



Universiteit
Leiden
The Netherlands

Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

Yan, Q.; Yan Q.

Citation

Yan, Q. (2017, October 18). *Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties*. Retrieved from <https://hdl.handle.net/1887/56255>

Version: Not Applicable (or Unknown)

License: [Licence agreement concerning inclusion of doctoral thesis in the Institutional Repository of the University of Leiden](#)

Downloaded from: <https://hdl.handle.net/1887/56255>

Note: To cite this publication please use the final published version (if applicable).

Cover Page



Universiteit Leiden



The handle <http://hdl.handle.net/1887/56255> holds various files of this Leiden University dissertation

Author: Yan, Qijun

Title: Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

Date: 2017-10-18

4 Classifications of p -divisible groups

4.1 General notions of frames and windows

Frames and windows were introduced by Zink in [Zin01] and greatly generalized by Lau in [Lau10]. Below we introduce these notions mainly following [CLA17].

Definition 4.1.1. A **Frame** $\underline{S} = (S, \text{Fil}S, R, \varphi, \varphi_1, \varpi)$ consists of the following data:

- a ring S and an ideal $\text{Fil}S$ of S such that $\text{Fil}S + pS$ is contained in the Jacobson radical of S .
- the quotient ring $R = S/\text{Fil}S$.
- a ring endomorphism $\varphi : S \rightarrow S$ whose reduction modulo p is the absolute Frobenius map $S/pS \rightarrow S/pS$ (in other words, the pair (S, φ) is a simple frame).
- a φ -linear map $\varphi_1 : \text{Fil}S \rightarrow S$.
- ϖ is an element in S such that $\varphi = \varpi\varphi_1$ on $\text{Fil}S$.

We say the frame $\underline{S}$ satisfies the **surjectivity condition** if the image of φ_1 generates the unit ideal of S .

Let $\underline{S}' = (S', \text{Fil}S', R', \varphi, \varphi_1, \varpi')$ be another frame. A **homomorphism of frames** from $\underline{S} \rightarrow \underline{S}'$ is a homomorphism of rings $f : S \rightarrow S'$ compatible with φ and φ_1 . Note that a morphism of frames here is called a **strict** homomorphism in literatures; see for example, [Lau10].

A frame $\underline{S}$ is called a **lifting frame** if every finite projective R -module lifts to a finite projective S -module. We shall only concern lifting frames in the sequel.

Definition 4.1.2. A **window** $\underline{M} = (M, \text{Fil}M, \varphi_M, \varphi_{M,1})$ over a lifting frame $\underline{S}$ consists of a finite projective S -module M , an S -submodule $\text{Fil}M \subset M$, and φ -linear maps $\varphi_M : M \rightarrow M$ and $\varphi_{M,1} : \text{Fil}M \rightarrow M$ subject to the following constraints

- (1) there exists a decomposition of S -modules $M = N \oplus L$ with $\text{Fil}M = N \oplus (\text{Fil}S)L$;
- (2) if $s \in \text{Fil}S$ and $m \in M$ then $\varphi_{M,1}(sm) = \varphi_1(s)\varphi_M(m)$;
- (3) for all $m \in \text{Fil}M$, $\varphi_M(m) = \varpi\varphi_{M,1}(m)$;

(4) $\varphi_{M,1}(\text{Fil}M) + \varphi_M(M)$ generate M as an S -module.

A **homomorphism of windows** is an S -linear map that preserves the filtration $\text{Fil}M$ and commutes with φ_M and $\varphi_{M,1}$.

A decomposition in (1) is called a **normal decomposition** of $\underline{M}$ (or of M).

Let $f : \underline{S} \rightarrow \underline{S}'$ be a homomorphism of frames. The **base change of a window** $\underline{M}$ over $\underline{S}$ is defined as $\underline{M}' = (M', \text{Fil}M', \varphi_{M'}, \varphi_{M',1})$, where

- $M' = M \otimes_S S'$, $\varphi_{M'} = \varphi_M \otimes_S \varphi : M' \rightarrow M'$;
- $\text{Fil}M'$ is the submodule of M' generated by $\text{Fil}S' \cdot M'$ and the image of $\text{Fil}M$ in M' ;
- $\varphi_{M',1}$ is determined by

$$\varphi_{M',1}(m \otimes x) = \varphi(x)\varphi_{M,1}(m), \quad m \otimes x \in \text{Fil}M \otimes S' \subset \text{Fil}M',$$

and condition (2) in Definition 4.1.2.

For **the dual of a window** the reader may refer to Definition 2.1.7 in [CLA17] or Section 2.1 in [Lau14], or Section 2 in [Lau10]. It will appear again but we shall not use it in detail.

Remark 4.1.3. We give several remarks on definitions above.

(a) Once we have a normal decomposition $M = N \oplus L$ of $\underline{M}$, we also have

$$\text{Fil}M = N + (\text{Fil}S)M.$$

It follows then that any decomposition $M = N \oplus L'$ of S -modules is a normal decomposition of $\underline{M}$.

(b) If $\underline{S}$ is a lifting frame, the requirement (1) is equivalent to

$$(1)' \quad (\text{Fil}S)M \subset \text{Fil}M \text{ and } M/\text{Fil}M \text{ is a projective } R\text{-module.}$$

(c) If $\underline{S}$ satisfies the surjectivity condition, the condition (2) implies (3).

Indeed, if $1 = \sum a_i \varphi_1(b_i) \in S$ with all $b_i \in \text{Fil}S$, then

$$\varphi_M(m) = \sum a_i \varphi_1(b_i) \varphi_M(m) = \sum a_i \varphi_{M,1}(b_i m) = \sum a_i \varphi(b_i) \varphi_{M,1}(m) = \varpi \varphi_{M,1}(m).$$

In this case, condition (4) means that $\varphi_{M,1}(\text{Fil}M)$ generates M and φ_M is determined by $\varphi_{M,1}$. In many cases condition (2) can be replaced by condition (3).

