# Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

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# 8 Ekedahl-Oort stratification

The definition of the Ekedahl-Oort stratification for a general Shimura variety of Hodge type is based on the theory of $G$-zips developed in [PWZ11] and [PWZ15]. We review first the definition and basic properties of $G$-zips. Notations are as in Section 6.4 and Section 6.5.

## 8.1 $G$-zips of a certain type $\chi$

**Definition** 8.1.1 ([PWZ15, 3.1]). Let $T$ be a scheme over $\kappa$.

(1) **A $G$-zip of type** $\chi$ over $T$ is a quadruple $\underline{\mathrm{I}} = (\mathrm{I}, \mathrm{I}_+, \mathrm{I}_-, \iota)$ consisting of a right $G$-torsor I over $T$, a $P_+$-torsor $\mathrm{I}_+ \subset \mathrm{I}$, and $P_-^{(p)}$-torsor $\mathrm{I}_- \subset \mathrm{I}$, and an isomorphism of $M^{(p)}$-torsors:
$$\iota : \mathrm{I}_+^{(p)}/U_+^{(p)} \cong \mathrm{I}_-/U_-^{(p)}.$$

(2) A morphism $\underline{\mathrm{I}} \to \underline{\mathrm{I}}' = (\mathrm{I}', \mathrm{I}'_+, \mathrm{I}'_-, \iota')$ of $G$-zips of type $\chi$ over $T$ consists of an equivariant morphism $\mathrm{I} \to \mathrm{I}'$ which sends $\mathrm{I}_+$ to $\mathrm{I}'_+$ and $\mathrm{I}_-$ to $\mathrm{I}'_-$, and which is compatible with the isomorphisms $\iota$ and $\iota'$.

With the natural notion of pullback, the $G$-zips of type $\chi$, as $T$ varies, form a category fibred in groupoids over the category of $\kappa$-schemes. We denote this fibred category by $G\text{-}\mathsf{Zip}^\chi$. It is shown in [PWZ15, Proposition 3.1] that $G\text{-}\mathsf{Zip}^\chi$ is a stack over $\kappa$.

The stack $G\text{-}\mathsf{Zip}^\chi$ has an interpretation as an algebraic stack as below. To $G$ and $\chi$ one can also associate a natural **algebraic zip datum** which we shall not specify (see [PWZ15, Definition ]), and to such an algebraic datum there is an associated **zip group** $E_\chi := E_{G,\chi}$, given on points of a $\kappa$-scheme $T$ by
$$E_\chi(T) := \{(mu_+, m^{(p)}u_-) \big| m \in M(T), u_+ \in U_+(T), u_- \in U_-^{(p)}(T)\} \subset P_+(T) \times P_-^{(p)}(T).$$
The group $E_\chi$ is a linear algebraic group over $\kappa$ and it has a left action on $G_\kappa$ by $(p_+, p_-) \cdot g := p_+ g p_-^{-1}$. With respect to this action one can form the quotient stack $[E_\chi \backslash G_\kappa]$.

**Theorem** 8.1.2 ([PWZ15, Proposition 3.11, Corollary 3.12]). The stacks $G\text{-}\mathsf{Zip}^\chi$ and $[E_\chi \backslash G_\kappa]$ are naturally equivalent. They are smooth algebraic stacks of dimension 0 over $\kappa$. In particular, the isomorphism class of $G$-zips over any algebraically closed field extension $k$ of $\kappa$ are in bijection with the $E_\chi(k)$-orbits in $G(k)$.

**Remark** 8.1.3. This realizes $G\text{-}\mathsf{Zip}^\chi$ as an algebraic quotient stacks. While the language of $G$-zips is more “motivic” (see [Zha13, Chapter 0, 0.3]), the realization as a quotient stack will help to give a combinatory description of the topological space of $G\text{-}\mathsf{Zip}^\chi$ (see the next section).

