Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

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7 Construction of a morphism from I_+ to $\mathcal{D}_1$

7.1 Recall of notations

We retain notations from Section 6.4 and Section 6.5. We specify from now on the field k and the group G in Section 3 to be κ and $\mathcal{G}_\kappa$ respectively. Now we can form the ind-scheme $\mathcal{L}G$ over κ and the following infinite dimensional group schemes, all over κ:

$$\mathcal{K} = \mathcal{L}^+G,\ \mathcal{K}_1,\ \mathcal{K}^+,\ \mathcal{K}^\diamond.$$

Also we can form the schemes

$$\mathcal{D} := \mathcal{D}(G, \mu(u)), \quad \mathcal{D}_1 := \mathcal{D}_1(G, \mu(u)),$$

where $\mathcal{D}$ is an infinite dimensional subscheme of $\mathcal{L}G$, $\mathcal{D}_1$ is a smooth scheme of finite type over κ, and $\mathcal{D}$ is a $\mathcal{K}_1$-torsor over $\mathcal{D}_1$.

For any κ-algebra A, we continue to denote by $\varphi : \mathfrak{S}(A) \to \mathfrak{S}(A)$ the absolute Frobenius of $\mathfrak{S}(A)$ while we use $\sigma : \mathfrak{S}(A) \to \mathfrak{S}(A)$ to denote the ring endomorphism of $\mathfrak{S}(A)$, which is the absolute Frobenius on A and sends the variable u to itself. We have seen in Section 3.3.2 that there are two natural Frobenii σ and φ on $\mathcal{L}G$ and on $\mathcal{L}^+G$. Seeing G as a subgroup of $\mathrm{GL}(\Lambda_\kappa)$, here we write down explicitly σ and φ. For any κ-algebra A, and any element $g \in \mathcal{L}^+G(A)$, seeing as an isomorphism $g : \Lambda_{\mathfrak{S}(A)} \cong \Lambda_{\mathfrak{S}(A)}$ preserving tensors $s_{\mathfrak{S}(A)}$, we have

$$\sigma(g) = \varepsilon_1 \circ \sigma^* g \circ \varepsilon_1, \quad \varphi(g) = \varepsilon_2 \circ \varphi^* g \circ \varepsilon_2, \tag{7.1.1}$$

where $\sigma^* g$ is the pull back along $\sigma : \mathfrak{S}(A) \to \mathfrak{S}(A)$ of g and similarly for $\varphi^* g$, and where

$$\varepsilon_1 : \Lambda_{\mathfrak{S}(A)}^{(\sigma)} \cong \Lambda_{\mathfrak{S}(A)}, \quad \varepsilon_2 : \Lambda_{\mathfrak{S}(A)}^{(\varphi)} \cong \Lambda_{\mathfrak{S}(A)}$$

are canonical isomorphisms since Λ is defined over $\mathbb{Z}_p$. We have similar descriptions for those $g \in \mathcal{L}G(A)$.

Note that we use $\sigma, \varphi : G(A) \to G(A)$ interchangeably for the relative (= absolute) Frobenius of $G(A)$.

7.2 Some torsors over $\mathcal{S}$ and S

Write $\mathcal{V} = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{S})$ and $\mathcal{C} \subset \mathcal{V}$ the Hodge filtration of $\mathcal{V}$. Recall that $\mathcal{V}$ comes with a finite set of tensors $s_{\mathrm{dR}} \subset \mathcal{V}^\otimes$. Recall also that we have fixed a cocharacter $\tilde{\chi} : \mathbb{G}_{m,W(\kappa)} \to \mathcal{G}_{W(\kappa)}$ over $W(\kappa)$ in Section 6.5 which is a lift of χ_κ. The cocharacter $()^\vee \circ \tilde{\chi}$ induces a $\mathbb{Z}$-grading of $W(\kappa)$-modules: $\Lambda^*_{W(\kappa)} = (\Lambda^*)^0_{\tilde{\chi}} \oplus (\Lambda^*)^1_{\tilde{\chi}}$. From this grading we obtain a descending filtration

$$\mathrm{Fil}^0_{\tilde{\chi}} := \Lambda^*_{W(\kappa)} \supset \mathrm{Fil}^1_{\tilde{\chi}} := (\Lambda^*)^1_{\tilde{\chi}} \supset \mathrm{Fil}^2_{\tilde{\chi}} := 0. \tag{7.2.1}$$

We can then define two schemes over $\mathcal{S}$ from this filtration,

$$\begin{aligned} \mathbb{I} &:= \mathrm{Isom}_{\mathcal{S}}\big([\Lambda^*_{W(\kappa)}, s_{W(\kappa)}] \otimes O_{\mathcal{S}}, [\mathcal{V}, s_{\mathrm{dR}}]\big), \\ \mathbb{I}_+ &:= \mathrm{Isom}_{\mathcal{S}}\big([\Lambda^*_{W(\kappa)}, s_{W(\kappa)}, \mathrm{Fil}^\bullet_{\tilde{\chi}}] \otimes O_{\mathcal{S}}, [\mathcal{V}, s_{\mathrm{dR}}, \mathcal{V} \supset \mathcal{C}]\big). \end{aligned}$$

Let $\mathcal{P}_+$ be the stabilizer of $\mathrm{Fil}^\bullet_{\tilde{\chi}}$ in $\mathcal{G}$. It is a parabolic subgroup of $\mathcal{G}$ over $W(\kappa)$, and has a similar description as P_+ (cf. Section 6.5).

Lemma 7.2.1. Let A be a p-adic $W(\kappa)$-algebra and write $X = \mathrm{Spec}A$. Then for any morphism of schemes $X \to \mathcal{S}$ over $W(\kappa)$, the fullback $\mathbb{I}_{+,X}$ is a $\mathcal{P}_+$-torsor over X. Moreover, for any closed point x of X, if we denote by $\hat{x}$: $\mathrm{Spec}\widehat{\mathcal{O}}_{X,x} \to \mathcal{S}$ the induced morphism, then the pullback to $\mathrm{Spec}\widehat{\mathcal{O}}_{X,x}$ of $\mathbb{I}_+$, denoted by $\mathbb{I}_{+,\hat{x}}$, is a trivial $\mathcal{P}_+$-torsor over $\mathrm{Spec}\widehat{\mathcal{O}}_{X,x}$.

Proof. Note that if G is a smooth group scheme over a scheme S, then a scheme Y of finite presentation over S is an fppf G-torsor over S if and only if the action of G on Y is free and transitive and Y is faithfully flat over S. Indeed if $Y \to S$ is faithfully flat, then it is an fppf cover of S and the scheme $Y \times_S Y$ is a trivial $G \times_S Y$-torsor over Y since $Y \times_S Y$ has a Y-point given by the diagonal map $Y \to Y \times_S Y$ and the action of $G \times_S Y$ on $Y \times_S Y$ is free and transitive.

