Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

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5 Adapted deformations of p-divisible groups

We continue to follow notations in Sections 4.2, 4.3, and 4.5. Let (R, φ) be a simple frame of a k-algebra R_0, which locally admits a finite p-basis. We fix a p-divisible group H_0 over R_0.

5.1 Adapted deformations

We have seen in Theorem 4.3.1 that H_0 corresponds to an $\underline{R}$-window, together with a natural connection, namely

$$\underline{M} = (M := \mathbb{D}^*(H_0)(R),\ \mathrm{Fil}M,\ \varphi_M,\ \varphi_M/p,\ \ \nabla_M).$$

We let $\lambda : \mathbb{G}_{m,R} \to \mathrm{GL}(M)$ be a cocharacter over R of weights 0 and 1, such that the reduction modulo p of λ, denoted by $\lambda_0 : \mathbb{G}_{m,R_0} \to \mathrm{GL}(M_0)$, induces the Hodge filtration of $M_0 := \mathbb{D}^*(H_0)(R_0)$ as the weight 1 submodule of M_0. Let $M = N \oplus L$ be the splitting induced by λ, where N and L are submodules of M with weights 1 and 0 respectively.

Write $\mathfrak{S} = \mathfrak{S}(R)$. In the following we shall construct a Kim-Kisin window $\underline{\mathcal{M}}$ over $\underline{\mathfrak{S}}$. The construction is analogous to the construction of Kisin's "G_W-adapted deformations" in [Kis13, Prop. (1.1.13)]. We first give each of the datum in $\underline{\mathcal{M}}$ as below.

- $\mathcal{M} = M \otimes_R \mathfrak{S}$, $\mathcal{N} = N \otimes_R \mathfrak{S}$, $\mathcal{L} = L \otimes_R \mathfrak{S}$;
- $\mathrm{Fil}\mathcal{M} = \mathcal{N} \oplus E \cdot \mathcal{L}$;
- $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is the following composition

$$\mathcal{M} \to \mathcal{M}^{(\varphi)} = \mathcal{N}^{(\varphi)} \oplus \mathcal{L}^{(\varphi)} \xrightarrow{\varphi(E)\cdot \mathrm{id}_{\mathcal{N}^{(\varphi)}} \oplus \mathrm{id}_{\mathcal{L}^{(\varphi)}}} \mathcal{N}^{(\varphi)} \oplus \mathcal{L}^{(\varphi)} = M^{(\varphi)} \otimes_R \mathfrak{S}$$

$$M^{(\varphi)} \otimes_R \mathfrak{S} \xrightarrow{\Gamma^{\mathrm{lin}} \otimes \mathrm{id}_{\mathfrak{S}}} M \otimes_R \mathfrak{S} = \mathcal{M}.$$

  Here if we denote by $F^{\mathrm{lin}} : M^{(\varphi)} \to M$ the linearization of the Frobenius $F : M \to M$ then Γ^{lin} is defined as

$$\Gamma^{\mathrm{lin}} = \frac{1}{p} F^{\mathrm{lin}}|_{N^{(\varphi)}} \oplus F^{\mathrm{lin}}|_{L^{(\varphi)}} : N^{(\varphi)} \oplus L^{(\varphi)} \to M. \tag{5.1.1}$$

- $\varphi_{\mathcal{M},1} = \frac{1}{\varphi(E)} \cdot \varphi_{\mathcal{M}}$;
- $\nabla_M : M \to M \otimes_R \hat{\Omega}_R$ is already given as part of the data in $\underline{M}$.

Lemma 5.1.1. The tuple $\underline{\mathcal{M}} = (\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1}, \nabla_M)$ is indeed a Kim-Kisin window.

Proof. The nontrivial part is to verify condition (4) in Definition 4.1.2, i.e., we need to check that the the linearization

$$\varphi^{\mathrm{lin}}_{\mathcal{M},1} : \mathrm{Fil}\mathcal{M}^{(\varphi)} \to \mathcal{M}$$

is surjective (equivalently an isomorphism). A simple calculation shows that one only needs to show that the map Γ^{lin} in (5.1.1) is an isomorphism. To see this we apply Lemma 5.1.3 below by letting $P = N^{(\varphi)}$ and $Q = L^{(\varphi)}$. $\square$

The following (probably well-known) lemma will be used in the proof of Lemma 5.1.3.

Lemma 5.1.2. Let (A, φ) be a simple frame of a k-algebra A_0. Then for any homomorphism $f : A_0 \to B_0$ of k-algebras, there exists a (canonical) homomorphism of simple frames $(A, \varphi) \to (W(B_0), \varphi)$ which lifts f.