## 8.2 Combinatory description of the topological space $G$-$\mathsf{Zip}^\chi$

Let $k$ be an algebraically closed field extension of $\kappa$. We shall see that the topological space of $[E_\chi \backslash G_\kappa]$ is identified with a subset ${}^JW$ of the Weyl group $W$, which will later play the part of "index set" of the Ekedahl-Oort stratification.

As a preparation for what to follow, we let $T \subset B$ be a maximal torus, respectively a Borel subgroup of $G_k$. Let $W = N_G T(k)/T(k)$ be the finite Weyl group of $G_k$ with respect to $T_k$, and let $I := I(B,T) \subset W$ be the set of simple reflections associated to the pair $(B,T)$. As the pair is unique up to a conjugation by $G(k)$, the Coxeter system $(W,I)$ is, up to unique isomorphism, independent of the choice of the pair $(B,T)$ (see [PWZ11, Section 2.3] for a detailed explanation). Moreover, for any field extension $k \to k'$, the canonical map

$$\big(W(G_k,B,T), I(G_k,B,T)\big) \to \big(W(G_{k'},B_{k'},T_{k'}), I(G_{k'},B_{k'},T_{k'})\big)$$

is an isomorphism.

Recall that the **length** of an element $w \in W$ is the smallest number $l(w)$ such that $w$ can be written as a product of $l(w)$ simple reflections. By definition of the **Bruhat order** $\leq$ on $W$, we have $w' \leq w$ if and only if for some (equivalently, for any) reduced expression of $w$ as a product of simple reflections, by leaving out certain factors one can get an expression of $w'$.

For any subset $J \subset I$ we denote by $W_J$ the subgroup of $W$ generated by $J$. For any $w \in W$, there is a unique element in the left coset $W_J w$ (resp. right coset $wW_J$) which is of minimal length; if $K \subset I$ is another subset, then there is a unique element in the double coset $W_J w W_K$ which is of minimal length. We denote by ${}^JW$ (resp. $W^K$, resp. ${}^JW^K$) the subset of $W$ consisting of $w \in W$ which are of minimal length in $W_J w$ (resp. $wW_K$, resp. $W_J w W_K$). Then we have ${}^JW^K = {}^JW \cap W^K$ and ${}^JW$ (resp. $W^K$, resp. ${}^JW^K$) is a set of representatives of $W_J \backslash W$ (resp. $W/W^K$, resp. $W_J\backslash W/W_K$). Denote by $w_0, w_{0,J}$ the unique element of maximal length in $W$ and in $W_J$ respectively, and let $x_J := w_0 w_{0,J}$. For any $w \in W$, we denote by $\dot{w}$ a lift of $w$ in $N_G T(k)$.

**Definition** 8.2.1. For any two elements $w, w' \in {}^JW$, we write $w' \preceq w$ if there exists a $y \in W_J$, such that $y^{-1} w' \sigma(x_J y x_J^{-1}) \leq w$.

The relation $\preceq$ defines a partial order on ${}^JW$ ([PWZ11, Corollary 6.3]). This partial order "$\preceq$" defined above induces a unique topology on ${}^JW$.

For any standard parabolic subgroup $P$ of $G_k$ (the word "standard" means $P$ contains $B$), there is an associated subset $J := J(P) \subset I$, defined as $J = \{g \in I \big| gT(k) \subset P(k)\}$, called the **type** of $P$. If we view $I$ as the set of simple roots of $G$ with respect to the pair $(B,T)$, then the type $J$ of $P$ is simply the set of simple roots whose inverse are roots of $P$. It is a basic fact that two parabolic subgroups of $G_k$ are conjugate if and only if they have the same type. For a cocharacter $\lambda : \mathbb{G}_{m,k} \to G_k$ over $k$, the type of $\lambda$ is defined to be the type of the parabolic subgroup $P = P_+(\lambda)$ (see Section 6.5 for the formation of $P_+(\lambda)$). It only depends on the $G(k)$-conjugacy class of $\lambda$.

