcal{O}(1)$ and $\mu \sim \mathcal{O}(N_c^2)$. Furthermore, while $T \sim \mathcal{O}(1)$, p, ε and s all scale as $\mathcal{O}(N_c^2)$.

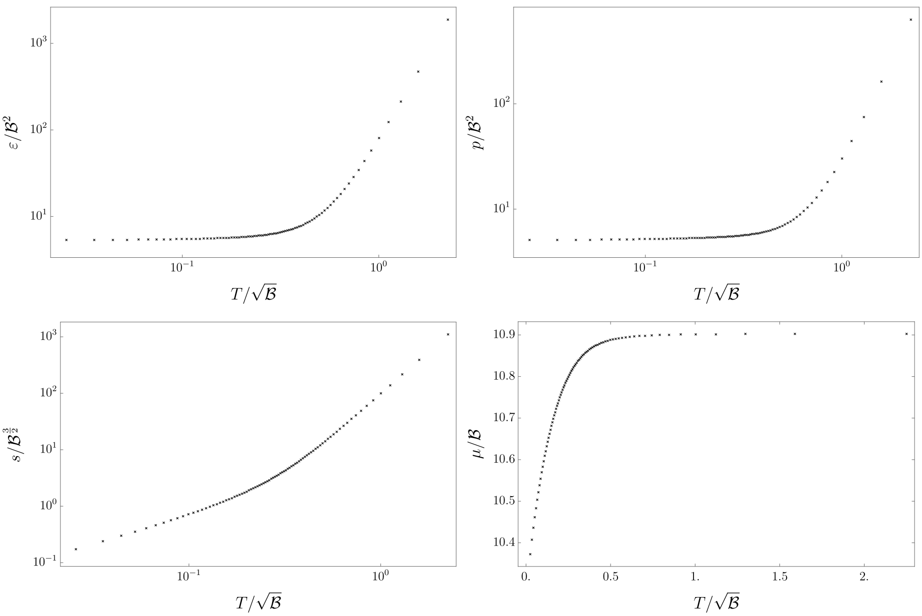

Figure 5.1. Dimensionless energy density $\varepsilon/\mathcal{B}^2$ (top-left), pressure $p/\mathcal{B}^2$ (top-right), entropy density $s/\mathcal{B}^{3/2}$ (bottom-left) and chemical potential $\mu/\mathcal{B}$ (bottom-right), in units of $N_c^2/(2\pi^2)$, plotted as a function of the dimensionless parameter $T/\sqrt{\mathcal{B}}$. The first three plots use logarithmic scales on both axes.

Using the above relations, we can perform two consistency checks on our holographic setup and numerical calculations of the background. First, the value of the pressure computed from the stress-energy tensor component $\langle T^{xx}\rangle =$

p can be compared with the value of the Euclidean on-shell action,

$$p = -i(\beta V_3)^{-1} S_{on-shell}, \tag{5.82}$$

where $\beta = 1/T$ and V_3 is the spatial volume of the theory. Secondly, we can compute $\varepsilon + p - \mu\rho$ from the stress-energy tensor evaluated near the boundary and by using the thermodynamic relation (5.20), check whether its values agrees with sT computed purely from horizon quantities. Both calculations show consistency of our setup in that we find $\langle T^{xx} \rangle = -i(\beta V_3)^{-1} S_{on-shell}$ and $\langle T^{tt} \rangle + \langle T^{zz} \rangle = sT$, within numerical precision.

We can now plot various thermodynamic quantities in a dimensionless manner by dividing them by appropriate powers of $\mathcal{B}$. The natural dimensionless parameter with respect to which we present our numerical results is $T/\sqrt{\mathcal{B}}$. The results for the energy density, pressure, entropy density and chemical potential are shown in Figure 5.1. The theory has two distinct regimes: the low- and the high-temperature regimes, or alternatively, the strong and weak magnetic field regimes, respectively. The high-temperature regime $T/\sqrt{\mathcal{B}} \gg 1$ is one to which MHD has been historically applied and to which the formulation of MHD, which assumes a weak-field separation between fluid and charge degrees of freedom can be applied. The claim presented in the Ref. [47] is that within the dual formulation, however, MHD applies for all values of $T/\sqrt{\mathcal{B}}$ provided that the theory remains in the hydrodynamic regime. The profiles of the thermodynamic functions in Figure 5.1 show a smooth crossover between the two regimes, which occurs around

$$T/\sqrt{\mathcal{B}} \approx 0.5 - 0.7\,. \tag{5.83}$$

By using numerical fits, the equation of state in the two limits behaves as expected on dimensional grounds [47]. We present the numerical results in Table 5.1.

In the limit of $\mathcal{B} \to 0$, the weak-field result approximately limits to the equation of state of a strongly coupled, thermal $\mathcal{N} = 4$ plasma, dual to a five dimensional AdS-Schwarzschild black brane with $p_{\mathcal{N}=4} = \frac{1}{8} N_c^2 \pi^2 T^4$; i.e.

	weak field ($T/\sqrt{\mathcal{B}} \gg 1$)	strong field ($T/\sqrt{\mathcal{B}} \ll 1$)
ε	$\frac{N_c^2}{2\pi^2}\left(74.1 \times T^4\right)$	$\frac{N_c^2}{2\pi^2}\left(5.62 \times \mathcal{B}^2\right)$
p	$\frac{N_c^2}{2\pi^2}\left(25.3 \times T^4\right)$	$\frac{N_c^2}{2\pi^2}\left(5.32 \times \mathcal{B}^2\right)$
s	$\frac{N_c^2}{2\pi^2}\left(99.4 \times T^3\right)$	$\frac{N_c^2}{2\pi^2}\left(7.41 \times \mathcal{B}T\right)$
μ	$\frac{N_c^2}{2\pi^2}\left(10.9 \times \mathcal{B}\right)$	$\frac{N_c^2}{2\pi^2}\left(2.88 \times \mathcal{B}\right)$

Table 5.1. Asymptotic behaviour of the equation of state in weak and strong field limits for $\bar{\alpha} = 1/137$.

$\lim_{\mathcal{B}\to 0} p_{weak} \approx 1.28 \times N_c^2 T^4$ and $p_{\mathcal{N}=4} \approx 1.23 \times N_c^2 T^4$. We also note that the value of the pressure at low temperature strongly depends on the renormalised (re-scaled) fine structure constant $\bar{\alpha}$, which we set to $\bar{\alpha} = 1/137$.

5.3.4 Transport coefficients

Next, we compute the seven transport coefficients, $\eta_\perp, \eta_\parallel, r_\perp, r_\parallel, \zeta_\perp, \zeta_\parallel$ and $\zeta_\times$, by using the Kubo formulae derived in [47, 278] and reviewed in Appendix 5.6.1. The procedure only requires us to turn on time-dependent fluctuations of the background fields without any spatial dependence, $G_{ab} \to G_{ab} + \delta G_{ab}(t)$ and $B_{ab} \to B_{ab} + \delta B_{ab}(t)$. The perturbations asymptote to the boundary sources $\delta g^{(0)}_{\mu\nu}$ and $\delta b^{(0)}_{\mu\nu}$ of the dual stress-energy tensor and the two-form current. In the absence of spatial dependence, the fluctuations decouple into five separate channels, from which the seven transport coefficients are computed, with each channel containing one independent dynamical second-order equation. The sets of decoupled fluctuations responsible for their respective transport coefficients are

$$
\begin{aligned}
\eta_\perp : &\quad \delta G_{xy}\,, \\
\eta_\parallel : &\quad \delta G_{xz},\ \delta B_{tx},\ \delta B_{xu}\,, \\
\zeta_\perp,\ \zeta_\parallel, \zeta_\times : &\quad \delta G_{tt},\ \delta G_{xx},\ \delta G_{yy},\ \delta G_{zz},\ \delta B_{tz},\ \delta G_{tu},\ \delta B_{zu},\ \delta G_{uu}\,, \\
r_\perp : &\quad \delta B_{xz},\ \delta G_{tx},\ \delta G_{xu}\,, \\
r_\parallel : &\quad \delta B_{xy}\,,
\end{aligned}
\tag{5.84}
$$

with only one out of the three bulk viscosities being independent. Each one of the transport coefficients can then be related to a membrane paradigm formula and computed entirely in terms of the horizon quantities. We summarise these relations here and discuss their derivation below:

$$\begin{aligned} \eta_\perp &= \frac{N_c^2}{2\pi^2}\left(\frac{r_h^3}{4v\sqrt{w}}\right) = \frac{1}{4\pi}s\,, \\ \eta_\parallel &= \frac{N_c^2}{2\pi^2}\left(\frac{r_h^3}{4w^{3/2}}\right) = \frac{1}{4\pi}\frac{v}{w}s\,, \\ r_\perp &= \frac{2\pi^2}{N_c^2}\left(\frac{\sqrt{w}}{2r_h}\right)\left(\frac{\mathfrak{b}_{xz}^{(-)}(1)}{\mathfrak{b}_{xz}^{(-)}(0)}\right)^2, \\ r_\parallel &= \frac{2\pi^2}{N_c^2}\left(\frac{v}{2r_h\sqrt{w}}\right), \\ \zeta_\perp = \frac{1}{4}\zeta_\parallel = -\frac{1}{2}\zeta_\times &= \frac{N_c^2}{2\pi^2}\left(\frac{r_h^3}{12v\sqrt{w}}\left(\frac{6+B^2}{6-B^2}\right)^2\left[\frac{\mathfrak{z}^{(-)}(1)}{\mathfrak{z}^{(-)}(0)}\right]^2\right), \end{aligned} \tag{5.85}$$

where $\mathfrak{b}^{(-)}$ and $\mathfrak{z}^{(-)}$ are the time-independent solutions of the fluctuations δB_{xz} and $Z_s = \delta G^x{}_x + \delta G^y{}_y - (2\mathcal{V}'/\mathcal{W}')\delta G^z{}_z$, respectively. The arguments denote that the functions are evaluated either at the horizon, $u = 1$, or the boundary, $u = 0$. Note that the value at the boundary is set by the Dirichlet boundary conditions.

What we see is that the ratio of the transverse shear viscosity (w.r.t to the background magnetic field) to entropy density is universal, resulting in $\eta_\perp/s = 1/4\pi$. Furthermore, the expressions for $\eta_\parallel$ and $r_\parallel$ only depend on the background quantities v and w, while $\zeta_\perp$, $\zeta_\parallel$ and $r_\perp$ also depend on the fluctuations of the fields.[16]

In order to derive the horizon formulae, we use a method similar to [301, 308]. Here, we will only explicitly show the derivation of the transverse resistivity $r_\perp$. The other formulae from Eq. (5.85) are derived in Appendix 5.7. First, we combine the equations of motion for the relevant fluctuations δB_{xz}, δG_{tx},

[16] For a holographic derivation of bulk viscosity in neutral relativistic hydrodynamics, see [300].

and δG_{xu} by eliminating the metric fluctuations into a single second-order differential equation

$$\delta B''_{xz} + \left(\frac{3}{2u} + \frac{F'}{F} - \mathcal{W}'\right)\delta B'_{xz} + \left(\frac{\omega^2}{4r_h^2 u^3 F^2} - \frac{B^2 e^{-4\mathcal{V}}}{u^3 F}\right)\delta B_{xz} = 0\,. \tag{5.86}$$

Since we are only computing first-order transport coefficients, it is sufficient to solve Eq. (5.86) to linear order in ω. To find the solution, we assume that there exists a time-independent solution $\mathfrak{b}_{xz}^{(-)}(u)$, which asymptotes to a constant both at the boundary and the horizon. At the boundary, this asymptotic value is related to the source of the two-form background gauge field, i.e. $\mathfrak{b}_{xz}^{(-)}(u \to 0) = \delta B_{xz}^{(0)}$. The time-dependent information is contained in the second solution, linearly-independent from $\mathfrak{b}_{xz}^{(-)}$. We refer to this solution as $\mathfrak{b}_{xz}^{(+)}$. It can be expressed in terms on integral over the Wronskian W_R of (5.86) as

$$\mathfrak{b}_{xz}^{(+)}(u) = \mathfrak{b}_{xz}^{(-)}(u)\int_u^1 du' \frac{W_R(u')}{\left(\mathfrak{b}_{xz}^{(-)}(u')\right)^2}\,, \tag{5.87}$$

where

$$W_R(u) = \exp\left[-\int_u^1 du'\left(\frac{3}{2u'} + \frac{F'(u')}{F(u')} - \mathcal{W}'(u')\right)\right] = \frac{1}{u^{3/2}Fe^{-\mathcal{W}}}\,. \tag{5.88}$$

The near-boundary and the near-horizon expansions of $\mathfrak{b}_{xz}^{(+)}$ are

$$\mathfrak{b}_{xz}^{(+)} = \begin{cases} \sqrt{w}\left[\mathfrak{b}_{xz}^{(-)}(0)\right]^{-1}\ln u + \mathcal{O}(\sqrt{u})\,, & \text{for } u \approx 0\,, \\ -r_h\left[2\pi T\mathfrak{b}_{xz}^{(-)}(1)\right]^{-1}\ln(1-u) + \mathcal{O}(1-u)\,, & \text{for } u \approx 1\,. \end{cases} \tag{5.89}$$

Finally, $\delta B_{xz}(\omega, u)$ is then a the following linear combination of the two solutions:

$$\delta B_{xz}(\omega, u) = \mathfrak{b}_{xz}^{(-)}(u) + \alpha(\omega)\mathfrak{b}_{xz}^{(+)}(u) + \mathcal{O}(\omega^2)\,. \tag{5.90}$$

The coefficient $\alpha(\omega)$ can be determined by imposing regular ingoing boundary conditions at the horizon, which corresponds to computing a retarded dual correlator [309, 310]:

$$\delta B_{xz}(u) = (1-u)^{-\frac{i\omega}{4\pi T}} \tilde{B}_{xz}\,, \tag{5.91}$$

The function $\tilde{B}_{xz}(u)$ is regular at the horizon. This choice of the boundary condition implies that near the horizon, δB_{xz} behaves as

$$\delta B_{xz}(u) = \mathfrak{b}_{xz}^{(-)}(u) + \alpha(\omega)\mathfrak{b}_{xz}^{(+)}(u) + \ldots = \mathfrak{b}_{xz}^{(-)}(1)\left(1 - \frac{i\omega}{4\pi T}\ln(1-u)\right) + \ldots\,. \tag{5.92}$$

Comparing Eq. (5.92) with the asymptotic behaviour of $\mathfrak{b}_{xz}^{(+)}$ in (5.89), we find $\alpha = (i\omega/2r_h)\left[\mathfrak{b}_{xz}^{(-)}(1)\right]^2$. Thus, the near-boundary expression for δB_{xz} becomes

$$\delta B_{xz}(u) = \mathfrak{b}_{xz}^{(-)}(0)\left(1 + \frac{i\omega}{2r_h}\sqrt{w}\left[\frac{\mathfrak{b}_{xz}^{(-)}(1)}{\mathfrak{b}_{xz}^{(-)}(0)}\right]^2 \ln u\right) + \mathcal{O}(u)\,. \tag{5.93}$$

By substituting this expression into the expectation value of the two-form current $\langle J^{\mu\nu}\rangle$, as found in Eq. (5.62), we obtain

$$\langle \delta J^{xz}\rangle = \lim_{u\to 0}\left(2u^{3/2}\sqrt{F}\delta B'_{xz}(u)\right) = \frac{2\pi^2}{N_c^2}\left(i\omega r_h^{-1}\sqrt{w}\left[\frac{\mathfrak{b}_{xz}^{(-)}(1)}{\mathfrak{b}_{xz}^{(-)}(0)}\right]^2\right)\delta b_{xz}^{(0)}\,. \tag{5.94}$$

Finally, using the Kubo formula for $r_\perp$, which is derived and presented in Eq. (5.119) of Appendix 5.6.1, we recover the expression presented in Eq. (5.85). All of the six remaining transport coefficients can be obtained by following the same procedure. We refer the reader to Appendix 5.7 for their detailed derivations.

