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Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

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8 Ekedahl-Oort stratification

The definition of the Ekedahl-Oort stratification for a general Shimura variety of Hodge type is based on the theory of G -zips developed in [PWZ11] and [PWZ15]. We review first the definition and basic properties of G -zips. Notations are as in Section 6.4 and Section 6.5.

8.1 G -zips of a certain type χ

Definition 8.1.1 ([PWZ15, 3.1]). Let T be a scheme over κ .

- (1) A **G -zip of type χ** over T is a quadruple $\mathbb{I} = (I, I_+, I_-, \iota)$ consisting of a right G -torsor I over T , a P_+ -torsor $I_+ \subset I$, and $P_-^{(p)}$ -torsor $I_- \subset I$, and an isomorphism of $M^{(p)}$ -torsors:

$$\iota : I_+^{(p)} / U_+^{(p)} \cong I_- / U_-^{(p)}.$$

- (2) A morphism $\mathbb{I} \rightarrow \mathbb{I}' = (I', I'_+, I'_-, \iota')$ of G -zips of type χ over T consists of an equivariant morphism $I \rightarrow I'$ which sends I_+ to I'_+ and I_- to I'_- , and which is compatible with the isomorphisms ι and ι' .

With the natural notion of pullback, the G -zips of type χ , as T varies, form a category fibred in groupoids over the category of κ -schemes. We denote this fibred category by $G\text{-Zip}^\chi$. It is shown in [PWZ15, Proposition 3.1] that $G\text{-Zip}^\chi$ is a stack over κ .

The stack $G\text{-Zip}^\chi$ has an interpretation as an algebraic stack as below. To G and χ one can also associate a natural **algebraic zip datum** which we shall not specify (see [PWZ15, Definition]), and to such an algebraic datum there is an associated **zip group** $E_\chi := E_{G, \chi}$, given on points of a κ -scheme T by

$$E_\chi(T) := \{(mu_+, m^{(p)}u_-) \mid m \in M(T), u_+ \in U_+(T), u_- \in U_-^{(p)}(T)\} \subset P_+(T) \times P_-^{(p)}(T).$$

The group E_χ is a linear algebraic group over κ and it has a left action on G_κ by $(p_+, p_-) \cdot g := p_+ g p_-^{-1}$. With respect to this action one can form the quotient stack $[E_\chi \backslash G_\kappa]$.

Theorem 8.1.2 ([PWZ15, Proposition 3.11, Corollary 3.12]). The stacks $G\text{-Zip}^\chi$ and $[E_\chi \backslash G_\kappa]$ are naturally equivalent. They are smooth algebraic stacks of dimension 0 over κ . In particular, the isomorphism class of G -zips over any algebraically closed field extension k of κ are in bijection with the $E_\chi(k)$ -orbits in $G(k)$.

Remark 8.1.3. This realizes $G\text{-Zip}^\chi$ as an algebraic quotient stacks. While the language of G -zips is more “motivic” (see [Zha13, Chapter 0, 0.3]), the realization as a quotient stack will help to give a combinatory description of the topological space of $G\text{-Zip}^\chi$ (see the next section).

8.2 Combinatory description of the topological space $G\text{-Zip}^\chi$

Let k be an algebraically closed field extension of κ . We shall see that the topological space of $[E_\chi \setminus G_\kappa]$ is identified with a subset JW of the Weyl group W , which will later play the part of “index set” of the Ekedahl-Oort stratification.

As a preparation for what to follow, we let $T \subset B$ be a maximal torus, respectively a Borel subgroup of G_k . Let $W = N_G T(k)/T(k)$ be the finite Weyl group of G_k with respect to T_k , and let $I := I(B, T) \subset W$ be the set of simple reflections associated to the pair (B, T) . As the pair is unique up to a conjugation by $G(k)$, the Coxeter system (W, I) is, up to unique isomorphism, independent of the choice of the pair (B, T) (see [PWZ11, Section 2.3] for a detailed explanation). Moreover, for any field extension $k \rightarrow k'$, the canonical map

$$(W(G_k, B, T), I(G_k, B, T)) \rightarrow (W(G_{k'}, B_{k'}, T_{k'}), I(G_{k'}, B_{k'}, T_{k'}))$$

is an isomorphism.