For a closed point x of X, we also denote by x : $\mathrm{Spec}\mathcal{O}_{X,x} \to \mathcal{S}$ the induced morphism. By definition of $\mathbb{I}_+$, the action of $\mathcal{P}_+$ on $\mathbb{I}_+$ is free and transitive. Hence due to the discussion above, for the first assertion in the lemma it suffices to show that for each closed point x of X, the pullback $\mathbb{I}_{+,x}$ is a $\mathcal{P}_+$-torsor over $\mathrm{Spec}\mathcal{O}_{X,x}$. Since A is p-adically complete, the image in $\mathcal{S}$ of x, denoted by s, necessarily lies in the special fibre S.

On the other hand, by [Zha13, Lemma 2.3.2, 2)], if s' is any closed point of $\mathcal{S}$ which is a specialization of s (s' exists since $\mathcal{S}$ is (locally) noetherian, see [Sta, Tag 01OU]), the fibre $\mathbb{I}_{+,\widehat{s'}}$ is a trivial $\mathcal{P}_+$-torsor over $\widehat{\mathcal{O}}_{\mathcal{S},s'}$. But since the canonical morphism $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{S},s} \to \mathcal{S}$ factors through $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{S},s'} \to \mathcal{S}$, the fibre $\mathbb{I}_{+,\hat{s}}$ is also a trivial $\mathcal{P}_+$-torsor over $\widehat{\mathcal{O}}_{\mathcal{S},s}$. In particular, $\mathbb{I}_{+,\hat{s}}$ is faithfully flat over $\widehat{\mathcal{O}}_{\mathcal{S},s}$. Now by faithfully flat descent for $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{S},s} \to \mathrm{Spec}\mathcal{O}_{\mathcal{S},s}$, the fibre $\mathbb{I}_{+,s}$ is also faithfully flat over $\mathrm{Spec}\mathcal{O}_{\mathcal{S},s}$. Now the first assertion follows from the fact that $x : \mathrm{Spec}\mathcal{O}_{X,x} \to \mathcal{S}$ factors through the canonical embedding $\mathrm{Spec}\mathcal{O}_{\mathcal{S},s} \to \mathcal{S}$ while the second assertion follows from the fact that $\hat{x} : \mathrm{Spec}\widehat{\mathcal{O}}_{X,x} \to \mathcal{S}$ factors through $\mathrm{Spec}\widehat{\mathcal{O}}_{\mathcal{S},s} \to \mathcal{S}$. ☐

Remark 7.2.2. A similar statement holds true for $\mathbb{I}$ (with respect to $\mathcal{G}$). The lemma above is in a sense an ad hoc one, since we believe that the following slightly stronger statement holds true: $\mathbb{I}_+$ is a $\mathcal{P}_+$-torsor over $\mathcal{S}$ and $\mathbb{I}$ is a $\mathcal{G}$-torsor over $\mathcal{S}$. In fact, the stronger assertion here concerning $\mathbb{I}$ follows readily from the combination of [Zha13, Lemma 2.3.2, 1)] and (3.2.1) in [Kim13]. We hope to give a whole proof of this stronger version in the future.

Write $\mathcal{V}_0 = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_0/S)$ and $\mathcal{C}_0 \subset \mathcal{V}_0$ the Hodge filtrations of $\mathcal{V}_0$. By functoriality of the de Rham cohomology one knows that $\mathcal{C}_0 \subset \mathcal{V}_0$ is the pullback to S of $\mathcal{C} \subset \mathcal{V}$. Denote by $\bar{s}_{\mathrm{dR}} \subset \mathcal{V}_0^{\otimes}$ the reduction of the set of tensors $s_{\mathrm{dR}} \subset \mathcal{V}^{\otimes}$, and by $\bar{s}$ the base change to $(\Lambda^*_\kappa)^{\otimes}$ of $s \subset (\Lambda^*)^{\otimes}$. Now we define the following schemes over S,

$$\mathrm{I} := \mathrm{Isom}_S\big([\Lambda^*_\kappa, \bar{s}] \otimes O_S, [\mathcal{V}_0, \bar{s}_{\mathrm{dR}})\big], \tag{7.2.2}$$

$$\mathrm{I}_+ := \mathrm{Isom}_S\big([\Lambda^*_\kappa, \bar{s}, \mathrm{Fil}^\bullet_{\chi,\kappa}] \otimes O_S, [\mathcal{V}_0, \bar{s}_{\mathrm{dR}}, \mathcal{V}_0 \supset \mathcal{C}_0]\big), \tag{7.2.3}$$

where the $\mathrm{Fil}^\bullet_{\chi,\kappa}$ denotes the base change to S of the filtration $\mathrm{Fil}^\bullet_{\tilde{\chi}}$ in (7.2.1). Clearly I, I_+ are base changes to S of $\mathbb{I}, \mathbb{I}_+$ respectively. By Theorem 2.4.1, 2) in [Zha13], I is a G-torsor over S and I_+ is a P_+-torsor over S.

7.3 Frobenius invariance of tensors s_{dR}

Let R_0 be a κ-algebra as in Lemma 4.2.1, i.e., R_0 locally admits a finite p-basis (e.g., R_0 is a smooth κ-algebra of finite type), and (R, φ) a simple frame of R_0. Assume from now on that R is noetherian. Let $x : \mathrm{Spec}R \to \mathcal{S}$ be a morphism of $W(\kappa)$-schemes. Write $H = \mathcal{A}_x[p^\infty]$ and $H_0 = H \otimes_R R_0$.

By Theorem 4.4.1, associated to the p-divisible group H is a tuple $(M, N, F = F_M, \nabla)$, where

- $M = \mathbb{D}^*(H_0)(R) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x/R) = \mathcal{V}_x$;
- $F = F_M : M \to M$ is the Frobenius endomorphism of M (c.f. Section 4.3.1);
- $N \subset M$ is the Hodge filtration of M, i.e., N is the preimage of $\mathcal{C}_x$ in M under the canonical isomorphism $M \cong \mathcal{V}_x$;
- $\nabla = \nabla_M : M \to M \otimes_R \hat{\Omega}_R$ is an integrable topologically nilpotent connection over $d_R : R \to \hat{\Omega}_R$, whose existence is stated in Theorem 4.3.1, and with respect to which F is horizontal.

By identifying the filtered modules $M \supset N$ with $\mathcal{V}_x \supset \mathcal{C}_x$ we also obtain a set of tensors $s_{\mathrm{dR},R} \subset M^{\otimes}$ over M.

Lemma 7.3.1. The Frobenius endomorphism $F_M : M \to M$, after inverting p, preserves the tensors $s_{\mathrm{dR},R}$ pointwise.