The plots of the (dimensionless) transport coefficients $\eta_\parallel$, $\zeta_\parallel$, $r_\perp$ and $r_\parallel$ as a function of $T/\sqrt{\mathcal{B}}$ are presented in Figure 5.2. The remaining three viscosities can easily be inferred from Eq. (5.85). In particular, $\eta_\perp/s = 1/(4\pi)$, $\zeta_\perp = \zeta_\parallel/4$ and $\zeta_\times = -\zeta_\parallel/2$. We note that all transport coefficients satisfy the positive entropy production bounds discussed in Section 5.1. It is interesting that the bulk viscosity inequality $\zeta_\perp\zeta_\parallel \geq \zeta_\times^2$ is saturated, i.e. $\zeta_\perp\zeta_\parallel = \zeta_\times^2$ in

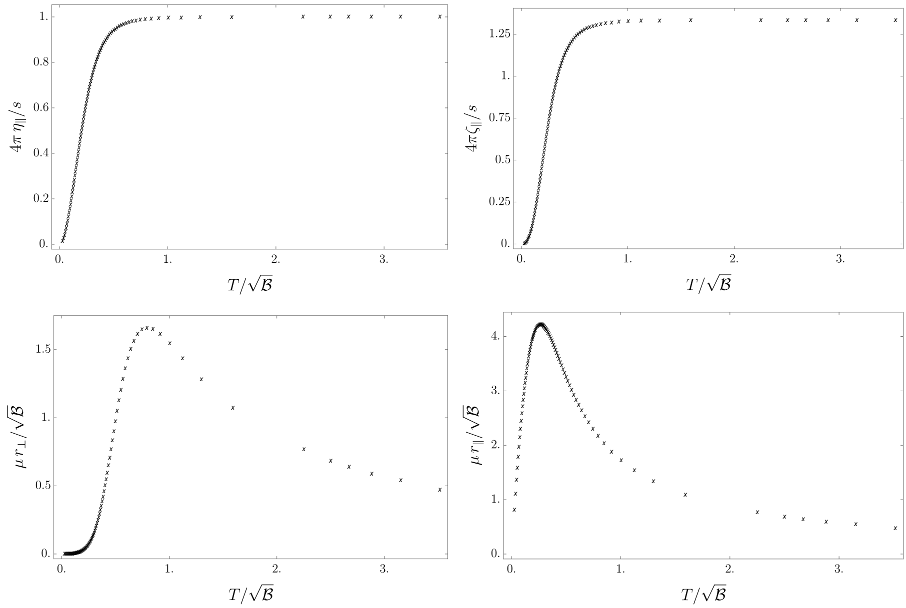

Figure 5.2. The plots of (dimensionless) first-order transport coefficients as a function of $T/\sqrt{\mathcal{B}}$.

the plasma studied here for all parameters of the theory.

We can now investigate the behaviour of the transport coefficients in the two extreme limits of $T/\sqrt{\mathcal{B}} \to 0$ and $T\sqrt{\mathcal{B}} \to \infty$, i.e. the strong- and the weak-field regimes, respectively. The leading-order power-law scaling (which we assume) and the coefficient follow from numerical fits. The results are presented in Table 5.2.

Since the entropy density s vanishes in the limit of zero temperature, all first-order transport coefficients vanish in the strong-field limit of $T \to 0$. This observation is consistent with predictions of [47], based on symmetry arguments. As a consequence, all (first-order) dissipative effects also vanish in the $T \to 0$ limit.

In the regime of a weak magnetic field, $T \gg \sqrt{\mathcal{B}}$, we find that both shear viscosities $\eta_\perp$ and $\eta_{||}$ converge to $\eta_\perp = \eta_{||} = s/(4\pi)$ as $\mathcal{B}/T^2 \to 0$. On the other hand, the longitudinal bulk viscosity limits to $\zeta_{||} \to 4\eta/3$, which is

	weak field ($T/\sqrt{\mathcal{B}} \gg 1$)	strong field ($T/\sqrt{\mathcal{B}} \ll 1$)
$\eta_\perp$	$\frac{s}{4\pi}$	$\frac{s}{4\pi}$
$\eta_\parallel$	$1.00 \times \frac{s}{4\pi}$	$\frac{s}{4\pi}\left(21.32 \times \frac{T^2}{\mathcal{B}}\right)$
$\zeta_\perp$	$0.33 \times \frac{s}{4\pi}$	$\frac{s}{4\pi}\left(16.34 \times \frac{T^3}{\mathcal{B}^{3/2}}\right)$
$\zeta_\parallel$	$1.33 \times \frac{s}{4\pi}$	$\frac{s}{4\pi}\left(65.37 \times \frac{T^3}{\mathcal{B}^{3/2}}\right)$
$\zeta_\times$	$-0.66 \times \frac{s}{4\pi}$	$-\frac{s}{4\pi}\left(32.69 \times \frac{T^3}{\mathcal{B}^{3/2}}\right)$
$r_\perp$	$\frac{\mathcal{B}}{\mu}\left(1.84 \times \frac{1}{T}\right)$	$\frac{\sqrt{\mathcal{B}}}{\mu}\left(2.34 \times \frac{T^3}{\mathcal{B}^{3/2}}\right)$
$r_\parallel$	$\frac{\mathcal{B}}{\mu}\left(1.71 \times \frac{1}{T}\right)$	$\frac{\sqrt{\mathcal{B}}}{\mu}\left(31.14 \times \frac{T}{\sqrt{\mathcal{B}}}\right)$

Table 5.2. Asymptotic behaviour of all first-order transport coefficients in weak- and strong-field limits. The temperature-dependent scaling of the shear viscosities at low temperature agrees with what was reported in Ref. [280].

consistent with expectations that as $\mathcal{B}/T^2 \to 0$, the plasma should becomes an uncharged relativistic conformal hydrodynamics (see e.g. [51] or Appendix 5.7). Furthermore, the resistivities $r_\perp$ and $r_\parallel$ also tend to zero in the limit.

We also note that the weak-field behaviour of $r_\perp$ and $r_\parallel$ is consistent with assumptions of standard (ideal) MHD, where conductivity is taken to infinity, $\sigma \approx 1/r \to \infty$, and where one adds corrections proportional to $1/\sigma$.[17] In other words, small weak-field resistivities are compatible with the assumption of ideal Ohm's law, which gave rise to Eq. (5.7) (see also our discussion around this equation in Section 5.1.). Furthermore, note that in standard MHD, only one resistivity (conductivity) is typically added to include dissipative corrections. What we see is that in our theory, the two resistivities take similar values in the weak-field limit in which standard MHD applies. However, in the strong-field limit, they assume drastically different values, including a different scaling with $T/\sqrt{\mathcal{B}}$. This observation therefore further points to the important role of anisotropic effects in MHD [47] and the necessity for using the formulation of [47, 278] as one moves from the weak to the strong-field regime.

The fact that $r_\perp$ and $r_\parallel$ tend to zero both in the limits of $T/\sqrt{\mathcal{B}} \to 0$

[17] See Ref. [47] for a discussion regarding the subtleties in relating resistivities to conductivities.

and $T\sqrt{\mathcal{B}} \to \infty$, along with the positivity of the entropy production bounds $r_\perp \geq 0$ and $r_\parallel \geq 0$ [47], implies that there always exists a maximum value of the resistivities at some intermediate $T/\sqrt{\mathcal{B}}$. It would be interesting to find the sizes of these maxima in experimentally realisable systems and probe the regimes of the "least conductive" plasmas. Finally, it would be interesting to further investigate the connection between maximal r and various discussions of lower bounds on conductivities, e.g. [38, 42, 311].

5.4 Magnetohydrodynamic waves in a strongly coupled plasma

We are now ready to use the information obtained from the holographic analysis of Section 5.3 to study dissipative dispersion relations of magnetohydrodynamic waves in a toy model of a strongly coupled plasma. We will use the theory of MHD [47], which is a phenomenological effective theory, and supplement it with microscopic details—the equation of state and transport coefficients—of the holographic setup investigated above. We will be particularly interested in the dependence of the MHD modes on the angle between momentum and magnetic field, as well as the ratio between temperature and the strength of the magnetic field. The 't Hooft coupling of interactions in the matter sector is not tuneable in our model, however, the electromagnetic coupling is. In all sections, except in Section 5.4.3, it will be set to $\alpha = 2\pi^2/137N_c^2$.

Before presenting the numerical result, we review the relevant facts about MHD modes. For a detailed derivation of these results, see Ref. [47] and for a discussion of the general procedure, see Refs. [51, 312]. First, we write the hydrodynamic variables u^μ, h^μ, T and μ in terms of oscillating modes perturbed around their near-equilibrium values, e.g.

$$u^\mu \to (1,0,0,0) + \delta u^\mu \, e^{-i\omega t + ikx\sin\theta + ikz\cos\theta}, \tag{5.95}$$

so that $\theta \in [0, \pi/2]$ measures the angle between the equilibrium magnetic field pointing in the z-direction and the wave momentum k in the x–z plane.

The dispersion relations $\omega(k)$ are then derived from the equations of MHD, i.e. Eqs. (5.8) and (5.9), with the external $H_{\mu\nu\rho} = 0$. The solutions depend on the angle θ, temperature T and the strength of the magnetic field (or the chemical potential of the magnetic flux number density), parametrised in our solutions by $\mathcal{B}$. Any dimensionless quantity will only depend on the single dimensionless ratio $T/\sqrt{\mathcal{B}}$. The resulting modes can be decomposed into two channels—odd and even under the reflection of $y \to -y$. The first channel is the transverse Alfvén channel. The second is the magnetosonic channel with two branches of solutions: slow and fast magnetosonic waves.

The linearised MHD equations of motion (5.8) and (5.9) need to be expanded in the hydrodynamic regime in powers of small $\omega/\Lambda_h \ll 1$ and $k/\Lambda_h \ll 1$, where Λ_h is the UV cut-off of the effective theory. In standard MHD, where $T \gg \sqrt{\mathcal{B}}$, then $\Lambda_h \approx T$, whereas in the strong-field regime of $T \ll \sqrt{\mathcal{B}}$, the cut-off can be set by the magnetic field, then $\Lambda_h \approx \sqrt{\mathcal{B}}$. As shown in [47], hydrodynamics can exist all the way to $T \to 0$, even when $\delta T = 0$. Such an expansion, performed to some order, gives rise to a polynomial equation in ω and k. For example, in the Alfvén channel, within first-order dissipative MHD,

$$\frac{\omega^2}{k^2} = \left(\frac{\mu\rho\cos^2\theta}{\varepsilon+p}\right) - i\left[\left(\frac{\mu r_\perp}{\rho} + \frac{\eta_\parallel}{\varepsilon+p}\right)\cos^2\theta + \left(\frac{\mu r_\parallel}{\rho} + \frac{\eta_\perp}{\varepsilon+p}\right)\sin^2\theta\right]\omega + \frac{\mu}{2\rho(\varepsilon+p)}\left(r_\perp\cos^2\theta + 2r_\parallel\sin^2\theta\right)\left(\eta_\perp\sin^2\theta + \eta_\parallel\cos^2\theta\right)k^2 \,. \tag{5.96}$$

The two solutions of the quadratic equation for ω are given by

$$\omega = -\frac{i}{2}(\mathcal{D}_{A,+})k^2 \pm \frac{k}{2}\sqrt{\mathcal{V}_A^2\cos^2\theta - (\mathcal{D}_{A,-})^2k^2}\,, \tag{5.97}$$

where $\mathcal{D}_{A,+}$ and $\mathcal{D}_{A,-}$ are

$$\mathcal{D}_{A,\pm} = \left(\frac{\mu r_\perp}{\rho} \pm \frac{\eta_\parallel}{\varepsilon+p}\right)\cos^2\theta + \left(\frac{\mu r_\parallel}{\rho} \pm \frac{\eta_\perp}{\varepsilon+p}\right)\sin^2\theta\,. \tag{5.98}$$

One can now series expand $\omega(k) = \mathcal{D}_0 k + \mathcal{D}_1 k^2$, or alternatively, plug this ansatz in Eq. (5.96) and solve order-by-oder in k. What we find is the Alfvén wave dispersion relation [47]:

$$\omega = \pm \mathcal{V}_A k \cos\theta - \frac{i}{2}\left\{ \frac{1}{\varepsilon + p}\left(\eta_\perp \sin^2\theta + \eta_\parallel \cos^2\theta\right) + \frac{\mu}{\rho}\left(r_\perp \cos^2\theta + r_\parallel \sin^2\theta\right) \right\} k^2 \,, \tag{5.99}$$

where the speed is given by $\mathcal{V}_A^2 = \mu\rho/(\varepsilon + p)$. The dispersion relation appears to be well-defined for any angle $\theta \in [0, \pi/2]$ between momentum and equilibrium magnetic field. In particular, if we were to take the $\theta \to \pi/2$ limit, (5.99) would yield two diffusive modes, both with dispersion relation

$$\omega = -\frac{i}{2}\left(\frac{\eta_\perp}{\varepsilon + p} + \frac{\mu r_\parallel}{\rho}\right) k^2 \,, \tag{5.100}$$

which are, however, unphysical and only result from an incorrect order of limits of k and θ.

As can be seen from the structure of the square-root in Eq. (5.97), the expansion in small k is only sensible so long as $k^2 \ll \mathcal{V}_A^2 \cos^2\theta / (\mathcal{D}_{A,-})^2$. Hence, even for a small finite k, this expansion is inapplicable for angles θ near $\theta = \pi/2$ where $\cos\theta$ becomes very small. In fact, for

$$\mathcal{V}_A^2 \cos^2\theta \leq (\mathcal{D}_{A,-})^2 k^2 \,, \tag{5.101}$$

the propagating modes cease to exist altogether and the two modes become purely imaginary (diffusive to $\mathcal{O}(k^2)$). The transmutation of two propagating Alfvén modes into two non-propagating modes occurs when the inequality in (5.101) is saturated, i.e. at the critical angle θ_c when $\text{Re}[\omega] = 0$:

$$\frac{\cos(\theta_c)}{\mathcal{D}_{A,-}(\theta_c)} = \frac{k}{\mathcal{V}_A}. \tag{5.102}$$

In other words, the plasma exhibits propagating (sound) modes for $0 \leq \theta < \theta_c$ and non-propagating (diffusive) modes for $\theta_c < \theta \leq \pi/2$. We plot the dependence of the critical angle θ_c on $k/\sqrt{\mathcal{B}}$ and $T/\sqrt{\mathcal{B}}$ for the Alfvén waves in our model in Figure 5.3. What we see is that for small $k/\sqrt{\mathcal{B}}$ and small

$T/\sqrt{\mathcal{B}}$, the transition to diffusive modes occurs closer to $\theta_c \approx \pi/2$. For any fixed and finite $T/\sqrt{\mathcal{B}}$, Eq. (5.102) indeed implies that $\theta_c \to \pi/2$ as $k \to 0$.

