Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

Yan, Q.; Yan Q.

2 Notations and conventions

2.1 General notions

(1) As mentioned earlier, for a prime number p in this paper, we assume that $p \geq 3$.

(2) All Dieudonné crystals and Diudonné modules are contravariant.

(3) If X is a scheme over a ring R (resp. over a scheme S) and R' is an R-algebra (resp. S' is an S-scheme), we often denote by $X_{R'}$ (resp. $X_{S'}$) the pullback of X along the ring homomorphism $R \to R'$ (resp. along the structure morphism $S' \to S$). Sometimes we suppress the subscripts R' (resp. S') if the base ring (resp. base scheme) is clear from the context.

(4) Similar convention as in (3) applies for modules, p-divisible groups, stacks, etc.

(5) If R is a ring of characteristic p or a $\mathbb{Z}_p$-flat p-adic ring and φ_R a Frobenius lift of R (i.e., the reduction mod p of φ_R is the absolute Frobenius of R/pR), we often suppress the subscript R when it is clear from the context. Here and everywhere in the sequel a p-adic ring is p-adically complete and separated.

(6) If $\varphi : R \to R$ is an endomorphism of rings and M is an R-module, we use the following notations interchangeably

$$M^{(\varphi)} := \varphi^* M := M \otimes_{R,\varphi} R.$$

(7) If R is a ring with $r \in R$, and M an R-module, we sometimes simply denote by $M[\frac{1}{r}]$ the module $M \otimes_R R[\frac{1}{r}]$ over $R[\frac{1}{r}]$ and $f[\frac{1}{r}] : M[\frac{1}{r}] \to N[\frac{1}{r}]$ the induced homomorphism of a homomorphism $f : M \to N$ of R-modules.

(8) By a linear algebraic group over a ring R we mean an affine group scheme of finite presentation over R. By a reductive group over R we mean a connected smooth affine group scheme over R such that for each geometric point s of $\mathrm{Spec}R$, the pullback G_s is a connected reductive group over s.

(9) Let k be a field of characteristic p. For any k-scheme S, we denote by $\mathrm{Frob}_S : S \to S$ the absolute Frobenius endomorphism of S, and by $(\ )^{(p)}$ the pull-back along Frob_S of a scheme or a sheaf or a morphism over S. For example, if G is a reductive group over k, we denote by $G^{(p)}$ the pull-back of G along $\mathrm{Frob}_{\mathrm{Spec}k}$; if P is a G-torsor over S, then $P^{(p)}$ is a $G^{(p)}$-torsor over S; the pull back $\lambda^{(p)} : \mathbb{G}_{m,k} \cong \mathbb{G}_m^{(p)} \to G^{(p)}$ along $\mathrm{Frob}_{\mathrm{Spec}k}$ of a cocharacter $\lambda : \mathbb{G}_{m,k} \to G$ (over k) is a cocharacter of $G^{(p)}$ (over k). In particular, if G is defined over $\mathbb{F}_p$, then $\lambda^{(p)}$ is again a cocharacter of G

since we then have canonical isomorphism $G^{(p)} \cong G$. When k is perfect, we have an automorphism of $X_*(G) := \mathrm{Hom}_k(\mathbb{G}_{m,k}, G)$, given by

$$\sigma : X_*(G) \longrightarrow X_*(G), \quad \lambda \longmapsto \lambda^{(p)}. \tag{2.1.1}$$

Let G be reductive model over $\mathbb{F}_p$ of G, i.e., $\mathrm{G} \otimes k = G$, and $\sigma : k \to k(x \mapsto x^p)$ the Frobenius element in the Galois group $\mathrm{Gal}(k/\mathbb{F}_p)$. Then the action σ in (2.1.1) is the same as the Galois action of $\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)$ on $X_*(G)$, i.e., we have

$$\lambda^{(p)} = \big(\mathbb{G}_{m,k} \xrightarrow{\mathrm{id}\otimes\sigma} \mathbb{G}_{m,k} \xrightarrow{\lambda} \mathrm{G} \otimes k \xrightarrow{\mathrm{id}\otimes\sigma^{-1}} \mathrm{G} \otimes k = G\big). \tag{2.1.2}$$

(10) If G is a group scheme over a field of characteristic p, and R is a k-algebra, then for any $x \in G(R)$, by $\sigma(x)$ we mean the image of x in $G^{(p)}(R)$ under the relative Frobenius $G \to G^{(p)}$. Note that there are two Frobenii σ and φ on $\mathcal{L}G$ and on $\mathcal{L}^+G$ (Section 3). The difference between them is discussed in Section 3.3.

2.2 Tensors and contragredient representations

We introduce below the "contragredient representations" following Wortmann's thesis ([Wor13]) but note that it is used slightly differently in this thesis.

Let R be a ring and M a finite locally free R-module. Denote by M^* the dual R-module of M and by $M^\otimes$ the direct sum of all R-modules that arise from M by applying the operation of taking duals, tensor products, symmetric powers and exterior powers a finite number of times. An element of $M^\otimes$ is called a **tensor** over M. For an isomorphism $f : M_1 \cong M_2$ of finite locally free R-modules, we have an induced isomorphism $(f^{-1})^* : M_1^* \to M_2^*$, and hence $f^\otimes : M_1^\otimes \to M_2^\otimes$.

For any M as above, there is a canonical isomorphism of group schemes, called the **contragredient representation** of $\mathrm{GL}(M)$, defined as

$$(\cdot)^\vee : \mathrm{GL}(M) \cong \mathrm{GL}(M^*), \quad g \longmapsto g^\vee := (g^{-1})^*.$$

Through the contragredient representation of M, $M^\otimes$ is natually identified with $(M^*)^\otimes$. If $s \subset M^\otimes$ is a set of tensors over M, which defines a subgroup $G \subset \mathrm{GL}(M)$, then they also defines a subgroup $\{g^\vee | g \in G\} \subset \mathrm{GL}(M^*)$, when considered as tensors over M^*. Since M is canonically identified with $(M^*)^*$, we also have the contragredient of $\mathrm{GL}(M^*)$,

$$(\cdot)^\vee : \mathrm{GL}(M^*) \to \mathrm{GL}(M),$$

which we shall use frequently in the thesis.