Proof. By Proposition 2, a) of [Bou06, Chap. IX, Sec. 1], the ring homomorphism (namely the ghost map)

$$\Phi_A : W(A) \to \prod_{i=0}^{\infty} A$$

in loc. cit. is injective and by c) in the same proposition the image of the ring homomorphism

$$\alpha := \prod_{i=0}^{\infty} \varphi^i : A \longrightarrow \prod_{i=0}^{\infty} A,\ a \longmapsto (a, \varphi(a), \varphi(\varphi(a)), \cdots)$$

lies in the image of Φ_A, and hence the composition $\beta := \Phi_A^{-1}|_{\Phi_A(W(A))} \circ \alpha$ gives a section (as a ring homomorphism) of the canonical projection $W(A) \to A$. Moreover, by formula (25) in loc. cit. β is compatible with Frobenius lifts. Now the composition map

$$(A, \varphi) \xrightarrow{\beta} (W(A), \varphi) \to (W(A_0), \varphi) \to (W(B_0), \varphi)$$

gives the desired homomorphism of simple frames. $\square$

When R_0 is a perfect field, the following lemma is a classical result. In the general situation as below, it is also known but for lack of good references, we give a proof.

Lemma 5.1.3. Let $M_0 = N_0 \oplus L_0$ be the normal decomposition induced by λ_{R_0} and let $M^{(\varphi)} = P \oplus Q$ be any decomposition whose reduction modulo p is identified with the decomposition $M_0^{(\varphi)} = N_0^{(\varphi)} \oplus L_0^{(\varphi)}$. Then the R-linear homomorphism

$$\Gamma^{\mathrm{lin}} := \frac{1}{p} F^{\mathrm{lin}}|_P \oplus F^{\mathrm{lin}}|_Q : M^{(\varphi)} \to M \tag{5.1.2}$$

is an isomorphism.

Proof. To see Γ^{lin} is an isomorphism we need only to show that $\Gamma^{\mathrm{lin}}\otimes\mathrm{id}_{R_\mathfrak{m}}$ is an isomorphism for each maximal ideal $\mathfrak{m}$ of R ($\mathfrak{m}$ necessarily contains p since R is p-adically complete). Denote by $\widehat{R}_\mathfrak{m}, \widehat{R}_{0,\mathfrak{m}}$ the $\mathfrak{m}$-adic completion of $R_\mathfrak{m}$, $R_{0,\mathfrak{m}}$ respectively, and $k(\mathfrak{m})$ their residue field. If we let $\overline{k(\mathfrak{m})}$ be an algebraic closure of $k(\mathfrak{m})$, then by Lemma 5.1.2, there exists a homomorphism of simple frames $(\widehat{R}_\mathfrak{m}, \varphi) \to (W(\overline{k(\mathfrak{m})}), \varphi)$, lifting the composition $\widehat{R}_{0,\mathfrak{m}} \to k(\mathfrak{m}) \to \overline{k(\mathfrak{m})}$. Since the Dieudonné functor $\mathbb{D}^*$ commutes with homomorphisms of simple frames, using the fact that the assertion holds for the classical case where R_0 is a perfect field (see for example [Zha13, Lemma 2.2.6] for a proof), one sees that $\Gamma^{\mathrm{lin}} \otimes \mathrm{id}_{W(\overline{k(\mathfrak{m})})}$ is an isomorphism. In particular, $\Gamma^{\mathrm{lin}} \otimes \mathrm{id}_{\overline{k(\mathfrak{m})}}$ is an isomorphism, and hence by faithfully flat descent, so is $\Gamma^{\mathrm{lin}} \otimes \mathrm{id}_{k(\mathfrak{m})}$. Now by Nakayama's lemma, Γ^{lin} is an isomorphism. ☐

It is easy to see that by taking the reduction modulo p of frames $\underline{\mathfrak{S}} \to \underline{R}$ one recovers the associated window $\underline{M}$ in (4.3.1) of H_0. Hence by the commutativity of diagram (4.5.3) the p-divisible group corresponding to Kim-Kisin window $\underline{\mathcal{M}}$ constructed above is a deformation over R of H_0. We call $\underline{\mathcal{M}}$ the adapted deformation of H_0 with respect to the pair (φ_R, λ). Here we emphasize the Frobenius lift φ_R because later we will study the effect of Frobenius lifts on $\underline{\mathcal{M}}$. To emphasize the pair (φ_R, λ) we may as well write

$$\Phi(\varphi_R, \lambda) := \Phi_{\mathcal{M}}(\varphi_R, \lambda) := \varphi_{\mathcal{M}}^{\mathrm{lin}}. \tag{5.1.3}$$

Remark 5.1.4. We give a few remarks concerning the construction above, which will become useful in the sequel.