Recall that the **length** of an element $w \in W$ is the smallest number $l(w)$ such that w can be written as a product of $l(w)$ simple reflections. By definition of the **Bruhat order** $\leq$ on W , we have $w' \leq w$ if and only if for some (equivalently, for any) reduced expression of w as a product of simple reflections, by leaving out certain factors one can get an expression of w' .

For any subset $J \subset I$ we denote by W_J the subgroup of W generated by J . For any $w \in W$, there is a unique element in the left coset $W_J w$ (resp. right coset $w W_J$) which is of minimal length; if $K \subset I$ is another subset, then there is a unique element in the double coset $W_J w W_K$ which is of minimal length. We denote by JW (resp. W^K , resp. ${}^JW^K$) the subset of W consisting of $w \in W$ which are of minimal length in $W_J w$ (resp. $w W_K$, resp. $W_J w W_K$). Then we have ${}^JW^K = {}^JW \cap W^K$ and JW (resp. W^K , resp. ${}^JW^K$) is a set of representatives of $W_J \backslash W$ (resp. W/W^K , resp. $W_J \backslash W/W_K$). Denote by $w_0, w_{0,J}$ the unique element of maximal length in W and in W_J respectively, and let $x_J := w_0 w_{0,J}$. For any $w \in W$, we denote by $\dot{w}$ a lift of w in $N_G T(k)$.

Definition 8.2.1. For any two elements $w, w' \in {}^JW$, we write $w' \preceq w$ if there exists a $y \in W_J$, such that $y^{-1} w' \sigma(x_J y x_J^{-1}) \leq w$.

The relation $\preceq$ defines a partial order on JW ([PWZ11, Corollary 6.3]). This partial order “ $\preceq$ ” defined above induces a unique topology on JW .

For any standard parabolic subgroup P of G_k (the word “standard” means P contains B), there is an associated subset $J := J(P) \subset I$, defined as $J = \{g \in I \mid gT(k) \subset P(k)\}$, called the **type** of P . If we view I as the set of simple roots of G with respect to the pair (B, T) , then the type J of P is simply the set of simple roots whose inverse are roots of P . It is a basic fact that two parabolic

subgroups of G_k are conjugate if and only if they have the same type. For a cocharacter $\lambda : \mathbb{G}_{m,k} \rightarrow G_k$ over k , the type of λ is defined to be the type of the parabolic subgroup $P = P_+(\lambda)$ (see Section 6.5 for the formation of $P_+(\lambda)$). It only depends on the $G(k)$ -conjugacy class of λ .

We let $J := J(\chi_k)$ be the type of the cocharacter $\chi_k : \mathbb{G}_{m,k} \rightarrow G_k$. Recall that the finite set $E_\chi(k) \backslash G(k)$ as orbit spaces is naturally equipped with the quotient topology of $G(k)$. Following [PWZ15, Section 3.5] (see also [Wor13, Section 6.4] for a clear statement) one can define a map

$$\pi : {}^J W \longrightarrow E_\chi(k) \backslash G(k) \cong [E_\chi \backslash G_\kappa]_k, \quad w \longmapsto E_\chi(k) \cdot \dot{w} \dot{w}_0 \sigma(\dot{w}_0, J). \quad (8.2.1)$$

Lemma 8.2.2. The map π in (8.2.1) is a bijection preserving partial order relations and hence is a homeomorphism of topological spaces. Hence there is a homeomorphism between ${}^J W$ and $|G\text{-Zip}_k^\chi|$.

8.3 Definition of Ekedahl-Oort stratification

Recall that to give a morphism of stacks over κ from S to $G\text{-Zip}^\chi$ is equivalent to give an object in $G\text{-Zip}^\chi(S)$, namely to give a G -zip of type χ over S . C. Zhang defines a classification map of Ekedahl-Oort strata $\zeta : S \rightarrow G\text{-Zip}^\chi$ by constructing a universal G -zip over S , from which he gives the definition of Ekedahl-Oort stratification of S by taking geometric fibres of ζ (see Definition 8.3.1 below). Wortmann gives in [Wor13] a slightly different construction (replacing the $\mathbb{Z}_p$ lattice Λ by its dual Λ^*). We recall Wortmann's construction in the following.