Lemma 4.1.4 ([Lau10, Lemma 2.6]). Let $\underline{S}$ be a frame as in Definition 4.1.2. Suppose we are given a finite projective S -module M , an S -submodule $\text{Fil}M \subset M$ and a normal decomposition $M = N \oplus L$. Then to give a pair $(\varphi_M, \varphi_{M,1})$ such that $\underline{M} = (M, \text{Fil}M, \varphi_M, \varphi_{M,1})$ is an $\underline{S}$ window is equivalent to give a φ -linear isomorphism $\Psi : N \oplus L \rightarrow M$ by the assignment

$$\Psi(n + l) = \varphi_{M,1}(n) + \varphi_M(l)$$

for $n \in N$ and $l \in L$.

Proof. We refer to [Lau10, Lemma 2.6] for the whole proof but only give here the inverse of this equivalence. Given a φ -linear isomorphism $\Psi : N \oplus L \rightarrow M$, the corresponding φ_M and $\varphi_{M,1}$ are given as follows

$$\varphi_M(n + l) = \varphi \Psi(n) + \Psi(l), \quad \varphi_{M,1}(n + al) = \Psi(n) + \varphi_1(a)\Psi(l) \quad (4.1.1)$$

for all $n \in N, l \in L$. □

4.2 Frobenius lifts and the frame $\underline{\mathfrak{S}}$

For the convenience of future discussions, we devote one subsection to the setting of algebras. Till the end of this section we let k be a perfect field of characteristic $p > 0$ and denote by $\varphi : W(k) \rightarrow W(k)$ the unique ring automorphism of $W(k)$ inducing the absolute Frobenius of k .

Frobenius lifts

Lemma 4.2.1. Let R_0 be a k algebra which (Zariski) locally admits a finite p -basis ([dJ95, Definition 1.1.1], or [BM07, Définition 1.1.1]). The following holds:

- (1) There exists a p -adic flat $W(k)$ -algebra R lifting R_0 (i.e., $R/pR \cong R_0$), which is formally smooth over $W(k)$ with respect to the p -adic topology. Such an R is unique up to (nonunique) isomorphisms and we call it a **lift of R_0** .
- (2) There is a ring endomorphism $\varphi = \varphi_R : R \rightarrow R$ lifting the absolute Frobenius of R_0 , which is compatible with $\varphi : W(k) \rightarrow W(k)$.
- (3) Let R_0, R and φ_R be as above and A_0 an étale R_0 algebra. Then there exists an R -algebra A , unique up to unique isomorphism, such that A lifts A_0 and the structure ring homomorphism $R \rightarrow A$ lifts the structure homomorphism $R_0 \rightarrow A_0$. Moreover, $\varphi_R : R \rightarrow R$ extends uniquely to a ring endomorphism of $\varphi_A : A \rightarrow A$, lifting the absolute Frobenius of A_0 .

- (4) If $\mathfrak{m}$ is a maximal idea of R , then φ extends uniquely to a ring endomorphism of the $\mathfrak{m}$ -adic completion $\widehat{R}_{\mathfrak{m}}$ of R , lifting the absolute Frobenius of the $\mathfrak{m}$ -adic completion $\widehat{R}_{0,\mathfrak{m}}$ of R_0 .

Proof. For the proof of (1) and (2), one may refer to [dJ95, Remarks 1.2.3], or [Kim15, Lemma 2.1], where deformation theory developed in [Ill71] is essentially used in the proof. Or one may see [BM07, 1.1] for an explicit construction of the lifts. The statement (3) is the first part of [Kim15, Lemma 2.5] (take $I = (p)$ and the R_0 in loc. cit. to be our R here). For (4), note first that $\varphi(\mathfrak{m}) \subset \mathfrak{m}$. This follows from the fact that $\mathfrak{m}$ contains p , and the fact that the morphism $\mathrm{Spec} R_0 \rightarrow \mathrm{Spec} R_0$ induced by the absolute Frobenius of R_0 is identity on topological spaces. Now we define $\varphi_{\widehat{R}_{\mathfrak{m}}} : \widehat{R}_{\mathfrak{m}} \rightarrow \widehat{R}_{\mathfrak{m}}$ by sending an element

$$(r_i)_i \in \varprojlim_i R/\mathfrak{m}^i = \widehat{R}_{\mathfrak{m}}$$

to $(\varphi(r_i))_i \in \widehat{R}_{\mathfrak{m}}$. The element $(\varphi(r_i))_i$ does lie in $\widehat{R}_{\mathfrak{m}}$ since $\varphi(\mathfrak{m}) \subset \mathfrak{m}$. One checks that $\varphi_{\widehat{R}_{\mathfrak{m}}}$ is the desired ring endomorphism. $\square$

Definition 4.2.2. (1) A ring endomorphism φ of R in Lemma 4.2.1, (2) is usually called a **Frobenius lift** of R over $W(k)$. But note that such a lift $\varphi : R \rightarrow R$ is in general NOT unique.

- (2) For any k -algebra R_0 (not necessarily admitting a finite p -basis), we call a pair (R, φ) satisfying conditions (1) and (2) in Lemma 4.2.1 a **simple frame** of R_0 over $W(k)$ (compare Definition 4.1.1 and [Kis06, (A.3)]).

- (3) A **homomorphism of simple frames** $(R, \varphi) \rightarrow (R', \varphi)$ is a ring homomorphism $f : R \rightarrow R'$ compatible with Frobenius lifts.

