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## **Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties**

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### **Citation**

Yan, Q. (2017, October 18). *Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties*. Retrieved from <https://hdl.handle.net/1887/56255>

Version: Not Applicable (or Unknown)

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**Note:** To cite this publication please use the final published version (if applicable).

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**Date:** 2017-10-18

## 3 Loop groups

### 3.1 Representability of loop groups

**Definition 3.1.1.** Let  $k$  be a field and  $G$  a linear algebraic group over  $k$ . The **(algebraic) loop group** of  $G$ , denoted by  $\mathcal{L}G$ , is the fpqc sheaf of groups whose  $A$ -valued points for a  $k$ -algebra  $A$  is given by  $\mathcal{L}G(A) = G(A((u)))$ , where  $A((u))$  is the ring of Laurent series with coefficients in  $A$ . Let  $\mathcal{K} := \mathcal{L}^+G \subset \mathcal{L}G$  be the subgroup of  $\mathcal{L}G$  with  $\mathcal{K}(A) = G(A[[u]])$ , where  $A[[u]]$  is the ring of formal power series with coefficients in  $A$ . It is called the **(strict) positive loop group** of  $G$ . We set  $\mathcal{K}_1$  to be the kernel of the reduction map  $\mathcal{K} \rightarrow G$ .

In fact, the functors  $\mathcal{L}^+(\cdot)$  and  $\mathcal{L}(\cdot)$  can be defined more generally. For example, if  $X$  is a scheme over  $k[[u]]$ , and  $\mathcal{X}$  a scheme over  $k((u))$ , we can define the **positive loop functor**  $\mathcal{L}^+X$  and the **loop functor**  $\mathcal{L}\mathcal{X}$  as follows: for any  $k$ -algebra  $A$ ,

$$\mathcal{L}^+X(A) := X(A[[u]]), \quad \mathcal{L}\mathcal{X}(A) := X(A((u))).$$

These are the so called the **twisted** versions in [PR08]) to distinguish the cases where  $X$  and  $\mathcal{X}$  come from  $k$ -schemes. Note that if  $X$  does come from a  $k$ -scheme, i.e.,  $X = X_0 \otimes_k k[[u]]$  for some  $k$ -scheme  $X_0$ , then we have  $\mathcal{L}^+G(X) = \mathcal{L}^+G(X_0)$ ; the similar assertion holds for the functor  $\mathcal{L}(\cdot)$  as well. We shall not need such generalities in the sequel.

In what follows in this section, we let  $G$  be a reductive group over  $k$ .

**Lemma 3.1.2.** The positive loop group  $\mathcal{L}^+G$  is represented by an affine group scheme (of infinite type) over  $k$ .

For the proof the preceding lemma, one may see Proposition 3.2.1 in B. Levin's thesis [Lev13]. The basic idea is to embed  $G$  into  $\mathrm{GL}_{n,k}$ , and then reduce to show that  $\mathcal{L}^+\mathrm{GL}_{n,k}$  is represented by an affine group scheme and that  $\mathcal{L}^+G \hookrightarrow \mathcal{L}^+\mathrm{GL}_{n,k}$  is a closed embedding. We refer to the discussion before Definition 1.1 in [PR08] for the equations defining this embedding.

Following the conventions in [PR08], we call an fpqc sheaf  $Y$  over  $k$  an **ind-scheme** if it is the inductive limit of the functors associated to a directed system  $\{Y_i\}$  of  $k$ -schemes. We say  $Y$  is **strict** if in addition, the transition morphisms are closed embeddings. A **group ind-scheme** is an ind-scheme which is a group object in the category of ind-schemes. We say  $Y$  is **ind-affine** (resp. **ind-finite type**, **ind-projective**, etc) if each  $Y_i$  can be taken to be affine (resp. ind-finite type, projective, etc).

**Proposition 3.1.3** ([Lev13, Proposition 3.2.4.]). The loop group  $\mathcal{L}G$  is represented by a strict ind-affine group ind-scheme over  $k$ .

