Split Jacobians and Lower Bounds on Heights

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Appendix

Computations

This appendix contains the source code for some of the software computations carried out for Chapter 1.

SAGE code that outputs the generic $(2,2)$-case j-invariants (page 23):

```
K.<a,b,c> = Frac(PolynomialRing(QQ,'a,b,c'))
R.<x> = PolynomialRing(K,'x')
S.<y> = PolynomialRing(R,'y')

P = x^3+a*x^2+b*x+c
Q = x^3+(b/c)*x^2+(a/c)*x+1/c

E1 = EllipticCurve([0, P.coefficients()[2], 0,
   P.coefficients()[1],P.coefficients()[0]])
E2 = EllipticCurve([0, Q.coefficients()[2], 0,
   Q.coefficients()[1],Q.coefficients()[0]])

#print the j-invariants
print "j(E1) =",factor(E1.j_invariant()),"\n\n"
print "j(E2) =",factor(E2.j_invariant())
```

SAGE code that outputs the generic $(3,3)$-case j-invariants (page 27):

```
K.<a,b,c,d,e> = Frac(PolynomialRing(QQ,'a,b,c,d,e'))
R.<x> = PolynomialRing(K,'x')
S.<y> = PolynomialRing(R,'y')
L = Frac(PolynomialRing(QQ,'a,b,c'))
L0 = PolynomialRing(QQ,'a,b,c')
M = PolynomialRing(QQ,'a,b,c,d,e')
N = PolynomialRing(L,'d,e',order='lex')

P = x^3+a*x^2+b*x+c
D1 = -2*P + x*P.derivative()
```

```
F1 = S([R(i) for i in
    (x^2*P(y)-y^2*P(x)).quo_rem(x-y)[0].coefficients()])

Res1 = F1.sylvester_matrix(D1(y)).det()
AllmostQ = Res1.quo_rem(D1)[0]
#this polynomial is divisible by Res(P(x),x)=-c
Q = AllmostQ/(-c)

T = (x+d)^2*(x+e)*Q(y)-(y+d)^2*(y+e)*Q(x)
F2 = S([R(i) for i in T.quo_rem(x-y)[0].coefficients()])
D2 = -2*(x+e)*Q - Q*(x+d) + (x+e)*(x+d)*Q.derivative()
Res2 = F2.sylvester_matrix(D2(y)).det()


AllmostP = Res2.quo_rem(D2)[0]
#this polynomial must be divisible by P, i.e.
#the following polynomial is identically zero

RemainderP = AllmostP.quo_rem(P)[1]

Equations = [N(M(RemainderP.coefficients()[0])),
   N(M(RemainderP.coefficients()[1])),
   N(M(RemainderP.coefficients()[2]))]

#the remainder is divisible by Res(Q(x),x+d) and Res(Q(x),x+e)
for i in range(3):
    Equations[i]=Equations[i].quo_rem(N(M(Q(-d)*Q(-e))))[0]

#print the equations
print "C is given by y^2=("+str(P)+")("+str(Q)+")"
print "\nThe two P^1->P^1 maps are"
print "f1: x->x^2/("+str(P)+"),\nf2: x->(x+d)^2*(x+e)/("+str(Q)+")"
print "\nd,e are determined by the following:\n"
for i in range(3):
    print str(i+1)+")",Equations[i],"= 0\n\n"

#print the Groebner basis
I = N.ideal(Equations)
GB = I.groebner_basis()
print "The solution is found by a Groebner basis computation."
print "The lex Groebner basis has",len(GB),"elements:\n"
for i in range(0,len(GB)):
    print "g"+str(i+1)+"=",GB[i],"\n\n"
```

```
#obtain d,e as elements of L
d1 = L(-GB[0]+N(M(d)))
e1 = L(-GB[1]+N(M(e)))
print "Therefore d = "+str(d1)+" and e = "+str(e1)

#the cubic defining E1
U = (x*P(y)-y^2).sylvester_matrix(Q(y)).det()
U = U/U.coefficients()[3]

#the cubic defining E2
V = (x*Q(y)-(y+d1)^2*(y+e1)).sylvester_matrix(P(y)).det()
V = V/V.coefficients()[3]

#print the j-invariants
E1 = EllipticCurve([0, U.coefficients()[2], 0,
   U.coefficients()[1], U.coefficients()[0]])

E2 = EllipticCurve([0, V.coefficients()[2], 0,
   V.coefficients()[1], V.coefficients()[0]])

print "\nThe two curves have modular invariants:\n"
print "j(E1) =",factor(E1.j_invariant()),"\n\n"
print "j(E2) =",factor(E2.j_invariant())
```