We let $J := J(\chi_k)$ be the type of the cocharacter $\chi_k : \mathbb{G}_{m,k} \to G_k$. Recall that the finite set $E_\chi(k)\backslash G(k)$ as orbit spaces is naturally equipped with the quotient topology of $G(k)$. Following [PWZ15, Section 3.5] (see also [Wor13, Section 6.4] for a clear statement) one can define a map

$$\pi : {}^J W \longrightarrow E_\chi(k)\backslash G(k) \cong \left|[E_\chi \backslash G_\kappa]_k\right|, \quad w \longmapsto E_\chi(k) \cdot \dot{w}\dot{w}_0\sigma(\dot{w}_{0,J}). \quad (8.2.1)$$

**Lemma** 8.2.2. The map $\pi$ in (8.2.1) is a bijection preserving partial order relations and hence is a homeomorphism of topological spaces. Hence there is a homeomorphism between ${}^J W$ and $\left|G\text{-}\mathsf{Zip}_k^\chi\right|$.

## 8.3 Definition of Ekedahl-Oort stratification

Recall that to give a morphism of stacks over $\kappa$ from $S$ to $G\text{-}\mathsf{Zip}^\chi$ is equivalent to give an object in $G\text{-}\mathsf{Zip}^\chi(S)$, namely to give a $G$-zip of type $\chi$ over $S$. C. Zhang defines a classification map of Ekedahl-Oort strata $\zeta : S \to G\text{-}\mathsf{Zip}^\chi$ by constructing a universal $G$-zip over $S$, from which he gives the definition of Ekedahl-Oort stratification of $S$ by taking geometric fibres of $\zeta$ (see Definition 8.3.1 below). Wortmann gives in [Wor13] a slightly different construction (replacing the $\mathbb{Z}_p$ lattice $\Lambda$ by its dual $\Lambda^*$). We recall Wortmann's construction in the following.

Recall that the de Rham cohomology $\mathcal{V}_0$ comes with a Hodge filtration $\mathcal{C}_0$. Denote by $\mathcal{D}_0 \subset \mathcal{V}_0$ the conjugate filtration of $\mathcal{V}_0$. Recall that we have fixed an embedding (cf. Section 6.4)

$$\iota : G \hookrightarrow \mathrm{GL}(\Lambda_{\mathbb{F}_p}) \cong \mathrm{GL}(\Lambda^*_{\mathbb{F}_p}).$$

The cocharacters $()^\vee \circ \chi$ and $()^\vee \circ \chi^{(p)}$ induce $\mathbb{Z}$-gradings (cf. Section 6.5)

$$\Lambda^*_\kappa = (\Lambda^*_\kappa)^0_\chi \oplus (\Lambda^*_\kappa)^1_\chi, \qquad \Lambda^*_\kappa = (\Lambda^*_\kappa)^0_{\chi^{(p)}} \oplus (\Lambda^*_\kappa)^1_{\chi^{(p)}}.$$

From this we get a descending and an ascending filtration

$$\begin{aligned} &\mathrm{Fil}^0_\chi := \Lambda^*_\kappa \supset \mathrm{Fil}^1_\chi := (\Lambda^*_\kappa)^1_\chi \supset \mathrm{Fil}^2_\chi := 0, \\ &\mathrm{Fil}^{\chi^{(p)}}_{-1} := 0 \subset \mathrm{Fil}^{\chi^{(p)}}_0 := (\Lambda^*_\kappa)^0 \subset \mathrm{Fil}^{\chi^{(p)}}_1 := \Lambda^*_\kappa. \end{aligned}$$