Proof. This essentially follows from the discussion in [Kis10, 1.5.4]. Indeed, for any maximal idea $\mathfrak{m}$ of R, by Lemma 4.2.1, (4) the Frobenius lift $\varphi : R \to R$ induces a simple frame $(\hat{R}_{\mathfrak{m}}, \varphi)$ of $\hat{R}_{\mathfrak{m}}$, compatible with (R, φ_R). Note that $\hat{R}_{\mathfrak{m}}$ is necessarily p-adically complete (since $\mathfrak{m}$ contains p). Hence it suffices to show the lemma after base change to $\hat{R}_{\mathfrak{m}}$ for all $\mathfrak{m}$. In particular, we may assume that R is a local ring.

Let $s' \in \mathcal{S}$ be the image of the closed point of Spec$R$, which necessarily lies in the special fibre $S \subset \mathcal{S}$. As in the proof of Lemma 7.2.1, there exists a closed point $s \in S$ such that s is a specialization in $\mathcal{S}$ of s'. Then the morphism $x : \mathrm{Spec}R \to \mathcal{S}$ facts through the canonical embedding $s : \mathrm{Spec}\hat{\mathcal{O}}_{\mathcal{S},s} \to \mathcal{S}$. We know from the proof of Corollary 2.3.9 in [Kis10] (the second paragraph) that $\hat{\mathcal{O}}_{\mathcal{S},s}$ is isomorphic to the deformation ring R_{G_W} of the p-divisible group $H_{0,s} := \mathcal{A}_s[p^\infty] \otimes_{\hat{\mathcal{O}}_{\mathcal{S},s}} k(s)$ (equipped with Tate tensors) over $k(s)$ induced by the embedding s. The ring R_{G_W} is defined in the the proof of Proposition 3.5 in loc. cit. but note that there is a notation difference since this ring is irrelevant of our R in question: in order to avoid confusion we will denote it by A.

We may identify $\hat{\mathcal{O}}_{\mathcal{S},s}$ with A. The ring A is in fact a quotient of Faltings's versal deformation ring B of $H_{0,s}$, which is isomorphic to a ring of power series over $W(k(s))$ (Section 1.5 of loc. cit.). One may arrange so that $A = B/I$, with $I \subset B$ an ideal generated by some formal variables of B. Then if we denote by $\varphi_B : B \to B$ the Frobenius lift of B obtained by sending all formal variables of B to their p-th powers, φ_B induces a Frobenius lift $\varphi_A : A \to A$, i.e., we obtain a homomorphism of simple frames $(B, \varphi_B) \to (A, \varphi_A)$. Write $p : B \to A$ for the canonical projection. Let (N, F_N, ∇_N) be the Dieudonné module of the tautological p-divisible group H_B over B as in [Kis10, Section 1.5] (denoted by (M, φ, ∇) in loc.cit.). Then the triple

$$(L := N \otimes_B A, F_L := F_N \otimes \varphi_A, \nabla_N \otimes \mathrm{id}_A)$$

corresponds to the Dieudonné module of $p^* H_B$ over A. Let $g : A \to R$ be the induced homomorphism and write $f = g \circ p$. Since $\varphi_R \circ f$ and $f \circ \varphi_B$ become the same after modulo p, we obtain from the definition of Dieudonné crystals a canonical isomorphism

$$\varepsilon : \varphi_R^* f^* N \cong \mathbb{D}^*(\varphi_R^* f^* H_B \otimes_R R_0) = \mathbb{D}^*(f^* \varphi_B^* H_B \otimes_R R_0) \cong f^* \varphi_B^* N.$$

Now we have the following identifications:

$$M = N \otimes_B R, \quad \nabla_M = \nabla_N \otimes_B \mathrm{id}_R, \quad F_M^{\mathrm{lin}} = (F_N^{\mathrm{lin}} \otimes \mathrm{id}_R) \circ \varepsilon = (F_L^{\mathrm{lin}} \otimes \mathrm{id}_R) \circ \varepsilon.$$

Note that the tensors $s_{\mathrm{dR},R}$ are obtained from the tensors s_{dR} over N by base change, i.e., $s_{\mathrm{dR},R} = s_{\mathrm{dR}} \otimes_B 1$. By [Kis10, 1.5.4], one finds

$$\varepsilon\big(s_{\mathrm{dR}} \otimes_B 1 \otimes_{R,\varphi_R} 1\big) = s_{\mathrm{dR}} \otimes_{B,\varphi_B} 1 \otimes_B 1 = s_{\mathrm{dR},A} \otimes_{A,\varphi_A} 1 \otimes_A 1 \in \big(\varphi_A^* L \otimes_A R\big)^{\otimes}$$

On the other hand, we know that $F_L^{\mathrm{lin}} \otimes \mathrm{id}_R$ sends the tensors $s_{\mathrm{dR},A} \otimes_{A,\varphi_A} 1$ to $s_{\mathrm{dR},A}$ (see [Wor13, Corollary 4.8] for a clear statement though the proof is essentially in [Kis10]). Hence F_M^{lin} sends the tensors $s_{\mathrm{dR},R} \otimes_{R,\varphi_R} 1$ to $s_{\mathrm{dR},R}$. In other words, $s_{\mathrm{dR},R}$ is F_M-invariant. □

Each element $x \in \mathbb{I}(R)$ by definition gives an isomorphism $\beta_x : \Lambda_R^* \cong M$, sending the tensors s_R to the tensors $s_{\mathrm{dR},R}$ pointwise. We can transform the Frobenius map F to Λ_R^* via β_R by defining

$$F_x := \beta_x^{-1} \circ F \circ \beta_x.$$

Note that we have a canonical isomorphism $\varepsilon : (\varphi^*\Lambda_R^*,\ \varphi^* s_R) \longrightarrow (\Lambda_R^*, s_R)$. Define

$$F_x^{\mathrm{lin}} := \big(V_R^* \xrightarrow{\varepsilon^{-1}} \varphi^* V_R^* = V_R^* \otimes_{R,\varphi} R \xrightarrow{F_x \otimes \mathrm{id}} V_R^*\big), \tag{7.3.1}$$

which by Lemma 7.3.1 preserves the tensors s_R, i.e., we have $(F_x^{\mathrm{lin}})^\vee \in \mathcal{G}(R[\frac{1}{p}])$. For the next lemma, one may recall the definition of the cocharacter $\tilde{\mu} : \mathbb{G}_{m,W(\kappa)} \to \mathcal{G}_{W(\kappa)}$ in (6.5.5).

Lemma 7.3.2. For any $x \in \mathbb{I}_+(R)$ we have $(F_x^{\mathrm{lin}})^\vee \in \mathcal{G}(R)\tilde{\mu}(p)$ and for any $x \in \mathbb{I}(R)$ we have $(F_x^{\mathrm{lin}})^\vee \in \mathcal{G}(R)\tilde{\mu}(p)\mathcal{G}(R)$.

