Adapted deformations and the Ekedahl-Oort stratifications of Shimura varieties

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1 Introduction

We fix a prime number $p > 2$ throughout this thesis.

1.1 Ekedahl-Oort stratification

We start by giving a short review of the development of the Ekedahl-Oort stratification of Shimura varieties, following [GK15].

The Ekedahl-Oort stratification was initially defined and studied by Ekedahl and Oort for $\mathcal{A}_{g,\mathbb{F}_p}$, the special fibre of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension of g, which can be seen as the Shimura variety associated to the group $\mathbf{GSp}_{2g}$. In [Oor01] Oort defined a stratification for $\mathcal{A}_{g,\mathbb{F}_p}$ by isomorphism classes of the BT1's (BT1 stands for truncated Barsotti Tate group of level 1). The strata are listed in loc. cit. by the so called "elementary sequences". Moonen later on gave a group-theoretical classification of BT1's with PEL-structures in [Moo01], where he describes the isomorphism classes as a certain Weyl cosets ${}^JW$ of the group G of the Shimura datum associated to the PEL-structure. In a series of papers, Moonen, Wedhorn, Pink and Ziegler showed that BT1's with G-structures of certain type χ give rise to an algebraic stack $G\text{-}\mathsf{Zip}^\chi$, which is isomorphic to a quotient stack $[E_\chi\backslash G]$, where E_χ is some "zip group" (see [MW04], [Wed01], [PWZ11], [PWZ15]).

In [VW13] Viehmann and Wedhorn generalized and studied the Ekedahl-Oort stratification for the special fibre of a PEL-type Shimura variety. In his thesis [Zha13], C. Zhang constructed a universal G-Zip for the special fibre S of a general Hodge-type Shimura variety, which induces a morphism of algebraic stacks [1]

$$\zeta : S \longrightarrow G\text{-}\mathsf{Zip}^\chi \cong [E_\chi\backslash G_\kappa], \tag{1.1.1}$$

and showed that the morphism is smooth. A detailed review of the construction is given in Section 8.3.

Definition 1.1.1. An **Ekedahl-Oort stratum** of S is defined to be a geometric fibre of the morphism ζ in (1.1.1).

In the case of PEL type, the strata thus defined coincide with the Ekedahl-Oort strata defined in [VW13]. The smoothness (hence openness) of ζ provides a description of the closure relation of the strata, in terms of the underlying topological space of the stack $G\text{-}\mathsf{Zip}^\chi$.

[1] Wortmann slightly modified Zhang's constructions in his thesis ([Wor13]) and we are here following his notations. We will also review his constructions in this thesis.

1.2 Motivations

Let G be a reductive group over $\mathbb{F}_p$ and fix a Borel pair (B, T) of G (namely a maximal torus T and a Borel subgroup B of G with $B \supset T$), all defined over $\mathbb{F}_p$. Let $\mathcal{L}G$ and $\mathcal{K} := \mathcal{L}^+G$ be the loop group and the strict loop group of G respectively (Section 3), and $\mathcal{K}_1$ the kernel of the reduction map $\mathcal{K} \to G$. Let k be an algebraically closed field of characteristic p and denote by $W(k)$ the ring of Witt vectors of k. Denote by $\sigma : k \to k$ the absolute Frobenius of k and use the same notation for the ring endomorphism $k[[u]] \to k[[u]]$, which is σ on k and fixes $u \in k[[u]]$. Write $W = N_GT(k)/T(k)$ for the Weyl group of G, with respect to T.