Then $P_+$ (resp. $P_-^{(p)}$) is the stabilizer of $\mathrm{Fil}^\bullet_\chi$ (resp. $\mathrm{Fil}^{\chi^{(p)}}_\bullet$) in $G_\kappa$. Denote by $\bar{s}_{\mathrm{dR}} \subset \bar{\mathcal{V}}^\otimes$ the reduction of the set of tensors $s_{\mathrm{dR}} \subset \mathcal{V}^\otimes$, and by $\bar{s}$ the base change to $(\Lambda^*_\chi)^\otimes$ of $s \subset (\Lambda^*)^\otimes$. Now we define

$$\begin{aligned}
\mathrm{I} &:= \mathrm{Isom}_S\big([\Lambda^*_\kappa, \bar{s}] \otimes O_S, [\mathcal{V}_0, \bar{s}_{\mathrm{dR}}]\big),\\
\mathrm{I}_+ &:= \mathrm{Isom}_S\big([\Lambda^*_\kappa, \bar{s}, \mathrm{Fil}^\bullet_\chi] \otimes O_S, [\mathcal{V}_0, \bar{s}_{\mathrm{dR}}, \mathcal{V}_0 \supset \mathcal{C}_0]\big),\\
\mathrm{I}_- &:= \mathrm{Isom}_S\big([\Lambda^*_\kappa, \bar{s}, \mathrm{Fil}_\bullet^{\chi^{(p)}}] \otimes O_S, [\mathcal{V}, \bar{s}_{\mathrm{dR}}, \mathcal{D}_0 \subset \mathcal{V}_0]\big),
\end{aligned}$$

where I and $\mathrm{I}_+$ were defined already in Section 7.2 and here we just repeat the definitions for the convenience of readers. The group $G_\kappa$ acts on I from the right by $t \cdot g := t \circ g^\vee$ for any $S$-scheme $T$ and any sections $g \in G(T)$ and $t \in \mathrm{I}(T)$. This action induces the action of $P_+$ (resp. $P_-^{(p)}$) on $\mathrm{I}_+$ (resp. $\mathrm{I}_-$). The Cartier isomorphism on $\mathcal{V}_0$ induces an isomorphism $\iota : \mathrm{I}_+^{(p)}/U_+^{(p)} \cong \mathrm{I}_-/U_-^{(p)}$ and it is shown in [Zha13, Theorem 2.4.1] (see [Wor13, 5.14] for necessary changes of the proof) that $\underline{\mathrm{I}} := (\mathrm{I}, \mathrm{I}_+, \mathrm{I}_-, \iota)$ is a $G$-zip of type $\chi$ over $S$, which induces a morphism of stacks

$$\zeta : S \longrightarrow G\text{-}\mathsf{Zip}^\chi \cong [E_\chi \backslash G_\kappa].$$

Moreover, Zhang shows that the morphism $\zeta$ is a smooth morphism of stacks over $\kappa$ ([Zha13, Theorem 3.1.2]).

**Definition** 8.3.1. For any geometric point $w : \mathrm{Spec} k \to G\text{-}\mathsf{Zip}^\chi$, the **Ekedahl-Oort stratum** of $S_k$ associated to $w$, denoted by $S_k^w$, is defined to be the fibre of $w$ under the classifying morphism $\zeta_k : S_k \to [E_\chi \backslash G_\kappa]$.

The **Ekedahl-Oort stratification** of $S_k$ is by definition the disjoint union

$$S_k = \coprod_{w \in {}^J W} S_k^w \tag{8.3.1}$$

by taking the fibre of the stratification

$$[E_\chi \backslash G_\kappa]_k = \coprod_{w \in {}^J W} [E_\chi \backslash O^w],$$

where each $O^w$ is the $E_\chi$-orbit in $S_k$ corresponding to $w \in W$ (cf. Lemma 8.2.2), and where each $S_k^w$ is the fibre along $\zeta_k$ of $[E_\chi \backslash O^w]$. Each $S_k^w$ is a locally closed subscheme of $S_k$.

**Remark** 8.3.2. As remarked in [Wor13, Remark 5.16] (the proof is actually already given there), though the definition of $\zeta$ depends on the choice of a cocharacter $\chi \in [\chi]_\kappa$, the resulting map $\zeta(k) : S(k) \to {}^J W$ is in fact independent of the choice of $\chi$.