We note that as already pointed out in [278], the limits of $k \to 0$ and $\theta \to \pi/2$ do not commute and we obtain different results depending on which expansion ($k \approx 0$ or $\theta \approx \pi/2$) is performed first. If one first takes the limit $\theta \to \pi/2$, then Eq. (5.96) becomes

$$-\omega^2 - i\left(\frac{\mu r_\parallel}{\rho} + \frac{\eta_\perp}{\varepsilon + p}\right)\omega k^2 + \frac{\mu r_\parallel \eta_\perp}{\rho(\varepsilon + p)} k^4 = 0\,, \tag{5.103}$$

which instead of Eq. (5.100) results in two non-degenerate diffusive modes

$$\omega = -i\frac{\eta_\perp}{\varepsilon + p}k^2\,, \qquad \omega = -i\frac{\mu r_\parallel}{\rho}k^2\,. \tag{5.104}$$

The dispersion relation (5.99) is therefore only sensible at a finite $T/\sqrt{\mathcal{B}}$ and infinitesimally small k/Λ_h.

In the magnetosonic channel, the story is entirely analogous to the one described for the Alfvén waves. By expanding around $k \approx 0$ first, we obtain the dispersion relation of [47]:

$$\omega = \pm v_M k - i\tau k^2\,, \tag{5.105}$$

where the speed of magnetosonic wave is given by

$$v_M^2 = \frac{1}{2}\Bigg\{(\mathcal{V}_A^2 + \mathcal{V}_0^2)\cos^2\theta + \mathcal{V}_S^2\sin^2\theta \pm\sqrt{[(\mathcal{V}_A^2 + \mathcal{V}_0^2)\cos^2\theta + \mathcal{V}_S^2\sin^2\theta]^2 + 4\mathcal{V}^4\cos^4\sin^2\theta}\Bigg\}. \tag{5.106}$$

The functions $\mathcal{V}_A$, $\mathcal{V}_0$, $\mathcal{V}_S$ and $\mathcal{V}$ appearing in (5.106) are

$$\begin{aligned}
\mathcal{V}_A^2 &= \frac{\mu\rho}{\varepsilon + p}, \\
\mathcal{V}_0^2 &= \frac{s}{T\chi_{11}}, \\
\mathcal{V}_S^2 &= \frac{(s - \rho\chi_{12})(s + \rho\chi_{21}) + \rho^2\chi_{11}\chi_{22}}{(\varepsilon + p)\chi_{11}}, \\
\mathcal{V}^4 &= \frac{s(s - \rho\chi_{12})(s + \rho\chi_{21})}{T(\varepsilon + p)\chi_{11}^2}.
\end{aligned} \tag{5.107}$$

The susceptibilities are[18]

$$\chi_{11} = \left(\frac{\partial s}{\partial T}\right)_\rho, \quad \chi_{12} = \left(\frac{\partial s}{\partial \rho}\right)_T, \quad \chi_{21} = \left(\frac{\partial \mu}{\partial T}\right)_\rho, \quad \chi_{22} = \left(\frac{\partial \mu}{\partial \rho}\right)_T. \tag{5.108}$$

The two types of magnetosonic waves, corresponding to $\pm$ solutions in (5.106), are known as the fast (with $+$) and the slow (with $-$) magnetosonic waves. We refer the reader to Appendix 5.8 for further details regarding the derivation of the magnetosonic modes. Each pair of the propagating slow magnetosonic modes also splits, in analogy with the Alfvén waves, into two non-propagating diffusive modes for $\theta \geq \theta_c$. The critical angle θ_c for magnetosonic modes is also defined as in the Alfvén channel: the angle at which $\mathrm{Re}[\omega] = 0$. We plot the numerically-computed dependence of the magnetosonic θ_c on $k/\sqrt{\mathcal{B}}$ and $T/\sqrt{\mathcal{B}}$ in Fig. 5.3. As can be seen from the plot, the critical angles for the two types of waves are independent. However, they show similar qualitative dependence on the parameters that characterise the waves.

We summarise the θ-dependent characteristics of MHD modes in Fig. 5.4. We observe the pattern of a transmutation of sound modes into diffusion to be different in the weak- and strong-field regimes. Namely, the two magnetosonic waves interchange their dispersion relations at small θ. Since the complicated expressions for dispersion relations greatly simplify at $\theta = 0$ and $\theta = \pi/2$, we

[18]Note that these susceptibilities are different to the ones used in [47], where independent thermodynamic quantities were T and μ, not T and ρ. For this reason we also use different notation.

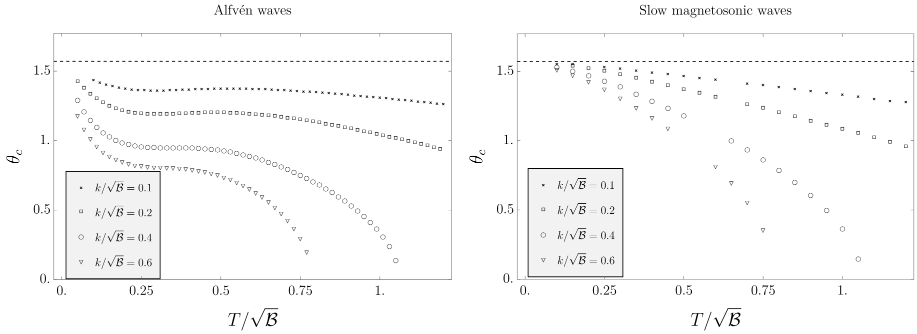

Figure 5.3. The critical angle θ_c for Alfvén waves (left) and slow magnetosonic waves (right), plotted as a function of $T/\sqrt{\mathcal{B}}$ for $k/\sqrt{\mathcal{B}} = \{0.1,\, 0.2,\, 0.4,\, 0.6\}$. The dashed line at the top of both sub-figures indicates the value of $\theta_c = \pi/2$.

state them below. The sound mode dispersion relations, denoted by S, are

$$\begin{aligned}
\text{S1}: \quad & \omega = \pm\mathcal{V}_S k - \frac{i}{2}\Bigg\{ \frac{\zeta_\perp + \eta_\perp}{\varepsilon + p} \\
& \quad + r_\perp \left(\frac{[(s-\rho\chi_{12})(\mu - T\chi_{21}) - \rho T\chi_{11}\chi_{22}] \times [\chi_{12} \leftrightarrow -\chi_{21}]}{T^2\chi_{11}\left[(s-\rho\chi_{12})(s+\rho\chi_{21}) + \rho^2\chi_{11}\chi_{22}\right]} \right) \Bigg\} k^2 \,, \\
\text{S2}: \quad & \omega = \pm\mathcal{V}_A k - \frac{i}{2}\left(\frac{\eta_\parallel}{\varepsilon + p} + \frac{\mu r_\perp}{\rho} \right) k^2 \,, \\
\text{S3}: \quad & \omega = \pm\mathcal{V}_0 k - \frac{i}{2}\frac{\zeta_\parallel}{sT} k^2 \,,
\end{aligned} \tag{5.109}$$

and the diffusive modes, denoted by D, are

$$\begin{aligned}
\text{D1}: \quad & \omega = -i\frac{\eta_\parallel}{sT} k^2 \,, \\
\text{D2}: \quad & \omega = -\frac{i r_\perp (\varepsilon + p)^2 \chi_{22}}{T^2\left[(s-\rho\chi_{12})(s+\rho\chi_{21}) + \rho^2\chi_{11}\chi_{22}\right]} k^2 \,, \\
\text{D3}: \quad & \omega = -i\frac{\eta_\perp}{\varepsilon + p} k^2 \,, \\
\text{D4}: \quad & \omega = -i\frac{r_\parallel \mu}{\rho} k^2 \,.
\end{aligned} \tag{5.110}$$

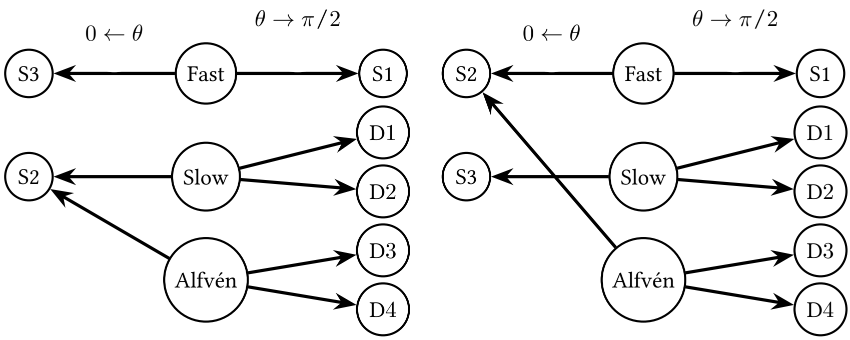

Figure 5.4. Diagrams depicting the θ-dependent pattern of transmutation from sound to diffusive modes for Alfvén waves and slow and fast magnetosonic waves. The left and right diagrams correspond to weak- and strong-field regimes. The relevant dispersion relation are stated in Eqs. (5.109) and (5.110).

In the regime of a large $T/\sqrt{\mathcal{B}}$, the results agree with those of [278]. Furthermore, using the asymptotic form of the thermodynamics quantities and transport coefficients in the $T/\sqrt{\mathcal{B}} \to \infty$ limit, one can show that these modes reduce to sound and diffusive modes of uncharged relativistic hydrodynamics.

In the strong-field regime, which cannot be described within standard MHD, the speeds of S1 and S3 become large and approach the speed of light in the limit of $T \to 0$. As discussed before, all diffusion constants vanish and the system becomes controlled by second-order MHD [47], which we do not investigate in this work. Further details of angle-dependent wave propagation are presented in Section 5.4.2.

5.4.1 Speeds and attenuations of MHD waves

Here, we plot the speeds (phase velocities) and first-order attenuation coefficients of the three types of MHD sound waves: the Alfvén and the fast and slow magnetosonic waves for the holographic strongly coupled plasma discussed above. These results assume an infinitesimally small value of momentum k, and follow from first expanding the polynomial equation of the type of (5.96) around $k \approx 0$ and writing the dispersion relation as $\omega = \pm vk - i\mathcal{D}k^2$. The speeds v (presented in Fig. 5.5) and attenuation coefficients $\mathcal{D}$ (presented in Fig. 5.6) are then plotted for all $0 \leq \theta \leq \pi/2$, which, as discussed above, is only physically sensible when $\theta_c \to \pi/2$, i.e. as $k \to 0$.

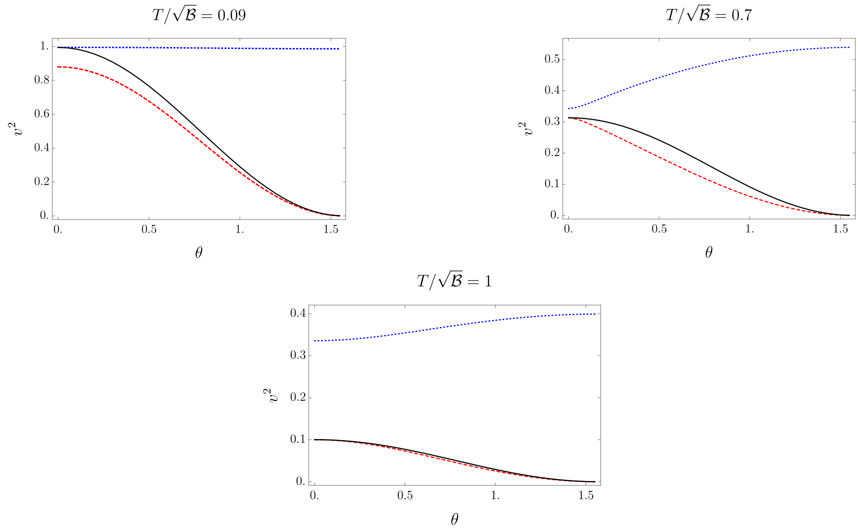

Figure 5.5. Angular dependence of the speeds of Alfvén (black, solid), fast (blue, dotted) and slow (red, dashed) magnetosonic waves in the strong-field, the crossover and the weak-field regimes.

The angular profiles of the speeds and the dissipative attenuation coefficients show distinct behaviour in the strong-, the crossover (cf. Eq. (5.83)) and the weak-field regimes. In particular, the speeds of sound enter the weak-field regime, where they reduce to well-known standard MHD results, rapidly after the temperature exceeds $T/\sqrt{\mathcal{B}} \approx 0.7$. There, Alfvén and slow magnetosonic waves travel with very similar speeds for all θ and their speeds coincide at $\theta = 0$ and $\theta = \pi/2$. The situation is qualitatively different in the strong-field regime where the profiles of speeds qualitatively match the strong-field predictions of [47], but of whicthe sh the behaviour was to our knowledge previously unknown. There, slow magnetosonic and Alfvén waves can travel faster at small θ, with speeds comparable to those of fast magnetosonic waves. At $\theta = 0$, the

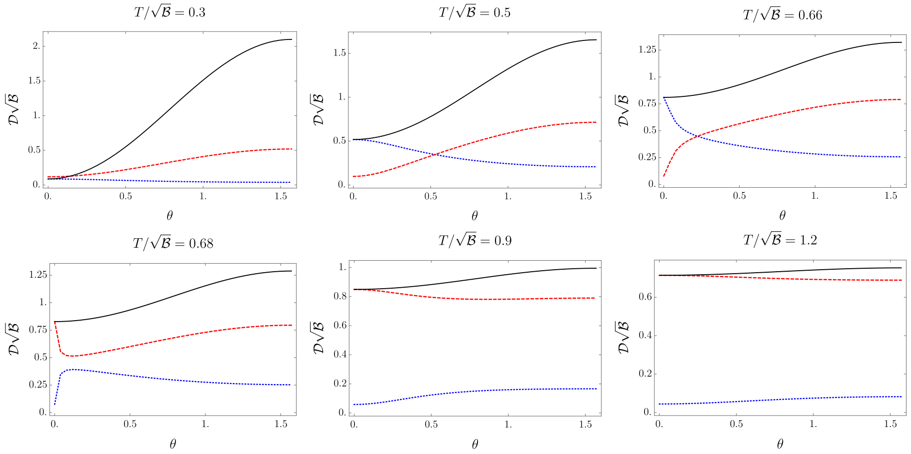

Figure 5.6. Angular dependence of the (dimensionless) attenuation coefficients of Alfvén (black, solid), fast (blue, dotted) and slow (red, dashed) magnetosonic waves, $\mathcal{D}\sqrt{\mathcal{B}}$, in the strong-field, the crossover and the weak-field regimes.

Alfvén speed equals that of fast, instead of slow, magnetosonic waves (cf. Fig. 5.4). It should also be noted that there exists a value of $T/\sqrt{\mathcal{B}}$ in the crossover regimes where all three speeds are equal at $\theta = 0$.