SAGE code that outputs (1.35) (page 47):

```
R.<u,v,w,a,b,c,r,s,t> =
    PolynomialRing(QQ,'u,v,w,a,b,c,r,s,t',order='lex')

Iu = R.ideal(a+r+s+t, -b+r*s+r*t+s*t, c+r*s*t,
   -u*(r-s)*(r-t)+2*r-s-t)
Iv = R.ideal(a+r+s+t, -b+r*s+r*t+s*t, c+r*s*t,
   -v*(r-s)*(r-t)-r^2+s^2+t^2-r*s-r*t+s*t)
Iw = R.ideal(a+r+s+t, -b+r*s+r*t+s*t, c+r*s*t,
   -w*(r-s)*(r-t)+r^2*s-r*s^2+r^2*t-r*t^2)

GBu = Iu.groebner_basis('singular:std')._singular_()
Lu = [f.sage_poly(R) for f in GBu.eliminate(prod([s,t]))]
GBv = Iv.groebner_basis('singular:std')._singular_()
Lv = [f.sage_poly(R) for f in GBv.eliminate(prod([s,t]))]
GBw = Iw.groebner_basis('singular:std')._singular_()
Lw = [f.sage_poly(R) for f in GBw.eliminate(prod([s,t]))]
```

```
print "Ideal Iu with s,t eliminated:"
for g in Lu:
    print str(factor(g))

print "\nIdeal Iv with s,t eliminated:"
for g in Lv:
    print str(factor(g))

print "\nIdeal Iw with s,t eliminated:"
for g in Lw:
    print str(factor(g))

print "\nWe solve the following for u,v,w:"
print Lu[2],"= 0"
print Lv[2],"= 0"
print Lw[2],"= 0"
```

SAGE code that outputs (1.36) (page 48):

```
R.<u,v,w,a,b,c,d,r,s,t> =
    PolynomialRing(QQ,'u,v,w,a,b,c,d,r,s,t',order='lex')

Iu = R.ideal(a+r+s+t, -b+r*s+r*t+s*t, c+r*s*t, d-(r-s)*(s-t)*(t-r),
    -u*d+r^2+s^2+t^2-r*s-r*t-s*t )
Iv = R.ideal(a+r+s+t, -b+r*s+r*t+s*t, c+r*s*t, d-(r-s)*(s-t)*(t-r),
    -v*d-r^3-s^3-t^3+r^2*s+r*t^2+s^2*t )
Iw = R.ideal(a+r+s+t, -b+r*s+r*t+s*t, c+r*s*t, d-(r-s)*(s-t)*(t-r),
    -w*d + r^3*t+r*s^3+s*t^3-r^2*t^2-r^2*s^2-s^2*t^2)

GBu = Iu.groebner_basis('singular:std')._singular_()
Lu = [f.sage_poly(R) for f in GBu.eliminate(prod([r,s,t]))]
GBv = Iv.groebner_basis('singular:std')._singular_()
Lv = [f.sage_poly(R) for f in GBv.eliminate(prod([r,s,t]))]
GBw = Iw.groebner_basis('singular:std')._singular_()
Lw = [f.sage_poly(R) for f in GBw.eliminate(prod([r,s,t]))]

print "Ideal Iu with r,s,t eliminated:"
for g in Lu:
    print str(factor(g))

print "\nIdeal Iv with r,s,t eliminated:"
for g in Lv:
    print str(factor(g))
```

```
print "\nIdeal Iw with r,s,t eliminated:"
for g in Lw:
    print str(factor(g))

print "\nWe solve the following for u,v,w:"
print Lu[1],"= 0"
print Lv[1],"= 0"
print Lw[1],"= 0"
```

The following MAGMA codes give the results on page 59.

