Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

Yan, Q.; Yan Q.

Author: Yan, Qijun
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6 Shimura varieties of Hodge type

6.1 Shimura data and Shimura varieties over $\mathbb{C}$

Definition 6.1.1. Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ and $\mathbf{X}$ a conjugacy class of homomorphism of algebraic groups over $\mathbb{R}$

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathbf{G}_{\mathbb{R}}.$$

(1) A pair $(\mathbf{G}, \mathbf{X})$ as above is called a **Shimura datum** if it satisfies the following conditions:

   (a) Let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}_{\mathbb{R}}$. Then the composite

   $$\mathbb{S} \to \mathbf{G}_{\mathbb{R}} \to \mathbf{G}_{\mathbb{R}}^{\mathrm{ad}} \to \mathrm{GL}(\mathfrak{g})$$

   defines a Hodge structure of type $(-1,1),(0,0),(1,-1)$.

   (b) $h(i)$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}^{\mathrm{ad}}$. This means that we require the real form of $\mathbf{G}^{\mathrm{ad}}$ defined by the involution $g \mapsto h(i)\bar{g}h(i)^{-1}$ to be compact.

   (c) $\mathbf{G}$ has no factor defined over $\mathbb{Q}$ whose real points form a compact group.

(2) A morphism $i : (\mathbf{G}_1, \mathbf{X}_1) \to (\mathbf{G}_2, \mathbf{X}_2)$ of Shimura data is a map of groups $\mathbf{G}_1 \to \mathbf{G}_2$, which induces a map $\mathbf{X}_1 \to \mathbf{X}_2$.

The requirement (a) in Definition 6.1.1 means that under the action of $\mathbb{C}^\times$ on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ by conjugation through h_0, we have a decomposition

$$\mathfrak{g}_{\mathbb{C}} = V^{-1,1} \oplus V^{0,0} \oplus V^{1,-1},$$

where $z \in \mathbb{C}^\times$ on $V^{p,q}$ via multiplication by $z^{-p}\bar{z}^{-q}$. This implies that for any $h_0 \in \mathbf{X}$ the stabilizer $\mathsf{K}_\infty \subset \mathbf{G}(\mathbb{R})$ of h_0 is compact modulo its center, and $\mathbf{G}(\mathbb{R})/\mathsf{K}_\infty \cong \mathbf{X}$ has a complex structure.

Let $\mathbb{A}_f$ denote the finite adeles over $\mathbb{Q}$, and $\mathbb{A}_f^p \subset \mathbb{A}_f$ the subgroup of adeles with trivial component at a prime p. Let $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset \mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup, where $\mathsf{K}_p \subset \mathbf{G}(\mathbb{Q}_p)$, and $\mathsf{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ are compact open subgroups. A theorem of Baily-Borel asserts that when K^p is small enough

$$\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})_{\mathbb{C}} := \mathbf{G}(\mathbb{Q})\backslash \mathbf{X} \times \mathbf{G}(\mathbb{A}_f)/\mathsf{K} \tag{6.1.1}$$

has a natural structure of an algebraic variety over $\mathbb{C}$. We will always assume in the following that K^p is small enough.

6.2 Shimura varieties over number fields

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum and $E = E(\mathbf{G}, \mathbf{X})$ the reflex field of $(\mathbf{G}, \mathbf{X})$ (see [Mil04, Definition 12.2]).

Remark 6.2.1. Any subfield F of $\bar{\mathbb{Q}}$ over which the group $\mathbf{G}$ splits contains E. This is because if T is a split maximal torus of $\mathbf{G}$ over F, then the set $W\backslash \mathrm{Hom}(\mathbb{G}_m, T)$ does not change if we pass from k to $\bar{\mathbb{Q}}$. It follows that E is a finite field extension of $\mathbb{Q}$; i.e., E is a number field.

Results of Shimura, Deligne, Milne and others imply that, up to isomorphism, $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G},\ \mathbf{X})_{\mathbb{C}}$ has a unique quasi-projective smooth model $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})$ over the number field E.