The attenuation coefficients, computed with all seven transport coefficients [47, 278], are computed for the first time for a concrete microscopically (holographically) realisable plasma and therefore difficult to compare with other past results. What we observe is that the Alfvén waves experience the strongest damping for all values of $T/\sqrt{\mathcal{B}}$. Beyond that, the qualitative behaviour again displays distinct angle-dependent features in the three regimes, which are apparent from Fig. 5.6. A noteworthy, but not surprising feature is that the strength of attenuation appears to be much more strongly dependent on the angle between momentum and magnetic field in the regime of small $T/\sqrt{\mathcal{B}}$. Furthermore, in the crossover regime, we find that the strengths of fast and slow magnetosonic mode attenuations interchange roles at $T/\sqrt{\mathcal{B}}$ increases. In plots at $T/\sqrt{\mathcal{B}} = 0.5$ and $T/\sqrt{\mathcal{B}} = 0.66$, there exists an angle θ at which the two attenuation strengths coincide.

5.4.2 MHD modes on a complex frequency plane

By assuming a finite value of momentum k, a full analysis of the spectrum requires us to take into account the transmutation of sound modes into non-propagating diffusive modes. The pattern of this behaviour, as a function of the angle between momentum and the direction of the equilibrium magnetic field θ, was summarised in Fig. 5.4. Motivated by holographic quasinormal mode (poles of two-point correlators) analyses, we plot the motion of the MHD modes on the complex frequency plane—here, as a function of θ and $T/\sqrt{\mathcal{B}}$. One should consider these plots as a prediction of how the first-order approximation to the hydrodynamic sector of the full quasinormal spectrum computed from the theory (5.43) is expected to behave.

In Fig. 5.7, we plot the typical θ-dependent trajectories of $\omega(\theta)$ on the complex ω-plane for Alfvén and magnetosonic modes in distinctly strong- and weak-field regimes. At all temperatures (except at $T = 0$ where $\mathcal{D} = 0$), the behaviour is consistent with our previous discussions, including the fact that the transmutation of Alfvén and slow magnetosonic waves into diffusive modes occurs at lower θ_c as $k/\sqrt{\mathcal{B}}$ increases.

In the crossover temperature regime (around $T/\sqrt{\mathcal{B}} \approx 0.6$), we find another manifestation of the interplay between fast and slow magnetosonic modes, which was noted in Section 5.4.1. While the speed of fast magnetosonic waves always exceeds that of slow waves, their attenuation strength exchange roles around $T/\sqrt{\mathcal{B}} \simeq 0.675$, which manifests in characteristically distinct behaviour for $\theta < \theta_c$. The behaviour is presented in Fig. 5.8 (see also Fig. 5.6). The θ-dependence of Alfvén waves remains qualitatively similar to those depicted in Fig. 5.7.

For a fixed $\theta < \theta_c$, where θ_c depends on k and $T/\sqrt{\mathcal{B}}$, we plot the typical behaviour of $\omega(k)$ as a function of $T/\sqrt{\mathcal{B}}$ in Fig. 5.9. At $T = 0$, all poles start from the non-dissipative regime (the real $\mathfrak{w}$ axis), with the speed of fast magnetosonic waves given by $v = 1$. As they move towards larger $T/\sqrt{\mathcal{B}}$, the Alfvén and slow magnetosonic modes again asymptote to each other, eventually

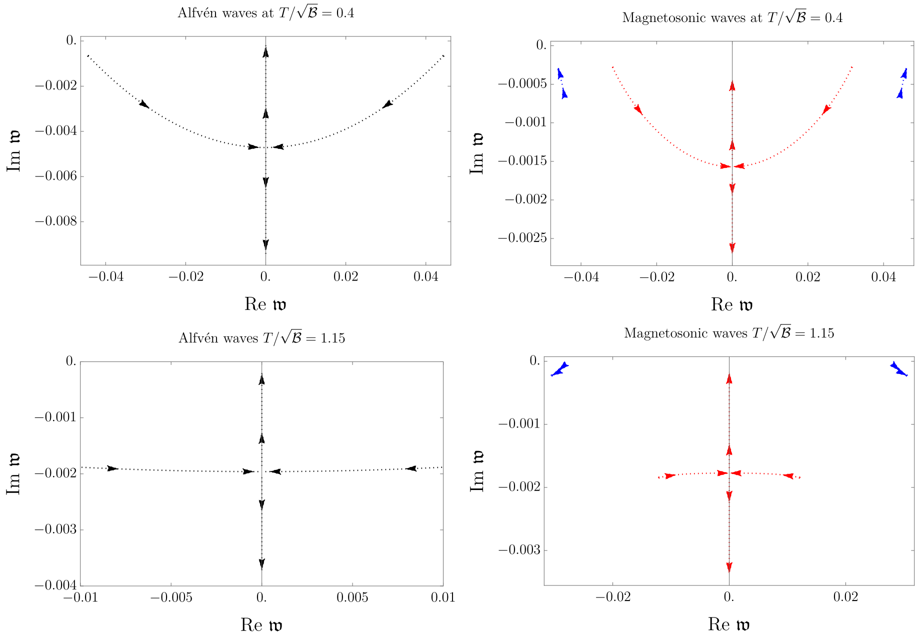

Figure 5.7. Dependence of the complex (dimensionless) frequency $\mathfrak{w} = \omega/\sqrt{\mathcal{B}}$ on θ, plotted for Alfén (black) and fast (blue) and slow (red) magnetosonic waves in the strong- and weak-field regimes with $T/\sqrt{\mathcal{B}} = 0.4$ and $T/\sqrt{\mathcal{B}} = 1.15$, respectively. The arrows represent the motion of poles as θ is tuned from 0 to $\pi/2$. Momentum is set to $k/\sqrt{\mathcal{B}} = 0.05$.

transforming into diffusive modes, while the speed of the fast magnetosonic modes gradually converges towards that of neutral conformal sound with $v = 1/\sqrt{3}$.

In the high temperature limit, the collision of the Alfvén and, independently, the slow magnetosonic modes on the imaginary axis occurs close to the real axis, which follows from the fact that for both types of waves,

$$\mathrm{Im}\,[\mathfrak{w}] \approx \frac{(\eta_\perp \sin^2\theta + \eta_\parallel \cos^2\theta)\sqrt{\mathcal{B}}}{2(\varepsilon + p)} \sim \frac{\sqrt{\mathcal{B}}}{T} \to 0\,, \tag{5.111}$$

as $T/\sqrt{\mathcal{B}} \to \infty$. The Alfvén waves then become the diffusive modes of uncharged conformal hydrodynamics with $\omega = -i\eta k^2/(2sT)$. As for our final

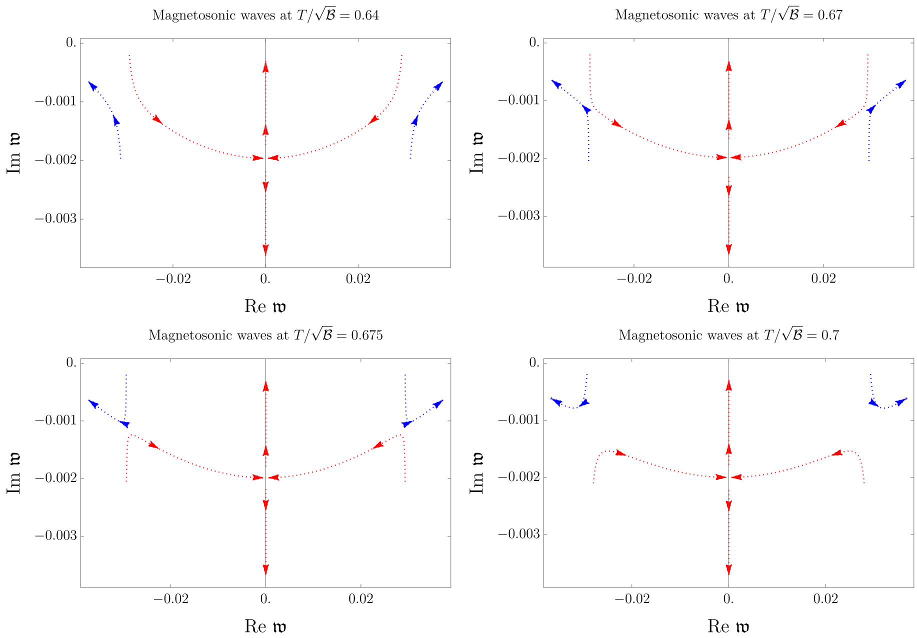

Figure 5.8. Dependence of the complex (dimensionless) frequency $\mathfrak{w} = \omega/\sqrt{\mathcal{B}}$ of fast (blue) and slow (red) magnetosonic modes on θ in the crossover regime. The arrows represent the motion of poles as θ is tuned from 0 to $\pi/2$. Momentum is set to $k/\sqrt{\mathcal{B}} = 0.05$.

plot, in Fig. 5.10, we present the dependence of the four diffusion constants and one sound attenuation coefficient on the temperature at $\theta = \pi/2$ (cf. Fig. 5.4 and Eqs. (5.109)–(5.110)). The modes D1, D3 and S1 reduce to dispersion relations of uncharged relativistic hydrodynamics. D2 and D4 are new.

5.4.3 Electric charge dependence

We end our discussion of MHD dispersion relations by investigating their dependence on the choice of the $U(1)$ coupling constant, which has so far been set to the (N_c-rescaled) $\bar{\alpha} = 1/137$. All dependence on $\bar{\alpha}$ enters into the expectation value of the stress-energy tensor through the term proportional to $\mathcal{H}_{\mu\nu}\mathcal{H}^{\mu\nu} \ln \mathcal{C}$ (cf. Eq. (5.61)), which contributes no terms linear in ω. For this

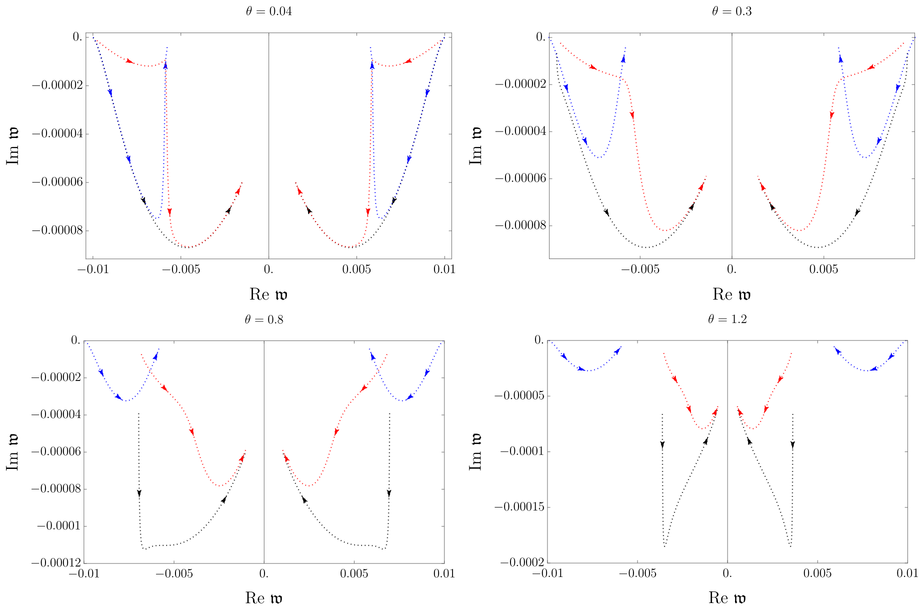

Figure 5.9. Dependence of the complex (dimensionless) frequency $\mathfrak{w} = \omega/\sqrt{\mathcal{B}}$ on $T/\sqrt{\mathcal{B}}$, plotted for Alfén (black) and fast (blue) and slow (red) magnetosonic waves for $\theta < \theta_c$. The arrows represent the motion of poles as $T/\sqrt{\mathcal{B}}$ is tuned from 0 towards the weak-field regime. Momentum is set to $k/\sqrt{\mathcal{B}} = 0.01$.

reason, while the equation of state strongly depends on $\bar{\alpha}$, the first-order transport coefficients do not. Hence, all speeds of sound and attenuation (and diffusive) coefficients depend on the choice of $\bar{\alpha}$ through the equation of state and susceptibilities.

What we observe is that the speeds of waves and attenuation coefficients strongly depend on the renormalised electromagnetic coupling, so unsurprisingly, the strength of electromagnetic interactions plays an important role in the phenomenology of MHD. For concreteness, we only present the detailed behaviour of the Alfvén waves (with speed $\mathcal{V}_A \cos\theta$), which reduce the neutral hydrodynamic diffusion mode D3 (and D4) at $\theta = \pi/2$. Both $\mathcal{V}_A$ and the diffusion constant of D3, $\mathcal{D}_{D3}$, strongly depend on $\bar{\alpha}$. On the other hand, it is interesting that the speed of S3 mode, $\mathcal{V}_0$, does not depend on $\bar{\alpha}$. We note that $\mathcal{V}_0$ is the $\theta = 0$ limit of fast and slow magnetosonic mode speeds in the weak-

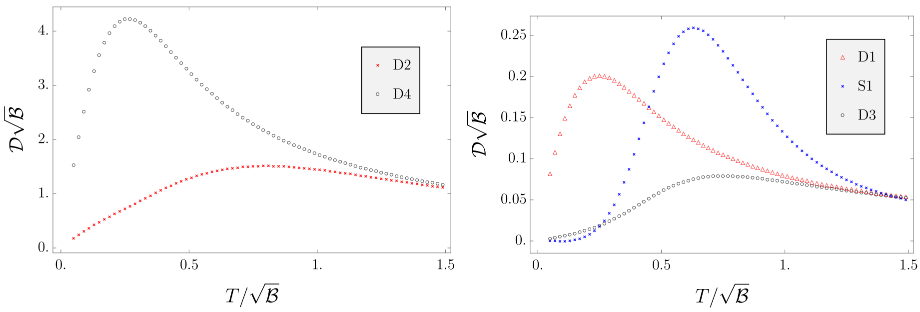

Figure 5.10. Plots of the four diffusion constants (D1, D2, D3, D4) and the sound attenuation (S1) as a function of $T/\sqrt{\mathcal{B}}$ at $\theta = \pi/2$. Black, red and blue curves depict dissipative coefficients that originate from the Alfvén, slow magnetosonic and fast magnetosonic waves, respectively.

and strong-field limits, For reasonable values of $\bar{\alpha}$, we plot the results in Fig. 5.11. To show the importance of a sensible choice of the renormalisation condition, we also vary the coupling over a larger range (to $\bar{\alpha} = 80/137$), where we see that the system develops unphysical behaviour with instabilities. As is apparent from Fig. 5.12, Alfvén waves become unstable at low $T/\sqrt{\mathcal{B}}$.

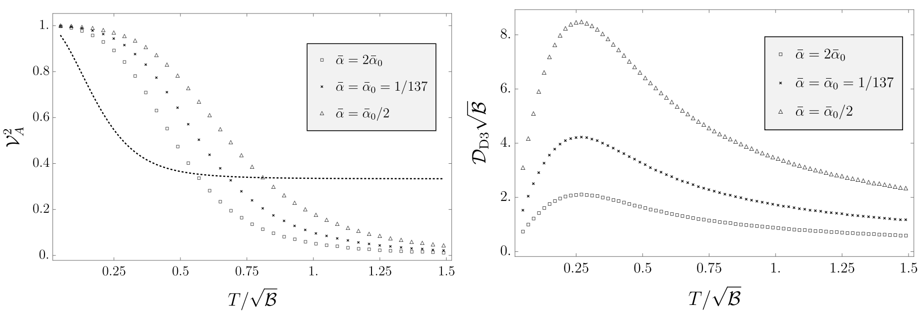

Figure 5.11. The plot of $\mathcal{V}_A^2$ and the diffusion constant $\mathcal{D}_{D3}$ at $\bar{\alpha} = \{\bar{\alpha}_0/2,\ \bar{\alpha}_0,\ 2\bar{\alpha}_0\}$, where $\bar{\alpha}_0 = 1/137$. The dashed line is the $\bar{\alpha}$-independent $\mathcal{V}_0^2$, which is plotted for comparison.