In [Vie14, Theorem 1.1] Viehmann gives a classification of the $\mathcal{K}$-σ conjugacy classes in $\mathcal{K}_1(k)\backslash\mathcal{L}G(k)/\mathcal{K}_1(k)$ in terms of certain Weyl groups. For a dominant cocharacter $\mu : \mathbb{G}_{m,k} \to T_k$ over k (with respect to B_k), if we denote by $C(G, \mu)$ the set of $\mathcal{K}$-σ conjugacy classes in

$$\mathcal{K}_1(k)\backslash\mathcal{K}(k)\mu(u)\mathcal{K}(k)/\mathcal{K}_1(k),$$

then the result in loc. cit. implies that there is a 1-1 correspondence between $C(G, \mu)$ and the subset ${}^JW$ of the Weyl group W, where J is the type of the cocharacter $\sigma^{-1}(\mu) : \mathbb{G}_{m,k} \to T_k$. The precise definitions of J and ${}^JW$ are reviewed in Section 8.2.

We are interested in the situation where G and μ come from a Shimura variety S of characteristic p (namely, the reduction modulo p of a Shimura variety). In this case, as discussed in the previous subsection, the finite set ${}^JW$ can be seen as the index set of the Ekedahl-Oort strata of this Shimura variety (see for example [VW13, Section 8.2]), and hence so can be $C(G, \mu)$. But note that such a bijection between ${}^JW$ and $C(G, \mu)$ is purely group-theoretic and a priory there is no direct connection from the Shimura variety in question to $C(G, \mu)$. The main goal of this thesis is to construct a map from S to a subquotient (stack, preferably) of $\mathcal{L}G$ whose geometric points can be identified with $C(G, \mu)$[2], such that the geometric fibres of this map give back the Ekedahl-Oort strata of S. This goal is basically achieved except that the target of the map we construct fails to be a stack (see Section 1.5).

What makes us believe that such a connection is possible are the following observations: (1). p-divisible groups over rings like $W(k)$ can be classified in term of Breuil-Kisin modules, or equivalently Breuil-Kisin windows in our term (we give a short review of these in Section 1.4), which are modules over the power series ring $W(k)[[u]]$ equipped with additional data; (2). Shimura varieties naturally carry families p-divisible groups (with extra structures) which determine the stratifications we want to define. This belief is also supported

[2] Our cocharacter μ is in fact a variant of μ in [Vie14], but as explained in Remark 8.4.2 the resulting finite sets $C(G, \mu)$ are the same.

by other evidence; see Theorem 1.4.2 and the paragraph after that. Our philosophy in short is: the Frobenii of Breuil-Kisin windows will give $C(G,\mu)$, and this turns out to be correct (see Remark 1.5.1.(1)).

In [VW13] the authors determined the closure relation of Ekedahl-Oort strata via that of some truncation strata in $\mathcal{L}G$ ([VW13, Corollary 7.2], [Vie14, Corollary 4.7]). Such a criteria again depends on the bijection between ${}^JW$ and $C(G,\mu)$. One of our goals is to remedy this situation by deducing this criteria in a similar manner as the proof of [Zha13, Proposition 3.1.6]. So far we have not fully achieved this but we are confident that it can be done (see Remark 1.5.1.(2)).

1.3 Special fibre of a Shimura variety (of Hodge type)

To state our main results, we need to fix some notations on the reduction of Shimura varieties (of Hodge type). All the technical terminologies and results used here are discussed or reviewed in Section 6.4 and Section 6.5.

Let $(\mathbf{G},\mathbf{X})$ be a Shimura data of Hodge type, i.e., it can be embedded into a Siegel-type Shimura data $(\mathbf{GSp},\mathbf{S}^{\pm})$. Assume that $\mathbf{G}$ has good reduction at a prime number $p>2$, i.e., $\mathbf{G}$ has a reductive model $\mathcal{G}$ over $\mathbb{Z}_p$, whose special fibre is denoted by G. Let $\mathsf{K}=\mathsf{K}_p\mathsf{K}^p\subset\mathbf{G}(\mathbb{A}_f)$ be an open compact subgroup, with $\mathsf{K}_p=\mathcal{G}(\mathbb{Z}_p)$ and $\mathsf{K}^p\subset\mathbf{G}(\mathbb{A}_f^p)$ sufficiently small, and let $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G},\mathbf{X})$ be the associated Shimura variety over the reflex field E of $(\mathbf{G},\mathbf{X})$.