A morphism $j : (\mathbf{G}_1, \mathbf{X}_1) \to (\mathbf{G}_2, \mathbf{X}_2)$ of Shimura data induces a morphism of schemes

$$\mathrm{Sh}_{\mathsf{K}_1}(\mathbf{G}_1, \mathbf{X}_1) \to \mathrm{Sh}_{\mathsf{K}_2}(\mathbf{G}_2, \mathbf{X}_2), \tag{6.2.1}$$

provided that the compact open subgroups are chosen so that K_1 maps into K_2. This map is defined over the composite of the reflex fields $E(\mathbf{G}_1, \mathbf{X}_1)$ and $E(\mathbf{G}_2, \mathbf{X}_2)$. If the morphism j is an embedding, i.e., the homomorphism $\mathbf{G}_1 \to \mathbf{G}_2$ associated to j is a closed embedding, then for any K_1, the subgroup K_2 can always be chosen so that (6.2.1) is a closed immersion of schemes.

Sometimes people also consider the pro-varieties

$$\mathrm{Sh}_{\mathsf{K}_p}(\mathbf{G}, \mathbf{X}) := \varprojlim \mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X}),$$

where K runs through compact open subgroups as above with a fixed factor K_p at p, and

$$\mathrm{Sh}(\mathbf{G}, \mathbf{X}) := \varprojlim \mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X}),$$

where K runs through all compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. But we shall not consider such towers by fixing soon a Shimura datum $(\mathbf{G}, \mathbf{X})$ and a compact open subgroup $\mathsf{K} \subset \mathbb{A}_f$.

6.3 Shimura datum of Hodge type

Suppose that V is a finite-dimensional $\mathbb{Q}$-vector space with a perfect alternating pairing ψ and write $\mathbf{GSp} = \mathbf{GSp}(V, \psi)$ the corresponding group of symplectic similitudes. Then we get a Shimura datum $(\mathbf{GSp}, \mathbf{S}^{\pm})$ with $\mathbf{S}^{\pm}$ the Siegel double space, which is defined to be the set of maps $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ such that

(1) The $\mathbb{C}^{\times}$ action on $V_{\mathbb{R}}$ gives rise to a Hodge structure of type $(-1, 0)$ and $(0, -1)$.

(2) $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

Definition 6.3.1. A Shimura datum $(\mathbf{G}, \mathbf{X})$ is said to be **of Hodge type** if there is an embedding of Shimura data $i : (\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathbf{GSp}, \mathbf{S}^{\pm})$ for some $(\mathbf{GSp}, \mathbf{S}^{\pm})$ as above.

The reflex field of $(\mathbf{GSp}, \mathbf{S}^{\pm})$ is just $\mathbb{Q}$. We fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}$ of V and write $V_{\hat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$ with $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$. If $V_{\hat{\mathbb{Z}}}$ is stable by $\mathsf{K} \subset \mathbf{G}(\mathbb{A}_f)$, then for K^p small enough $\mathrm{Sh}_{\mathsf{K}}(\mathbf{GSp}, \mathbf{S}^{\pm})$ can be seen as the moduli space of abelian varieties.

6.4 Good reduction of Shimura varieties of Hodge type

We will mainly study the special fibre of the integral model of some Shimura variety of Hodge type. Let us fix some notations and some basic assumptions, following [KW14, 4.1] and [Kis13, (1.3.3)].

We fix a Shimura datum $(\mathbf{G}, \mathbf{X})$ of Hodge type and let $i : (\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathbf{GSp}, \mathbf{S}^{\pm})$ be an embedding of Shimura data (we do not fix this embedding for the moment). Fix a prime $p > 3$ and let $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset \mathbf{G}(\mathbb{A}_f)$ be an open compact subgroup such that $\mathsf{K}_p \subset \mathbf{G}(\mathbb{Q}_p)$ is a hyperspecial subgroup and such that $\mathsf{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ is sufficiently small. The condition that K_p is hyperspecial means that there is a reductive group $\mathcal{G}$ over $\mathbb{Z}_{(p)}$, which we fix from now on, such that $\mathsf{K}_p = \mathcal{G}(\mathbb{Z}_p)$. Then by [Kis10, (2.3.1), (2.3.2)] there is an $\mathbb{Z}_{(p)}$-lattice Λ of V such that the embedding i is induced by some embedding $i : \mathcal{G} \to \mathrm{GL}(\Lambda)$. By Zarhin's trick we may assume after changing (V, ψ) and λ that ψ induces a perfect $\mathbb{Z}_{(p)}$-pairing on Λ (cf. [Kis10, (1.3.3)]), still denoted by ψ. Now we have an embedding $\iota : \mathcal{G} \to \mathbf{GSp}(\Lambda, \psi)$ of reductive group schemes over $\mathbb{Z}_{(p)}$. The condition that K^p is sufficiently small guarantees that the double quotient in (6.1.1) has a structure of smooth quasi-projective complex variety and hence admits a canonical model $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})$ over the reflex field E of $(\mathbf{G}, \mathbf{X})$.