In all to us known literature, the unavoidable choice of the constant $\mathcal{C}$, which sets $\bar{\alpha}$ is made in a different way. Either $\mathcal{C}$ is chosen so that the loga-

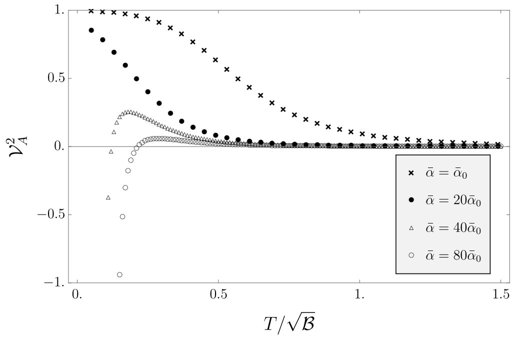

Figure 5.12. The plot of the Alfvén $\mathcal{V}_A^2$ at a varying $\bar{\alpha}$ ranging from $\bar{\alpha} = \bar{\alpha}_0$ to $\bar{\alpha} = 80\bar{\alpha}_0$, where $\bar{\alpha}_0 = 1/137$. We see that as $\bar{\alpha}$ increases, the waves develop an instability in the strong-field regime.

rithmic terms vanish altogether, or so that it sets the UV scale to be set by the magnetic field. The latter choice is convenient when studying strong magnetic fields, as e.g. in [284, 292]. Here, we would like to point out some of the consequences of setting $\mathcal{C}$ to either of the two standard options. Following the first choice, which eliminates the logarithmic terms, the resulting thermodynamics quantities are

$$\begin{aligned} \varepsilon &= \frac{N_c^2}{2\pi^2}\left(-\frac{3}{4}f_4^b r_h^4\right) , \\ p &= \frac{N_c^2}{2\pi^2}\left[\left(-\frac{1}{4}f_4^b + \frac{v_4^b}{v}\right) r_h^4 - \frac{\mathcal{B}^2}{4}\right] , \\ \mu\rho &= \frac{N_c^2}{2\pi^2}\left(\frac{3v_4^b}{v} r_h^4 - \frac{\mathcal{B}^2}{4}\right) . \end{aligned} \tag{5.112}$$

The second choice, results in

$$\begin{aligned} \varepsilon &= \frac{N_c^2}{2\pi^2}\left(-\frac{3}{4}f_4^b r_h^4 + \frac{\mathcal{B}^2}{4}\ln\mathcal{B}\right) , \\ p &= \frac{N_c^2}{2\pi^2}\left[\left(-\frac{1}{4}f_4^b + \frac{v_4^b}{v}\right) r_h^4 - \frac{\mathcal{B}^2}{4} + \frac{\mathcal{B}^2}{4}\ln\mathcal{B}\right] , \\ \mu\rho &= \frac{N_c^2}{2\pi^2}\left(\frac{3v_4^b}{v} r_h^4 - \frac{\mathcal{B}^2}{4} - \frac{\mathcal{B}^2}{4}\ln\mathcal{B}\right) . \end{aligned} \tag{5.113}$$

What we would like to claim is that while these two convenient renormalisation conditions are suitable for studying certain physical setups involving static electromagnetic fields, they lead to unphysical results when the boundary $U(1)$ gauge field is dynamical. By comparing the renormalised stress-energy tensor (5.74)–(5.76) to expressions in (5.112) and (5.113), we find that the two choices correspond to the renormalised coupling being $e_r^2 \to \infty$ and $e_r^2 \sim \ln \mathcal{B}$, respectively. The first choice is clearly unusual and unphysical. The problem with the second choice is that in certain regimes, $\ln \mathcal{B}$ can become negative and e_r imaginary, which is again unphysical. Both lead to instabilities and superluminal propagation, which were absent in from our results for $\bar{\alpha}$ near $1/137$. We plot the Alfvén speed parameter $\mathcal{V}_A$ for the two couplings from (5.112) and (5.113) in Fig. 5.13.

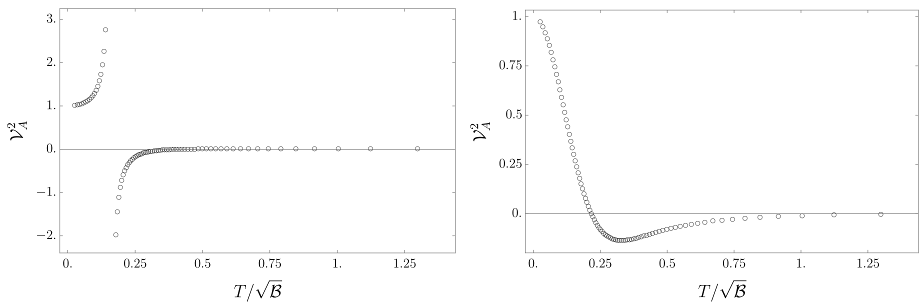

Figure 5.13. The θ-independent factor $\mathcal{V}_A$ of the Alfvén wave speed plotted for the renormalised $e_r^2 \to \infty$ (left) from Eq. (5.112) and for $e^2 \sim \ln \mathcal{B}$ (right) from Eq. (5.113).

5.5 Discussion

This work should be considered as the first holographic step in a long road towards a better understanding of magnetohydrodynamics in plasmas outside of the regime of validity of standard MHD, be it in the presence of strong magnetic fields or in a strongly interacting (or dense) plasma with a complicated equation of state and transport coefficients—all claimed to be describable within the recent (generalised global) symmetry-based formulation of MHD of Ref. [47]. In order to supply a hydrodynamical theory of MHD with the necessary microscopic information of a strongly coupled plasma, we resorted to the simplest, albeit experimentally inaccessible option: holography. Nevertheless, our hope is that in analogy with the myriad of works on holographic conformal hydrodynamics, which have led to important new insights into strongly interacting realistic fluids, holography can also help us understand MHD in the presence of strong fields, high density and of strongly interacting gauge theories, such as QCD.

With this view, we constructed the simplest theory dual to the operator structure and Ward identities used in MHD of [47], investigated the relevant aspects of the holographic dictionary and used it to compute the equation of state and transport coefficients of the dual plasma state. This information was then used to analyse the dependence of MHD waves—Alfvén and magnetosonic waves—on tuneable parameter of the state: the strength of the magnetic field, temperature, the angle between momentum of propagation and the equilibrium magnetic field direction, as well as the strength of the $U(1)$ electromagnetic gauge coupling. We believe that the latter feature of our model—dynamical electromagnetism on the boundary—which in the S-dual language of two-form gauge fields in the bulk allows for standard (Dirichet) quantisation, could in its own right be used for holographic studies of $U(1)$-gauged systems, unrelated to MHD.

Our results have revealed several new qualitative features of MHD waves, particularly in the regime of a strong magnetic field, which was previously inaccessible to standard MHD methods. Various properties of the equation of state, transport coefficients and dispersion relations may now be compared to those in experimentally realisable plasmas, or at the least, used as a toy model for future studies of MHD. The scalings are collected in Tables 5.1 and 5.2. Here, we summarise some of the most interesting observations:

- The equation of state and transport coefficients strongly depend on the strength of the magnetic field, i.e. on whether the plasma is in weak-field, crossover, or strong-field regime.
- In the weak-field limit of $T/\sqrt{\mathcal{B}} \gg 1$, the system is well-described by

standard MHD (see [278] for a full description) with small resistivities (large conductivity regime, which is assumed by ideal Ohm's law) and small effects of anisotropy. As $T/\sqrt{\mathcal{B}} \to \infty$, the plasma becomes an uncharged, conformal fluid with a single independent transport coefficient, $\eta = s/4\pi$. In the strong-field limit of $T/\sqrt{\mathcal{B}} \ll 1$, the plasma limits to a non-dissipative regime with all first-order transport coefficients (along with sound attenuations and diffusion constants) tending to zero. Effects of anisotropy are large.

• Resistivities have a global maximum in the intermediate $T/\sqrt{\mathcal{B}}$ regime, which indicates a regime of least conductive plasma. If the assumptions of standard MHD are correct at $T/\sqrt{\mathcal{B}} \gg 1$ and the symmetry-based predictions of [47] are correct at $T/\sqrt{\mathcal{B}} \ll 1$, such a regime should be generically exhibited by any plasma.

• Out of the three bulk viscosities, $\zeta_\perp$, $\zeta_\parallel$ and $\zeta_\times$, only one is independent and they saturate the positivity of the entropy production inequality, i.e. they are related by $\zeta_\perp\zeta_\parallel = \zeta_\times^2$. One may speculate on how general this result is and whether it is related to the suppression of entropy production at strong coupling [149, 150] or perhaps some form holographic universality at infinite (or strong) coupling.

• Various qualitative features of slow and fast magnetosonic modes are exchanged in the weak- and strong-field regimes at small angle θ between momentum and equilibrium magnetic field direction, such as their asymptotic tendency to the speed of Alfvén waves and the strength of sound attenuation.

• For a finite momentum, propagating (sound to $\mathcal{O}(k^2)$) Alfvén and slow magnetosonic modes transmute into pairs of non-propagating diffusive (to $\mathcal{O}(k^2)$) modes at large angles between the direction of momentum propagation and the equilibrium magnetic field, $\theta_c < \theta \le \pi/2$, where θ_c is some momentum- and $T/\sqrt{\mathcal{B}}$-dependent critical angle (cf. Eq. (5.102) for Alfvén waves).

• The phenomenology of MHD modes strongly depends on the strength of electromagnetic coupling and can, for large ranges of the coupling, lead to unstable or superluminal propagation.

Beyond the types of waves studied in this work, it would be particularly interesting to better understand the role finite charge density, as studied in [278], within the formalism of [47]. The important question then is how the phenomenology of such MHD waves, which typically experience gapped propagation (Langmuir waves) and instabilities (e.g. the infamous Weibel instability), becomes altered by strong interactions, strong fields and more 'exotic' field content.

Finally, the holographic setup studied here will need to undergo extensive further tests and analyses in order to unambiguously establish its connection to plasma physics and MHD. In particular, it is essential to study the quasinormal spectrum of the theory to verify that the hydrodynamic modes indeed describe the small-ω and small-k expansion of the leading infrared poles. Furthermore, it will be interesting to understand the role of higher-frequency spectrum and its interplay with MHD modes. We leave all these and many other interesting questions to the future.

5.6 Appendices

5.6.1 Kubo formulae for first-order transport coefficients

In this appendix, we outline the derivation of the Kubo formulae that have been used to compute the seven first-order transport coefficients in (5.12) and (5.14)–(5.19) in Section 5.3.4 [47, 278]. We derive the Kubo formulae by using the variational background field method (see e.g. [51] for a review), which amounts to varying the background metric $g_{\mu\nu}$ and background two-form gauge field $b_{\mu\nu}$, sourcing $T^{\mu\nu}$ and $J^{\mu\nu}$, by writing

$$g_{\mu\nu} \to \eta_{\mu\nu} + \int \frac{d\omega}{2\pi} e^{-i\omega t} \delta h_{\mu\nu}(\omega), \qquad b_{\mu\nu} \to b^{\text{eq}}_{\mu\nu} + \int \frac{d\omega}{2\pi} e^{-i\omega t} \delta b_{\mu\nu}(\omega)\,, \tag{5.114}$$

where $\delta h_{\mu\nu}$ and $\delta b_{\mu\nu}$ are small variation, η is the flat Minkowski metric and $b^{\text{eq}}_{\mu\nu} = 2\mu\, u^{[\mu} h^{\nu]}$. These variations of background fields can be viewed as a sources that generate a variation in the hydrodynamic variables T, ρ (which we use here instead of μ in [47]), u^μ and h^μ:

$$T(t) \to T + \delta T(t)\,, \qquad \rho \to \rho + \delta\rho(t)\,, \tag{5.115}$$

$$u^\mu \to u^\mu_{\text{eq}} + \delta u(t)\,, \qquad h^\mu \to h^\mu_{\text{eq}} + \delta h^\mu(t)\,, \tag{5.116}$$

where we choose the equilibrium configuration to be $u^\mu_{\text{eq}} = \delta^\mu_t$ and $h^\mu_{\text{eq}} = \delta^\mu_z$. The normalisation and orthogonality conditions for the two vectors ($u_\mu u^\mu = -1$, $h_\mu h^\mu = 1$, $u_\mu h^\mu = 0$) imply

$$\delta u^t = \frac{1}{2}\delta h_{tt}\,, \qquad \delta h^t = \delta u^z + \delta h_{tz}\,, \qquad \delta h^z = -\frac{1}{2}\delta h_{zz}\,. \tag{5.117}$$