Fix a place v of E above p. Denote by $\mathcal{O}_E$ the ring of integers of E. Write $\mathcal{O}_{E,v}=W(\kappa)$ for the completion of $\mathcal{O}_E$ at v. Kisin ([Kis10]) and Vasiu ([Vas99]) have showed the existence of the integral canonical model $\mathcal{S}_{\mathsf{K}}(\mathbf{G},\mathbf{X})$ of $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G},\mathbf{X})$ over $\mathcal{O}_{E,v}$. Denote by S the special fibre of $\mathcal{S}_{\mathsf{K}}(\mathbf{G},\mathbf{X})$, which is a quasi-projective and smooth scheme over κ.

Denote by $[\chi]_{\mathbb{C}}$ the unique $\mathbf{G}(\mathbb{C})$-conjugacy class of the inverse of any of the Hodge cocharacters $\lambda:\mathbb{G}_m\to\mathbf{G}_{\mathbb{C}}$ determined by $(\mathbf{G},\mathbf{X})$. Denote by $[\chi]_\kappa$ the reduction over κ of $[\chi]_{\mathbb{C}}$ (Section 6.5), which is the $G(\kappa)$-conjugacy class of some cocharacter $\chi:\mathbb{G}_{m,\kappa}\to G_\kappa$ of G_κ over κ. We fix such a cocharacter χ and define the cocharacter

$$\mu:=\mathrm{Frob}_{G_\kappa/\kappa}\circ\chi:\mathbb{G}_{m,\kappa}\to G_\kappa, \tag{1.3.1}$$

where $\mathrm{Frob}_{G_\kappa/\kappa}:G_\kappa\to G_\kappa^{(p)}=G_\kappa$ is the relative Frobenius of G_κ over κ (see (9) in Section 2.1). The cocharacter χ defines a parabolic subgroup P_+ of G_κ, defined over κ (cf. Section 6.5) We can and do lift χ to be a cocharacter $\tilde{\chi}:\mathbb{G}_{m,W(\kappa)}\to\mathcal{G}_{W(\kappa)}$ of $\mathcal{G}_{W(\kappa)}$ over $W(\kappa)$, which in turn defines a parabolic subgroup $\mathcal{P}_+$ of $\mathcal{G}_{W(\kappa)}$, defined over $W(\kappa)$. Note that our "μ" is different from the "μ" in [Wor13]: we remind the reader to be careful on this point.

1.4 Review of Breuil-Kisin windows

Let k be an algebraically closed field extension of κ. Breuil-Kisin modules are typically used to classify p-divisible groups over a totally ramified extension R of $W(k)$ (of arbitrary ramification index). But for our application, R is simply $W(k)$ itself and hence no nontrivial ramification happens.

We give below the definition of Breuil-Kisin modules and Breuil-Kisin windows, together with a classification result. In the thesis we also use relative Breuil-Kisin modules developed by W. Kim (we call them Kim-Kisin modules, see Section 4.5) but we will not need them now since the purpose here is to provide some background knowledge so that we can present our main results in the next subsection.

Let $\mathfrak{S} = W(k)[[u]]$ and let $\varphi = \varphi_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ be the ring endomorphism of $\mathfrak{S}$ which is the unique Frobenius lift φ on $W(k)$ and sends u to u^p. We write $E(u) = u + p$, which generates the kernel of the projection $\mathfrak{S} \to W(k)$ (as $W(k)$-algebras) sending u to $-p \in W(k)$.

Definition 1.4.1. (1) A **Breuil-Kisin $\mathfrak{S}$-module** is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module and $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ is a φ-linear map, such that the cokernel of the linearization

$$\varphi_{\mathfrak{M}} \otimes_{\mathfrak{S},\varphi} \mathrm{id} : \varphi^*\mathfrak{M} := \mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \to \mathfrak{M}$$

is killed by $E(u)$.