Recall that the de Rham cohomology $\mathcal{V}_0$ comes with a Hodge filtration $\mathcal{C}_0$. Denote by $\mathcal{D}_0 \subset \mathcal{V}_0$ the conjugate filtration of $\mathcal{V}_0$. Recall that we have fixed an embedding (cf. Section 6.4)

$$\iota : G \hookrightarrow \mathrm{GL}(\Lambda_{\mathbb{F}_p}) \cong \mathrm{GL}(\Lambda_{\mathbb{F}_p}^*).$$

The cocharacters $()^\vee \circ \chi$ and $()^\vee \circ \chi^{(p)}$ induce $\mathbb{Z}$ -gradings (cf. Section 6.5)

$$\Lambda_\kappa^* = (\Lambda_\kappa^*)_\chi^0 \oplus (\Lambda_\kappa^*)_\chi^1, \quad \Lambda_\kappa^* = (\Lambda_\kappa^*)_{\chi^{(p)}}^0 \oplus (\Lambda_\kappa^*)_{\chi^{(p)}}^1.$$

From this we get a descending and an ascending filtration

$$\begin{aligned} \mathrm{Fil}_\chi^0 &:= \Lambda_\kappa^* \supset \mathrm{Fil}_\chi^1 := (\Lambda_\kappa^*)_\chi^1 \supset \mathrm{Fil}_\chi^2 := 0, \\ \mathrm{Fil}_{-1}^{\chi^{(p)}} &:= 0 \subset \mathrm{Fil}_0^{\chi^{(p)}} := (\Lambda_\kappa^*)^0 \subset \mathrm{Fil}_1^{\chi^{(p)}} := \Lambda_\kappa^*. \end{aligned}$$

Then P_+ (resp. $P_-^{(p)}$) is the stabilizer of $\mathrm{Fil}_\chi^\bullet$ (resp. $\mathrm{Fil}_{\bullet}^{\chi^{(p)}}$) in G_κ . Denote by $\bar{s}_{\mathrm{dR}} \subset \bar{\mathcal{V}}^\otimes$ the reduction of the set of tensors $s_{\mathrm{dR}} \subset \mathcal{V}^\otimes$, and by $\bar{s}$ the base

change to $(\Lambda_\chi^*)^\otimes$ of $s \subset (\Lambda^*)^\otimes$. Now we define

$$\begin{aligned} \mathbf{I} &:= \mathrm{Isom}_S([\Lambda_\kappa^*, \bar{s}] \otimes \mathcal{O}_S, [\mathcal{V}_0, \bar{s}_{\mathrm{dR}}]), \\ \mathbf{I}_+ &:= \mathrm{Isom}_S([\Lambda_\kappa^*, \bar{s}, \mathrm{Fil}_\chi^\bullet] \otimes \mathcal{O}_S, [\mathcal{V}_0, \bar{s}_{\mathrm{dR}}, \mathcal{V}_0 \supset \mathcal{C}_0]), \\ \mathbf{I}_- &:= \mathrm{Isom}_S([\Lambda_\kappa^*, \bar{s}, \mathrm{Fil}_\chi^{\chi^{(p)}}] \otimes \mathcal{O}_S, [\mathcal{V}, \bar{s}_{\mathrm{dR}}, \mathcal{D}_0 \subset \mathcal{V}_0]), \end{aligned}$$

where $\mathbf{I}$ and $\mathbf{I}_+$ were defined already in Section 7.2 and here we just repeat the definitions for the convenience of readers. The group G_κ acts on $\mathbf{I}$ from the right by $t \cdot g := t \circ g^\vee$ for any S -scheme T and any sections $g \in G(T)$ and $t \in \mathbf{I}(T)$. This action induces the action of P_+ (resp. $P_-^{(p)}$) on $\mathbf{I}_+$ (resp. $\mathbf{I}_-$). The Cartier isomorphism on $\mathcal{V}_0$ induces an isomorphism $\iota : \mathbf{I}_+^{(p)}/U_+^{(p)} \cong \mathbf{I}_-/U_-^{(p)}$ and it is shown in [Zha13, Theorem 2.4.1] (see [Wor13, 5.14] for necessary changes of the proof) that $\underline{\mathbf{I}} := (\mathbf{I}, \mathbf{I}_+, \mathbf{I}_-, \iota)$ is a G -zip of type χ over S , which induces a morphism of stacks

$$\zeta : S \longrightarrow G\text{-Zip}^\chi \cong [E_\chi \backslash G_\kappa].$$

Moreover, Zhang shows that the morphism ζ is a smooth morphism of stacks over κ ([Zha13, Theorem 3.1.2]).