Example 4.2.3. (1) Let $\tilde{R}$ be a smooth integral $\mathbb{Z}_{(p)}$ -algebra of finite type and R the p -adic completion of $\tilde{R}$. Then R is a formally smooth flat $\mathbb{Z}_p$ -algebra which lifts $\tilde{R}/p\tilde{R}$.

- (2) The crystalline Dieudonné functor $\mathbb{D}^*$ is compatible with change of simple frames. This will be used frequently in the sequel.

The frame $\mathfrak{S}$

Let R_0 and k be as in Lemma 4.2.1 and (R, φ) a simple frame of R_0 . We associate a lifting frame

$$\mathfrak{S}(R) := (\mathfrak{S}(R), E \cdot \mathfrak{S}(R), R, \varphi, \varphi_1, \varphi(E))$$

to (R, φ) by setting:

- $\mathfrak{S}(R) = R[[u]]$, $E = E(u) = u + p \in \mathfrak{S}(R)$;

- $\varphi = \varphi_{\mathfrak{S}(R)} : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R)$ is an extension of φ_R by sending u to u^p and $\varphi_1(Ex) = \varphi(x)$.

Every morphism of simple frames $f : (R, \varphi) \rightarrow (R', \varphi)$ induces a morphism of lifting frames $\underline{\mathfrak{S}}(f) : \underline{\mathfrak{S}}(R) \rightarrow \underline{\mathfrak{S}}(R')$.

Example 4.2.4. (1) For any étale R_0 -algebra A_0 , we have a morphism of lifting frames $\underline{\mathfrak{S}}(R) \rightarrow \underline{\mathfrak{S}}(A)$ (cf. (3) in Lemma 4.2.1).

- (2) If $\mathfrak{m}$ is a maximal idea of R , then we have a morphism of lifting frames $\underline{\mathfrak{S}}(R) \rightarrow \underline{\mathfrak{S}}(\widehat{R}_{\mathfrak{m}})$, where $\widehat{R}_{\mathfrak{m}}$ is the completion of R with respect to $\mathfrak{m}$ (cf. (4) in Lemma 4.2.1).

From now on we use the notation $\mathfrak{S}(R)$ for $R[[u]]$ for an arbitrary ring R , and we often just write $\mathfrak{S}$ instead of $\mathfrak{S}(R)$ when there is no risk of confusion.

4.3 Classification of p -divisible groups over R_0

We continue to let R_0 and k be as in Lemma 4.2.1 and (R, φ) a simple frame of R_0 . Let $\underline{R}$ be a lifting frame naturally associated to the simple frame (R, φ) , defined as follows

$$\underline{R} := (R, pR, \varphi, \varphi_1, p), \quad \text{with } \varphi_1 = \frac{1}{p}\varphi.$$

We denote by $\widehat{\Omega}_R$ the module of p -adically continuous differentials of R , i.e.,

$$\widehat{\Omega}_R := \varprojlim_n \Omega_{(R/p^n R)/W(k)}^1.$$

It is a projective R -module of finite type due to the finite p -basis assumption on R_0 . We denote by $\mathbf{Win}(R, \nabla)$ the category of tuples $(M, \text{Fil}M, \varphi_M, \varphi_{M,1}, \nabla_M)$, where a tuple $(M, \text{Fil}M, \varphi_M, \varphi_{M,1})$ is a window over the frame $\underline{R}$, and $\nabla_M : M \rightarrow M \otimes_R \widehat{\Omega}_R$ is an integrable topologically quasi-nilpotent connection over the p -adically continuous derivation $d_R : R \rightarrow \widehat{\Omega}_R$ of R , with respect to which φ_M is horizontal, i.e., $\nabla_M \circ \varphi_M = (\varphi_M \otimes d_\varphi) \circ \nabla_M$.

For any p -divisible group H_0 over R_0 , we denote by $\mathbb{D}^*(H_0)$ the Dieudonné crystal of H_0 . Denote by $\mathbb{D}^*(H_0)(R)$ its evaluation at the canonical PD-thickening $R \twoheadrightarrow R_0$ and by $\mathbb{D}^*(H_0)(R_0)$ its evaluation at the trivial PD thickening $\text{id}_{R_0} : R_0 \rightarrow R_0$. Write

$$M = \mathbb{D}^*(H_0)(R), \quad M_0 = \mathbb{D}^*(H_0)(R_0).$$

Denote by $F_M : M \rightarrow M$, $F_{M_0} : M_0 \rightarrow M_0$ the Frobenius endomorphism of M and of M_0 respectively. We write $\text{Fil}M$ for the preimage of the Hodge filtration of M_0 under the canonical projection $M \twoheadrightarrow M_0$. Denote by $(\mathbf{BT}/R_0)$ the category of p -divisible groups over R_0 . The following classification result is known.

Theorem 4.3.1. For any p -divisible group H_0 over R_0 , there exists a natural connection $\nabla_M : M \rightarrow M \otimes_R \hat{\Omega}_R$ such that the tuple

$$\underline{M} = (M, \text{Fil}M, F_M, F_M/p, \nabla_M) \quad (4.3.1)$$

is an object in $\mathbf{Win}(R, \nabla)$. Moreover, such an assignment gives an equivalence of categories

$$(\mathbf{BT}/R_0) \xrightarrow{\cong} \mathbf{Win}(R, \nabla).$$

Proof. This follows from the combination of [dJ95, Theorem 4.1.1] and [CLA17, Theorem 2.6.4]. Note that [CLA17, Theorem 2.6.4] deals with filtered Dieudonné crystals as defined in [CLA17, Definition 2.4.1] but when the scheme T in loc. cit. is of characteristic p , this notion is equivalent to the version without filtrations as defined in [dJ95, Definition 2.3.2]. $\square$