(2) A **Breuil-Kisin $\mathfrak{S}$ window** is a triple $\underline{\mathcal{M}} := (\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}})$, where $\mathcal{M}$ is a free $\mathfrak{S}$-module and $\mathrm{Fil}\mathcal{M} \subset \mathcal{M}$ a $\mathfrak{S}$-submodule of $\mathcal{M}$ of finite type, and where $\varphi_{\mathcal{M}} : \mathrm{Fil}\mathcal{M} \to \mathcal{M}$ is a φ-linear map such that

(a) $E \cdot \mathcal{M} \subset \mathrm{Fil}\mathcal{M}$;

(b) $\mathcal{M}/\mathrm{Fil}\mathcal{M}$ is a free $W(k)$-module;

(c) the subset $\varphi_{\mathcal{M}}(\mathrm{Fil}\mathcal{M})$ generates $\mathcal{M}$ as an $\mathfrak{S}$-module.

A Breuil-Kisin window is also called a filtered Breuil-Kisin module in the literature, but we prefer the current terminology as it fits well with the windows theory developed by Lau and Zink.

Denote by $\mathbf{BT}(W(k))$ the category of p-divisible groups over $W(k)$.

Theorem 1.4.2 (Kisin). There are the following equivalence of categories:

$$\begin{array}{ccccc} \mathbf{BT}(W(k)) & \xleftrightarrow{\cong} & \{\text{Breuil-Kisin } \mathfrak{S}\text{-modules}\} & \xleftrightarrow{\cong} & \{\text{Breuil-Kisin } \mathfrak{S}\text{-windows}\} \\ & & (\mathfrak{M},\ \varphi_{\mathfrak{M}}) & \longmapsto & (\varphi^*\mathfrak{M},\ \mathrm{Fil}\varphi^*\mathfrak{M},\ \varphi_{\mathfrak{M}} \otimes 1), \end{array}$$

where $\mathrm{Fil}\varphi^*\mathfrak{M}$ is defined in the proof of Proposition 4.5.4 (we will not need it explicitly in this Introduction). The essential part of the Theorem is the first equivalence which is proved quite indirectly and hence we omit its description here.

Moreover, if $(\mathcal{M}, \mathrm{Fil}\mathcal{M}, \varphi_{\mathcal{M}})$ is the associated Breuil-Kisin window of a p-divisible group H over $W(k)$, then $(\mathcal{M}, \varphi_{\mathcal{M}}) \otimes \mathfrak{S}/(u)$ is canonically identified with the (contravariant) Dieudonné module $\mathbb{D}^*(H)(W(k))$ of H, together with its Frobenius map, and $(\mathcal{M}, \mathrm{Fil}\mathcal{M}) \otimes \mathfrak{S}/(E(u))$ is canonically identified with $\mathbb{D}^*(H)(W(k))$ together with its Hodge filtration (cf. [BBM82, Corollaire 3.3.5]).

There is also a notion of torsion Breuil-Kisin modules, and hence torsion Breuil-Kisin windows. If $\underline{\mathcal{M}}$ is the associated Breuil-Kisin window of H, then the reduction modulo p of $\underline{\mathcal{M}}$ is the associated torsion Breuil-Kisin window associated to the BT1, $H[p]$.

1.5 Main results

(1) We define a subquotient scheme $\mathcal{D}_1$ of $\mathcal{L}G$ (Section 7.1) and construct a morphism of schemes over κ (Section 7.4)

$$\theta : \mathrm{I}_+ \to \mathcal{D}_1.$$

More details on this morphism will be given in the next subsection.