We only explain here the basic idea of the proof. In the case  $G = \mathrm{GL}_{n,k}$ , this is well-known (see [BL94]). For each  $N \geq 0$ , denote by  $\mathrm{GL}_{n,k}^{(N)}$  the subfunctor of  $\mathcal{L}\mathrm{GL}_{n,k}$  given by

$$\mathrm{GL}_{n,k}^{(N)}(R) = \{M \in \mathrm{GL}_n(R((u))) \mid \text{both } M \text{ and } M^{-1} \text{ have at most } N \text{ poles}\}.$$

One shows that each  $\mathrm{GL}_{n,k}^{(N)}$  is represented by an affine scheme (not group scheme, unless  $N = 0$ ) and the transition morphisms are closed immersions. Clearly one has then

$$\mathcal{L}\mathrm{GL}_{n,k} = \bigcup_{N=0}^{\infty} \mathrm{GL}_{n,k}^{(N)}.$$

For the general case, again we take an embedding  $G \hookrightarrow \mathrm{GL}_{n,k}$  and set

$$G^{(N)} = \mathcal{L}G \cap \mathrm{GL}_{n,k}^{(N)}. \quad (3.1.1)$$

Then one finishes the proof by showing that each  $G^{(N)}$  is a closed subscheme of  $\mathrm{GL}_{n,k}^{(N)}$ .

### 3.2 A subfunctor $\mathcal{D}(G, x)$ of the loop group $\mathcal{L}G$

Take  $x \in \mathcal{L}G(k)$  and let  $\mathcal{C}(G, x)$  be the subfunctor of  $\mathcal{L}G$  which associates to a  $k$ -algebra  $R$  the subset  $\mathcal{K}(R)x\mathcal{K}(R)$  of  $\mathcal{L}G(R)$ . Denote by  $\mathcal{D}'(G, x)$  the fpqc sheafification of  $\mathcal{C}(G, x)$ . We show in this subsection that  $\mathcal{D}'(G, x)$  is represented by a subscheme of  $\mathcal{L}G$ .

Let  $G^{(i)}$  be as in (3.1.1), then we have  $\mathcal{L}G = \varinjlim G^{(i)}$ . Since the point  $x$  lies in some  $G^{(i)}(k)$ , a natural idea is to consider the group action of schemes

$$\rho : \mathcal{K} \times \mathcal{K} \times G^{(i)} \rightarrow G^{(i)}, (g, h, z) \mapsto gzh^{-1} \quad (3.2.1)$$

and the orbit map

$$\rho_x : \mathcal{K} \times \mathcal{K} \rightarrow G^{(i)}, (g, h) \mapsto gxh^{-1}.$$

Indeed, if the orbit map  $\rho_x$  has locally closed image, one would like to hope that the sheaf  $\mathcal{D}'(G, x)$  is represented by a subscheme (maybe not reduced) of  $G^{(i)}$  with underlying topological space  $|\mathrm{Im}(\rho_x)|$ . This idea turns out to be correct. However, since both  $\mathcal{K}$  and  $G^{(i)}$  are infinitely dimensional and it is not clear whether the morphisms  $\rho, \rho_x$  are of finite type or not, many of the techniques and results of algebraic group actions on algebraic schemes (among them is Chevalley's theorem on constructible sets) do not apply directly. For this reason we need to pass to the affine Grassmannians of  $G$ .

The **affine Grassmannian** of  $G$ , denoted by  $\mathrm{Gr}$ , is the fpqc sheafification of the presheaf which, to every  $k$ -algebra  $R$ , associates the quotient  $\mathcal{L}G(R)/\mathcal{K}(R)$ .

Let  $\mathcal{L}G = \varinjlim G^{(i)}$  be as above. If we denote by  $(G^{(i)}/\mathcal{K})^\circ$  the presheaf  $R \mapsto G^{(i)}(R)/\mathcal{K}(R)$  and  $\mathrm{Gr}^{(i)}$  the fpqc sheafification of  $(G^{(i)}/\mathcal{K})^\circ$ , then each  $\mathrm{Gr}^{(i)}$  is represented by a finite type  $k$ -scheme and we have  $\mathrm{Gr} = \varinjlim \mathrm{Gr}^{(i)}$ . In other words, the affine Grassmannian  $\mathrm{Gr}$  is an ind-scheme of ind-finite type. Moreover, since  $G$  is reductive, each  $\mathrm{Gr}^{(i)}$  is projective and hence  $\mathrm{Gr}$  is ind-projective. In the case of  $G = \mathrm{GL}_n$ , each  $\mathrm{Gr}^{(i)}$  is a closed subscheme of a finite disjoint union of usual Grassmannians.