(a) There is another way to see the construction of φ-linear maps $\varphi_{\mathcal{M}}$ and $\varphi_{\mathcal{M},1}$. Indeed the φ-linear map

$$\Gamma := \frac{1}{p} F|_N \oplus F|_L : M \to M$$

is a φ-linear isomorphism since its linearization is (5.1.1). Hence the base change $\Psi := \Gamma \otimes_R \varphi_{\mathfrak{S}}$ is also a φ-linear isomorphism. It is direct to check that the pair $(\varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1})$ constructed above is exactly the one induced by Ψ via Lemma 4.1.4. If we denote by $\lambda^{(\varphi)} : \mathbb{G}_{m,R} \to \mathrm{GL}(M^{(\varphi)})$ the pull back along φ_R of λ. Then it is clear that

$$\Phi(\varphi, \lambda) = \lambda^{(\varphi)}(E) \circ \Psi. \tag{5.1.4}$$

(b) It is direct to check that we have

$$\Phi(\varphi, \lambda) = \left(\varphi^*\mathcal{N} \oplus \varphi^*\mathcal{L} \xrightarrow{f} \varphi^*\mathcal{N} \oplus \varphi^*\mathcal{L} \xrightarrow{F^{\mathrm{lin}}\otimes \mathrm{id}_{\mathfrak{S}}} M \otimes_R \mathfrak{S}\right), \quad \text{with} \tag{5.1.5}$$

$$f = \frac{\varphi(E)}{p} \mathrm{id}|_{\varphi^*\mathcal{N}} \oplus \mathrm{id}|_{\varphi^*\mathcal{L}} \tag{5.1.6}$$

Strictly speaking, such a decomposition only makes sense after base change to $\mathfrak{S}[\frac{\varphi(E)}{p}] \subset \mathfrak{S}[\frac{1}{p}]$.

5.2 Functoriality of adapted deformations

Let R_0' be another k-algebra, locally admitting a finite p-basis and $R_0 \to R_0'$ a homomorphism of k-algebras. Let $(R', \varphi_{R'})$ a simple frame of R_0' and $\alpha : (R, \varphi_R) \to (R', \varphi_{R'})$ a homomorphism of simple frames over $W(k)$, which induces the structure homomorphism $R_0 \to R_0'$. By Example 4.2.3 (2) we have

$$\mathbb{D}^*(H_0 \otimes R_0')(R') \cong M \otimes_R R',$$

and hence one checks easily that the adapted deformation of $H_0 \otimes R_0'$ w.r.t. the pair $(\varphi_{R'}, \lambda_{R'})$ is the base change of $\underline{\mathcal{M}}$ along the homomorphism $\underline{\mathfrak{S}} \to \underline{\mathfrak{S}(R')}$ induced from α. In particular, we have

$$\Phi(\varphi_{R'}, \lambda_{R'}) = \Phi(\varphi_R, \lambda) \otimes R'.$$

5.3 Comparisons of different Frobenii and different decompositions

Notations are as in Section 5.1. Denote by $(\mathcal{M}_0, \varphi^{\mathrm{lin}}_{\mathcal{M}_0})$ the reduction modulo p of the pair $(\mathcal{M}, \varphi^{\mathrm{lin}}_{\mathcal{M}})$. Note that by our convention in (5.1.3) we write $\varphi^{\mathrm{lin}}_{\mathcal{M}} = \Phi(\varphi_R, \lambda)$. Similarly we write $\varphi^{\mathrm{lin}}_{\mathcal{M}_0} = \Phi_0(\varphi_R, \lambda)$. We are mainly interested in the Frobenius $\Phi_0(\varphi_R, \lambda)$ and for our purpose we shall study how the pair (φ_R, λ) should affect $\Phi_0(\varphi_R, \lambda)$.