(2) We define an fpqc quotient sheaf $\mathcal{D}_1/\mathcal{K}^\diamond$ of $\mathcal{D}_1$ (obtained from an action of some group scheme $\mathcal{K}^\diamond$ on $\mathcal{D}_1$, Section 7.1) such that the geometric points $\mathcal{D}_1(k)/\mathcal{K}^\diamond(k)$ are identified with $C(G, \mu)$, and we show that θ induces a morphism of fpqc sheaves (Theorem 7.5.2)

$$\eta : S \to \mathcal{D}_1/\mathcal{K}^\diamond.$$

(3) We show that the geometric fibres of η are exactly the Ekedahl-Oort strata of S by establishing for each algebraically closed field extension k of κ a bijection

$$\omega : \mathcal{D}_1(k)/\mathcal{K}^\diamond(k) \to E_\chi(k)\backslash G(k),$$

fitting into the commutative diagram (Proposition 8.4.1),

$$\begin{tikzcd} & \mathcal{D}_1(k)/\mathcal{K}^\diamond(k) \arrow[dd, "\omega"] \\ S(k) \arrow[ur, "\eta"] \arrow[dr, "\zeta"'] & \\ & E_\chi(k)\backslash G(k) \end{tikzcd}$$

Remark 1.5.1. (1) To construct the global θ, we proceed by constructing local maps from Zariski open affines of I_+ to $\mathcal{D}_1$, and then gluing the local maps together (Section 7.4). For the local maps we use our constructions of Kim-Kisin windows (we also call them "adapted deformations") in Section 5.1. But we found afterwards that if one only concerns the map θ itself, the constructions of Kim-Kisin windows can be totally avoided (the relative crystalline Dieudonné theory developed in [dJ95] is still needed): this point will be reflected in the formula (1.6.1) below. But what we think is more important is the conceptual interpretation of these maps: θ is given by the Frobenius of Kim-Kisin windows and the Ekedahl-Oort stratification of S can be defined through Breuil-Kisin windows via η. The maps θ and η conceptually explain why Shimura varieties can be related to loop groups directly and why the new invariants $C(G, \mu)$ in [Vie14] can be seen as the index set of Ekedahl-Oort strata of S.

(2) We don't know yet whether η is smooth but it is expected to be so. Once this is proven, we can determine the closure relations of the Ekedahl-Oort strata of S as in [VW13, Corollary 7.2] for free (in fact there is a slight difference since we are using a different "μ").

(3) The morphism η cannot be refined to a morphism from S to the quotient stack $[\mathcal{D}_1/\mathcal{K}^\diamond]$. Although the quotient stacks $[\mathcal{D}_1/\mathcal{K}^\diamond]$ and $[E_\chi\backslash G]$ have the same topological spaces, they are not isomorphic as stacks for obvious reasons: the latter is algebraic while the former is not.

1.6 Strategy

We now describe the main idea of the construction of θ and the bijection ω. To do this, we need to be more precise and hence the text below will be slightly technical.

Let $\mathcal{D}$ be the fpqc sheafification of the subfunctor of $\mathcal{L}G$ which sends a $\mathbb{F}_p$-algebra R to

$$\mathcal{K}^+(R)\mu(u)\mathcal{K}^+(R) \subset \mathcal{L}G(R).$$

It can be shown that $\mathcal{D}$ is represented by an infinite dimensional formally smooth scheme over κ (Proposition 3.2.3). Then $\mathcal{D}_1$ is defined to be the quotient of $\mathcal{D}$ by the free action of $\mathcal{K}_1$ by right multiplication. We show in Section 3.4 that $\mathcal{D}_1$ is represented by a smooth κ-scheme of finite type.

We refer to Section 7.2 for the $\mathcal{P}_+$-torsor $\mathbb{I}_+$ over $\mathcal{S}$ and its reduction I_+ over S. Given an element $x \in \mathrm{I}_+(k)$, a lift $\tilde{x} \in \mathbb{I}_+(W(k))$ gives an isomorphism

$$\beta_{\tilde{x}} : \Lambda^*_{W(k)} \cong M := \mathbb{D}^*(\mathcal{A}_x[p^\infty]) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}}/W(k))$$