From now on we fix this embedding ι (and hence i). By [Kis10, Proposition (1.3.2)], $\mathcal{G}$ is then the schematic stabilizer of a set of tensors $s \subset \Lambda^{\otimes}$. From the discussion in Section 2.2, we may identify $\Lambda^{\otimes}$ with $(\Lambda^*)^{\otimes}$, and hence we have

$$\mathcal{G} = \{g \in \mathrm{GL}(\Lambda) \big| g^{\vee}(s) = s \text{ pointwise}\}. \tag{6.4.1}$$

Set $\tilde{\mathsf{K}}_p = \mathbf{GSp}(\mathbb{Z}_p)$. By [Kis10, (2.1.2)] there exists an open compact subgroup $\tilde{\mathsf{K}}^p \subset \mathbf{GSp}(\mathbb{A}_f)$ containing K^p such that ι induces an embedding of Shimura varieties over E

$$\varepsilon^0 : \mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X}) \hookrightarrow \mathrm{Sh}_{\tilde{\mathsf{K}}}(\mathbf{GSp}, \mathbf{S}^{\pm}).$$

Moreover, if $\tilde{\mathsf{K}}^p$ is sufficiently small, $\mathrm{Sh}_{\tilde{\mathsf{K}}}(\mathbf{GSp}, \mathbf{S}^{\pm})$ has a quasi-projective smooth integral canonical model over $\mathbb{Z}_{(p)}$, denoted by

$$\tilde{\mathcal{S}} := \tilde{\mathcal{S}}_{\tilde{\mathsf{K}}}(\mathbf{GSp}, \mathbf{S}^{\pm}),$$

which has an explicit moduli interpretation ([Kis10, (1.3.4)]) and admits a universal abelian scheme $\mathcal{A} \to \tilde{\mathcal{S}}$. We always assume that K^p is sufficiently small for what follows.

Fix a place v of the reflex field E of $(\mathbf{G}, \mathbf{X})$ above p. Denote by $\mathcal{O}_E$ the ring of integers of E and $\mathcal{O}_{E,(v)}$ its localization at v. Write $\kappa := k(v)$ the residue field of $\mathcal{O}_{E,(v)}$ and $\mathcal{O}_{E,v}$ the completion of $\mathcal{O}_E$ at v. The existence of the hyperspecial subgroup K_p implies that E is unramified at p ([Mil94, Corollary 4.7]), and hence we have

$$\mathcal{O}_{E,v} = W(\kappa),$$

we shall use these two notations interchangeably. Kisin showed the existence of the integral canonical model

$$\mathcal{S} := \mathcal{S}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})$$

of $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})$ over $\mathcal{O}_{E,(v)}$ by taking $\mathcal{S}$ to be the normalization of the closure of $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})$ in $\tilde{\mathcal{S}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$ and proving that it is smooth over $\mathcal{O}_{E,(v)}$ ([Kis13, (1.3.4), (1.3.5)]). In particular, we obtain a finite morphism of schemes over $\mathcal{O}_{E,(v)}$

$$\varepsilon : \mathcal{S} \to \tilde{\mathcal{S}}. \tag{6.4.2}$$

We call the pull-back to $\mathcal{S}$ of the universal abelian scheme $\mathcal{A}$ the **universal abelian scheme** of $\mathcal{S}$, still denoted by $\mathcal{A}$. Denote by

$$S := S_{\mathsf{K}}, \quad \tilde{S} := \tilde{S}_{\tilde{\mathsf{K}}}, \quad \varepsilon_0 : S \to \tilde{S}, \quad \mathcal{A}_0$$

the pull-back to $\mathrm{Spec}\kappa$ of $\mathcal{S}$, $\tilde{\mathcal{S}}$, ε and $\mathcal{A}$ respectively. We also call $\mathcal{A}_0$ the **universal abelian scheme** of S. In particular, S is a quasi-projective, smooth scheme over κ.