Consider the left action

$$\theta : \mathcal{K} \times (G^{(i)}/\mathcal{K})^\circ \rightarrow (G^{(i)}/\mathcal{K})^\circ, (g, x\mathcal{K}(R)) \mapsto gx\mathcal{K}(R)$$

of  $\mathcal{K}$  on  $(G^{(i)}/\mathcal{K})^\circ$  as functors. Then  $\theta$  extends naturally to a left action of  $\mathcal{K}$  on  $\mathrm{Gr}^{(i)}$ , which we denote again by  $\theta$ . Now we have a commutative diagram of morphisms of presheaves

$$\begin{array}{ccc} \mathcal{K} \times \mathcal{K} \times G^{(i)} & \xrightarrow{\rho} & G^{(i)} \\ \downarrow \mathbf{pr}_1 \times p & & \downarrow p \\ \mathcal{K} \times (G^{(i)}/\mathcal{K})^\circ & \xrightarrow{\theta} & (G^{(i)}/\mathcal{K})^\circ \\ \downarrow \mathrm{id} \times i & & \downarrow i \\ \mathcal{K} \times \mathrm{Gr}^{(i)} & \xrightarrow{\theta} & \mathrm{Gr}^{(i)} \end{array}$$

where  $p : G^{(i)} \rightarrow (G^{(i)}/\mathcal{K})^\circ$  is the natural projection and  $i : (G^{(i)}/\mathcal{K})^\circ \hookrightarrow \mathrm{Gr}^{(i)}$  is the natural inclusion (note that the presheaf  $(G^{(i)}/\mathcal{K})^\circ$  is separated). Denote by  $\bar{x}$  the image of  $x$  in  $(G^{(i)}/\mathcal{K})^\circ(k) \subset \mathrm{Gr}^{(i)}(k)$  and by  $\theta_{\bar{x}} : \mathcal{K} \rightarrow \mathrm{Gr}^{(i)}$  the orbit map of  $\theta$  at  $\bar{x}$ .

**Lemma 3.2.1.** The topological image  $|\theta_{\bar{x}}(\mathcal{K})|$  is a locally closed subset of  $\mathrm{Gr}^{(i)}$ .

*Proof.* Note that we have  $\mathcal{K} = \varprojlim G_j$ , with

$$G_j := \mathrm{Res}_{(k[u]/u^j)/k} G_{k[u]/u^j},$$

the Weil restriction of  $G_{k[u]/u^j}$  over  $k$ . One may check that the action of  $\mathcal{K}$  on  $(G^{(i)}/\mathcal{K})^\circ$  factors through  $G_{2i}$  (first do the case of  $G = \mathrm{GL}_n$  and for the general case, embed  $G$  into some  $\mathrm{GL}_n$  as in the definition of  $G^{(i)}$ ). Hence the action of  $\mathcal{K}$  on  $\mathrm{Gr}^{(i)}$  also factors through  $G_{2i}$ . Since  $G_{2i}$  is again an algebraic group over  $k$ , one gets the result by applying Proposition 1.52. (b) in [Mil15] for the induced orbit map  $G_{2i} \rightarrow \mathrm{Gr}^{(2i)}$  in question.  $\square$

We define  $O_{\bar{x}}$  to be the unique reduced subscheme of  $\mathrm{Gr}^{(i)}$  with underlying topological space  $|\theta_{\bar{x}}(\mathcal{K})|$ . The subscheme  $O_{\bar{x}}$  is usually called the **Schubert**

cell of  $\bar{x}$  in  $\text{Gr}$ . It is clear that the orbit map  $\theta_{\bar{x}}$  factors through  $O_{\bar{x}}$  because the Weil restriction  $G_{2i}$  in the proof of Lemma 3.2.1 is again a smooth linear algebraic group.

**Lemma 3.2.2.** The orbit map  $\theta_{\bar{x}} : \mathcal{K} \rightarrow O_{\bar{x}}$  is a surjective morphism of fpqc sheaves.