**Remark** 8.3.3. We list here some properties of Ekedahl-Oort stratification following [Wor13] and [KW14].

(1) Each Ekedahl-Oort stratum $S_k^w$ is smooth and hence reduced, since the classifying map $\zeta : S \to [E_\chi \backslash G_\kappa]$ is smooth.

(2) Each stratum $S_k^w$ is either empty or equidimensional of dimension $l(w)$ ([Zha13, Proposition 3.1.6]).

(3) The closure relationship between Ekedahl-Oort strata is given by

$$\overline{S_k^w} = \bigcup_{w' \preceq w} S_k^{w'},$$

where $\preceq$ is the Bruhat order ([Zha13, Proposition 3.1.6]). This follows from the topological structure of ${}^J W$ and the fact that taking inverse under $\left|\zeta\right| : |S_k| \to \left|[E_\chi \backslash G_\kappa]_k\right|$ commutes with taking closure as the map $\zeta : S \to [E_\chi \backslash G_\kappa]$ is smooth and hence is open.

(4) The set ${}^J W$ contains, with respect to the partial order $\preceq$, a unique maximal element $w_{\max} := w_{0,J} w_0$, and a unique minimal element $w_{\min} := 1$. By (3), the associated stratum $S_k^{w_{\max}}$ is the unique open Ekedahl-Oort stratum and is dense in $S_k$, and $S_k^{w_{\min}}$ is closed and contained in the closure of each stratum $S_k^w$.

(5) For Shimura variety of PEL type each Ekedahl-Oort stratum is nonempty ([VW13, Theorem 10.1]). For general Hodge type Shimura varieties, the non-emptiness is claimed by Chia-Fu Yu (the preprint is not available yet).

## 8.4 Comparison of $\mathcal{D}_1(k)/\mathcal{K}^\diamond(k)$ and $E_\chi(k)\backslash G_\kappa(k)$

We let $k$ be a perfect field extension of $\kappa$. Denote by $[[g]]$ the $\mathcal{K}^+(k)$-orbit of $g$; that is, we have

$$\begin{aligned}[[g]] &= \{\mathcal{K}_1(k)h^{-1}g\varphi(h)\mathcal{K}_1(k) | h \in \mathcal{K}(k)\} \\ &= \{\mathcal{K}_1(k)h^{-1}g\sigma(h)\mathcal{K}_1(k) | h \in \mathcal{K}(k)\},\end{aligned}$$

where the second "=" is due to Lemma 3.3.2.

Recall that the functor $\mathcal{C}(G, \mu(u))$, following the notations of Section 3.2, is the subfunctor of $\mathcal{L}G$ sending a $\kappa$-algebra $R$ to the set $\mathcal{K}(R)\mu(u)\mathcal{K}(R)$. For simplicity of notations, we write

$$\mathcal{C} := \mathcal{C}(G, \mu(u)).$$

We want to define such a map of sets

$$\begin{array}{rcl} \omega : \mathcal{C}(k) & \longrightarrow & E_\chi(k)\backslash G(k) \\ h_1\mu(u)h_2 & \longmapsto & E_\chi \cdot (\sigma^{-1}(\bar{h}_2)\bar{h}_1), \end{array} \tag{8.4.1}$$

where for an element $h \in \mathcal{K}(k)$, we write $\bar{h} = h \bmod u$.

The following proposition is an equicharacteristic analogue of [Wor13, Proposition 6.7]. We follow his strategy but use a slightly different method to prove Proposition 8.4.1. (1).

**Proposition** 8.4.1. For any perfect field extension $k$ of $\kappa$, we have the following:

(1) The map $\omega$ in (8.4.1) is a well-defined map.