4.4 Classification of p -divisible groups over R (via Dieudonné theory)

We continue to let R_0 and k be as in Lemma 4.2.1 and (R, φ) a simple frame of R_0 . Let $\underline{R}^0$ be another lifting frame naturally associated to (R, φ) , defined as follows

$$\underline{R}^0 := (R, 0, \varphi, 0, p).$$

The frame $\underline{R}^0$ does not satisfy the surjectivity condition. We denote by $\mathbf{Win}^0(R, \nabla)$ the category of tuples $(M, \text{Fil}M, \varphi_M, \varphi_{M,1}, \nabla_M)$, where a tuple $(M, \text{Fil}M, \varphi_M, \varphi_{M,1})$ is a window over the frame $\underline{R}^0$, and $\nabla_M : M \rightarrow M \otimes_R \hat{\Omega}_R$ is an integrable topologically quasi-nilpotent connection over the p -adically continuous derivation $d_R : R \rightarrow \hat{\Omega}_R$ of R such that

- (1) $\text{Fil}M \subset M$ is a direct summand, lifting $\ker(\varphi_M \otimes \varphi_{R_0}) \subset M \otimes_R R_0$.
- (2) φ_M is horizontal with respect to ∇_M (i.e., $\nabla_M \circ \varphi_M = (\varphi_M \otimes d\varphi) \circ \nabla_M$).

A morphism in $\mathbf{Win}^0(R, \nabla)$ is a morphism of windows over $\underline{R}^0$, which is compatible with connections.

For each p -divisible group H over R , we write $H_0 = H \otimes_R R_0$. Note that the Dieudonné module $M := \mathbb{D}^*(H_0)(R)$ is equipped with the Hodge filtration $\text{Fil}M \subset M$ which lifts the (Hodge) filtration $\ker(\varphi_M \otimes \varphi_{R_0}) \subset M \otimes_R R_0$. If we let (F_M, ∇_M) be as in Theorem 4.3.1, then we obtain a natural functor

$$P : (\mathbf{BT}/R) \longrightarrow \mathbf{Win}^0(R, \nabla). \quad (4.4.1)$$

by sending a p -divisible group H over R to the tuple $(M := \mathbb{D}^*(H_0)(R), \text{Fil}M, \varphi_M, 1/p\varphi_M, \nabla_M)$ described above.

A combination of Theorem 4.3.1 and Grothendieck-Messing deformation theory ([?, Chapter V, Theorem (1.6)]) gives the following classification result on p -divisible groups over R .

Theorem 4.4.1. The functor P in (4.4.1) is an equivalence of categories.

Breuil's ring S

Breuil's ring S to be defined below is closely related to the Kim-Kisin windows to be discussed in the next subsection. We will give a classification of p -divisible groups over R in term of S -modules with extra structures and then discuss the relation between such a classification and that given by Theorem 4.4.1.

Write $\mathfrak{S} = \mathfrak{S}(R)$ and let $\varpi_1 : \mathfrak{S} \rightarrow R$ be the R -algebra homomorphism sending the formal variable u to p (here we see $\mathfrak{S}$ as an R -algebra through the embedding $R \hookrightarrow \mathfrak{S}$). Let S be the p -adic completion of the divided power envelope of $\mathfrak{S}$ with respect to the kernel of ϖ_1 (namely the ideal $E = E(u) = u + p$).

The ring S is $\mathbb{Z}_p$ -flat and is a subring of $S[\frac{1}{p}]$ (see [Kim15, Section 3.3] for an explicit description of S). Let $\pi_1 : S \rightarrow R$ be the natural projection of R -algebras sending u to p and denote its kernel by $\mathrm{Fil}S$. Then $\mathrm{Fil}S$ is topologically generated by the divided powers $\{\frac{E(u)^n}{n!}\}_{n \geq 1}$ of $E(u)$. The Frobenius lift $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ extends uniquely to a Frobenius lift $\varphi = \varphi_S : S \rightarrow S$ of S , and we have $\varphi(\mathrm{Fil}S) \subset pS$. The tuple

$$\underline{S} := (S, \mathrm{Fil}S, \varphi, \frac{1}{p}\varphi, p)$$

is a frame satisfying the surjectivity condition (in fact, $\frac{\varphi(E)}{p}$ is unit in S). In addition to the natural projection $\pi_1 : S \rightarrow R$ which sends u to p , there is also another natural projection $\pi_2 : S \rightarrow R$ of R -algebras which sends u to 0. The kernel of π_2 , denoted by $\mathrm{Fil}'S$, is topologically generated by u and all $\{\frac{u^n}{n!}\}_{n \geq 1}$. Moreover, $\mathrm{Fil}'S$ is φ -stable and hence p_2 induces a homomorphism of simple frames $(S, \varphi) \rightarrow (R, \varphi)$. Note that π_1 and π_2 are two sections of the natural embedding $R \hookrightarrow S$ and they induce two morphisms of PD thickenings

$$(S \twoheadrightarrow R_0) \xrightarrow{\pi_1} (R \twoheadrightarrow R_0), \quad (S \twoheadrightarrow R_0) \xrightarrow{\pi_2} (R \twoheadrightarrow R_0), \quad (4.4.2)$$

which are two sections of the PD morphism $(R \twoheadrightarrow R_0) \hookrightarrow (S \twoheadrightarrow R_0)$. Here we use $(S \twoheadrightarrow R_0) \xrightarrow{\pi_1} (R \twoheadrightarrow R_0)$ to denote a commutative diagram

$$\begin{array}{ccc} S & \twoheadrightarrow & R_0 \\ \downarrow \pi_1 & & \parallel \\ R & \twoheadrightarrow & R_0. \end{array}$$