Lemma 5.3.1. Let φ_1, φ_2 be two Frobenius lifts of R and λ_1, λ_2 two cocharacters of $\mathrm{GL}(M)$ over R of weights $0, 1$ such that the weight submodules of M are free R-modules and that the reductions modulo p of λ_1, λ_2 induce the Hodge filtration of M_0 as weight 1-submodule. Suppose that $\lambda_1 = \lambda_2 \bmod p$. Then there exists a unique automorphism $\eta : \mathcal{M}_0^{(\varphi)} \to \mathcal{M}_0^{(\varphi)}$ whose reduction modulo u is identity, such that

$$\Phi_0(\varphi_1, \lambda_1) = \Phi_0(\varphi_2, \lambda_2) \circ \eta.$$

Proof. The uniqueness of η follows from the fact that $\Phi_0(\varphi_1, \lambda_1)$, $\Phi_0(\varphi_2, \lambda_2)$ become isomorphisms after inverting $u \in \mathfrak{S}(R_0)$. We show below the existence of η. Write

$$c = \varphi_1(E) = \varphi_2(E) = u^p + p.$$

Since $\Phi(\varphi_1, \lambda_1)$ and $\Phi(\varphi_2, \lambda_2)$ become isomorphisms after inverting c there exists a unique isomorphism

$$\tilde{\eta} : (\varphi_1^*\mathcal{M})[\frac{1}{c}] \to (\varphi_2^*\mathcal{M})[\frac{1}{c}] \text{ such that } \Phi(\varphi_1, \lambda_1)[\frac{1}{c}] = \Phi(\varphi_2, \lambda_2)[\frac{1}{c}] \circ \tilde{\eta}.$$

To show the existence of η it is sufficient to show that the reduction modulo p of $\tilde{\eta}$, which is a priori only an automorphism of $\mathcal{M}_0^{(\varphi)}[\frac{1}{u}]$, comes from an automorphism of $\mathcal{M}_0^{(\varphi)}$. We show this in two steps: in the first step we let $\lambda_1 = \lambda_2$ while in the second step we let $\varphi_1 = \varphi_2$.

Step 1: Assume $\lambda = \lambda_1 = \lambda_2$. Let $M = N \oplus L$ be the decomposition induced by λ.

Recall that from Remark 5.1.4, (b), for each $i = 1, 2$ we have

$$\Phi(\varphi_i, \lambda) = (F_i^{\mathrm{lin}} \otimes \mathrm{id}_{\mathfrak{S}}) \circ f_i,$$

where $F_i^{\mathrm{lin}} : \varphi^* M \to M$ is the linearization of the Frobenius $F_i : M \to M$ of M with respect to the Frobenius lift φ_i, and f_i is defined as

$$f_i = \frac{c}{p}\mathrm{id}|_{\varphi_i^* \mathcal{N}} \oplus \mathrm{id}|_{\varphi_i^* \mathcal{L}}.$$

By Dieudonné theory there exists a unique isomorphism $\chi : \varphi_1^* M \to \varphi_2^* M$ of R-modules such that

$$F_1^{\mathrm{lin}} = F_2^{\mathrm{lin}} \circ \chi \quad \text{with} \quad \chi \otimes \mathrm{id}_{R_0} = \mathrm{id}_{M_0^{(\varphi)}}.$$

Hence we have the following commutative diagram, where we omit $[\frac{1}{c}]$ everywhere.

$$\begin{tikzcd}
\varphi_1^*\mathcal{M} \arrow[r, equal] \arrow[d, "\tilde{\eta}"] & \varphi_1^*\mathcal{N} \oplus \varphi_1^*\mathcal{L} \arrow[r, "f_1"] & \varphi_1^*\mathcal{N} \oplus \varphi_1^*\mathcal{L} \arrow[r, equal] & \varphi_1^* M \otimes_R \mathfrak{S} \arrow[d, "\chi \otimes \mathrm{id}_{\mathfrak{S}}"] \\
\varphi_2^*\mathcal{M} \arrow[r, equal] & \varphi_2^*\mathcal{N} \oplus \varphi_2^*\mathcal{L} \arrow[r, "f_2"] & \varphi_2^*\mathcal{N} \oplus \varphi_2^*\mathcal{L} \arrow[r, equal] & \varphi_2^* M \otimes_R \mathfrak{S}
\end{tikzcd} \tag{5.3.1}$$