(2) For any $g_1, g_2 \in \mathcal{C}(k)$, we have $\rho(g_1) = \rho(g_2)$ if and only if $[[g_1]] = [[g_2]]$; in other words, the map $\omega$ induces a bijection

$$\omega : \mathcal{C}(k)/\mathcal{K}^+(k) = \mathcal{D}_1(k)/\mathcal{K}^\diamond(k) \to E_\chi(k)\backslash G(k). \tag{8.4.2}$$

(3) We have the following commutative diagram

$$\begin{tikzcd} & \mathcal{D}_1(k)/\mathcal{K}^\diamond(k) \arrow[dd, "\omega"] \\ S(k) \arrow[ur, "\eta(k)"] \arrow[dr, "\zeta(k)"'] & \\ & E_\chi(k)\backslash G(k) \end{tikzcd} \tag{8.4.3}$$

*Proof.* (1) To show $\rho$ is well defined, it is enough to show that the orbit $E_\chi \cdot (\sigma^{-1}(\bar{h}_2)\bar{h}_1)$ is independent of the choice of $h_1$ and $h_2$. Suppose that

$$h_1\mu(u)h_2 = h_1'\mu(u)h_2', \qquad h_1, h_2, h_1', h_2' \in \mathcal{K}(k).$$

Write $c_1 = h_1^{-1}h_1'$ and $c_2 = \sigma^{-1}(h_2 h_2'^{-1})$, then we have

$$c_2(\sigma^{-1}(h_2')h_1')c_1^{-1} = \sigma^{-1}(h_2)h_1, \qquad c_2 = y^{-1}\sigma^{-1}(c_1)y, \tag{8.4.4}$$

where we set $y := \sigma^{-1}(\mu(u)) = \chi(u^p)$ (the latter equality here follows from Lemma 3.3.1). The first part of (8.4.4) implies that we need only to show that $(\bar{c}_2, \bar{c}_1) \in E_\chi(k)$. But since $E_\chi$ is a subscheme of $G \times_\kappa G$ and the formation of $P_\pm, U_\pm$ and $M$ commute with base change of $\chi$, it suffices to show that étale locally the point $(\bar{c}_2, \bar{c}_1) \in (G \times_\kappa G)(k)$ lies in $E_\chi(k)$. Hence by passing to a finite (separable) extension of $\kappa$ we may suppose that the cocharacter $\chi$ factors through a split maximal torus $T$ of $G$. Then a similar discussion as in the beginning of Section 7.5 implies the following inclusions of schemes (cf. (7.5.3))

$$y\mathcal{L}^+U_+y^{-1} \subset \mathcal{K}_1, \qquad y^{-1}\mathcal{L}^+U_-y \subset \mathcal{K}_1. \tag{8.4.5}$$

By [DG, Exposé XXVI] Theorem 5.1, as a scheme over $\kappa$, $G_\kappa$ is a union of open subschemes $s \cdot P_-P_+ = s \cdot U_-U_+M$, where $s$ runs over $s \in U_+(k)$.

In particular $c_2 \in \mathcal{L}^+G(k) = G(k[[u]])$ as a $k[[u]]$-point of $G$ must lie in some $s \cdot U_-U_+M$. And hence $c_2$ is of the form $c_2 = u_1vu_2m$, with

$$u_1 \in U_+(k) \subset \mathcal{L}^+U_+(k),\ u_2 \in \mathcal{L}^+U_+(k),\ v \in \mathcal{L}^+U_-(k),\ m \in \mathcal{L}^+M(k).$$

Given such a decomposition, by the second part of (8.4.4) we have

$$yu_1vu_2y^{-1}m = yu_1vu_2my^{-1} = \sigma^{-1}(c_1) \in \mathcal{L}^+G(k) \tag{8.4.6}$$

By the first part of (8.4.5), $yu_1y^{-1}, yu_2y^{-1}$ lies in $\mathcal{K}_1(k)$, and hence we have $yvy^{-1} \in \mathcal{L}^+G(k)$, which implies that the element $yvy^{-1} \in \mathcal{L}U_-(k)$ actually lies in $\mathcal{L}^+U_-(k)$ since $U_-$ is a closed subscheme of $G$. Then we have $v \in \mathcal{K}_1(k)$ by the second part of (8.4.5). Now it is clear that $\bar{c}_2 = \bar{u}_1\bar{u}_2\bar{m}$ lies in $P_+(k)$ with Levi component $\bar{m} \in M(k)$ and

$$\bar{c}_1 = \sigma(\overline{yvy^{-1}})\sigma(m)$$

lies in $\sigma\big(U_-(k)M(k)\big) = P_-^{(p)}(k)$, with Levi component $\sigma(m)$.