The embedding $R \rightarrow S$ induces homomorphisms of frames

$$\begin{aligned} \underline{R} = (R, pR, \varphi, \tfrac{1}{p}\varphi, p) &\longrightarrow \underline{S} = (S, \text{Fil}S, \varphi, \tfrac{1}{p}\varphi, p); \\ \underline{R}^0 = (R, 0, \varphi, 0, p) &\longrightarrow \underline{S}. \end{aligned} \quad (4.4.3)$$

The second map in (4.4.2) induce a homomorphism of frames

$$\underline{S} \longrightarrow \underline{R}^0. \quad (4.4.4)$$

Denote by $\mathbf{Win}(S, \nabla^0)$ the category of tuples $(M, \text{Fil}M, \varphi_M, \varphi_{M,1}, \nabla_{M \otimes_{S, \pi_2} R})$, where a tuple $(M, \text{Fil}M, \varphi_M, \varphi_{M,1})$ is a window over $\underline{S}$, and $\nabla_{M \otimes_{S, \pi_2} R} : M \otimes_{S, \pi_2} R \rightarrow M \otimes_{S, \pi_2} R \otimes_R \widehat{\Omega}_R$ is an integrable topologically quasi-nilpotent connection over the p -adically continuous derivation $d_R : R \rightarrow \widehat{\Omega}_R$ of R , with respect to which $\varphi_{M \otimes_{S, \pi_2} R} := \varphi_M \otimes \varphi_R$ is horizontal.

For each object $\underline{M} = (M, \text{Fil}M, \varphi_M, \varphi_{M,1}, \nabla_M)$ in $\mathbf{Win}^0(R, \nabla)$, we can associate to $\underline{M}$ an object $(M, \text{Fil}M, \varphi_M, \varphi_{M,1}, \nabla_{M \otimes_{S, \pi_2} R})$ in $\mathbf{Win}(S, \nabla^0)$ by setting $(M, \text{Fil}M, \varphi_M, \varphi_{M,1})$ to be the base change of the window $(M, \text{Fil}M, \varphi_M, \varphi_{M,1})$ along the homomorphism of frames $\underline{R}^0 \rightarrow \underline{S}$ in (4.4.3), and $\nabla_M = \nabla_{M \otimes_{S, \pi_2} R}$. Note that by definition $\text{Fil}M = \text{Fil}M \otimes_R S + \text{Fil}SM$ and hence $\text{Fil}M$ is also the preimage of $\text{Fil}M$ under the projection $M = M \otimes_R S \xrightarrow{\text{id}_M \otimes \pi_1} M$. Such associations are functorial and hence induces a functor

$$X : \mathbf{Win}^0(R, \nabla) \longrightarrow \mathbf{Win}(S, \nabla^0).$$

Write $Q = X \circ P : (\mathbf{BT}/R) \longrightarrow \mathbf{Win}(S, \nabla^0)$.

Theorem 4.4.2 ([Kim15, Theorem 3.17]). The functor $X : (\mathbf{BT}/R) \longrightarrow \mathbf{Win}(S, \nabla^0)$ is an equivalence of categories.

Remark 4.4.3. Note that we are in the simplest situation of [Kim15] since in our situation here no ramification happens and hence we can identify R with $\mathfrak{S}/(E(u))$. To be precise, our $R, \mathfrak{S}/(E(u))$ play the role of R_0, R in [Kim15] respectively and $p \subset \mathfrak{S}/(E(u))$ plays the role of ϖ in loc. cit. Though in our case we can identify R with $\mathfrak{S}/(E(u))$, sometimes it is still necessary to distinguish them. In the general situation as in loc. cit., π_1 and π_2 have different targets.

It follows immediately from Theorem 4.4.1 and Theorem 4.4.2 that the functor X is also an equivalence of categories. To give an explicit inverse functor of X , we need the following fact: for the S -module M in an object $\underline{M}$ of $\mathbf{Win}(S, \nabla^0)$, we have a canonical isomorphism of R -modules

$$M \otimes_{S, \pi_1} R \cong M \otimes_{S, \pi_2} R. \quad (4.4.5)$$

This is due to the crystalline interpretation of $\mathbf{M}$ via Theorem 4.4.2. Indeed, we may assume $\mathbf{M} = \mathbb{D}^*(H_0)(S)$ for some p -divisible group H_0 over R_0 . Then we have a canonical isomorphism

$$\mathbf{M} \otimes_{S, \pi_1} R \cong \mathbb{D}^*(H_0)(R) \cong \mathbf{M} \otimes_{S, \pi_2} R.$$

Define a functor

$$Y : \mathbf{Win}(S, \nabla^0) \longrightarrow \mathbf{Win}^0(R, \nabla) \quad (4.4.6)$$

$$(\mathbf{M}, \text{Fil}\mathbf{M}, \varphi_{\mathbf{M}}, \varphi_{\mathbf{M}, 1}, \nabla_{\mathbf{M} \otimes_{S, \pi_2} R}) \longmapsto (M := \mathbf{M} \otimes_{S, \pi_2} R, \text{Fil}M, \varphi_M \otimes \varphi_R, \frac{1}{p}(\varphi_M \otimes \varphi_R), \nabla_M),$$

where $\nabla_{\mathbf{M} \otimes_{S, \pi_2} R}$ is sent to itself (since we set $M := \mathbf{M} \otimes_{S, \pi_2} R$), and $\text{Fil}M$ is the image of $\text{Fil}\mathbf{M}$ under the projection

$$\mathbf{M} \rightarrow \mathbf{M} \otimes_{S, \pi_1} R \cong \mathbf{M} \otimes_{S, \pi_2} R = M.$$

It is trivial to check that X and Y are inverse to each other.