Definition 8.3.1. For any geometric point $w : \mathrm{Spec} k \rightarrow G\text{-Zip}^\chi$, the **Ekedahl-Oort stratum** of S_k associated to w , denoted by S_k^w , is defined to be the fibre of w under the classifying morphism $\zeta_k : S_k \rightarrow [E_\chi \backslash G_\kappa]$.

The **Ekedahl-Oort stratification** of S_k is by definition the disjoint union

$$S_k = \bigsqcup_{w \in {}^JW} S_k^w \tag{8.3.1}$$

by taking the fibre of the stratification

$$[E_\chi \backslash G_\kappa]_k = \bigsqcup_{w \in {}^JW} [E_\chi \backslash O^w],$$

where each O^w is the E_χ -orbit in S_k corresponding to $w \in W$ (cf. Lemma 8.2.2), and where each S_k^w is the fibre along ζ_k of $[E_\chi \backslash O^w]$. Each S_k^w is a locally closed subscheme of S_k .

Remark 8.3.2. As remarked in [Wor13, Remark 5.16] (the proof is actually already given there), though the definition of ζ depends on the choice of a cocharacter $\chi \in [\chi]_\kappa$, the resulting map $\zeta(k) : S(k) \rightarrow {}^JW$ is in fact independent of the choice of χ .

Remark 8.3.3. We list here some properties of Ekedahl-Oort stratification following [Wor13] and [KW14].

- (1) Each Ekedahl-Oort stratum S_k^w is smooth and hence reduced, since the classifying map $\zeta : S \rightarrow [E_\chi \backslash G_\kappa]$ is smooth.

- (2) Each stratum S_k^w is either empty or equidimensional of dimension $l(w)$ ([Zha13, Proposition 3.1.6]).
- (3) The closure relationship between Ekedahl-Oort strata is given by

$$\overline{S_k^w} = \bigcup_{w' \preceq w} S_k^{w'},$$

where $\preceq$ is the Bruhat order ([Zha13, Proposition 3.1.6]). This follows from the topological structure of JW and the fact that taking inverse under $|\zeta| : |S_k| \rightarrow |[E_\chi \backslash G_\kappa]_k|$ commutes with taking closure as the map $\zeta : S \rightarrow [E_\chi \backslash G_\kappa]$ is smooth and hence is open.

- (4) The set JW contains, with respect to the partial order $\preceq$, a unique maximal element $w_{\max} := w_{0,J}w_0$, and a unique minimal element $w_{\min} := 1$. By (3), the associated stratum $S_k^{w_{\max}}$ is the unique open Ekedahl-Oort stratum and is dense in S_k , and $S_k^{w_{\min}}$ is closed and contained in the closure of each stratum S_k^w .
- (5) For Shimura variety of PEL type each Ekedahl-Oort stratum is nonempty ([VW13, Theorem 10.1]). For general Hodge type Shimura varieties, the non-emptiness is claimed by Chia-Fu Yu (the preprint is not available yet).

8.4 Comparison of $\mathcal{D}_1(k)/\mathcal{K}^\diamond(k)$ and $E_\chi(k) \backslash G_\kappa(k)$

We let k be a perfect field extension of κ . Denote by $[[g]]$ the $\mathcal{K}^+(k)$ -orbit of g ; that is, we have

$$\begin{aligned} [[g]] &= \{\mathcal{K}_1(k)h^{-1}g\varphi(h)\mathcal{K}_1(k) | h \in \mathcal{K}(k)\} \\ &= \{\mathcal{K}_1(k)h^{-1}g\sigma(h)\mathcal{K}_1(k) | h \in \mathcal{K}(k)\}, \end{aligned}$$

where the second “=” is due to Lemma 3.3.2.

Recall that the functor $\mathcal{C}(G, \mu(u))$, following the notations of Section 3.2, is the subfunctor of $\mathcal{L}G$ sending a κ -algebra R to the set $\mathcal{K}(R)\mu(u)\mathcal{K}(R)$. For simplicity of notations, we write

$$\mathcal{C} := \mathcal{C}(G, \mu(u)).$$

We want to define such a map of sets

$$\begin{aligned} \omega : \mathcal{C}(k) &\longrightarrow E_\chi(k) \backslash G(k) \\ h_1\mu(u)h_2 &\longmapsto E_\chi \cdot (\sigma^{-1}(\bar{h}_2)\bar{h}_1), \end{aligned} \tag{8.4.1}$$

where for an element $h \in \mathcal{K}(k)$, we write $\bar{h} = h \bmod u$.