6.5 Reduction of Hodge cocharacters

We retain notations from Section 6.4. Then for any $h \in \mathbf{X}$, there is an associated Hodge cocharacter

$$\nu_h : \mathbb{G}_m \to \mathbf{G}_{\mathbb{C}}$$

by the formula $\nu_h(z) := h_{\mathbb{C}}(z, 1)$. More precisely, for any $\mathbb{C}$-algebra R, we have $R \otimes_{\mathbb{R}} \mathbb{C} = R \times c^*(R)$ where c denotes complex conjugation. Then on R-points the cocharacter ν_h is given by

$$R^\times \hookrightarrow (R \times c^*(R))^\times = (R \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{S}(R) \xrightarrow{h} \mathbf{G}_{\mathbb{C}}(R).$$

Denote by $[\chi]_{\mathbb{C}}$ the unique $\mathbf{G}(\mathbb{C})$-conjugacy class in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}, \mathbf{G}_{\mathbb{C}})$ which contains all the ν_h. Every element $h \in \mathbf{X}$ defines a Hodge decomposition

$V_{\mathbb{C}} = V^{(-1,0)} \oplus V^{(0,-1)}$ via the embedding $\mathbf{X} \hookrightarrow \mathbf{S}^{\pm}$. By definition $\nu_h(z)$ acts on $V^{(-1,0)}$ through multiplication by z and on $V^{(0,-1)}$ as the identity. Consequently $V_{\mathbb{C}}$ has only weight 0 and -1 with respect to any cocharacter $\lambda \in [\chi]_{\mathbb{C}}$.

Denote by G the special fibre of $\mathcal{G}$, i.e.,

$$G = \mathcal{G}_{\mathbb{F}_p} = \{g \in \mathrm{GL}(\Lambda_{\mathbb{F}_p}) | g^{\vee}(s_{\mathbb{F}_p}) = s_{\mathbb{F}_p} \text{ pointwise}\}. \tag{6.5.1}$$

It is a reductive group over $\mathbb{F}_p$. We describe in the following how to obtain a $G(\kappa)$-conjugacy class of cocharacters of G_κ over κ from the Shimura datum $(\mathbf{G}, \mathbf{X})$, as it plays an important role for the study of reductions of Shimura varieties.

As a preparation for what follows, we let $\mathsf{Z} := \mathbf{Hom}_{\mathbb{Z}_{(p)}}(\mathbb{G}_{m,\mathbb{Z}_{(p)}}, \mathcal{G})$ be the fppf sheaf of cocharacters, and $\mathsf{Ch} := \mathcal{G} \backslash \mathsf{Z}$ the quotient sheaf of Z by the adjoint action of $\mathcal{G}$. Then by [DG, Chapter XI, Corollary 4.2], the sheaf Z is represented by a smooth separated scheme over $\mathbb{Z}_{(p)}$, and it is shown in [Zha13, Proposition 2.2.2] that Ch is represented by a disjoint union of finite étale (and hence proper) scheme over $\mathbb{Z}_{(p)}$.

Note that the reductive group $\mathcal{G}_{\mathbb{Z}_p}$ is quasi-split over $\mathbb{Z}_p$ and splits over a finite étale extension of $\mathbb{Z}_p$, as is the case for any reductive group over $\mathbb{Z}_p$ ([Con11, Corollary 5.2.14]). Identifying the conjugacy class $[\chi]_{\bar{\mathbb{Q}}}$ with a point in $\mathsf{Ch}(\bar{\mathbb{Q}})$, then this point has a unique lift c in $\mathsf{Ch}(E)$. Let c_v be the pull back of c in $\mathsf{Ch}(E_v)$. Then by the properness of Ch, c_v lifts uniquely to a point $\tilde{c}_v \in \mathsf{Ch}(\mathcal{O}_{E,v})$. Let $\bar{c}_v$ be the reduction of $\tilde{c}_v$ in $\mathsf{Ch}(k)$. As G_κ is quasi-split over κ, it follows from [Kot84, Lemma 1.1.3] that there exists a cocharacter $\chi : \mathbb{G}_m \to G_\kappa$ such that the $G(\kappa)$-conjugacy class of χ, denoted by $[\chi]_\kappa$, is identified with the point $\bar{c}_v \in \mathsf{Ch}(\kappa)$. It is clear that $[\chi]_\kappa$ is uniquely determined by the Shimura datum $(\mathbf{G}, \mathbf{X})$.