After writing δT, $\delta\rho$, δu^μ and δh^μ in terms of $\delta h_{\mu\nu}$ and $\delta b_{\mu\nu}$, we can insert these solution into

$$\mathcal{T}^{\mu\nu} \equiv \sqrt{-g}\,\langle T^{\mu\nu}\rangle|_{g,b}\,, \qquad \mathcal{J}^{\mu\nu} \equiv \sqrt{-g}\,\langle J^{\mu\nu}\rangle|_{g,b}\,, \tag{5.118}$$

which give

$$\begin{aligned}
\operatorname{Im}\mathcal{T}^{xx} + \operatorname{Im}\mathcal{T}^{yy} &= \omega\zeta_\perp(\delta h_{xx} + \delta h_{yy}) + \omega\zeta_\times^{(1)}\delta h_{zz} + \mathcal{O}(\omega^2, \delta h^2, \delta b^2)\,,\\
\operatorname{Im}\mathcal{T}^{zz} &= \frac{1}{2}\omega\zeta_\parallel\delta h_{zz} + \frac{1}{2}\omega\zeta_\times^{(2)}\,(\delta h_{xx} + \delta h_{yy}) + \mathcal{O}(\omega^2, \delta h^2, \delta b^2),\\
\operatorname{Im}\mathcal{T}^{xy} &= \omega\eta_\perp\delta h_{xy} + \mathcal{O}(\omega^2, \delta h^2, \delta b^2),\\
\operatorname{Im}\mathcal{T}^{xz} &= \omega\eta_\parallel\delta h_{xz} + \mathcal{O}(\omega^2, \delta h^2, \delta b^2),\\
\operatorname{Im}\mathcal{J}^{xy} &= 2\omega r_\parallel\delta b_{xy} + \mathcal{O}(\omega^2, \delta h^2, \delta b^2),\\
\operatorname{Im}\mathcal{J}^{xz} &= 2\omega r_\perp\delta b_{xz} + \mathcal{O}(\omega^2, \delta h^2, \delta b^2)\,,
\end{aligned} \tag{5.119}$$

where we have not imposed the Onsager relation equating $\zeta_\times^{(1)}$ with $\zeta_\times^{(2)}$ [47, 278]. By using the linear response formulae relating the variations of one-point functions to retarded two-point Green's functions,

$$\begin{aligned}
\delta\mathcal{T}^{\mu\nu}(\omega,\mathbf{k}) &= -\frac{1}{2}G^{\mu\nu,\lambda\sigma}_{TT}(\omega,\mathbf{k})\delta h_{\lambda\sigma}(\omega,\mathbf{k}) - \frac{1}{2}G^{\mu\nu,\lambda\sigma}_{TJ}(\omega,\mathbf{k})\delta b_{\lambda\sigma}(\omega,\mathbf{k}),\\
\delta\mathcal{J}^{\mu\nu}(\omega,\mathbf{k}) &= -\frac{1}{2}G^{\mu\nu,\lambda\sigma}_{JT}(\omega,\mathbf{k})\delta h_{\lambda\sigma}(\omega,\mathbf{k}) - \frac{1}{2}G^{\mu\nu,\lambda\sigma}_{JJ}(\omega,\mathbf{k})\delta b_{\lambda\sigma}(\omega,\mathbf{k}),
\end{aligned} \tag{5.120}$$

it is then easy to extract the relevant Kubo formulae for the seven transport coefficients [47, 278], which we used in this work:

$$\eta_{\|} = \lim_{\omega\to 0} \frac{G^{xz,xz}_{TT}(\omega,0)}{-i\omega}\,, \qquad \eta_{\perp} = \lim_{\omega\to 0} \frac{G^{xy,xy}_{TT}(\omega,0)}{-i\omega}\,, \tag{5.121}$$

$$\zeta_{\|} = \lim_{\omega\to 0} \frac{G^{zz,zz}_{TT}(\omega,0)}{-i\omega}\,, \qquad \zeta_{\perp}+\eta_{\perp} = \lim_{\omega\to 0} \frac{G^{xx,xx}_{TT}(\omega,0)}{-i\omega}\,, \tag{5.122}$$

as well as

$$\zeta_{\times} = \lim_{\omega\to 0} \frac{G^{zz,xx}_{TT}(\omega,0)}{-i\omega} = \lim_{\omega\to 0} \frac{G^{xx,zz}_{TT}(\omega,0)}{-i\omega}\,. \tag{5.123}$$

and

$$r_{\|} = \lim_{\omega\to 0} \frac{G^{xy,xy}_{JJ}(\omega,0)}{-i\omega}\,, \qquad r_{\perp} = \lim_{\omega\to 0} \frac{G^{xz,xz}_{JJ}(\omega,0)}{-i\omega}\,. \tag{5.124}$$

5.7 Further details regarding the derivation of the transport coefficients

Here, we show the details of the derivation of horizon formulae of all remaining transport coefficient: $\eta_\perp$, $\eta_\|$, $\zeta_\perp$, $\zeta_\|$, $\zeta_\times$ and $r_\|$. The computation are analogous to the calculation of $r_\perp$ in Section 5.3.4.

(i) *Shear viscosity* $\eta_\perp$

The only relevant bulk fluctuation for $\eta_\perp$ is δG_{xy} with the equation of motion

$$\delta {G^y_x}'' + \left(\frac{3}{2u} + \frac{F'}{F} + 2\mathcal{V}' + \mathcal{W}'\right)\delta {G^y_x}' + \frac{\omega^2}{4r_h^2u^3F^2}\delta G^y_x = 0\,. \tag{5.125}$$

The solution to leading order in the frequency ω can be found analytically and its near-boundary expansion gives

$$\delta G^y_x = \delta h_{xy}\left(1 + \frac{i\omega u^2}{2r_h v\sqrt{w}} + \mathcal{O}(u^3)\right)\,, \tag{5.126}$$

where δh_{xy} is the Dirichlet background condition and the boundary theory source. If we plug in this solution into to the stress-energy tensor, we find that

$$\begin{aligned}\langle\delta T^{xy}\rangle &= \frac{N_c^2}{2\pi^2}\left(\frac{r_h^4 e^{2\mathcal{V}}\sqrt{uF}}{2v}\delta G_x^{y\prime}\right)+\dots\\ &= \frac{N_c^2}{2\pi^2}\left(\frac{i\omega r_h^3}{4v\sqrt{w}}\right)\delta h_{xy}+\dots\,,\end{aligned} \tag{5.127}$$

Using Eq. (5.119), we find that

$$\eta_\perp = \frac{N_c^2}{2\pi^2}\left(\frac{r_h^3}{4v\sqrt{w}}\right) = \frac{1}{4\pi}s\,, \tag{5.128}$$

as stated in Eq. (5.85).

(ii) *Shear viscosity* $\eta_\parallel$

Similarly to the computation of $r_\perp$, the xu-component of the two-form gauge field fluctuation equation can be used to reduced the two coupled second order differential equations coupling δG_{xz} and δB_{tx} to a single equation:

$$\delta G^z_{\ x}{}'' + \left(\frac{3}{2u}+\frac{F'}{F}+3\mathcal{W}'\right)\delta G^z_{\ x}{}' + \frac{\omega^2}{4r_h^2u^3F^2}\delta G^z_{\ x} = 0\,. \tag{5.129}$$

The solution to linear order in ω can again be found analytically and in the near-boundary region yields

$$\delta G^z_{\ x} = \delta h^z_{\ x}\left(1+\frac{i\omega}{r_h w^{3/2}}u^2+\mathcal{O}(u^3)\right). \tag{5.130}$$

The relevant component of the stress-energy tensor is then

$$\langle T^{xz}\rangle = \frac{N_c^2}{2\pi^2}\left(\frac{i\omega r_h^3}{4w^{3/2}}\right)\delta h_{xz}+\dots\,, \tag{5.131}$$

which gives

$$\eta_\parallel = \frac{N_c^2}{2\pi^2}\left(\frac{r_h^3}{4w^{3/2}}\right) = \frac{1}{4\pi}\frac{v}{w}s\,, \tag{5.132}$$

as stated in Eq. (5.85).

(iii) *Resistivity* $r_\parallel$

The only equation of motion in this channel is

$$\delta B''_{xy} + \left(\frac{3}{u} + \frac{F'}{F} - 2\mathcal{V}' + \mathcal{W}'\right)\delta B'_{xy} + \frac{\omega^2}{4r_h^2 u^3 F^2}\delta B_{xy} = 0\,, \tag{5.133}$$

which leads to the near-boundary solution

$$\delta B_{xy} = \delta B^{(0)}_{xy}\left(1 + \frac{i\omega v}{2r_h\sqrt{w}}\ln\, u + \mathcal{O}(u)\right). \tag{5.134}$$

The two-form current can then be written as

$$\langle \delta J^{xy}\rangle = \frac{2\pi^2}{N_c^2}\left(\frac{i\omega v}{r_h\sqrt{w}}\right)\delta b_{xy} + \dots\,, \tag{5.135}$$

which yields

$$r_{\parallel} = \frac{2\pi^2}{N_c^2}\left(\frac{v}{2r_h\sqrt{w}}\right), \tag{5.136}$$

as stated in Eq. (5.85).

(iii) *Bulk viscosities* $\zeta_\perp$, $\zeta_\parallel$ *and* $\zeta_\times$

By counting the number of the relevant degrees of freedom, it turns out that there is only one dynamical mode in this decoupled systems coming from $4 \times$ (2[nd]-order ODE for $\delta g_{tt}, \delta g_{aa}, \delta g_{zz}, \delta b_{tz}$) $- 3 \times$ (1[st]-order ODE for $\delta g_{tu}, \delta g_{uu}$, δb_{zu}). To find this dynamical mode, we start by solving the algebraic equation for δg_{tu}, δg_{uu} and δb_{zu} from the tu and uu components of Einstein's equation combined with the zu component of Maxwell's equations. Plugging these solutions into the four second-order equations involving δg_{tt}, δg_{aa}, δg_{zz} and δb_{tz}, we find that the remaining two non-trivial equations involve only δg_{aa} and δg_{zz}. The single resulting equation of motion can then be expressed in terms of the gauge-invariant variable $Z_s(u)$ defined as

$$Z_s(u) = \delta G^a{}_a - \frac{2\mathcal{V}'}{\mathcal{W}'}\delta G^z{}_z\,, \tag{5.137}$$

where $\delta g_{aa} = \delta g_{xx} + \delta g_{yy}$. The equation of motion for Z_s can be written

$$Z''_s(u) + C_1(\omega,u)Z'_s(u) + C_2(\omega,u)Z_s(u) = 0\,, \tag{5.138}$$

where

$$\begin{aligned} C_1 &= \frac{3}{2u} + \frac{F'}{F} + \frac{2\mathcal{W}''}{W'} + 2\mathcal{V}' + \mathcal{W}' - 2\left(\frac{2\mathcal{V}'' + \mathcal{W}''}{2\mathcal{V}' + \mathcal{W}'}\right), \\ C_2 &= -\frac{b^2 e^{-4\mathcal{V}}}{3u^3 F\mathcal{W}'}\left(\frac{F'}{F} + 4\mathcal{W}'\right) + \frac{\omega^2}{4r_h^2 u^3 F^2} - \frac{2F'^2}{3F^2\mathcal{W}'}(\mathcal{V}' - \mathcal{W}') \\ &\quad + \frac{4\mathcal{V}'F'(\mathcal{V}' - \mathcal{W}')^2}{3F\mathcal{W}'(2\mathcal{V}' + \mathcal{W}')} + \frac{8\mathcal{V}'^2(\mathcal{V}' + 2\mathcal{W}')(\mathcal{V}' - \mathcal{W}')}{2\mathcal{W}'(2\mathcal{V}' + \mathcal{W}')}. \end{aligned} \tag{5.139}$$

Now, suppose that the time-independent solution for Z_s is $\mathfrak{Z}^{(-)}$, so that $\mathfrak{Z}^{(-)}(u \to 0) = Z^{(0)} \equiv \delta h_{aa} - 2\delta h_{zz}$ (note that $\mathcal{V}'/\mathcal{W}' \to 1$ and $u \to 0$). The second solution, denoted as $\mathfrak{Z}^{(+)}$, contains the time-dependent information and can be found from the Wronskian

$$\mathfrak{Z}^{(+)}(u) = \mathfrak{Z}^{(-)}(u)\int_u^1 du' \, \frac{W_R(u')}{\left(\mathfrak{Z}^{(-)}(u')\right)^2}, \quad W_R = \left(\frac{2\mathcal{V}' + \mathcal{W}'}{\mathcal{W}'}\right)^2 \frac{e^{2\mathcal{V}+\mathcal{W}}}{u^{3/2}F}. \tag{5.140}$$

We then find that the asymptotic solution for $\mathfrak{Z}^{(+)}$ are

$$\mathfrak{Z}^{(+)} = \frac{9}{2v\sqrt{w}}\left[\mathfrak{Z}^{(-)}(0)\right]^{-1} u^2 + \mathcal{O}(u^3), \tag{5.141}$$

near the boundary $u = 0$ and

$$\mathfrak{Z}^{(+)} = -9r_h\left(\frac{6+B^2}{6-B^2}\right)^2\left[2\pi T\mathfrak{Z}^{(-)}(1)\right]^{-1}\ln(1-u) + \mathcal{O}(1-u), \tag{5.142}$$

near the horizon $u = 1$. As the full solution is a linear combination, $Z_s(u) = \mathfrak{Z}^{(-)} + \alpha\mathfrak{Z}^{(+)}$, the ingoing boundary condition set the frequency-dependent function $\alpha(\omega)$ to be

$$\alpha(\omega) = \frac{i\omega}{2r_h}\left(\frac{(6+B^2)}{3(6-B^2)}\right)^2\left[\mathfrak{Z}^{(-)}(1)\right]^2, \tag{5.143}$$

which allows us to write the solution for Z_s near the boundary as

$$Z_s = Z^{(0)}\left(1 + \frac{i\omega}{4r_h v\sqrt{w}}\left(\frac{6+B^2}{6-B^2}\right)^2\left[\frac{\mathfrak{Z}^{(-)}(1)}{\mathfrak{Z}^{(-)}(0)}\right]^2 u^2\right) + \dots. \tag{5.144}$$

This expression can then be used to compute the bulk viscosities, for which we follow the approach by [313] and their analysis of the Green's function in sound channel. In summary, we first find the expression for $\langle \delta T^{xx} + \delta T^{yy} \rangle$ and $\langle \delta T^{zz} \rangle$ in terms of δh_{tt}, δh_{aa}, δh_{zz} and δb_{tz}, and then relate the near-boundary data of the bulk modes δG_{tt}, δG_{aa}, δG_{zz} and δB_{tz} to those of Z_s. Then, we impose the radial gauge, $\delta G_{u\mu} = 0$ and $\delta B_{u\mu} = 0$ and solve the equations of motion near the boundary. The first-order equations of motion gives the following relations

$$\begin{aligned} h_{aa}^{(2)} + h_{tt}^{(2)} + h_{zz}^{(2)} + \frac{B^2 h_{aa}^{(0)}}{36 v^2} &= 0 \,, \\ 2\left(h_{aa}^{(2)} + h_{zz}^{(2)}\right) + \frac{v_4^b}{v}\left(h_{aa}^{(0)} - 2h_{zz}^{(0)}\right) - f_4^b\left(h_{aa}^{(0)} + h_{zz}^{(0)}\right) &= 0 \,. \end{aligned} \tag{5.145}$$

The coefficients $h_{\mu\nu}^{(n)}$ are defined through the near-boundary expansion of the metric fluctuation. By using the second-order dynamical equation and the radial gauge, the solutions are

$$\begin{aligned} \delta G^a{}_a &= h_{aa}^{(0} + h_{aa}^{(2)} u^2 + \frac{h_{aa}^{(0)} B^2}{10 v^2} u^2 \ln u + \mathcal{O}(\omega^2, u^3), \\ \delta G^t{}_t &= h_{tt}^{(0)} + h_{tt}^{(2)} u^2 - \frac{h_{aa}^{(0)} B^2}{20 v^2} u^2 \ln u + \mathcal{O}(\omega^2, u^3), \\ \delta G^z{}_z &= h_{zz}^{(0)} + h_{zz}^{(2)} u^2 - \frac{h_{aa}^{(0)} B^2}{20 v^2} u^2 \ln u + \mathcal{O}(\omega^2, u^3), \end{aligned} \tag{5.146}$$

where $h_{\mu\nu}^{(0)} \equiv \delta h_{\mu\nu}$ is the metric perturbation used throughout the paper. By combining Eqs. (5.145) and (5.146), and using definition of gauge-invariant mode Z_s, we find that

$$Z_s = Z^{(0)} + \frac{Z^{(0)} \omega^2}{6 r_h^2} + Z^{(2)} u^2 + \frac{h_{aa}^{(0)} B^2}{5 v^2} u^2 \ln u + \mathcal{O}(\omega^4) \,, \tag{5.147}$$

where

$$Z^{(0)} = h_{aa}^{(0)} - 2h_{zz}^{(0)}, \quad Z^{(2)} = -3h_{zz}^{(2)} - \frac{v_4^b}{v} Z^{(0)} + f_4^b\left(h_{aa}^{(0)} + h_{zz}^{(0)}\right) \,. \tag{5.148}$$

It is most convenient to extract the transport coefficients from $\langle \delta T^{zz} \rangle$:

$$\langle \delta T^{zz} \rangle = -\frac{N_c^2}{2\pi^2} \frac{r_h^4 e^{2\mathcal{W}}}{w} \left(\frac{1}{2} \sqrt{uF} \left(\delta G_a^{a\prime} + \delta G_t^{t\prime} \right) + \left(\frac{3}{2u} + \frac{\sqrt{u}F'}{2F} + 2\sqrt{uF}\mathcal{V}' \right) \delta G_z^z \right) + \dots$$
$$= -\frac{N_c^2}{2\pi^2} \left(\frac{i\omega r_h^3}{12v\sqrt{w}} \left(\frac{6+B^2}{6-B^2} \right)^2 \left[\mathfrak{Z}^{(-)}(1)/\mathfrak{Z}^{(-)}(0) \right]^2 \right) (\delta h_{aa} - 2h_{zz}) + \dots .$$

The ellipses denote various time-independent terms, which are irrelevant for computing the first-order transport coefficients of interest. Using the Kubo formula (5.119), we find that

$$\begin{aligned} \zeta_{\parallel} &= \frac{N_c^2}{2\pi^2} \left(\frac{r_h^3}{3v\sqrt{w}} \left(\frac{6+B^2}{6-B^2} \right)^2 \left[\mathfrak{Z}^{(-)}(1)/\mathfrak{Z}^{(-)}(0) \right]^2 \right) \\ &= \frac{s}{4\pi} \left(\frac{4}{3} \left(\frac{6+B^2}{6-B^2} \right)^2 \left[\mathfrak{Z}^{(-)}(1)/\mathfrak{Z}^{(-)}(0) \right]^2 \right), \end{aligned} \tag{5.149}$$

and $\zeta_\times^{(2)} = -\zeta_\parallel/2$.