4.5 Kim-Kisin windows and Kim-Kisin modules

Breuil-Kisin modules, as defined in [Kis06, (2.2.1)], play a vital rule in the development of integral p -adic Hodge theory. They are typically used to classify p -divisible groups over a totally ramified extension R of $W(k)$ (of arbitrary finite ramification index). Such a classification was conjectured in a precise form by Breuil (see [Bre98]) and was first proved by Kisin in [Kis06]. A similar classification result was generalized by Brinon and Trihan ([BT08]) to the case where R is a p -adic discrete valuation ring with imperfect residue field admitting a finite p -basis. The case where R is a regular local ring with perfect residue field is studied in a series of paper by Cais, Lau, Vasiu and Zink ([VZ10], [Lau14], [CLA17]), using the theory of displays and windows. W. Kim in [Kim15] generalized the classification results aforementioned to a relative setting (e.g., R is a p -adic ring with $R/(p)$ locally admitting a finite p -basis), where he essentially used the method developed in [CL09]. We call the relative version of Breuil-Kisin modules Kim-Kisin modules (cf. [Kim15, Definition 6.1]). In the following we are dealing with the simplest relative situation in the sense that no nontrivial ramification occurs.

We retain the notations in Section 4.2 and let $\underline{\mathfrak{S}} := \underline{\mathfrak{S}}(R)$ be the lifting frame associated to a simple frame (R, φ) .

Definition 4.5.1. A **Kim-Kisin $\mathfrak{S}$ window**, or simply a **Kim-Kisin window**

$$\underline{\mathcal{M}} = (\mathcal{M}, \text{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M}, 1}, \nabla_M)$$

is a 5-tuple where

- (1) $(\mathcal{M}, \text{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1})$ is an $\mathfrak{S}$ -window.
- (2) M is defined to be $\mathcal{M} \otimes_{\mathfrak{S}} \mathfrak{S}/(u)$ and $\nabla_M : M \rightarrow M \otimes_R \widehat{\Omega}_R$ is an integrable topologically quasi-nilpotent connection over the p -adically continuous derivation $d_R : R \rightarrow \widehat{\Omega}_R$, with respect to which the φ -linear endomorphism $F_M := \varphi_{\mathcal{M}} \otimes_{\mathfrak{S}} \varphi_R$ of M is horizontal.

Remark 4.5.2. The following remarks will be needed in the sequel.

- (a) For a Kim-Kisin $\mathfrak{S}$ window $\underline{\mathcal{M}}$, we have $\varphi_{\mathcal{M},1} = \frac{1}{\varphi(E)}\varphi_{\mathcal{M}}$ on $\text{Fil}\mathcal{M}$ since $\varphi(E) \in \mathfrak{S}$ is not a zero divisor, and hence the map $\varphi_{\mathcal{M},1}$ is determined by $\varphi_{\mathcal{M}}$; see (c) in Remark 4.1.3.
- (b) For any normal decomposition $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$ of $\underline{\mathcal{M}}$, by definition the φ -linear map $\varphi_{\mathcal{M}}$ has the decomposition

$$\varphi_{\mathcal{M}} = (\mathcal{N} \oplus \mathcal{L} \xrightarrow{E \cdot \text{id}_{\mathcal{N}} \oplus \text{id}_{\mathcal{L}}} \mathcal{N} \oplus \mathcal{L} \xrightarrow{\Gamma} \mathcal{M}), \quad (4.5.1)$$

where

$$\Gamma = \frac{1}{E} \varphi_{\mathcal{M}}|_{\mathcal{M}} \oplus \varphi_{\mathcal{M}}|_{\mathcal{L}}$$

is a φ -linear isomorphism (Lemma 4.1.4). The linearization of (4.5.1) is the following

$$\varphi_{\mathcal{M}}^{\text{lin}} = (\mathcal{N}^{(\varphi)} \oplus \mathcal{L}^{(\varphi)} \xrightarrow{\varphi(E) \cdot \text{id} \oplus \text{id}} \mathcal{N}^{(\varphi)} \oplus \mathcal{L}^{(\varphi)} \xrightarrow{\Gamma^{\text{lin}}} \mathcal{M}), \quad (4.5.2)$$

where $\varphi_{\mathcal{M}}^{\text{lin}}, \Gamma^{\text{lin}}$ are the linearizations of $\varphi_{\mathcal{M}}$ and Γ respectively, and where Γ^{lin} is an $\mathfrak{S}$ -isomorphism.

Definition 4.5.3. A **Kim-Kisin $\mathfrak{S}$ module** $\underline{\mathfrak{M}}$ is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_M)$ where

- (1) $\mathfrak{M}$ is a finite projective $\mathfrak{S}$ -module;
- (2) $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -linear map such that the cokernel of the linearization $(1 \otimes \varphi_{\mathfrak{M}}) : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is annihilated by $E \in \mathfrak{S}$.
- (3) M is defined as

$$M := \mathfrak{M} \otimes_{\mathfrak{S}, \varphi} \mathfrak{S}/(u) = \varphi^* \mathfrak{M} \otimes_{\mathfrak{S}, \varpi_2} R$$

and $\nabla_M : M \rightarrow M \otimes_R \widehat{\Omega}_R$ is an integrable topologically quasi-nilpotent connection which commutes with the φ -linear endomorphism of M ,

$$F_M := (\varphi_{\mathfrak{M}} \otimes 1) \otimes_{\mathfrak{S}, \varpi_2} \varphi_R.$$

Denote by $\mathbf{Win}(\mathfrak{S}, \nabla^0)$ the category of Kim-Kisin-windows and by $\mathbf{Mod}(\mathfrak{S}, \nabla^0)$ the category of Kim-Kisin $\mathfrak{S}$ -modules, both with respect to (R, φ_R) . Here we emphasize the simple frame (R, φ_R) , especially the Frobenius lift φ_R , because later we are going to study the effect of different Frobeni on Kim-Kisin windows.