The following proposition is an equicharacteristic analogue of [Wor13, Proposition 6.7]. We follow his strategy but use a slightly different method to prove Proposition 8.4.1. (1).

Proposition 8.4.1. For any perfect field extension k of κ , we have the following:

- (1) The map ω in (8.4.1) is a well-defined map.
- (2) For any $g_1, g_2 \in \mathcal{C}(k)$, we have $\rho(g_1) = \rho(g_2)$ if and only if $[[g_1]] = [[g_2]]$; in other words, the map ω induces a bijection

$$\omega : \mathcal{C}(k)/\mathcal{K}^+(k) = \mathcal{D}_1(k)/\mathcal{K}^\diamond(k) \rightarrow E_\chi(k) \backslash G(k). \quad (8.4.2)$$

- (3) We have the following commutative diagram

$$\begin{array}{ccc} & & \mathcal{D}_1(k)/\mathcal{K}^\diamond(k) \\ & \nearrow \eta(k) & \downarrow \omega \\ S(k) & & \\ & \searrow \zeta(k) & \\ & & E_\chi(k) \backslash G(k) \end{array} \quad (8.4.3)$$

Proof. (1) To show ρ is well defined, it is enough to show that the orbit $E_\chi \cdot (\sigma^{-1}(\bar{h}_2)\bar{h}_1)$ is independent of the choice of h_1 and h_2 . Suppose that

$$h_1\mu(u)h_2 = h'_1\mu(u)h'_2, \quad h_1, h_2, h'_1, h'_2 \in \mathcal{K}(k).$$

Write $c_1 = h_1^{-1}h'_1$ and $c_2 = \sigma^{-1}(h_2h_2'^{-1})$, then we have

$$c_2(\sigma^{-1}(h'_2)h'_1)c_1^{-1} = \sigma^{-1}(h_2)h_1, \quad c_2 = y^{-1}\sigma^{-1}(c_1)y, \quad (8.4.4)$$

where we set $y := \sigma^{-1}(\mu(u)) = \chi(u^p)$ (the latter equality here follows from Lemma 3.3.1). The first part of (8.4.4) implies that we need only to show that $(\bar{c}_2, \bar{c}_1) \in E_\chi(k)$. But since E_χ is a subscheme of $G \times_\kappa G$ and the formation of $P_\pm, U_\pm$ and M commute with base change of χ , it suffices to show that étale locally the point $(\bar{c}_2, \bar{c}_1) \in (G \times_\kappa G)(k)$ lies in $E_\chi(k)$. Hence by passing to a finite (separable) extension of κ we may suppose that the cocharacter χ factors through a split maximal torus T of G . Then a similar discussion as in the beginning of Section 7.5 implies the following inclusions of schemes (cf. (7.5.3))

$$y\mathcal{L}^+U_+y^{-1} \subset \mathcal{K}_1, \quad y^{-1}\mathcal{L}^+U_-y \subset \mathcal{K}_1. \quad (8.4.5)$$

By [DG, Exposé XXVI] Theorem 5.1, as a scheme over κ , G_κ is a union of open subschemes $s \cdot P_-P_+ = s \cdot U_-U_+M$, where s runs over $s \in U_+(k)$.

In particular $c_2 \in \mathcal{L}^+G(k) = G(k[[u]])$ as a $k[[u]]$ -point of G must lie in some $s \cdot U_- U_+ M$. And hence c_2 is of the form $c_2 = u_1 v u_2 m$, with

$$u_1 \in U_+(k) \subset \mathcal{L}^+U_+(k), \quad u_2 \in \mathcal{L}^+U_+(k), \quad v \in \mathcal{L}^+U_-(k), \quad m \in \mathcal{L}^+M(k).$$

Given such a decomposition, by the second part of (8.4.4) we have

$$y u_1 v u_2 y^{-1} m = y u_1 v u_2 m y^{-1} = \sigma^{-1}(c_1) \in \mathcal{L}^+G(k) \quad (8.4.6)$$