*Proof.* It follows from Lemma 9.27 in [Mil15] that the induced orbit map  $G_{2i} \rightarrow O_{\bar{x}}$  is surjective as fpqc sheaves. But since  $G$  is smooth, the projection map  $\mathcal{K}(R) \rightarrow G_{2i}(R)$  is surjective for every  $k$ -algebra  $R$ . Hence  $\theta_{\bar{x}}$  is a surjective map of fpqc sheaves.  $\square$

Let  $\mathcal{D}(G, x)$  be the pull-back of  $O_{\bar{x}} \subset \text{Gr}^{(i)}$  along the projection map  $\pi := i \circ p : G^{(i)} \rightarrow \text{Gr}^{(i)}$ . Then the orbit map  $\rho_x$  factors through  $\mathcal{D}(G, x)$ . Our aim now is show that  $\mathcal{D}'(G, x)$  is represented by the scheme  $\mathcal{D}(G, x)$ .

**Proposition 3.2.3.** The fpqc sheaf  $\mathcal{D}'(G, x)$  is represented by the formally smooth subscheme  $\mathcal{D}(G, x)$  of  $\mathcal{L}G$ . Moreover, the equation  $\mathcal{D}(G, x)(l) = \mathcal{K}(l)x\mathcal{K}(l)$  holds for any algebraically closed field extension  $l$  of  $k$ . In particular, the subscheme  $\mathcal{D}(G, x) \subset G^{(i)}$  has underlying topological space  $|\text{Im}(\rho_x)|$ .

*Proof.* As the induced map of sheaves  $\mathcal{D}'(G, x) \rightarrow \mathcal{D}(G, x)$  from the inclusion  $\mathcal{C}(G, x) \subset \mathcal{D}(G, x)$  is injective, for the first part of the claim it suffices to show that the orbit map  $\rho_x : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{D}(G, x)$  is surjective as fpqc sheaves. Indeed, for any  $k$ -algebra  $R$  and any point  $y \in \mathcal{D}(G, x)(R)$ , by Lemma 3.2.2 fpqc locally its image

$$\pi(y) = y\mathcal{K}(R) \in (G^{(i)}/\mathcal{K})^\circ(R) \subset \text{Gr}^{(i)}$$

comes from a translation of  $\bar{x}$  by an element of  $\mathcal{K}(R)$ . In other words, there exists a faithfully flat  $R$ -algebra  $R'$  such that  $gx\mathcal{K}(R') = y\mathcal{K}(R') \in (G^{(i)}/\mathcal{K})^\circ(R')$ . This implies that the orbit map  $\rho_x : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{D}(G, x)$  is a surjective map of fpqc sheaves. In the case of  $R = l$  for an algebraically closed field extension  $l$  of  $k$ , the same argument shows that  $\mathcal{D}(G, x)(l) = \mathcal{K}(l)x\mathcal{K}(l)$  (here we use the fact that  $O_{\bar{x}}(l) = \mathcal{K}(l) \cdot \bar{x}$ ).  $\square$

### 3.3 Two Frobenii on the loop group

From now on, we let the field  $k$  be of characteristic  $p$  till the end of this section. For each  $i$ , we still let  $G^{(i)}$  be as in (3.1.1) and denote by  $\sigma_{G^{(i)}} : G^{(i)} \rightarrow (G^{(i)})^{(p)}$  the relative Frobenius of  $G^{(i)}$  over  $k$ . Then all these  $\sigma_{G^{(i)}}$  induce a homomorphism

$$\sigma = \sigma_{\mathcal{L}G} : \mathcal{L}G \rightarrow (\mathcal{L}G)^{(p)} := \varinjlim_i (G^{(i)})^{(p)}.$$

On the other hand, for each  $k$ -algebra  $A$ , and each  $y \in \mathcal{L}G(A)$ , if we see  $y$  as a morphism  $y : \operatorname{Spec} A((u)) \rightarrow G$ , then the composition  $\sigma_G \circ y$  gives an element in  $\mathcal{L}G^{(p)}(A)$ . This induces a homomorphism  $\varphi(A) : \mathcal{L}G(A) \rightarrow \mathcal{L}G^{(p)}(A)$ . Since  $\varphi(A)$  is functorial in  $A$ , we obtain another homomorphism

$$\varphi : \mathcal{L}G \rightarrow \mathcal{L}G^{(p)} \cong (\mathcal{L}G)^{(p)}.$$