Proposition 4.5.4. There is an equivalence of categories

$$\begin{aligned} \mathbf{Win}(\mathfrak{S}, \nabla^0) &\longrightarrow \mathbf{Mod}(\mathfrak{S}, \nabla^0), \\ (\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1}, \nabla_M) &\longmapsto (\mathrm{Fil}\mathcal{M}, E \cdot \varphi_{\mathcal{M},1}, \nabla_M), \end{aligned}$$

where the second ∇_M is justified via the isomorphism $\varphi_{\mathcal{M},1} \otimes 1 : \varphi^* \mathrm{Fil}\mathcal{M} \rightarrow \mathcal{M}$. This equivalence preserves exactness and duality.

Proof. This follows from an exactly same argument as in [CLA17, lemma 2.1.15], though connections are not concerned in loc. cit. We sketch here how to obtain the inverse of the equivalence of categories.

For any Kim-Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_M)$. There is a unique $\mathfrak{S}$ -linear map $\psi : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ such that

$$\varphi_{\mathfrak{M}} \circ \psi = E(u) \cdot \mathrm{id}_{\mathfrak{M}}, \quad \psi \circ \varphi_{\mathfrak{M}} = E(u) \cdot \mathrm{id}_{\varphi^* \mathfrak{M}}.$$

We obtain a Kim-Kisin window $\underline{\mathcal{M}} = (\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1}, \nabla_M)$ by setting

$$\begin{aligned} \mathcal{M} &:= \varphi^* \mathfrak{M}, \quad \mathrm{Fil}\mathcal{M} := \psi(\mathfrak{M}), \quad \varphi_{\mathcal{M}} := \varphi_{\mathfrak{M}} \otimes \mathrm{id}_{\mathfrak{S}}; \\ \varphi_{\mathcal{M},1}(\psi(x)) &= x \otimes 1, \quad \text{for any } x \in \mathfrak{M}. \end{aligned}$$

□

Theorem 4.5.5 ([Kim15, Corollary 6.7]). There is an equivalence of categories

$$(\mathbf{BT}/R) \rightarrow \mathbf{Mod}(\mathfrak{S}, \nabla^0).$$

It is compatible with duality. Moreover, if $(R, \varphi) \rightarrow (R', \varphi)$ is morphism of simple frames, then the equivalence commutes with base change of frames along $\underline{\mathfrak{S}}(R) \rightarrow \underline{\mathfrak{S}}(R')$.

Then it follows immediately that:

Corollary 4.5.6. There is a category equivalence

$$Z : (\mathbf{BT}/R) \rightarrow \mathbf{Win}(\mathfrak{S}, \nabla^0).$$

It is compatible with duality. Moreover, if $(R, \varphi) \rightarrow (R', \varphi)$ is morphism of simple frames, then the equivalence commutes with base change of frames from $\underline{\mathfrak{S}}(R) \rightarrow \underline{\mathfrak{S}}(R')$.

Remark 4.5.7. (1) In Corollary (4.5.6) there is no obvious map from either of the left and the right hand sides to the other.

(2) The classification of p -divisible groups over R in terms of Kim-Kisin $\mathfrak{S}$ -modules gives rise to a classification of (p -power order) finite flat group schemes over R in terms of torsion Kim-Kisin $\mathfrak{S}$ -modules (by no means trivial, see [Kim15, 9.8]). We will not explicitly use it. But note that taking the reduction modulo p of the Kim-Kisin module of a p -divisible group H over R one obtains exactly the Kim-Kisin module associated with the p -kernel of H . We will take such operations frequently in the future without further explanation.

The following lemma describes the basic relations between Kim-Kisin modules and its crystalline Dieudonné modules.

Lemma 4.5.8. Let $(\mathcal{M}, \text{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1}, \nabla_M)$ be the Kim-Kisin window associated to a p -divisible group H over R . We have the following canonical identifications.

- (1) $(\mathcal{M}, \varphi_{\mathcal{M}}) \otimes_{\mathfrak{S}, \varpi_2} R$ is canonically identified with the (contravariant) Dieudonné module $\mathbb{D}^*(H)$ of H , together with its Frobenius map.
- (2) $(\mathcal{M}, \text{Fil}\mathcal{M}) \otimes_{\mathfrak{S}, \varpi_1} R$ is canonically identified with $\mathbb{D}^*(H)$ together with its Hodge filtration (cf. [BBM82, Corollaire 3.3.5]).

The combination of Proposition 4.5.4, Theorem 4.5.5, and Theorem 4.3.1 gives below a commutative diagram.

$$\begin{array}{ccc}
 (\mathbf{BT}/R) & \xrightarrow{\cong} & \mathbf{Mod}(\mathfrak{S}, \nabla^0) \xrightarrow{\cong} \mathbf{Win}(\mathfrak{S}, \nabla^0) \\
 \downarrow \text{mod } p & & \downarrow \text{mod } u \\
 (\mathbf{BT}/R_0) & \xrightarrow{\cong} & \mathbf{Win}(R, \nabla)
 \end{array} \tag{4.5.3}$$

where we simply use “ $\cong$ ” to denote an equivalence of categories, and where the right vertical functor “mod u ” denotes the base change of windows along the homomorphism of lifting frames

$$\mathfrak{S} \xrightarrow{\text{mod } u} R.$$

Proof. This follows from the discussions between Remark 3.13 and Lemma 3.14 in [Kim15, Corollary 6.7], [Kim15, Corollary 6.7] and [Kim15, Corollary 6.7] together with the erratum [Kim15b]. $\square$

4.6 A comparison of classification results

In the remaining of this section, we compare the classification result given in Theorem 4.4.1 and that given in Corollary 4.5.6 by establishing an explicit functor from $\mathbf{Win}(\mathfrak{S}, \nabla^0)$ to $\mathbf{Win}^0(R, \nabla)$.