Proof. We show only the first part of the claim since it implies the second part.

For any $x \in \mathbb{I}_+(R)$, the weight decomposition $\Lambda_R^* = \Lambda_R^{*,1} \oplus \Lambda_R^{*,0}$ (cf. (6.5.2)) induced by the cocharacter $\tilde{\chi}_R : \mathbb{G}_{m,R} \to \mathcal{G}_R$ induces via β_x a decomposition $M = N \oplus L$. It suffices to show the following:

$$F_x^{\mathrm{lin}} = \Gamma_x^{\mathrm{lin}} \circ \big(\tilde{\mu}(p)\big)^\vee, \quad \text{with } \ \Gamma_x^{\mathrm{lin}} := \beta_x^{-1} \circ \Gamma^{\mathrm{lin}} \circ \varphi^*(\beta_x) \circ \varepsilon, \tag{7.3.2}$$

where Γ^{lin} is defined as in (5.1.1). Indeed this can been seen from the following commutative diagram

$$\begin{tikzcd}
V_R^* \arrow[r, "(\tilde{\mu}(p))^\vee"] \arrow[d, "\varepsilon"'] & V_R^* \arrow[r, "\Gamma_x^{\mathrm{lin}}"] \arrow[d, "\varepsilon"] & V_R^* \arrow[dd, "\beta_x"] \\
\varphi^* V_R^* \arrow[r, "\varphi^*(\tilde{\chi}(p)^\vee)"] \arrow[d, "\varphi^*(\beta)"'] & \varphi^* V_R^* \arrow[d, "\varphi^*(\beta)"'] & \\
N^{(\varphi)} \oplus L^{(\varphi)} \arrow[r, "p\cdot\mathrm{id}\oplus\mathrm{id}"] & N^{(\varphi)} \oplus L^{(\varphi)} \arrow[r, "\Gamma^{\mathrm{lin}}"] & M
\end{tikzcd}$$

□

7.4 The morphism $\theta : \mathrm{I}_+ \to \mathcal{D}_1$

As suggested by the title, in this subsection we construct a morphism of κ-schemes from I_+ to $\mathcal{D}_1$. We also give an explicit description of θ on geometric points, where we skip the notion of Kim-Kisin windows and adapted deformations.

Main Construction $\theta : \mathrm{I}_+ \to \mathcal{D}_1$

Affine open coverings for $\mathbb{I}_+$.

Take an affine open covering $\{\mathrm{Spec}\tilde{R}_i \to \mathcal{S}\}_{i\in I}$ of the integral model $\mathcal{S}$ over $W(\kappa)$ such that

$$\mathcal{C}_{\tilde{R}_i} \subset \mathcal{V}_{\tilde{R}_i} \text{ are free modules over } \tilde{R}_i \text{ for each } i. \tag{7.4.1}$$

Since $\mathcal{S}$ is quasi-projective (hence separated), the intersection of $\mathrm{Spec}\tilde{R}_i$ and $\mathrm{Spec}\tilde{R}_j$ is again affine, and hence is the spectrum of some $W(\kappa)$-algebra, say $\tilde{R}_{ij}$. For each i, j, since the torsors $\mathbb{I}_{+,\tilde{R}_i}, \mathbb{I}_{+,\tilde{R}_{ij}}$ are affine over $\tilde{R}_i$ and over $\tilde{R}_{ij}$ respectively, we can write

$$\mathbb{I}_{+,\tilde{R}_i} = \mathrm{Spec}\tilde{A}_i,\ \mathbb{I}_{+,\tilde{R}_{ij}} = \mathrm{Spec}\tilde{A}_{ij} \tag{7.4.2}$$

for some $W(\kappa)$-algebras $\tilde{A}_i$ and $\tilde{A}_{ij}$. Denote by $R_{i,0}, A_{i,0}, R_{ij,0}, A_{ij,0}$ the reduction modulo p of $\tilde{R}_i, \tilde{A}_i, \tilde{R}_{ij}, \tilde{A}_{ij}$ respectively. Clearly, we obtain affine open coverings

$$\{\mathrm{Spec}R_{i,0} \to S\}_{i\in I}, \quad \{\mathrm{Spec}A_{i,0} \to \mathrm{I}_+\}_{i\in I} \tag{7.4.3}$$

for S and for I_+ respectively, with

$$\mathrm{Spec}R_{i,0} \cap \mathrm{Spec}R_{j,0} = \mathrm{Spec}R_{ij,0}, \quad \mathrm{Spec}A_{i,0} \cap \mathrm{Spec}A_{j,0} = \mathrm{Spec}A_{ij,0}.$$

In the following we do the construction in four steps.

Step 1: Preparations

Write $H_{i,0} := \mathcal{A}[p^\infty]_{A_{i,0}}$ and let A_i, A_{ij} be the p-adic completions of $\tilde{A}_i, \tilde{A}_{ij}$ respectively. The induced morphism $x_i : \mathrm{Spec}A_i \to \mathbb{I}_+$ gives an isomorphism of free A_i-modules,

$$\beta_{x_i} : \Lambda^*_{A_i} \cong M_i,$$

which respects the filtrations and sends the tensors s_{A_i} to the tensors s_{dR,A_i} pointwise. Here as usual M_i is defined as

$$M_i := \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{A_i}/A_i) \cong \mathbb{D}^*(H_{i,0})(A_i).$$

We write

$$\mathfrak{S}_i = \mathfrak{S}(A_i), \quad \mathfrak{S}_{ij} = \mathfrak{S}(A_{ij}).$$

Denote by $M_{i,0}, \mathfrak{S}_{i,0}, \mathfrak{S}_{ij,0}, x_{i,0}$ and $\beta_{x_{i,0}}$ the reduction modulo p of $M_i, \mathfrak{S}_i, \mathfrak{S}_{ij}, x_i$ and β_{x_i} respectively. For each $i \in I$ we choose a Frobenius lift φ_i for A_i and denote by $\chi_{x_i} : \mathbb{G}_{m,A_i} \to \mathrm{GL}(M_i)$ the cocharacter of $\mathrm{GL}(M_i)$ over A_i induced by $\chi_{A_i} : \mathbb{G}_{m,A_i} \to \mathcal{G}_{A_i}$ via β_{x_i}. Consider the adapted deformation $\underline{\mathcal{M}_i} = (\mathcal{M}_i, \Phi(\varphi_i, \chi_{x_i}), \ldots)$ of $H_{i,0}$ w.r.t. the pair (φ_i, χ_{x_i})[3]. We refer to Section 5 for details on the Kim-Kisin window $\underline{\mathcal{M}_i}$ but recall that we use

$$\Phi_0(\varphi_i, \chi_{x_i}) : \mathcal{M}_{i,0}^{(\varphi)} \to \mathcal{M}_{i,0} \tag{7.4.4}$$

to denote the reduction modulo p of the Frobenius $\Phi(\varphi_i, \chi_{x_i}) : \mathcal{M}_i^{(\varphi_i)} \to \mathcal{M}_i$, where

$$\mathcal{M}_i := M_i \otimes_{A_i} \mathfrak{S}_i, \quad \mathcal{M}_{i,0} := M_{i,0} \otimes_{A_{i,0}} \mathfrak{S}_{i,0}.$$

Step 2: Construction of local morphisms $\theta_i : \mathrm{Spec} A_{i,0} \to \mathcal{D}_1$.