Let $m_1, \cdots, m_r \in N$ be an R- basis of N, and $m_{r+1}, \cdots, m_{r+s} \in L$ an R-basis of L. Let

$$\begin{aligned}
\mathfrak{B}^{(\varphi_i)} &:= (\varphi_i^* m_1, \ldots, \varphi_i^* m_{r+s}), \\
\mathfrak{B}^{(\varphi_i)} \otimes 1 &:= (\varphi_i^* m_1 \otimes 1, \ldots, \varphi_i^* m_{r+s} \otimes 1),
\end{aligned}$$

be the induced basis for $\varphi_i^* M$ and for $\varphi_i^* \mathcal{M}$ respectively. Then f_1, f_2 have the same matrix representation

$$X = \begin{pmatrix} \frac{c}{p}\mathbf{I}_r & 0 \\ 0 & \mathbf{I}_s \end{pmatrix}$$

in the sense that f_i sends the basis $\mathfrak{B}^{(\varphi_i)} \otimes 1$ to $(\mathfrak{B}^{(\varphi_i)} \otimes 1)X$. Let

$$Y = \begin{pmatrix} \mathbf{I}_r + pA & pB \\ pC & \mathbf{I}_s + pD \end{pmatrix} \tag{5.3.2}$$

be the matrix representation of χ under the basis $\mathfrak{B}^{(\varphi_1)}$ and $\mathfrak{B}^{(\varphi_2)}$, where A, B, C, D are matrices with entries in R. A direct computation shows that $\tilde{\eta}$ has matrix representation

$$Z = \begin{pmatrix} \mathbf{I}_r + pA & \frac{p^2}{c}B \\ cC & \mathbf{I}_s + pD \end{pmatrix}$$

If we denote by Z_0 the reduction modulo p of Z, then it is clear that Z_0 lies in the kernel of the reduction map

$$\mathrm{GL}_{r+s}(\mathfrak{S}(R_0)) \xrightarrow{\text{mod } u} \mathrm{GL}_{r+s}(R_0).$$

Step 2. Assume $\varphi = \varphi_1 = \varphi_2$. Let

$$M = N_1 \oplus L_1, \quad M = N_2 \oplus L_2$$

be the normal decompositions of M induced by λ_1 and λ_2 respectively. Write

$$\mathcal{N}_i = N_i \otimes \mathfrak{S}, \quad \mathcal{L}_i = L_i \otimes \mathfrak{S}.$$

Then we have the following commutative diagram

$$\begin{tikzcd}
\varphi^*\mathcal{M} \arrow[r, equal] \arrow[d, "\tilde{\eta}"] & \varphi^*\mathcal{N}_1 \oplus \varphi^*\mathcal{L}_1 \arrow[r, "f_1"] & \varphi^*\mathcal{N}_1 \oplus \varphi^*\mathcal{L}_1 \arrow[r, equal] & \varphi^*M \otimes_R \mathfrak{S} \arrow[d, equal] \\
\varphi^*\mathcal{M} \arrow[r, equal] & \varphi^*\mathcal{N}_2 \oplus \varphi^*\mathcal{L}_2 \arrow[r, "f_2"] & \varphi^*\mathcal{N}_2 \oplus \varphi^*\mathcal{L}_2 \arrow[r, equal] & \varphi^*M \otimes_R \mathfrak{S}
\end{tikzcd} \tag{5.3.3}$$

Here f_i by definition is given by

$$f_i = \frac{c}{p}\mathrm{id}|_{\varphi^*\mathcal{N}_i} \oplus \mathrm{id}|_{\varphi^*\mathcal{L}_i}.$$

Let $m_{i,1}, \cdots, m_{i,r} \in N_i$ be an R- basis of N_i, and $m_{i,r+1}, \cdots, m_{i,r+s} \in L_i$ an R-basis of L_i such that

$$m_{1,j} = m_{2,j} \text{ mod } p, \quad j = 1, \ldots, r+s.$$

Here we use the assumption $\lambda_1 = \lambda_2$ mod p. We let

$$\mathfrak{B}_1^{(\varphi)} := (\varphi^* m_{1,1}, \ldots, \varphi^* m_{1,r+s}),$$
$$\mathfrak{B}_1^{(\varphi)} \otimes 1 := (\varphi^* m_{1,1} \otimes 1, \ldots, \varphi^* m_{1,r+s} \otimes 1),$$

be the induced basis for φ^*M and for $\varphi^*\mathcal{M}$ which are on the top of diagram (5.3.3). In a same way we can define $\mathfrak{B}_2^{(\varphi)}$, $\mathfrak{B}_2^{(\varphi)} \otimes 1$, and take them as the basis for φ^*M and for $\varphi^*\mathcal{M}$ which are on the bottom of diagram (5.3.3), respectively. Now the right vertical identity homomorphism has the same matrix representation as Y in (5.3.2). Then we finish the proof by following a similar matrix computation as in Step 1.

$\square$