By the first part of (8.4.5), $y u_1 y^{-1}, y u_2 y^{-1}$ lies in $\mathcal{K}_1(k)$, and hence we have $y v y^{-1} \in \mathcal{L}^+G(k)$, which implies that the element $y v y^{-1} \in \mathcal{L}U_-(k)$ actually lies in $\mathcal{L}^+U_-(k)$ since U_- is a closed subscheme of G . Then we have $v \in \mathcal{K}_1(k)$ by the second part of (8.4.5). Now it is clear that $\bar{c}_2 = \bar{u}_1 \bar{u}_2 \bar{m}$ lies in $P_+(k)$ with Levi component $\bar{m} \in M(k)$ and

$$\bar{c}_1 = \sigma(\overline{y v y^{-1}}) \sigma(m)$$

lies in $\sigma(U_-(k)M(k)) = P_-^{(p)}(k)$, with Levi component $\sigma(m)$.

- (2) The “only if” part is clear from (1). For the “if” part, suppose $\rho(g_1) = \rho(g_2)$ and we write $x = \mu(u)$. Thanks to (1) and Lemma 3.3.2 we may assume $g_1 = h_1 x$ and $g_2 = h_2 x$. Choose $p_+ = u_+ m$, $p_- = u_- \sigma(m)$ with $m \in M(k)$, $u_+ \in U_+(k)$, $u_- \in U_-^{(p)}(k)$ such that $\bar{h}_2 = p_+ \bar{h}_1 p_-^{-1}$, i.e., $\mathcal{K}_1(k) h_2 = \mathcal{K}_1(k) p_+ h_1 p_-^{-1}$. We have seen from (8.4.5) that

$$x^{-1} u_-^{-1} x = \sigma(x_1 \sigma^{-1}(u_-^{-1}) x_1^{-1}) \in \mathcal{K}_1(k), \quad x \sigma(u_+) x^{-1} = \sigma(x_1 u_+ x_1^{-1}) \in \mathcal{K}_1(k),$$

where $x_1 := \chi(u^p)$. Hence, together with the fact that $\sigma(m)^{-1}$ commutes with x and $\mathcal{K}_1 \subset \mathcal{K}$ is normal, we find

$$\begin{aligned} [[h_2 x]] &= [[u_+ m h_1 \sigma(m)^{-1} u_-^{-1} x]] = [[u_+ m h_1 \sigma(m)^{-1} x]] \\ &= [[m h_1 \sigma(m)^{-1} x \sigma(u_+)]] = [[m h_1 \sigma(m)^{-1} x]] \\ &= [[m h_1 x \sigma(m)^{-1}]] = [[h_1 x]]. \end{aligned}$$

- (3) The commutativity of the diagram (8.4.3) follows from the construction of $\eta : S \rightarrow \mathcal{D}_1/\mathcal{K}^\circ$ and $\zeta : S \rightarrow |E_\chi \setminus G_\kappa|$. □

Remark 8.4.2. Recall that the cocharacter “ μ ” in Wortmann is our $\chi^{(p)}$. Let k be an algebraically closed field extension of κ and let (B, T) be a Borel pair of G_k such that χ_k factors through T_k and is dominant. Since T_k is necessarily split, the action

$$\sigma : X_*(G_k) := \text{Hom}_k(\mathbb{G}_{m,k}, G_k) \rightarrow X_*(G_k)$$

sending $\lambda \in X_*(G_k)$ to $\lambda^{(p)}$ (see (2.1.1)) is trivial, and hence we have

$$\sigma^{-1}(\mu_k) = \mu_k = (\text{Frob}_{G_{\kappa}/\kappa})_k \circ \chi_k = \text{Frob}_{G_k/k} \circ \chi_k = p\chi_k, \quad (8.4.7)$$

where the last equation is easy to see since T_k is split over k . It follows that $\sigma^{-1}(\mu_k)$ and $\sigma^{-1}(\chi_k^{(p)}) = \chi_k$ have the same type J (the type of a cocharacter is insensitive to multiplications). Recall that $\mathcal{D}_1(k)/\mathcal{K}^\circ(k)$ is by definition the $C(G, \mu_k)$ defined in the Introduction. Now by [Vie14, Theorem 1.1], both $C(G, \mu_k)$ and $C(G, \chi_k^{(p)})$ can be identified with the subset JW of W . And hence we can also identify $C(G, \chi_k^{(p)})$ with $C(G, \mu_k)$. This shows that our variation of μ is essentially harmless.