In particular, if  $G$  is defined over  $\mathbb{F}_p$ , then for each  $i$ ,  $(G^{(i)})^{(p)}$  is canonically isomorphic to  $G^{(i)}$ . And hence we have canonical isomorphisms

$$\mathcal{L}G^{(p)} \cong (\mathcal{L}G)^{(p)} \cong \mathcal{L}G.$$

This is the case which we will mostly concern in the sequel. Note that there is a difference between  $\varphi$  and  $\sigma$  as illustrated as follows: if we take  $G = \mathbb{G}_m$  and let  $R$  be a  $k$ -algebra, then for any  $f = \sum a_i u^i \in \mathcal{L}\mathbb{G}_m(R)$  we have

$$\varphi(f) = \sum a_i^p u^{pi}, \quad \sigma(f) = \sum a_i^p u^i. \quad (3.3.1)$$

In fact, given a perfect  $k$ -algebra  $R$  (i.e., the absolute Frobenius of  $R$  is bijective), the homomorphism  $\sigma(R) : \mathcal{L}G(R) \rightarrow (\mathcal{L}G)^{(p)}(R)$  is an isomorphism since for each  $i$ , the homomorphism  $G^{(i)}(R) \rightarrow (G^{(i)})^{(p)}(R)$  is an isomorphism. But this is almost never the case for  $\varphi(R)$ .

**Lemma 3.3.1.** Suppose that  $G$  is defined over  $\mathbb{F}_p$  and  $R$  a perfect  $k$ -algebra. Take  $x \in \mathcal{L}G(R)$ , which corresponds to a morphism  $\operatorname{Spec} R((u)) \xrightarrow{x} G$ . Then under the homomorphism  $\sigma^{-1} \circ \varphi : \mathcal{L}G(R) \rightarrow \mathcal{L}G(R)$ , the image  $(\sigma^{-1} \circ \varphi)(x)$  corresponds to the morphism

$$\operatorname{Spec} R((u)) \xrightarrow{\pi} \operatorname{Spec} R((u)) \xrightarrow{x} G,$$

where  $\pi : \operatorname{Spec} R((u)) \rightarrow \operatorname{Spec} R((u))$  corresponds to the  $R$ -endomorphism of  $R((u))$  which sends  $u$  to  $u^p$ .

*Proof.* The claim is clear for  $G = \operatorname{GL}_n$ ; see (3.3.1). For the general case, we let  $\iota : G \subset \operatorname{GL}_n$  be a closed embedding of group schemes over  $\mathbb{F}_p$ .

Denote by  $\sigma_G : G \rightarrow G^{(p)} \cong G$  the relative Frobenius of  $G$  over  $k$ . Then we have the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\sigma_G} & G \\ \downarrow \iota & & \downarrow \iota \\ \operatorname{GL}_n & \xrightarrow{\sigma_{\operatorname{GL}_n}} & \operatorname{GL}_n \end{array}$$

Recall that  $\sigma_G \circ x$  corresponds to  $\varphi(x)$  in  $\mathcal{L}G(R)$ . Similarly,  $\iota \circ \sigma_G \circ x = \sigma_{\operatorname{GL}_n} \circ \iota \circ x$  corresponds to the image of  $\varphi(x)$  in  $\mathcal{L}\operatorname{GL}_n(R)$ .

On the other hand, we also have commutative diagram

$$\begin{array}{ccc} \mathcal{L}G(R) & \xrightarrow{\sigma_{\mathcal{L}G}^{-1}} & \mathcal{L}G(R) \\ \mathcal{L}\iota \downarrow & & \downarrow \mathcal{L}\iota \\ \mathcal{L}\mathrm{GL}_n(R) & \xrightarrow{\sigma_{\mathcal{L}\mathrm{GL}_n}^{-1}} & \mathcal{L}\mathrm{GL}_n(R) \end{array}$$

where  $\mathcal{L}\iota$  is the homomorphism induced from  $\iota$ . Since we know that the statement is true for  $G = \mathrm{GL}_n$ ,

$$\mathcal{L}\iota(\sigma_{\mathcal{L}G}^{-1} \circ \varphi(x)) = \sigma_{\mathcal{L}\mathrm{GL}_n}^{-1}(\mathcal{L}\iota(\varphi(x)))$$

corresponds to the morphism

$$\mathrm{Spec}R((u)) \xrightarrow{\pi} \mathrm{Spec}R((u)) \xrightarrow{x} G \xrightarrow{\iota} \mathcal{L}\mathrm{GL}_n.$$

If  $(\sigma^{-1} \circ \varphi)(x) = (\sigma_{\mathcal{L}G}^{-1} \circ \varphi)(x)$  corresponds to a morphism  $y : \mathrm{Spec}R((u)) \rightarrow G$ . Then we have  $\iota \circ y = \iota \circ (x \circ \pi)$  and hence  $y = x \circ \pi$ .  $\square$

**Lemma 3.3.2.** Let  $G$  be a linear algebraic group over a field  $k$  of characteristic  $p > 0$  and let  $\mathcal{K}$  and  $\mathcal{K}_1$  be as in Definition 3.1.1. Then for any  $k$ -algebra  $R$  and any element  $g \in \mathcal{K}(R)$ , we have  $\varphi(g)\mathcal{K}_1(R) = \sigma(g)\mathcal{K}_1(R)$ .