Similarly, we can extract $\zeta_\perp$ and $\zeta_\times^{(1)}$ from

$$\langle \delta T^{xx} \rangle + \langle \delta T^{yy} \rangle = -\frac{N_c^2}{2\pi^2} \frac{r_h^4 e^{2\mathcal{V}}}{v} \left[\frac{1}{2} \sqrt{uF} \left(\delta G_a^{a\prime} + \delta G_t^{t\prime} + \delta G_z^{z\prime} \right) + \left(\frac{3}{2u} + \frac{\sqrt{u}F'}{2F} + \sqrt{uF}(\mathcal{V}' + \mathcal{W}') \right) \delta G_z^z \right] + \dots$$

$$= \frac{N_c^2}{2\pi^2} \left(\frac{i\omega r_h^3}{12v\sqrt{w}} \left(\frac{6+B^2}{6-B^2} \right)^2 \left[\mathfrak{Z}^{(-)}(1)/\mathfrak{Z}^{(-)}(0) \right]^2 \right) (\delta h_{aa} - 2h_{zz}) + \dots .$$

This gives $\zeta_\perp = \zeta_\parallel/4 = -\zeta_\times^{(1)}/2$. Hence, we find that $\zeta_\times^{(1)} = \zeta_\times^{(2)}$, which is the manifestation of the Onsager relation imposed in [47, 278]. This completes the derivation of expressions stated in Eq. (5.85).

As a simple check of our results, in the zero magnetic field limit, one can show that

$$\zeta_{\parallel} = \lim_{\omega\to 0} \partial_\omega G^{zz,zz}_{TT}(\omega,0) = \frac{4}{3}\lim_{\omega\to 0} \partial_\omega G^{xy,xy}_{TT}(\omega,0)\,, \tag{5.150}$$

which implies, as expected, that the bulk viscosity of conformal relativistic hydrodynamics vanishes (see e.g. [51]). For another check, one can write the relation $\zeta_{\parallel} = 2\,\zeta_{\perp}$ in the language of two-point functions and obtain the relation $\lim_{\omega\to 0}\left[\partial_\omega G^{aa,aa}_{TT}(\omega,0) - \frac{1}{2}\partial_\omega G^{zz,zz}_{TT}(\omega,0)\right] = 0$, which is also satisfied by conformal relativistic hydrodynamics.[19] Interestingly, this relations holds for all strengths of the magnetic field in the model studied in this work. One may wonder whether this relation between different bulk viscosities points to a more general property of (strongly interacting) MHD plasmas.

5.8 Dispersion relations of magnetosonic waves

In the magnetosonic channel, the polynomial equation in ω and k, which needs to be solved in order for us to find the dispersion relations $\omega(k)$ is a quartic equations in ω, which can be written in the following form:

$$\text{Det}\left[-i\omega\mathbb{1} + \mathbb{M}\right] = 0\,, \tag{5.151}$$

with $\mathbb{1}$ the 4×4 identity matrix and the non-zero components M_{ij} of the matrix $\mathbb{M}$ given by

[19] For a neutral relativistic fluid, one can show that $\text{Im}\langle\delta T^{xx}\rangle + \text{Im}\langle\delta T^{yy}\rangle = \omega\left(\frac{\eta}{3}+\zeta\right)\delta h_{aa} + \omega\left(\zeta - \frac{2}{3}\eta\right)\delta h_{zz}$ and $\text{Im}\langle\delta T^{zz}\rangle = \frac{1}{2}\omega\left(\zeta-\frac{2}{3}\eta\right)\delta h_{aa} + \omega\left(\frac{2}{3}\eta+\frac{1}{2}\zeta\right)\delta h_{zz}$. In a conformal fluid with $\zeta = 0$, one find that $\lim_{\omega\to 0}\partial_\omega G^{aa,aa}_{TT}(\omega,0) = \lim_{\omega\to 0}\frac{1}{4}\partial_\omega G^{zz,zz}_{TT}(\omega,0) = -\eta/3$.

$$
\begin{aligned}
&M_{11} = r_\perp k^2 \sin^2\theta \mathcal{A}_{11}\,, && M_{12} = -r_\perp k^2 \mathcal{A}_{12}\,, && M_{13} = ik\sin\theta \mathcal{A}_{13}\,,\\
&M_{14} = ik\frac{s\cos\theta}{\chi_{11}} && M_{21} = -r_\perp k^2 \sin^2\theta \mathcal{A}_{21}\,, && M_{22} = r_\perp k^2 \mathcal{A}_{22}\,,\\
&M_{23} = ik\rho\sin\theta && M_{31} = ik\sin\theta\mathcal{A}_{31}\,, && M_{32} = ik\mathcal{A}_{32}\,,\\
&M_{33} = \mathcal{A}_{33}k^2, && \mathcal{M}_{34} = \eta_\perp k^2 && \mathcal{A}_{34}M_{41} = i\frac{k}{T}\cos\theta\,,\\
&M_{43} = \eta_\perp k^2 \mathcal{A}_{43}\,, && M_{44} = k^2\mathcal{A}_{44}\,.
\end{aligned}
$$

The coefficients $\mathcal{A}_{ij}$ are

$$
\begin{aligned}
&\mathcal{A}_{11} = \frac{1}{2T^2\chi_{11}}(\mu + T\chi_{12})(\mu - T\mu_{21}),\\
&\mathcal{A}_{12} = \frac{1}{2T\rho\chi_{11}}(\mu + T\chi_{12})(\mu\cos^2\theta + \rho\chi_{22}\sin^2\theta),\\
&\mathcal{A}_{13} = \frac{s-\rho\chi_{12}}{\chi_{11}}\,, && \mathcal{A}_{21} = \frac{\mu - T\chi_{21}}{2T}\,,\\
&\mathcal{A}_{22} = \frac{1}{2\rho}\left(\mu\cos^2\theta + \rho\chi_{22}\sin^2\theta\right), && \mathcal{A}_{31} = \frac{s+\rho\chi_{21}}{\varepsilon + p}\,,\\
&\mathcal{A}_{32} = \frac{2\rho}{(\varepsilon+p)\sin\theta}\mathcal{A}_{22}\,,\\
&\mathcal{A}_{33} = \left(\frac{\eta_\parallel\cos^2\theta + (\eta_\perp + \zeta_\perp)\sin^2\theta}{\varepsilon+p}\right),\\
&\mathcal{A}_{34} = \frac{\cos\theta\sin\theta}{\varepsilon+p}\,, && \mathcal{A}_{43} = \frac{\varepsilon+p}{sT}\mathcal{A}_{34}\,,\\
&\mathcal{A}_{44} = \frac{2\zeta_\parallel\cos^2\theta + \eta_\parallel\sin^2\theta}{sT}\,.
\end{aligned}
$$

By computing the determinant in (5.151), the resulting quartic equation is

$$
\omega^4 + c_3\omega^3 + c_2\omega^2 + c_1\omega + c_0 = 0\,, \tag{5.152}
$$

where c_i are functions of thermodynamics quantities, transport coefficients, k and θ. The expressions for c_i in terms of $\mathcal{A}_{ij}$ in are

$$
\begin{aligned}
c_3 &= ik^2 \left(\mathcal{A}_{33} + \mathcal{A}_{44} + A_{22} r_\perp + A_{11} r_\perp \sin^2\theta\right), \\
c_2 &= -\frac{k^2}{T\chi_{11}}\left(s\cos^2\theta + T\chi_{11}\sin\theta(\mathcal{A}_{32}\rho + \mathcal{A}_{13}\mathcal{A}_{31}\sin\theta)\right) \\
&\quad - k^4\Big[\mathcal{A}_{22}\mathcal{A}_{44} r_\perp + \mathcal{A}_{33}(\mathcal{A}_{44} + \mathcal{A}_{22} r_\perp) - \mathcal{A}_{34}\eta_\perp^2 \\
&\qquad + r_\perp\left(\mathcal{A}_{11}(\mathcal{A}_{33} + \mathcal{A}_{44}) + r_\perp \sin^2\theta(\mathcal{A}_{11}\mathcal{A}_{22} - \mathcal{A}_{12}\mathcal{A}_{21})\right)\Big], \\
c_1 &= -i\frac{k^4}{T\chi_{11}}\Bigg\{ s(r_\perp\mathcal{A}_{22} + \mathcal{A}_{33})\cos^2\theta - \eta_\perp\cos\theta\sin\theta(sT\mathcal{A}_{31} \\
&\quad + \chi_{11}\mathcal{A}_{13}\mathcal{A}_{34}) + \chi_{11}T\sin\theta\Big[\rho\mathcal{A}_{32}\mathcal{A}_{44} + r_\perp\mathcal{A}_{32}\sin^2\theta(\mathcal{A}_{13}\mathcal{A}_{21} + \rho\mathcal{A}_{11}) \\
&\qquad + \mathcal{A}_{31}\sin\theta\left(\mathcal{A}_{13}\mathcal{A}_{44} + r_\perp\mathcal{A}_{13}\mathcal{A}_{22} + r_\perp\rho\mathcal{A}_{12}\right)\Big]\Bigg\} \\
&\qquad - ir_\perp k^6\Bigg\{\mathcal{A}_{22}(\mathcal{A}_{33}\mathcal{A}_{44} - \mathcal{A}_{34}\eta_\perp^2) + \sin^2\theta\Big[- r_\perp\mathcal{A}_{12}\mathcal{A}_{21}(\mathcal{A}_{33} + \mathcal{A}_{44}) \\
&\qquad + \mathcal{A}_{11}\sin^2\theta\left(\mathcal{A}_{33}\mathcal{A}_{44} + r_\perp\mathcal{A}_{22}\mathcal{A}_{33} + r_\perp\mathcal{A}_{22}\mathcal{A}_{44} - \eta_\perp^2\mathcal{A}_{34}\right)\Big]\Bigg\}, \\
c_0 &= \left(\frac{s\rho\mathcal{A}_{23}\cos^2\theta\sin^2\theta}{T\chi_{11}}\right)k^4 + \frac{r_\perp k^6}{T\chi_{11}}\Bigg\{ s\mathcal{A}_{22}\mathcal{A}_{33}\cos^2\theta \\
&\quad + \chi_{11}\mathcal{A}_{32}\mathcal{A}_{44}(\mathcal{A}_{13}\mathcal{A}_{21} + \rho\mathcal{A}_{11})\sin^3\theta + \chi_{11}\mathcal{A}_{13}\mathcal{A}_{31}\mathcal{A}_{44}(\mathcal{A}_{13}\mathcal{A}_{22} + \rho\mathcal{A}_{12}) \\
&\quad + \eta_\perp\cos\theta\sin\theta\Big[sT\mathcal{A}_{22}\mathcal{A}_{31} + \chi_{11}\mathcal{A}_{13}\mathcal{A}_{22}\mathcal{A}_{34} \\
&\qquad + \chi_{11}\rho\mathcal{A}_{12}\mathcal{A}_{34} + sT\mathcal{A}_{21}\mathcal{A}_{32}\sin\theta\Big]\Bigg\} \\
&\quad + r_\perp^2(\mathcal{A}_{12}\mathcal{A}_{21} - \mathcal{A}_{11}\mathcal{A}_{22})(\mathcal{A}_{33}\mathcal{A}_{34}\eta_\perp^2)k^8\sin^2\theta\,.
\end{aligned}
\tag{5.153}
$$

In principle, Eq. (5.152) gives a closed-form solution for the four $\omega(k)$. In practice, the solutions are extremely lengthy so it is more convenient to solve it numerically (our equations of state and transport coefficients are in any case given numerically), or by using various expansions, e.g. small k/T or small $k/\sqrt{\mathcal{B}}$.

More generally, we should study Quantum Field Theory from many points of view because of its many applications (and) in order to understand it better. Hopefully, we can learn how to improve its presentation – reformulate it.

N. Seiberg

6
Conclusion and outlook

In this thesis, three different classes of strongly interacting quantum field theories with holographic duals, classified by their global symmetries, have been studied. This work has focused on the transport coefficients which govern the low energy excitations of these systems. The results in this thesis contribute to three broad aspects of strongly interacting quantum field theory. Let me point out what they are, what have we learned about them in the course of this thesis and outline possible future research directions.

Possible universal relations of the transport coefficients

Two types of possible universal relations are investigated in chapter 3 and 4 of this thesis. The first relation is the bound on the ratio of the shear viscosity to the entropy (the KSS bound), which is widely believed to hold for a large class of systems. In a theory without translational symmetry, however, it was observed that the supposed "shear viscosity" extracted from the 2-point correlation function $\langle T^{xy}T^{xy}\rangle$ violates this bound. In chapter 3, we show that this quantity is not a shear viscosity due to the following reason. The holographic fluid which breaks translational symmetry is not a conventional relativistic fluid but a *forced fluid*, due to the presence of scalar fields which break translational symmetry. Thus, the constitutive relation and, consequently, the Kubo formula are modified. With the new constitutive relation, we can single out the *true shear viscosity* and show that it still violates the viscosity bound even at very weak momentum relaxation rate.

The second type of universal relations are relations between the anomalous conductivities and the anomaly coefficients. These are considered in chapter 4. We develop an independent proof for a large class of strongly interacting quantum field theories that the relations between the two quantities are exact, without relying on specific details of the Lagrangian. While, there is much evidences suggesting that the relations between anomalous conductivities and anomaly coefficients are exact, including a more conventional field theoretic proof, it is reassuring to find independent demonstration of the non-renomalisation nature of anomalous transports.