(2) The "only if" part is clear from (1). For the "if" part, suppose $\rho(g_1) = \rho(g_2)$ and we write $x = \mu(u)$. Thanks to (1) and Lemma 3.3.2 we may assume $g_1 = h_1x$ and $g_2 = h_2x$. Choose $p_+ = u_+m$, $p_- = u_-\sigma(m)$ with $m \in M(k)$, $u_+ \in U_+(k)$, $u_- \in U_-^{(p)}(k)$ such that $\bar{h}_2 = p_+\bar{h}_1p_-^{-1}$, i.e., $\mathcal{K}_1(k)h_2 = \mathcal{K}_1(k)p_+h_1p_-^{-1}$. We have seen from (8.4.5) that

$$x^{-1}u_-^{-1}x = \sigma(x_1\sigma^{-1}(u_-^{-1})x_1^{-1}) \in \mathcal{K}_1(k), \quad x\sigma(u_+)x^{-1} = \sigma(x_1u_+x_1^{-1}) \in \mathcal{K}_1(k),$$

where $x_1 := \chi(u^p)$. Hence, together with the fact that $\sigma(m)^{-1}$ commutes with $x$ and $\mathcal{K}_1 \subset \mathcal{K}$ is normal, we find

$$\begin{array}{rcl} [[h_2x]] & = & [[u_+mh_1\sigma(m)^{-1}u_-^{-1}x]] & = & [[u_+mh_1\sigma(m)^{-1}x]] \\ & = & [[mh_1\sigma(m)^{-1}x\sigma(u_+)]] & = & [[mh_1\sigma(m)^{-1}x]] \\ & = & [[mh_1x\sigma(m)^{-1}]] & = & [[h_1x]]. \end{array}$$

(3) The commutativity of the diagram (8.4.3) follows from the construction of $\eta : S \to \mathcal{D}_1/\mathcal{K}^\diamond$ and $\zeta : S \to |E_\chi\backslash G_\kappa|$.

□

**Remark** 8.4.2. Recall that the cocharacter "$\mu$" in Wortmann is our $\chi^{(p)}$. Let $k$ be an algebraically closed field extension of $\kappa$ and let $(B, T)$ be a Borel pair of $G_k$ such that $\chi_k$ factors through $T_k$ and is dominant. Since $T_k$ is necessarily split, the action

$$\sigma : X_*(G_k) := \mathrm{Hom}_k(\mathbb{G}_{m,k}, G_k) \to X_*(G_k)$$

sending $\lambda \in X_*(G_k)$ to $\lambda^{(p)}$ (see (2.1.1)) is trivial, and hence we have

$$\sigma^{-1}(\mu_k) = \mu_k = (\mathrm{Frob}_{G_\kappa/\kappa})_k \circ \chi_k = \mathrm{Frob}_{G_k/k} \circ \chi_k = p\chi_k, \tag{8.4.7}$$

where the last equation is easy to see since $T_k$ is split over $k$. It follows that $\sigma^{-1}(\mu_k)$ and $\sigma^{-1}(\chi_k^{(p)}) = \chi_k$ have the same type $J$ (the type of a cocharacter is insensitive to multiplications). Recall that $\mathcal{D}_1(k)/\mathcal{K}^\diamond(k)$ is by definition the $C(G, \mu_k)$ defined in the Introduction. Now by [Vie14, Theorem 1.1], both $C(G, \mu_k)$ and $C(G, \chi_k^{(p)})$ can be identified with the subset ${}^JW$ of $W$. And hence we can also identify $C(G, \chi_k^{(p)})$ with $C(G, \mu_k)$. This shows that our variation of $\mu$ is essentially harmless.