For each $i \in I$, we construct a morphism of κ-schemes $\theta_i : \mathrm{Spec} A_{i,0} \to \mathcal{D}_1$, equivalently, an element $\theta_i(x_{i,0}) \in \mathcal{D}_1(A_{i,0})$, in the following way. Define

$$\Phi_0(\varphi_i, x_i) := \big((\beta_{x_{i,0}} \otimes \mathrm{id}_{\mathfrak{S}_{i,0}})^{-1} \circ \Phi_0(\varphi_i, \chi_{x_i}) \circ \varphi(\beta_{x_{i,0}} \otimes \mathrm{id}_{\mathfrak{S}_{i,0}})\big)^{\vee} \tag{7.4.5}$$

which after inverting $u \in \mathfrak{S}_{i,0}$ lies in $\mathrm{GL}\big(\Lambda_{\mathfrak{S}_{i,0}}[\frac{1}{u}]\big)$, where we use the contragredient representation

$$(\cdot)^{\vee} : \mathrm{GL}\big(\Lambda^*_{\mathfrak{S}_{i,0}}[\frac{1}{u}]\big) \longrightarrow \mathrm{GL}\big(\Lambda_{\mathfrak{S}_{i,0}}[\frac{1}{u}]\big).$$

By Lemma 7.4.1 below, $\Phi_0(\varphi_i, x_i)$ lies in $\mathcal{D}(A_{i,0})$. Now we define

$$\theta_i(x_{i,0}) := \overline{\Phi_0(\varphi_i, x_i)}, \tag{7.4.6}$$

where $\overline{\Phi_0(\varphi_i, x_i)}$ denotes the image in $\mathcal{D}_1(A_{i,0})$ of $\Phi_0(\varphi_i, x_i) \in \mathcal{D}(A_{i,0})$. By Lemma 5.3.1, the element $\theta_i(x_{i,0})$ is independent of the choice of the Frobenius lift φ_i of A_i.

Lemma 7.4.1. $\Phi_0(\varphi_i, x_i)$ lies in $\mathcal{D}(A_{i,0})$.

Proof. We show this by giving a formula for $\Phi_0(\varphi_i, x_i)$. We use the notation $\Gamma^{\mathrm{lin}}_{x_i}$ as in (7.3.2). Define

$$\alpha(x_i) := (\Gamma^{\mathrm{lin}}_{x_i} \bmod p)^{\vee} \in G(A_{i,0}).$$

[3] Here in the Kim-Kisin window $\underline{\mathcal{M}_i}$ we omit all terms except the underlying $\mathfrak{S}(A_i)$-module $\mathcal{M}_i$ and its Frobenius $\Phi(\varphi_i, \chi_{x_i}) : \mathcal{M}_i^{(\varphi_i)} \to \mathcal{M}_i$.

Then by Remark 5.1.4. (a) we have

$$\Phi_0(\varphi_i, x_i) = \alpha(x_i)\mu(u) \in \mathcal{D}(A_{i,0}). \tag{7.4.7}$$

$\square$

Step 3: Gluing of the local morphisms

We shall show that for all $i, j \in I$, we have

$$\theta_i(x_{i,0})|_{A_{ij,0}} = \theta_j(x_{j,0})|_{A_{ij,0}}.$$

By the way we construct the morphisms θ_i and θ_j, we need only to show that the two homomorphisms

$$\Phi_0(\varphi_i, \chi_{x_i}) \otimes_{\mathfrak{S}_{i,0}} \mathrm{id}_{\mathfrak{S}_{ij,0}},\ \Phi_0(\varphi_j, \chi_{x_j}) \otimes_{\mathfrak{S}_{j,0}} \mathrm{id}_{\mathfrak{S}_{ij,0}} : \mathcal{M}_{ij,0}^{(\varphi)} \to \mathcal{M}_{ij,0}$$

differ by an automorphism of $\mathcal{M}_{ij,0}^{(\varphi)}$ whose reduction modulo u is the identity map, where

$$\mathcal{M}_{ij,0} := \mathcal{M}_{i,0} \otimes_{\mathfrak{S}_{i,0}} \mathfrak{S}_{ij,0} = \mathcal{M}_{j,0} \otimes_{\mathfrak{S}_{j,0}} \mathfrak{S}_{ij,0}.$$

This is in fact clear from the functoriality of adapted deformations. Indeed, if we still denote by φ_i the Frobenius lift of A_{ij} induced by φ_i. Then by Section 5.2 we have

$$\begin{aligned} \Phi(\varphi_i, \chi_{x_i}) \otimes_{\mathfrak{S}_i} \mathrm{id}_{\mathfrak{S}_{ij}} &:= \Phi(\varphi_i, \chi_{x_i, A_{ij}}) \\ \Phi(\varphi_j, \chi_{x_j}) \otimes_{\mathfrak{S}_j} \mathrm{id}_{\mathfrak{S}_{ij}} &:= \Phi(\varphi_j, \chi_{x_j, A_{ij}}) \end{aligned}$$

But since we have $\beta_{x_i, A_{ij}} = \beta_{x_j, A_{ij}}$, and hence $\chi_{x_i, A_{ij}} = \chi_{x_j, A_{ij}}$, the conclusion follows from Lemma 5.3.1. Finally we obtain a morphism of κ-schemes

$$\theta : \mathrm{I}_+ \longrightarrow \mathcal{D}_1.$$

Step 4: θ is independent of the affine open covering of $\mathcal{S}$

This follows from a standard argument. Given another affine open covering of $\mathcal{S}$ satisfying the requirements (7.4), which by the construction above, gives a morphism $\theta_1 : \mathrm{I}_+ \to \mathcal{D}_1$. Note that the union of these two affine open coverings is again an affine open covering of $\mathcal{S}$, satisfying (1), (2) in Section 7.4, and hence induces another morphism $\theta_2 : \mathrm{I}_+ \to \mathcal{D}_1$. But we have then $\theta = \theta_2, \theta_1 = \theta_2$, and therefore $\theta = \theta_1$. This finishes the proof.