The result of chapter 3 is a clear demonstration that the relation between the shear viscosity and the entropy density is not a universal relation in theories without translational symmetry, such as those in condensed matter. It is an interesting question to investigate whether the KSS bound originates from something more fundamental, such as the quantum bound on the rate of entropy production which can still be applied in this scenario. There is a good amount of more recent works such as [42, 314–319] which explored this fundamental aspect and also its experimentally measurable consequences. In this regard, the anomalous transport is a promising area of research, strengthened by recent experimental realisations [320, 321]: the first direct measurements of anomaly coefficients! Since the materials in these experiments are described by weakly coupled QFT, it would be fascinating to see the same effect persist in strongly coupled systems to confirm the non-renormalisation nature of the anomalous conductivities and also to explore the consequences of quantum anomalies in a far-from-equilibrium behaviour.

Decoding hydrodynamic information from gravity

The results of chapter 3-5 can be obtained due to the fact that the transport coefficients are encoded in a theory of gravity. The most studied quantity is, of course, the shear viscosity. In many examples, it can be thought of as the absorption cross section of the low-energy graviton by the black hole. Moreover, it is encoded in the fluid description of the stretch horizon, thanks to the existence of radially conserved currents.

In chapters 4 and 5, we find new radially conserved currents associated to anomalous conductivities and MHD transport coefficients. Although, not all transport coefficients can be extracted from this method (namely the shear viscosity of the forced fluid in chapter 3 and the transverse resistivity and bulk viscosities in chapter 5), we present a simple way to extract these quantities despite the fact that they depend strongly on the bulk of AdS space. These formulae will be beneficial to future research as they greatly simplify the computation of the transport coefficients, thus allowing us to easily explore the landscape of strongly interacting quantum field theories.

From their relation to the radially conserved currents, we can categorise the transport coefficients into two kinds: those that only require the radially conserved currents and those that are not. With this information and the assumption that the radial coordinate of AdS corresponds to the energy scale, it is tempting to conjecture that the second kind of transport coefficients are controlled by specific details of renormalisation group flow while the first kind are encoded in some sort of "zero modes" which are independent of renormalisation group flow trajectories. Consequently, the transport coefficients of the first kind only depend on the IR fixed point and this could potentially lead us to new universal relations.

The fact that different parts of spacetime encode different QFT information is very intriguing. By decoding hydrodynamic data from certain regions of the spacetime, it is tempting to assume that every piece of spacetime contains information about quantum field theory. There are many recent works that try to come up with a unified framework to interpret any given region spacetime to the information in quantum field theory (see e.g. [322–326]. The approach we take in this thesis, on the other hand, is much simpler, as we only interested in finding regions of spacetime that encode the transport coefficients. Perhaps further investigation in this direction could give us profound insight regarding the precise field theory interpretation of AdS radial coordinate and the field theory data encoded behind the black hole horizon.

This research direction ties together ideas from theory of gravity, renormalisation group and leads to possibility that the gravity is a phenomenon that emerges from quantum field theory. It is exciting that we are able to witness developments that could revolutionised the way we understand QFT.

New symmetry, new hydrrodynamics and new phenomena

On a less other-worldly note, I would like to emphasise the power of the hydrodynamic constructions, particularly in chapter 5. We are able to construct unique effective theories, characterised purely by equations of state and a few transport coefficients, only by considering the global symmetry. This type of effective theory is applicable not only to low-energy dynamics of quantum theories but also to classical theories with the same global symmetries. Thus, the classification of modified hydrodynamics is beneficial not only to the realm of strongly interacting quantum fluids considered in this thesis, but also to the other areas of physics which involve collective behaviour.

Magnetohydrodynamics, studied in chapter 5, is an effective theory of a strongly interacting matter and a dynamical electromagnetic field with the higher-form symmetry as its guiding principle While the standard MHD formalism, which treats the matter sector and the electromagnetic field separately, is a very successful theory, it is only valid in the regime where the magnetic field is weak. The formalism in chapter 5 does not distinguish between the two sectors. Thus, it allows us to explore the scenario where both the magnetic field and the interaction strength within the matter sector are strong. We focus on a theory where the matter sector is a strongly interacting quantum field theory with a holographic dual and explore the entire range of the magnetic field strength.

By doing this, we find several new qualitative features of waves in MHD as we vary the strength of the magnetic field and the angle between the wave propagation direction and magnetic field lines. Among many observations, we emphasise the role of the dissipative terms in turning the MHD sound modes into new diffusive modes as one varies the propagation direction. This is a phenomena which cannot be observed in an ideal magnetohydrodynamics, which is commonly applied in the phenomenology of plasma. Our results also reveal various qualitative features of MHD waves at strong magnetic field that are vastly different from those at weak field.

The role of holographic duality in this chapter is twofold. First, it can be used to create a state of plasma, which is thermodynamically stable throughout the entire range of the magnetic field strength. Second, the computation of transport coefficients in the gauge/gravity duality framework is much easier compared to the weakly coupled QFT computation. We expect that many quantitative features in chapter 5 also persist in the perturbative regime where the interaction within the matter sector is weak. It is one of our top priorities to confirm this conjecture. This would allow us to make easier contact with experiment (such as nuclear fusion experiment) or astronomical observations of phenomena which are believed to be described by magnetohydrodynamics (see e.g. Fig.6.1).

Last but not least, I would like to point out another promising aspect of global symmetry in hydrodynamics which emerges from attempts to combine dissipative effect with the Wilsonian effective action. It turns out that, as a consequence of fluctuation-dissipation relations, the effective action is invariant under an emergent supersymmetry (see e.g. [329] for early developments and [275, 330] for more recent ones). It would be very interesting to explore consequences of this large global symmetry group on observable effects, holographic duality and strongly interacting quantum field theory.

Let me conclude these adventures in the realm of strongly interacting QFT. (Global) Symmetry is a beautiful concept that guides us through this unknown land. It is a backbone of modern theoretical physics, including everything in this

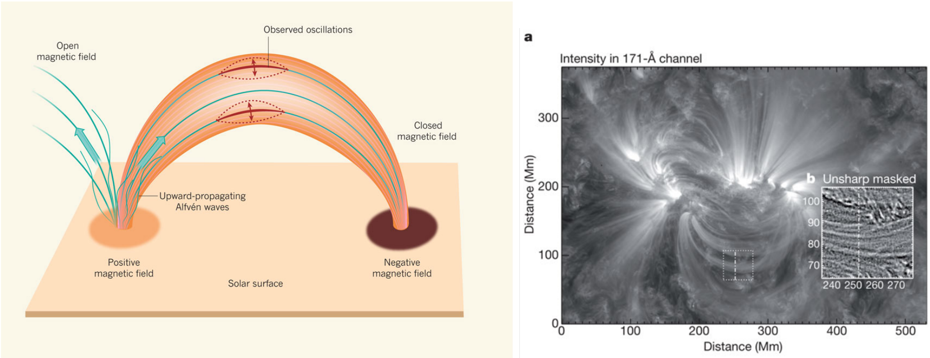

Figure 6.1. One of the MHD modes, the Alfvén wave, can be thought of as an oscillation mode of the magnetic field lines as depicted in (LEFT). It was postulated to be responsible for heat transfer in the solar corona and was recently observed by Hinode spacecraft in [327]. The images are taken from [327, 328]

thesis. Regardless of how we change our view of QFT in the future, I believe that the global symmetry will always play a central role.

Samenvatting

In dit proefschrift wordt de niet-perturbatieve dualiteit tussen de zwaartekracht en ijktheorie gebruikt om de transporteigenschappen van een grote groep sterk-wisselwerkende kwantumsystemen te onderzoeken. Deze methodiek, die ontstaan is vanuit de snaartheorie, stelt in staat om de kwantumveldentheorie te beschrijven via de zwaartekracht. Op deze manier kan een andere parameterruimte worden bestudeerd, die niet noodzakelijkerwijs kan worden onderzocht via conventionele methodes. Het blijkt dat de aangeslagen toestand bij lage energie in dergelijke theorieën kan worden beschreven via een symmetrie-geleide, effectieve theorie: de hydrodynamica.

In hoofdstuk 3 construeren we een consistent hydrodynamisch model waar translatiesymmetrie wordt verbroken. Dit wordt tot stand gebracht via de koppeling van de sterk-wisselwerkende materie met externe, positie-afhankelijke scalaire velden. We definiëren nauwkeurig de schuifviscositeit in dit systeem en onderzoeken de geldigheid van de welbekende ondergrens op de ratio van deze viscositeit en de entropiedichtheid, in de hoop deze te kunnen toepassen op systemen in de fysica van gecondenseerde materie waar translatiesymmetrie is verbroken.

In hoofdstuk 4 onderzoeken we het voorgaande systeem met twee aanvullende, globale U(1)-symmetrieën, wanneer het wordt onderworpen aan een zwak extern, niet-dynamisch magnetisch veld. Een Noether-stroom wordt gebroken door een kwantumanomalie, die aanleiding geeft tot een nieuw transportfenomeen, genaamd anomaal transport. We ontdekken dat de geleidbaarheden die dit transport drijven, worden beschreven op de waarnemingshorizon van het zwarte gat in de zwaartekracht dualiteitstheorie. Via deze vergelijkingen voor de waarnemingshorizon tonen we aan dat - voor alle theorieën met een holografische, duale beschrijving bij eindige temperatuur - de functionele vorm van de anomale geleidbaarheden wordt vastgesteld door de anomalieco-ëfficiënten, ongeacht de waarde van de koppelingsconstante, of de details van het renormalizatieproces.

In hoofdstuk 5 onderzoeken we een sterk-wisselwerkende theorie, gekoppeld met een dynamisch, extern U(1)-ijkveld. Een dergelijke situatie wordt dikwijls aangetroffen in de plasmafysica en magnetohydrodynamica. We gebruiken het nieuwe concept van gegeneraliseerde, globale symmetrieën, welke geassocieerd zijn met snaar-achtige objecten, om de daadwerkelijke globale symmetrie van dit systeem te identificeren. In combinatie met het nieuwe formalisme van de hydrodynamica met deze gegeneraliseerde, globale symmetrie, implementeren we dit nieuwe concept in de ijk/zwaartekracht-dualiteit. Het resultaat is dat we - voor het eerst - de dynamica van een dergelijk sterk-wisselwerkend systeem bij lage energie kunnen onderzoeken, voor elke mogelijke waarde van de magnetische veldsterkte.

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Curriculum Vitæ

Nick Poovuttikul was born in Bangkok, Thailand on May 1990. He stayed in Bangkok until he graduated from high school in 2008. During that time he participated in a few physics competitions and got a scholarship from the Thai government to study physics. At Imperial College London, he obtained a bachelor degree in Physics with Theoretical Physics and a master degree in Quantum Fields and Fundamental Forces. He worked on a condensed matter problem for his final year project with Dr. Derek K.K. Lee and and wrote a literature review on fermions in gauge/gravity duality for his master degree dissertation under the guidance of Prof. Dr. Toby Wiseman. While in London, he was a bassist in a Thai pop rock band at Thai Square Putney; a member of the editorial board for an annual journal of "Samaggi Samagom", the Thai association in the UK, from 2011-2012; and Vice president of Imperial College's Thai society in 2011. He also did a few volunteering jobs for education in Thailand during the summer breaks.

He moved to Leiden, the Netherlands, to start his PhD under the supervision of Prof. Dr. Jan Zaanen in autumn 2013. He publised his work in peer-reviewed journals, volunteered to organise a holography journal club while grading homeworks for master students attending the theory of general relativity course at Leiden. He tried to make friends and promote his work by giving talks and poster presentations and attending several schools, meetings, workshops and conferences in the Netherlands, Brazil, Belgium, Germany, Portugal, Denmark, Spain, Greece, England, Sweden and Thailand.

After his PhD, he will be a postdoctoral researcher in the high energy theoretical physics group at the University of Iceland. Eventually, bound by his so-called "scholarship" contract, he will go back to do research in one of the universities in Thailand.

List of Publications

1. A. Parnachev, N. Poovuttikul, "Topological entanglement entropy, ground state degeneracy and holography," JHEP **1510** (2015) 092 [arXiv:1504.08244 [hep-th]].

2. P. Burikham, N. Poovuttikul, "Shear viscosity in holography and effective theory of transport without translational symmetry," Phys.Rev. D **94** (2016) 106001 [arXiv:1601.04624 [hep-th]].

3. S. Grozdanov, N. Poovuttikul, "Universality of anomalous conductivities in theories with higher-derivative holographic duals," JHEP **1609** (2016) 046 [arXiv:1603.08770 [hep-th]].

4. S. Grozdanov, N. Poovuttikul, "Magnetohydrodynamic waves in a strongly coupled holographic plasma," [arXiv:1707.04182 [hep-th]].

5. T. Andrade, M. Baggioli, A. Krikun, N. Poovuttikul, "Pinning of longitudinal phonons in holographic helical crystals," [arXiv:1708.08306 [hep-th]].

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Stellingen

behorende bij het proefschrift
Transport coefficients and low energy excitations of a strongly interacting holographic fluid

1. In the situation where additional degrees of freedom are added to a hydrodynamical system, the shear viscosity is defined through the constitutive relation, not the Kubo formula derived for a relativisic fluid.

   Chapter 3

2. Before declaring results from holography to be “universal", one has to make sure that these do not rely on the fine-tuning of the equation of state or the coupling constants.

   Chapter 4

3. Magnetohydrodynamics is a theory of dissipative strings, not of charged particles and photons.

   Chapter 5

4. Gauge/gravity duality can be useful for studying macroscopic properties but is useless for extracting microscopic information unless one is interested in supersymmetric quantum field theories.

   Chapter 1

5. Despite its widespread usage, the ideal fluid is unstable: it is highly sensitive to initial conditions. For example, in the simplest model of the Earth’s atmosphere, the deviations grow by a factor of 10^5 in 2 months.

   V.I. Arnold & B.A. Khesin,
   Topological Methods in Hydrodynamics, Springer 1998

6. Hydrodynamical fluctuations, such as long-time tails, cannot be observed in a field theory with a classical gravity dual. Instead, they are intriguingly connected to quantum gravity corrections. This connection has largely been ignored in the literature.

   e.g. P. Kovtun & L. Yaffe, Phys.Rev. D **68** 02 (2003) [arxiv:hep-th/0303010]
   and S. Caron-Hout & O. Saremi, JHEP **11** 013 (2010) [arxiv:0909.4525]

7. From the condensed matter point of view, one of the most interesting aspects of entanglement entropy is topological entanglement entropy. In the holographic literature, a topological term is often mentioned in the introductions as a motivation to study the entanglement entropy but has almost never been the main focus.

   A. Parnachev & A. Pakman JHEP **07** 097 (2008) [arxiv:0805.1891]
   and A. Parnachev & N. Poovuttikul, JHEP **10** 092 (2015) [arxiv:1504.08244]

8. A popular toy model for weak momentum relaxation, where only the conservation of momentum is modified (taking the form $\partial_\mu T^{\mu i} = -\Gamma T^{ti}$), cannot reliably capture dissipative properties. This is because it ignores, by construction, the correction from momentum relaxation to the constitutive relation.

   S.Hartnoll, P.Kovtun, M.Muller & S.Sachdev,
   Phys.Rev. B **76** 144502 (2007) [arXiv:0901.0924]

9. Despite being trained to be logical, scientists are still highly susceptible to arguments from authority.

Nick Poovuttikul
Leiden, 16 November 2017