Let $\varpi_2 : \mathfrak{S} \rightarrow R$ be the homomorphism of R -algebras sending u to 0. Clearly we have a homomorphism of simple frames $(\mathfrak{S}, \varphi) \rightarrow (S, \varphi)$ induced by the embedding $\mathfrak{S} \rightarrow S$, and $(\mathfrak{S}, \varphi) \rightarrow (S, \varphi)$ induced by ϖ_2 . The compositions of the embedding $\mathfrak{S} \rightarrow S$ with π_1 and π_2 are equal to ϖ_1 and ϖ_2 respectively.

We now define a functor from $\mathbf{Win}(\mathfrak{S}, \nabla^0)$ to $\mathbf{Win}(S, \nabla^0)$ as follows.

$$W : \mathbf{Win}(\mathfrak{S}, \nabla^0) \longrightarrow \mathbf{Win}(S, \nabla^0)$$

$$(\mathcal{M}, \text{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1}, \nabla_{\mathcal{M} \otimes_{\mathfrak{S}, \varpi_2} R}) \longmapsto (\mathbf{M} := \mathcal{M} \otimes_{\mathfrak{S}} S, \text{Fil}\mathbf{M}, \varphi_{\mathbf{M}} := \varphi_{\mathcal{M}} \otimes \varphi_S, \frac{1}{p}\varphi_{\mathbf{M}}, \nabla_{\mathbf{M} \otimes_{S, \varpi_2} R}),$$

where the connection $\nabla_{\mathbf{M} \otimes_{S, \varpi_2} R}$ on the right hand side makes sense since by definition of $\mathbf{M}$ we have $\mathbf{M} \otimes_{S, \pi_2} R = \mathcal{M} \otimes_{\mathfrak{S}, \varpi_2} R$.

Theorem 4.6.1. The functor W is well defined and is an equivalence of categories. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} (\mathbf{BT}/R) & \xrightarrow{Z} & \mathbf{Win}(\mathfrak{S}, \nabla^0) \\ & \searrow Q & \downarrow W \\ & & \mathbf{Win}(S, \nabla^0) \end{array} \quad (4.6.1)$$

Proof. To see that W is well defined, we need to check that $\varphi_{\mathbf{M}} := \frac{1}{p}\varphi_{\mathbf{M}}$ makes sense. To see this, let $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$ be a normal decomposition of $\mathcal{M}$, then we have $\text{Fil}\mathcal{M} = \mathcal{N} \otimes_{\mathfrak{S}} S + \text{Fil}S \cdot \mathcal{M}$. Since $\varphi_{\mathcal{M}}(\mathcal{N}) \subset \varphi(E)\mathcal{M}$, we have

$$\varphi_{\mathbf{M}}(\mathcal{N} \otimes_{\mathfrak{S}} S) \subset \varphi(E)\mathbf{M} = \frac{\varphi(E)}{p} \cdot p\mathbf{M} = p\mathbf{M}.$$

Here we use the fact that $\frac{\varphi(E)}{p}$ is a unit in S . Now it follows from the inclusion $\varphi(\text{Fil}S) \subset pS$ that the functor W is well defined.

The fact that W is an equivalence of categories is due to [Kim15, Proposition 6.6] and [Kim15, Corollary 6.7]. Indeed, using the notations in loc. cit., the equivalence between $\text{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ and $\text{MF}_S(\varphi, \nabla)$ induces trivially an equivalence $W^0 : \text{Mod}_{\mathfrak{S}}(\varphi, \nabla^0) \rightarrow \text{MF}_S(\varphi, \nabla^0)$, which translated into our language via the equivalence in Proposition 4.5.4 is the W defined above.

The commutativity of (4.6.1) is due to the way in which the equivalence in Corollary 4.5.6 is deduced (see the remark below). $\square$

Remark 4.6.2. In fact, Theorem 4.5.5 is deduced in [Kim15] by first proving Theorem 4.4.2 and then showing the equivalence

$$\mathbf{Mod}(\mathfrak{S}, \nabla^0) \longrightarrow \mathbf{Win}(\mathfrak{S}, \nabla^0) \xrightarrow{W} \mathbf{Win}(S, \nabla^0).$$

Now a direct computation gives the following corollary.

Corollary 4.6.3. The composition of W and Y gives an equivalence of categories

$$\mathrm{Pr} : \mathbf{Win}(\mathfrak{S}, \nabla^0) \longrightarrow \mathbf{Win}^0(R, \nabla) \quad (4.6.2)$$

$$(\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M},1}, \nabla_{\mathcal{M} \otimes_{S, \varpi_2} R}) \longmapsto (M := \mathcal{M} \otimes_{\mathfrak{S}, \varpi_2} R, \mathrm{Fil}M, \varphi_M := \varphi_{\mathcal{M}} \otimes \varphi_R, \frac{1}{p} \varphi_M, \nabla_M),$$

where $\mathrm{Fil}M$ is the image of $\mathrm{Fil}\mathcal{M}$ under the projection

$$\mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathfrak{S}, \varpi_1} R \cong \mathcal{M} \otimes_{\mathfrak{S}, \varpi_2} R = M.$$

Here the isomorphism $\mathcal{M} \otimes_{\mathfrak{S}, \varpi_1} R \cong \mathcal{M} \otimes_{\mathfrak{S}, \varpi_2} R$ is due to (4.4.5) and is thus also canonical.

Remark 4.6.4. It is not clear to us how to give a direct construction of the inverse functor of Pr , even when R is $W(k)$ itself.