Remark 7.4.2. (1) We remark here that to give the morphism $\theta : \mathrm{I}_+ \to \mathcal{D}_1$ one may simply use abstract formulas for $\Phi_0(\varphi_i, x_i)$ as in (7.4.7), without relating to adapted deformations (or Kim-Kisin windows), as it seems to be much more direct. In fact, one shall see this more clearly in Section 7.4 below. Nonetheless, the connection with adapted deformations provides a conceptual interpretation of the abstract objects $\Phi_0(\varphi_i, x_i)$, and hence of the morphism $\theta : \mathrm{I}_+ \to \mathcal{D}_1$.

(2) Consider the morphism $x : \mathrm{Spec}A \to \mathbb{I}_+$ as in Step 1 (here we omit the subscript i), that gives a p-divisible group $H = \mathcal{A}_x[p^\infty]$ over A. A natural question one may ask is: is the adapted deformation $\underline{\mathcal{M}}$ "the" Kim-Kisin window corresponding to H? The answer is "yes", up to an isomorphism in $\mathbf{Win}(\mathfrak{S}, \nabla^0)$. This follows from Corollary 4.5.6. But note that the construction of $\underline{\mathcal{M}}$ really depends on the point $x \in \mathbb{I}_+(A)$ (not only the induced point in $\mathcal{S}(A)$), and hence is not canonical. As a complement of the remark in (1), it is necessary to point out that it is the explicit constructions of the adapted deformations that allows us to do the computation in Lemma 5.3.1 so that we can eventually construct the map $\theta : \mathrm{I}_+ \to \mathcal{D}_1$.

Description of $\theta : \mathrm{I}_+ \to \mathcal{D}_1$ on geometric points

Let k be an algebraically closed field extension of κ. Write $\mathfrak{S} = \mathfrak{S}(W(k))$ and $\mathfrak{S}_0$ for its reduction modulo p. Given an element $x \in \mathrm{I}_+(k)$, a lift $\tilde{x} \in \mathbb{I}_+(W(k))$ of x gives an isomorphism

$$\beta_{\tilde{x}} : \Lambda^*_{W(k)} \cong M := \mathbb{D}^*(\mathcal{A}_x[p^\infty])(W(k)) \cong \mathrm{H}^1_{\mathrm{cris}}(\mathcal{A}_{\tilde{x}}/W(k)) \tag{7.4.8}$$

compatible with torsors and filtrations. Similarly we have $\beta_x = \beta_{\tilde{x}} \otimes \mathrm{id}_k$. As in (7.4.5) we can then form

$$\Phi_0(\varphi, \tilde{x}) := \big((\beta_x \otimes \mathrm{id}_{\mathfrak{S}_0})^{-1} \circ \Phi_0(\varphi, \chi_{\tilde{x}}) \circ \varphi(\beta_x \otimes \mathrm{id}_{\mathfrak{S}_0})\big)^\vee \in \mathcal{D}(k), \tag{7.4.9}$$

where $\varphi : W(k) \to W(k)$ is the unique Frobenius lift of $W(k)$.

Lemma 7.4.3. The following holds:

$$\theta(k)(x) := \overline{\Phi_0(\varphi, \tilde{x})}, \tag{7.4.10}$$

where $\overline{\Phi_0(\varphi, \tilde{x})}$ denotes the image in $\mathcal{D}_1(k)$ of $\Phi_0(\varphi, \tilde{x})$.

Proof. The corresponding morphism $x : \mathrm{Spec}k \to \mathrm{I}_+$ factors through some $\mathrm{Spec}A_{i,0}$ in the covering $\{\mathrm{Spec}A_{i,0} \to \mathrm{I}_+\}$ of I_+ (cf. (7.4)). We may drop the subscript i in $A_{i,0}$ and write $A_0 = A_{i,0}$. We fix a Frobenius lift $\varphi_A : A \to A$ of A. Then there is a unique homomorphism of simple frames $(A, \varphi_A) \to (W(k), \varphi)$ over $W(\kappa)$ lifting the ring homomorphism $A_0 \to k$ corresponding to x. Denote by $\breve{x} : \mathrm{Spec}W(k) \to \mathbb{I}_+$ the corresponding lift of x. Let $\tilde{\mathbf{x}} : \mathrm{Spec}A \to \mathbb{I}_+$ be the structure morphism. By our construction of θ in Theorem 7.4, $\theta(x)$ is defined to be the image in $\mathcal{D}_1(k)$ of

$$\Phi_0(\varphi_A, \tilde{\mathbf{x}}) \otimes_{\mathfrak{S}(A_0)} \mathfrak{S}(k) \in \mathcal{D}(k).$$

But by the functoriality of the formation of $\Phi_0(\varphi_A, \chi_{\tilde{\mathbf{x}}})$ we have

$$\Phi_0(\varphi_A, \tilde{\mathbf{x}}) \otimes_{\mathfrak{S}(A_0)} \mathfrak{S}(k) = \Phi_0(\varphi, \breve{x}).$$

Now by Lemma 5.3.1, the two elements in $\mathcal{D}(k)$, namely $\Phi_0(\varphi, \breve{x})$ and $\Phi_0(\varphi, \tilde{x})$, have the same image in $\mathcal{D}_1(k)$. $\square$

We can also define $\Gamma_{\tilde{x}}^{\mathrm{lin}}$ and $\alpha(\tilde{x})$ as in Lemma 7.4.1 and have the following formula

$$\Phi_0(\varphi, \tilde{x}) = \alpha(\tilde{x})\mu(u) \in \mathcal{D}(k).$$

7.5 The morphism $\eta : S \to \mathcal{D}_1/\mathcal{K}^\diamond$

Our main aim in this subsection is to show that the composition morphism of fpqc sheaves $\mathrm{I}_+ \xrightarrow{\theta} \mathcal{D}_1 \to \mathcal{D}_1/\mathcal{K}^\diamond$ is P_+-invariant, and hence induces a morphism of fpqc sheaves $\eta : S \to \mathcal{D}_1/\mathcal{K}^\diamond$. To do this, we first investigate the behavior of $\theta(k) : \mathrm{I}_+(k) \to \mathcal{D}_1(k)$ under the $P_+(k)$-action on $\mathrm{I}_+(k)$, where as in Section 7.4, k is an algebraically closed field extension of κ.

The following is a continuation of our discussion in Section 7.4. We will need some group theoretic results first.