For any $\lambda \in [\chi]_\kappa$, $\mathrm{Lie} G_\kappa$ has only weights 0 and 1 with respect to the adjoint action of $\mathbb{G}_m$ through λ. If λ, via the embedding $i : G \to \mathbf{GSp}(\Lambda_\kappa)$, induces a weight decomposition $\Lambda_\kappa = \Lambda_\kappa^{-1} \oplus \Lambda_\kappa^0$, then the cocharacter $(\ )^{\vee} \circ \lambda$ induces a weight decomposition

$$\Lambda_\kappa^* = \Lambda_\kappa^{*,1} \oplus \Lambda_\kappa^{*,0}, \text{ with } \Lambda_\kappa^{*,1} = (\Lambda_\kappa^{-1})^* \text{ and } \Lambda_\kappa^{*,0} = (\Lambda_\kappa^0)^*, \tag{6.5.2}$$

through the contragredient representation $(\cdot)^{\vee} : \mathrm{GL}(M) \cong \mathrm{GL}(M^*)$ (cf. Section 2.2).

We fix a cocharacter $\chi = \chi_\kappa : \mathbb{G}_{m,\kappa} \to G_\kappa$ over κ in $[\chi]_\kappa$ and define the cocharacter

$$\mu := \mathrm{Frob}_{G_\kappa/\kappa} \circ \chi : \mathbb{G}_{m,\kappa} \to G_\kappa, \tag{6.5.3}$$

where $\mathrm{Frob}_{G_\kappa/\kappa} : G_\kappa \to G_\kappa^{(p)} = G_\kappa$ is the relative Frobenius of G_κ over κ, and where we identify G_κ with $G_\kappa^{(p)}$ since it is defined over $\mathbb{F}_p$. The cocharacter χ

determines a couple of subgroups of G_κ. We introduce here all these subgroups though not all of them will be used immediately.

Denote by P_+ (resp.P_-) the parabolic subgroup of G_κ which is characterized by the property that $\mathrm{Lie}(P_+)$ (resp. $\mathrm{Lie}(P_-)$) is the sum of the non-negative (resp. non-positive) weight spaces with respect to the adjoint operation of χ on $\mathrm{Lie}(G_\kappa)$. Let U_+ (resp. U_-) be the unipotent radical of P_+ (resp. P_-) and let M be the common Levi subgroup of P_+ and P_-, which is also the centralizer of χ in G_κ.

Since the scheme Z is smooth, χ_κ lifts to a cocharacter

$$\tilde{\chi} : \mathbb{G}_{m,W(\kappa)} \to \mathcal{G}_{W(\kappa)} \tag{6.5.4}$$

of $\mathcal{G}_{W(\kappa)}$, which we fix from now on. We define a cocharacter $\tilde{\mu}$ over $W(\kappa)$ of $\mathcal{G}_{W(\kappa)}$ as

$$\tilde{\mu} := \tilde{\chi}^{(p)} : \mathbb{G}_{m,W(\kappa)} \cong \mathbb{G}^{(p)}_{m,W(\kappa)} \to \mathcal{G}^{(p)}_{W(\kappa)} \cong \mathcal{G}_{W(\kappa)}, \tag{6.5.5}$$

where we identify $\mathcal{G}^{(p)}_{W(\kappa)}$ with $\mathcal{G}_{W(\kappa)}$ since the latter is defined over $\mathbb{Z}_p$ and similarly for $\mathbb{G}_{m,W(\kappa)}$.

As in practice it is often convenient to work over the completion $\mathcal{O}_{E,v} = W(\kappa)$, the “integral model $\mathcal{S}$” appearing in the sequel will always mean the pull back $\mathcal{S} \otimes_{\mathcal{O}_{(v)}} \mathcal{O}_v$ unless otherwise stated.