*Proof.* We may assume  $G = \mathrm{GL}_n$ . For any  $g \in \mathcal{K}(R) = G(R[[u]])$ , write  $g = g_0(1 + uM)$  with  $g_0 \in G(R)$  and  $M \in \mathrm{M}_n(R)$ . It is easy to see that in fact  $1 + uM$  lies in  $\mathcal{K}_1(R)$ . Hence we have  $\varphi(g)\mathcal{K}_1 = \sigma(g)\mathcal{K}_1(R)$  since they are both equal to  $\sigma(g_0)\mathcal{K}_1(R)$ .  $\square$

### 3.4 Fpqc sheaves $\mathcal{D}(G, x)/\mathcal{K}^+$ and $\mathcal{D}_1(G, x)/\mathcal{K}^\diamond$

Let  $G, k$  be as in the preceding lemma. We have a semidirect product of affine group schemes  $\mathcal{K}^+ := (\mathcal{K}_1 \times_k \mathcal{K}_1) \rtimes_k \mathcal{K}$  induced by the right action defined for each  $k$ -algebra  $R$  by

$$\begin{array}{ccc} \mathcal{K}_1(R) \times \mathcal{K}_1(R) \times \mathcal{K}(R) & \longrightarrow & \mathcal{K}_1(R) \times \mathcal{K}_1(R) \\ ((\alpha, \beta), \gamma) & \longmapsto & (\gamma^{-1}\alpha\gamma, \varphi(\gamma)^{-1}\beta\varphi(\gamma)) \end{array}$$

More explicitly, for any two elements  $(\alpha, \beta, \gamma), (\alpha', \beta', \gamma') \in \mathcal{K}^+(R)$ , the multiplication is given by

$$(\alpha', \beta', \gamma') \cdot (\alpha, \beta, \gamma) = (\gamma^{-1}\alpha'\gamma\alpha, \varphi(\gamma)^{-1}\beta'\varphi(\gamma)\beta, \gamma'\gamma).$$

We refer to Section 3.2 for the notations  $\mathcal{D}(G, x)$  and  $\mathcal{D}'(G, x)$  and for what follows we simply write

$$\mathcal{D} := \mathcal{D}(G, x) = \mathcal{D}'(G, x). \quad (3.4.1)$$



There is an action of the (infinite-dimensional) affine group scheme  $\mathcal{K}^+$  on  $\mathcal{D}$  given on local sections by

$$\begin{aligned} \mathcal{D} \times \mathcal{K}^+ &\longrightarrow \mathcal{D} \\ (t, (\alpha, \beta, \gamma)) &\longmapsto \alpha^{-1} \gamma^{-1} t \varphi(\gamma) \beta \end{aligned}$$

Let  $\mathcal{D}_1 := \mathcal{D}/\mathcal{K}_1$ , (resp.  $G^{(i)}/\mathcal{K}_1$ ) be the quotient of  $\mathcal{D}$  (resp.  $G^{(i)}$ ) by  $\mathcal{K}_1$ , which by definition is the fpqc sheafification of the presheaf

$$R \mapsto \mathcal{K}(R)z\mathcal{K}(R)/\mathcal{K}_1(R), \quad (\text{resp. } R \mapsto G^{(i)}(R)/\mathcal{K}_1(R))$$