Let $\mathcal{U}_+$ be the unipotent radical of $\mathcal{P}_+$ and fix a maximal torus $\mathcal{T}$ of $\mathcal{G}_{W(k)}$ such that the cocharacter $\tilde{\chi}_{W(k)} : \mathbb{G}_{m,W(k)} \to \mathcal{G}_{W(k)}$ factors through $\mathcal{T}$. Let $\mathcal{M}$ be the centralizer of $\tilde{\chi}_{W(k)}$ in $\mathcal{P}_+$, which is a Levi subgroup of $\mathcal{P}_+$. Note that $\mathcal{T}$ necessarily splits over $W(k)$ since $W(k)$ is strictly Henselian. Let Φ be the set of roots of $\mathcal{G}_{W(k)}$ with respect to $\mathcal{T}$. Recall that for every root $\alpha \in \Phi$, there is an embedding of algebraic groups $\mathcal{U}_\alpha : \mathbb{G}_{a,W(k)} \to G_{W(k)}$, unique up to an element in $W(k)^\times$ (here $W(k)^\times$ acts on $\mathbb{G}_{a,W(k)}$ by multiplication), such that if A is a $W(k)$-algebra we have $t\mathcal{U}_\alpha(x)t^{-1} = \mathcal{U}_\alpha\big(\alpha(t)x\big)$ for all $x \in \mathbb{G}_a(A) = A$ and all $t \in \mathcal{T}(A)$. We will also use the notation $\mathcal{U}_\alpha$ for the image of $\mathcal{U}_\alpha$. Then we have the following isomorphisms of closed subschemes of $\mathcal{G}_{W(k)}$,

$$\prod_{\langle \alpha, \chi \rangle > 0} \mathcal{U}_\alpha \xrightarrow{\sim} \mathcal{U}_{+,W(k)}, \qquad \prod_{\langle \alpha, \chi \rangle < 0} \mathcal{U}_\alpha \xrightarrow{\sim} \mathcal{U}_{-,W(k)}, \tag{7.5.1}$$

sending an element $(u_\alpha)_\alpha$ in $\prod_{\langle \alpha, \chi \rangle > 0} \mathcal{U}_\alpha$ to the product $\prod_{\langle \alpha, \chi \rangle > 0} u_\alpha \in \mathcal{U}_{+,W(k)}$, where $\alpha \in \Phi$ runs over all possible roots of $\mathcal{G}_{W(k)}$, and where the products can be taken in any fixed order; the second isomorphism in (7.5.1) is defined in a similar way. In particular, for each $\alpha_1, \alpha_2 \in \Phi$ with $\langle \alpha_1, \tilde{\chi}_{W(k)} \rangle > 0$, $\langle \alpha_2, \tilde{\chi}_{W(k)} \rangle < 0$ we have

$$\tilde{\chi}(p)\mathcal{U}_{\alpha_1}(W(k))\tilde{\chi}(p)^{-1} \subset K_1(k), \quad \tilde{\chi}(p)^{-1}\mathcal{U}_{\alpha_2}(W(k))\tilde{\chi}(p) \subset K_1(k), \tag{7.5.2}$$

where $K_1(k)$ denotes the kernel of the reduction modulo p map $\mathcal{G}(W(k)) \xrightarrow{\bmod p} G(k)$.

Now a combination of (7.5.1) and (7.5.2) gives the inclusions

$$\tilde{\chi}(p)\mathcal{U}_+(W(k))\tilde{\chi}(p)^{-1} \subset K_1(k), \quad \tilde{\chi}(p)^{-1}\mathcal{U}_-(W(k))\tilde{\chi}(p) \subset K_1(k). \tag{7.5.3}$$

Lemma 7.5.1. Let $p_+ \in P_+(k)$ be such that $(x, p_+) \in (\mathrm{I}_+ \times_S P_+)(k)$. Then there exists an element $c \in \mathcal{K}^\diamond(k)$ such that $\theta(x \cdot p_+) = \theta(x) \cdot c$. [4] In particular, the map $\mathrm{I}_+(k) \xrightarrow{\theta(k)} \mathcal{D}_1(k) \to \mathcal{D}_1(k)/\mathcal{K}^\diamond(k)$ is P_+-invariant, and hence induces a map

$$\eta(k) : S(k) \to \mathcal{D}_1(k)/\mathcal{K}^\diamond(k).$$

Proof. Note that we have a decomposition of algebraic groups over κ, $P_+ = U_+ \rtimes M$, and hence we may uniquely write $p_+ = u_+ m$, with $u_+ \in U_+(k)$, $m \in M(k)$. We shall find such a c in the statement of the lemma by calculation. By 7.3.2, if we write $g_{\tilde{x}} = \left(F_{\tilde{x}}^{\mathrm{lin}}\right)^\vee$ and $\Delta_{\tilde{x}} = \left(\Gamma_{\tilde{x}}^{\mathrm{lin}}\right)^\vee$, then we have $g_{\tilde{x}} = \Delta_{\tilde{x}} \tilde{\mu}(p) \in \mathcal{G}(W(k)[\frac{1}{p}])$, where $\Delta_{\tilde{x}}$ lies in $\mathcal{G}(W(k))$ and the reduction modulo p of $\Delta_{\tilde{x}}$ is $\alpha(\tilde{x})$.

Let $\tilde{p}_+ = \tilde{u}_+ \tilde{m} \in (\mathcal{U}_+ \rtimes \mathcal{M})(W(k))$ be a lift of p_+, such that $(\tilde{x}, \tilde{p}_+)$ lies in $(\mathbb{I}_+ \times_{\mathcal{S}} \mathcal{P}_+)(W(k))$. Then we have

$$\begin{aligned} g_{\tilde{x}\tilde{p}_+} &= (\tilde{p}_+)^{-1} g_{\tilde{x}} \sigma(\tilde{p}_+) \\ &= (\tilde{p}_+)^{-1} \Delta_{\tilde{x}} \tilde{\mu}(p) \sigma(\tilde{u}_+) \sigma(\tilde{m}) \\ &= (\tilde{p}_+)^{-1} \Delta_{\tilde{x}} \sigma(\tilde{m}) \Omega \tilde{\mu}(p), \quad \text{where} \\ \Omega :&= \tilde{\mu}(p) \sigma(\tilde{m})^{-1} \sigma(\tilde{u}_+) \sigma(\tilde{m}) \tilde{\mu}(p)^{-1}. \end{aligned}$$

It follows from (7.5.3) that Ω actually lies in $K_1(k)$. Hence we have

$$\theta(xp_+) = p_+^{-1} \alpha(\tilde{x}) \sigma(m) \mu(u) \in \mathcal{D}_1(k).$$

To continue our calculation, we need to use the equicharacteristic analogue of (7.5.3). To be precise, we have

$$\mu(u) u_+ \mu(u)^{-1} \in \mathcal{K}_1(k) \text{ for every element } u_+ \in U_+(k) \qquad (*).$$

We have then

$$\begin{aligned} \theta(xp_+) &= p_+^{-1} \alpha(\tilde{x}) \sigma(m) \mu(u) \\ &= \alpha(\tilde{x}) \mu(u) \sigma(u_+)^{-1} \cdot (1,\ p_+) \\ &= \alpha(\tilde{x}) y \alpha(\tilde{x})^{-1} \theta(x) \cdot (1,\ p_+) \\ &= \theta(x) \cdot \left(\alpha(\tilde{x}) y^{-1} \alpha(\tilde{x})^{-1},\ p_+\right), \end{aligned}$$

where we let $y := \mu(u) \sigma(u_+)^{-1} \mu(u)^{-1}$ and $(*)$ is used to deduce that the element $\alpha(\tilde{x}) y^{-1} \alpha(\tilde{x})^{-1}$ lies in $\mathcal{K}_1(k)$. We write

$$d(\tilde{x}, p_+) := \alpha(\tilde{x}) y \alpha(\tilde{x})^{-1} = \Phi_0(\varphi, \tilde{x}) \sigma(u_+) \Phi_0(\varphi, \tilde{x})^{-1}.$$

[4] We intentionally mention the element c as it will serve for Theorem 7.5.2 below.