Remark A.1 The notations λ, μ, and ζ are replaced by a, b, and z, respectively. The points of G and the corresponding translation morphisms on $\mathbb{P}^8$ can be found easily, using formulas (1.40) and (1.41).

```
RR<x> := PolynomialRing(Integers());
L<z> := NumberField(1+x+x^2);
K<a,b> := FunctionField(L, 2);
/* M is the group of translations by points of the graph of the
   3-torsion isomorphism that is given by S->S and T->-T,
   where S = [1 : 0 : -1], T = [-z : 1 : 0] */
M := MatrixGroup <9, K | [
0,0,0,0,1,0,0,0,0,
0,0,0,0,0,1,0,0,0,
0,0,0,1,0,0,0,0,0,
0,0,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,0,1,
0,0,0,0,0,0,1,0,0,
0,1,0,0,0,0,0,0,0,
0,0,1,0,0,0,0,0,0,
1,0,0,0,0,0,0,0,0
],[
1,0,0,0,0,0,0,0,0,
0,z,0,0,0,0,0,0,0,
0,0,z^2,0,0,0,0,0,0,
0,0,0,z^2,0,0,0,0,0,
0,0,0,0,1,0,0,0,0,
0,0,0,0,0,z,0,0,0,
0,0,0,0,0,0,z,0,0,
0,0,0,0,0,0,0,z^2,0,
0,0,0,0,0,0,0,0,1]>;
```

```
R<X1,X2,X3,X4,X5,X6,X7,X8,X9> := PolynomialRing(K,9);
InvariantsOfDegree(M,R,3);
/* invariants of degree < 3 are no longer invariants if we multiply
   the matrices by z or z^2; one could add these matrices to M,
   but the degree 3 invariants are the same either way */
```

We reduce the obtained invariants $P_1, \ldots, P_{21}$ modulo the ideal $I = I(A)$. This can be done by adding $P_i(X_1, \ldots, X_9) - T_i$ to the ideal and computing a Gröbner basis in $K[T_1, \ldots, T_{21}, X_1, \ldots, X_9]$.

```
I := ideal <R |
    X1^3 + X2^3 + X3^3 + 3*b*X1*X2*X3,
    X1^2*X2 + X4^2*X5 + X7^2*X8 + 3*a*X1*X4*X8,
    X1*X2^2 + X4*X5^2 + X7*X8^2 + 3*a*X1*X5*X8,
    X2^3 + X5^3 + X8^3 + 3*a*X2*X5*X8,
    X1^2*X3 + X4^2*X6 + X7^2*X9 + 3*a*X1*X4*X9,
    X1*X2*X3 + X4*X5*X6 + X7*X8*X9 + 3*a*X1*X5*X9,
    X2^2*X3 + X5^2*X6 + X8^2*X9 + 3*a*X2*X5*X9,
    X1*X3^2 + X4*X6^2 + X7*X9^2 + 3*a*X1*X6*X9,
    X2*X3^2 + X5*X6^2 + X8*X9^2 + 3*a*X2*X6*X9,
    X3^3 + X6^3 + X9^3 + 3*a*X3*X6*X9,
    X1^2*X4 + X2^2*X5 + X3^2*X6 + 3*b*X1*X2*X6,
    X1*X4^2 + X2*X5^2 + X3*X6^2 + 3*b*X1*X5*X6,
    X4^3 + X5^3 + X6^3 + 3*b*X4*X5*X6,
    X1^2*X7 + X2^2*X8 + X3^2*X9 + 3*b*X1*X2*X9,
    X1*X4*X7 + X2*X5*X8 + X3*X6*X9 + 3*b*X1*X5*X9,
    X4^2*X7 + X5^2*X8 + X6^2*X9 + 3*b*X4*X5*X9,
    X1*X7^2 + X2*X8^2 + X3*X9^2 + 3*b*X1*X8*X9,
    X4*X7^2 + X5*X8^2 + X6*X9^2 + 3*b*X4*X8*X9,
    X7^3 + X8^3 + X9^3 + 3*b*X7*X8*X9,
    X2*X4 + -1*X1*X5,
    X3*X4 + -1*X1*X6,
    X3*X5 + -1*X2*X6,
    X2*X7 + -1*X1*X8,
    X3*X7 + -1*X1*X9,
    X5*X7 + -1*X4*X8,
    X6*X7 + -1*X4*X9,
    X3*X8 + -1*X2*X9,
    X6*X8 + -1*X5*X9>;
```

