Split Jacobians and Lower Bounds on Heights

Djukanovic, M.

Citation

Djukanovic, M. (2017, November 1). *Split Jacobians and Lower Bounds on Heights*. Retrieved from https://hdl.handle.net/1887/54944

Author: Djukanovic, M.
Title: Split Jacobians and Lower Bounds on Heights
Issue Date: 2017-11-01

STELLINGEN

behorende bij het proefschrift

Split Jacobians and Lower Bounds on Heights

van Martin Djukanović

Let $n \geq 2$ be an integer and let K be a number field. In the following statements, all varieties and morphisms are defined over K.

1) Let C be a smooth curve of genus two with Jacobian $\operatorname{Jac}(C)$, let E_1 be an elliptic curve, and let $\phi_1 \colon C \to E_1$ be a covering of degree n that is optimal, i.e. a covering that does not factor through an isogeny. Then, possibly after extending K, there exist another elliptic curve E_2, an optimal covering $\phi_2 \colon C \to E_2$ of degree n, and an isogeny $\operatorname{Jac}(C) \to E_1 \times E_2$ whose kernel is $\varepsilon_1(E_1[n]) = \varepsilon_2(E_2[n]) \subset \operatorname{Jac}(C)[n]$, where $\varepsilon_i \colon E_i \hookrightarrow \operatorname{Jac}(C)$ are the embeddings induced by ϕ_i.

2) Let (E_1, O_1) and (E_2, O_2) be elliptic curves and let $\alpha \colon E_1[n] \to E_2[n]$ be an isomorphism (of finite K-group schemes) that is anti-symplectic with respect to the Weil pairing and denote its graph by Γ_α. Let Θ denote the divisor $E_1 \times \{O_2\} + \{O_1\} \times E_2$, that induces a principal polarization on $E_1 \times E_2$. Finally, let $\varphi \colon E_1 \times E_2 \to J$ denote the isogeny such that $\operatorname{Ker}(\varphi) = \Gamma_\alpha$. Then there exists a divisor C on J with arithmetic genus two that induces a principal polarization on J and satisfies $\varphi^*(C) \sim n\Theta$. If C is irreducible then it is a curve of genus two and $J \cong \operatorname{Jac}(C)$. If C is reducible then it is a sum $F_1 + F_2$ of two elliptic curves that meet in a rational 2-torsion point, such that $J \cong F_1 \times F_2$. Moreover, the curves E_1, E_2, F_1, F_2 are all isogenous.

We say that the curves E_1 and E_2 are glued along their n-torsion. If C is irreducible, we say that $\operatorname{Jac}(C)$ is (n, n)-split.

3) With assumptions as in 1), if $n = 3$ and both ϕ_1 and ϕ_2 have a point of ramification index three, then E_1 and E_2 are isomorphic and their modular invariants are either 1728 or $-873722816/59049$.

4) With assumptions as in 2), if E_1 and E_2 are such that the product of their (minimal) discriminants is a square in K or such that they both have a rational point of order two, then they can be glued along their 2-torsion via a K-rational isomorphism $\alpha \colon E_1[2] \to E_2[2]$.

5) With assumptions and notations as in 2), suppose that n is odd. Then the principally polarized abelian surface J is isomorphic to $F_1 \times F_2$ if and only if the divisor $\varphi^*(C)$ contains a (necessarily K-rational) point of $(E_1 \times E_2)[2]$ that is not a point of order two on $E_1 \times \{O_2\}$ or $\{O_1\} \times E_2$. If $n = 3$ and $J \cong F_1 \times F_2$, this point is not (O_1, O_2).

6) With notations as above, the Lang-Silverman conjecture holds for (n, n)-split Jacobians $\mathrm{Jac}(C)$ if and only if it holds for elliptic curves that can be glued along their n-torsion with another elliptic curve to form $\mathrm{Jac}(C)$.

7) Let Tr_∞ denote the archimedean trace and let Δ denote the minimal discriminant. The Lang-Silverman conjecture holds for Jacobians that are (n, n)-isogenous to a product $E_1 \times E_2$ of elliptic curves such that at least one of the following is satisfied for $i = 1, 2$:
    i) $\mathrm{Tr}_\infty(E_i) > \frac{1}{7} \log N_{K/\mathbb{Q}}(\Delta_{E_i})$;
    ii) The Szpiro ratio σ_{E_i} is uniformly bounded.

8) Let $n \geq 2$ be an integer, let S be a finite set of $m \geq 3$ elements, and let $\mathcal{T}(S)$ denote the set of total orderings of S. Suppose that $f\colon \mathcal{T}(S)^n \to \mathcal{T}(S)$ is a function that satisfies:
    i) For all $a, b \in S$ and for all $\mathcal{O} = (O_1, \ldots, O_n) \in \mathcal{T}(S)^n$, if $a < b$ is in $\cap_{i=1}^n O_i$ then $a < b$ is in $f(\mathcal{O})$;
    ii) For all $a, b \in S$, if $a < b$ is in $f(\mathcal{O})$ for some $\mathcal{O} = (O_1, \ldots, O_n)$ then $a < b$ is in $f(\widetilde{\mathcal{O}})$ for all $\widetilde{\mathcal{O}} = (\widetilde{O}_1, \ldots, \widetilde{O}_n)$ such that $\{a < b, b < a\} \cap O_i \cap \widetilde{O}_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$.

   Then f is a projection $(O_1, \ldots, O_n) \mapsto O_i$ for some $i \in \{1, \ldots, n\}$. This statement is known as Arrow's theorem. It is not a statement about all possible functions with codomain $\mathcal{T}(S)$.

9) There exist reasonably well behaved functions $f\colon I^{m \times n} \to \mathcal{T}(S)$, with S, m, n as in 8) and $I = [0, 1]$ or $I = \{0, 1\}$.

10) There are good practical reasons to introduce a notation other than 2π for the real number that is the length of the unit circle. The set of non-horrible choices has measure zero and it contains the symbol π.

11) It is difficult, and perhaps foolish, to get rid of a notation established by Euler.

12) Theorems of the form *"If A then B"* have a very limited practical use if A happens to be false.