Finally, we may take

$$c = \big(d(\tilde{x}, p_+),\ p_+\big) \in \mathcal{K}^\diamond(k). \tag{7.5.4}$$

$\square$

Theorem 7.5.2. The composition of morphisms of fpqc sheaves $\mathrm{I}_+ \to \mathcal{D}_1 \to \mathcal{D}_1/\mathcal{K}^\diamond$ is P_+-invariant, and hence induces a morphism of fpqc sheaves, $\eta : S \to \mathcal{D}_1/\mathcal{K}^\diamond$, fitting the following commutative diagram

$$\begin{tikzcd} \mathrm{I}_+ \arrow[r, "\theta"] \arrow[d] & \mathcal{D}_1 \arrow[d] \\ S \arrow[r, "\eta"] & \mathcal{D}_1/\mathcal{K}^\diamond. \end{tikzcd}$$

Proof. To keep coherence with notations we write $x : \mathrm{I}_+ \times_\kappa P_+ \to \mathrm{I}_+$ for the first projection and $p_+ : \mathrm{I}_+ \times_\kappa P_+ \to P_+$ for the second projection and $x \cdot p_+ : \mathrm{I}_+ \times_\kappa P_+ \to \mathrm{I}_+$ for the P_+-action on I_+. We need to show the commutativity of the diagram

$$\begin{tikzcd} \mathrm{I}_+ \times_\kappa P_+ \arrow[r, "\theta\circ(x\cdot p_+)"] \arrow[d, "\theta\circ x"] & \mathcal{D}_1 \arrow[d] \\ \mathcal{D}_1 \arrow[r] & \mathcal{D}_1/\mathcal{K}^\diamond. \end{tikzcd} \tag{7.5.5}$$

Since each $\mathrm{Spec}A_{i,0}$ in the covering $\{\mathrm{Spec}A_{i,0} \to \mathrm{I}_+\}_{i\in I}$ is P_+-invariant, we may assume that $\mathrm{I}_+ = U$, where U runs over all $\mathrm{Spec}A_{i,0}$ in the covering. For the discussion below we may suppress the subscript i in $A_{i,0}, \theta_i$, etc. We fix a Frobenius lift $\varphi_A : A \to A$ of A and write $\tilde{\mathbf{x}} : \mathrm{Spec}A \to \mathbb{I}_+$ for the structure morphism.

We define an element in $\mathcal{L}G(U \times_\kappa P_+)$ as follows

$$d = \Phi_0(\varphi_A, \tilde{\mathbf{x}})\sigma(u_+)\Phi_0(\varphi_A, \tilde{\mathbf{x}})^{-1}.$$

where $\sigma(u_+)$ is the morphism

$$U \times_\kappa P_+ \xrightarrow{p_+} P_+ \to U_+ \to U_+^{(p)},$$

and where $\Phi_0(\varphi_A, \tilde{\mathbf{x}}) \in \mathcal{D}(U)$ is defined in (7.4.5), but here we see it as an element in $\mathcal{D}(U \times_\kappa P_+)$ by precomposing the first projection $x : U \times_\kappa P_+ \to U$.

Claim: d lies in $\mathcal{K}_1(U \times_\kappa P_+)$.

Indeed, since $U \times_\kappa P_+$ is reduced and $\mathcal{K}_1$ is a closed subscheme of $\mathcal{L}G$, the morphism $d : U \times_\kappa P_+ \to \mathcal{L}G$ lies in $\mathcal{K}_1$ if and only if its set theoretic image lies in $\mathcal{K}_1$. But we have seen from the proof of Lemma 7.5.1 that the image of d on geometric points do lie in $\mathcal{K}_1$. Hence the claim is proved.

Denote by

$$\Gamma_{\theta\circ(x\cdot p_+)},\ \Gamma_{\theta\circ x} : U \times_\kappa P_+ \to \mathcal{D}_1 \times_\kappa (U \times_\kappa P_+)$$

the graph of $\theta \circ (x \cdot p_+)$, $\theta \circ x : U \times_\kappa P_+ \to \mathcal{D}_1$ respectively. We shall show the commutativity of the following diagram

$$\begin{tikzcd}
 & \mathcal{D}_1 \times_\kappa (U \times_\kappa P_+) \arrow[dd, "\cdot c"] \\
U \times_\kappa P_+ \arrow[ur, "\Gamma_{\theta\circ(x\cdot p_+)}"] \arrow[dr, "\Gamma_{\theta\circ x}"'] & \\
 & \mathcal{D}_1 \times_\kappa (U \times_\kappa P_+),
\end{tikzcd} \tag{7.5.6}$$

where the right vertical morphism $\cdot c$ is the translation by

$$c = (d, p_+) \in \mathcal{K}^\diamond(U \times_\kappa P_+),$$

which is an automorphism of $\mathcal{D}_1 \times_\kappa (U \times_\kappa P_+)$ over $U \times_\kappa P_+$. Note that the commutativity of (7.5.6) implies that $\theta \circ (x \cdot p_+) = \theta \circ x$ since we also have the commutative diagram

$$\begin{tikzcd}
\mathcal{D}_1 \times_\kappa (U \times_\kappa P_+) \arrow[r, "\mathrm{pr}_1"] \arrow[dd, "\cdot c"] & \mathcal{D}_1 \arrow[dr] & \\
 & & \mathcal{D}_1/\mathcal{K}^\diamond. \\
\mathcal{D}_1 \times_\kappa (U \times_\kappa P_+) \arrow[r, "\mathrm{pr}_1"] & \mathcal{D}_1 \arrow[ur] &
\end{tikzcd}$$

But as $U \times_\kappa P_+$ and $\mathcal{D}_1 \times_\kappa (U \times_\kappa P_+)$ are of finite type over κ and $U \times_\kappa P_+$ is smooth, hence geometrically reduced, it is enough to check the commutativity of (7.5.6) on geometric points (see for example Exercise 14.15 of [GW10]). Now the conclusion follows from Lemma 7.5.1 and the formula for the c in (7.5.4) $\square$