This leaves nine linearly independent invariants $F_1, \ldots, F_9$.

```
F1 := X1*X2*X4 + X3*X7*X9 + X5*X6*X8;
F2 := X1*X3*X7 + X2*X4*X5 + X6*X8*X9;
F3 := X2^2*X7 + X3*X5*X6 + X6*X7^2;
F4 := X3^2*X4 + X3*X8^2 + X4*X5*X7;
F5 := X3^2*X8 + X3*X4^2 + X5*X7*X8;
F6 := X3*X5*X7;
F7 := X2*X3*X5 + X2*X7^2 + X6^2*X7;
F8 := 3*a*X2*X5*X8 + X5^3 + -1*X6^3 + 3*b*X7*X8*X9 + 2*X8^3 + X9^3;
F9 := 3*a*X3*X6*X9 + 3*b*X4*X5*X6 + X5^3 + 2*X6^3 + -1*X8^3 + X9^3;
```

We reduce the polynomial

$$P := d_1F_1 + \cdots + d_9F_9 - (c_1X_1 + \cdots + c_9X_9)^3 \in K(c_i, d_j)[X_1, \ldots, X_9]$$

modulo $I = I(A)$ and we eliminate the variables d_i from the ideal generated by the coefficients of P mod I.

```
R2<d1,d2,d3,d4,d5,d6,d7,d8,d9,c1,c2,c3,c4,c5,c6,c7,c8,c9> :=
   PolynomialRing(K,18);
I2 := ideal<R2 |
   3*c1^2*c5 + 6*c1*c2*c4 - d7,
   3*c1^2*c8 + 6*c1*c2*c7,
   // many generators are omitted here
   -3*c2*c3^2 + 3*c8*c9^2,
   c1^3 - c3^3 - c7^3 + c9^3 + d1 + d2
>;
J := EliminationIdeal(I2,9);
```

Finally, the points of $Z(J)$ are found:

```
P8<c1,c2,c3,c4,c5,c6,c7,c8,c9> := ProjectiveSpace(K,8);
X := Scheme(P8, [
   c8*c9^4,
   c8^2*c9^2,
   c7*c9^4,
   c7*c8*c9^2 + -b*c8^3*c9,
   // many generators are omitted here
   c3^2*c7*c9 + -1/2*c3*c5^2*c9 + -1*c3*c5*c6*c8,
   c1*c3*c6 + 1/2*c3^2*c4 + -1/2*c4^2*c8 + -1*c4*c5*c7
]);
Degree(X) eq 9;
Points(X);
```

These solutions define the following nine linear forms:

```
L1 := z^2*X1 + z*X5 + X9;
L2 := z*X1 + z^2*X5 + X9;
L3 := X3 + X4 + X8;
L4 := z^2*X3 + z*X4 + X8;
L5 := z*X3 + z^2*X4 + X8;
L6 := X2 + X6 + X7;
L7 := z^2*X2 + z*X6 + X7;
L8 := z*X2 + z^2*X6 + X7;
L9 := X1 + X5 + X9;
```

We note that L_9 is the one that is fixed by $-\mathbb{1}_A$, so that it defines the divisor D whose image under $A \to A/G$ principally polarizes A/G. We note that D does not contain O. Finally, we check under which conditions D contains points of $A[2]$ that do not correspond to points of order two on E_λ or E_μ:

```
R3<T,X1,X2,X3,X4,X5,X6,X7,X8,X9,a,b> := PolynomialRing(L,12);
I3 := ideal <R3 |
   X5^3 + -9/4*a*b*X5^2*X9 + -3/4*a*X6^2*X9 + -3/4*b*X8^2*X9 +
       -1/4*X9^3,
   X5^2*X6 + 3/2*a*X5^2*X9 + 1/2*X8^2*X9,
   X5*X6^2 + 3/2*a*X5*X6*X9 + 1/2*X8*X9^2,
   X6^3 + 3/2*a*X6^2*X9 + 1/2*X9^3,
   X5^2*X8 + 3/2*b*X5^2*X9 + 1/2*X6^2*X9,
   X5*X8^2 + 3/2*b*X5*X8*X9 + 1/2*X6*X9^2,
   X8^3 + 3/2*b*X8^2*X9 + 1/2*X9^3,
   X6*X8 + -1*X5*X9,
   X1 + -1*X5,
   X2 + -1*X5,
   X3 + -1*X6,
   X4 + -1*X5,
   X7 + -1*X8,
   X9*T-1,
   L9>;
GroebnerBasis(EliminationIdeal(I3,10))[1];
```

The output is a polynomial that defines a curve of genus zero:

```
A<a,b> := AffineSpace(Rationals(),2);
Genus(Curve(A, 3*a^2*b^2 + a^3 - 3*a*b + b^3 + 2));
```

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