Since  $G^{(i)}/\mathcal{K}_1$  is represented by a proper  $k$ -scheme of finite type, by a similar argument as in the proof of Lemma 3.2.1 and Lemma 3.2.2, one sees that  $\mathcal{D}_1$  as a  $\mathcal{K} \times_k \mathcal{K}$ -orbit of  $\bar{x}$  is represented by a smooth  $k$ -scheme of finite type, where  $\bar{x}$  is the  $k$ -point of  $G^{(i)}/\mathcal{K}_1$  induced by  $x$ . In fact,  $\mathcal{D}_1$  is a  $G = \mathcal{K}/\mathcal{K}_1$ -torsor in the étale topology over  $O_{\bar{x}'} \subset \text{Gr}^{(i)}$  (here  $\bar{x}'$  denotes the image of  $x$  in  $\text{Gr}^{(i)}(k)$ ), as it is the pull-back of the  $G$ -torsor  $G^{(i)}/\mathcal{K}_1$  over  $\text{Gr}^{(i)}$ . In particular,  $\mathcal{D}_1$  is a smooth  $k$ -scheme of finite type.

Write  $\mathcal{K}^\diamond := \mathcal{K}_1 \rtimes_k \mathcal{K}$ , seen as a quotient group of  $\mathcal{K}^+$  modulo the second direct summand  $\mathcal{K}_1$ . Recall that  $\mathcal{K} = \varprojlim G_j$ , with  $G_j$  the restriction of  $G_{k[u]/u^j}$  over  $k$ . If we denote by  $\mathcal{H}_i$  the kernel of the natural reduction modulo  $u$  map  $G_j \rightarrow G$ , then we have

$$\mathcal{K}^\diamond = \varprojlim \mathcal{K}_i^\diamond, \quad \text{with } \mathcal{K}_i^\diamond := \mathcal{H}_i \times_k G_i.$$

Let us consider the right action induced by  $\rho$  in (3.2.1)

$$\rho_1 : \mathcal{D}_1 \times_k \mathcal{K}^\diamond \longrightarrow \mathcal{D}_1, \quad (t\mathcal{K}_1, (\alpha, \gamma)) \longmapsto \alpha^{-1} \gamma^{-1} t \varphi(\gamma) \mathcal{K}_1.$$

One may check that the action of  $\mathcal{K}^\diamond$  on  $\mathcal{D}_1$  factors through  $\mathcal{K}_{2i+1}^\diamond$  (see the hint in the proof of Lemma 3.2.1), which is an affine smooth  $k$ -scheme.

Again by a similar argument as in Lemma 3.2.1, Lemma 3.2.2 and Proposition 3.2.3 we have the following lemma.

**Lemma 3.4.1.** (1) The  $\mathcal{K}^\diamond$ -orbit  $O_{\bar{y}}$  of a point  $\bar{y} \in \mathcal{D}_1(k)$  is represented by a smooth  $k$ -scheme of finite type.

(2) If  $\bar{y}$  comes from an element  $y \in \mathcal{D}(k)$  the pull-back of  $O_{\bar{y}}$  under the natural projection  $\mathcal{D} \rightarrow \mathcal{D}_1$  is the  $\mathcal{K}^+$ -orbit  $O_y$  of  $y$  in  $\mathcal{D}$ .

(3) For any algebraically closed field extension  $l$  of  $k$ , we have  $O_y(l) = y \cdot \mathcal{K}^+(l)$ .

(4) For any algebraically closed field extension  $l$  of  $k$ , there is a commutative

diagram of bijective maps

$$\begin{array}{ccc}
\{\mathcal{K}^\diamond\text{-orbits of } \mathcal{D}_{1,l}\} & \xleftarrow{1-1} & \{\mathcal{K}^+\text{-orbits of } \mathcal{D}_l\} \\
\updownarrow 1-1 & & \updownarrow 1-1 \\
\mathcal{D}_1(l)/\mathcal{K}^\diamond(l) & \xlongequal{\quad\quad\quad} & \mathcal{D}(l)/\mathcal{K}^+(l),
\end{array}$$

where the top horizontal map commutes with the operation of taking closures.

*Proof.* Here we only show (4). Indeed there is a 1-1 correspondence

$$\{\mathcal{K}^\diamond\text{-orbits of elements in } \mathcal{D}_{1,l}(k)\} \xleftarrow{1-1} \{\mathcal{K}^+\text{-orbits of elements in } \mathcal{D}_l(k)\}. \tag{3.4.2}$$

But the left hand side of (3.4.2) is insensitive to algebraically closed field extensions. Hence every algebraic  $\mathcal{K}^+$ -orbit of  $\mathcal{D}$  is a  $\mathcal{K}^+$ -orbit of some  $k$ -point of  $\mathcal{D}$ .  $\square$