compatible with filtrations and sending tensors $s_{W(k)}$ to tensors s_{dR}. By transport of structure via $\beta_{\tilde{x}}$, we obtain a (linearized) Frobenius on $\Lambda^*_{W(k)}$, denoted by $F^{\mathrm{lin}}_{\tilde{x}} : \varphi^*\Lambda^*_{W(k)} \to \Lambda^*_{W(k)}$. We define

$$\Gamma^{\mathrm{lin}}_{\tilde{x}} : \Lambda^*_{W(k)} \cong \varphi^*\Lambda^*_{W(k)} = \varphi^*\Lambda^*_{W(k),1} \oplus \varphi^*\Lambda^*_{W(k),0} \xrightarrow{1/pF^{\mathrm{lin}}_{\tilde{x}} \oplus F^{\mathrm{lin}}_{\tilde{x}}} \Lambda^*_{W(k)},$$

where $\Lambda^*_{W(k)} \cong \varphi^*\Lambda^*_{W(k)}$ is the canonical isomorphism since Λ is defined over $\mathbb{Z}_p$. This is an isomorphism preserving tensors $s_{W(k)}$, and hence we obtain an element in $G(k)$, namely

$$\alpha(\tilde{x}) := (\Gamma^{\mathrm{lin}}_{\tilde{x}} \bmod p)^{\vee} \in G(k).$$

Here we use the contragredient representation $(\cdot)^{\vee} : \mathrm{GL}(\Lambda^*_{\kappa}) \cong \mathrm{GL}(\Lambda_{\kappa})$ introduced in Section 2.2. Finally, $\theta(x)$ is defined to the the image in $\mathcal{D}_1(k)$ of

$$\Phi_0(\varphi, \tilde{x}) := \alpha(\tilde{x})\mu(u) \in \mathcal{D}_1(k). \tag{1.6.1}$$

It turns out that $\theta(x)$ is independent of the choice of $\tilde{x}$. The unique Frobenius lift φ of $W(k)$ appears in $\Phi_0(\varphi, \tilde{x})$ to keep coherence of notations in the relative setting where the Frobenius lifts are not unique.

The element $\Phi_0(\varphi, \tilde{x})$ in $\mathcal{D}(k)$ has a realization as the reduction modulo p of the Frobenius of some Breuil-Kisin window isomorphic to the one associated to $\mathcal{A}_{\tilde{x}}[p^{\infty}]$. In other words, $\Phi_0(\varphi, \tilde{x})$ can be realized as the Frobenius of some torsion Breuil-Kisin window isomorphic to the one associated to the BT1, $\mathcal{A}_{\tilde{x}}[p]$. The ambiguity "isomorphic to" here (for Frobenius maps it is the same as "$\mathcal{K}$-σ conjugate to") still exists in $\mathcal{D}_1(k)$, but will be killed after passing to the quotient $\mathcal{D}_1(k)/\mathcal{K}^{\diamond}(k)$. Such a realization is the hard part of this thesis.

The element $\mu(u) \in \mathcal{L}G(k)$, instead of $\chi^{(p)}(u)$ ($\chi^{(p)}$ is the "μ" in [Wor13], see (9) in Section 2.1 for its definition), appears in $\Phi_0(\varphi, \tilde{x})$ is due to the fact that the Frobenius lift $\varphi : \mathfrak{S} \to \mathfrak{S}$ sends the formal variable u to u^p. This also explains why the cocharacter "μ" in this thesis is different from the "μ" in [Wor13].

For an algebraically closed field extension k of κ, the map ω is defined as follows.

$$\begin{array}{rcl} \omega : \mathcal{D}_1(k)/\mathcal{K}^{\diamond}(k) & \longrightarrow & E_{\chi}(k)\backslash G_{\kappa}(k) \\ h_1\mu(u)h_2 & \longmapsto & E_{\chi}(k) \cdot (\sigma^{-1}(\bar{h}_2)\bar{h}_1), \end{array}$$

where for an element $h \in \mathcal{K}(k)$, we write $\bar{h} = h \bmod u$.

The map ω above (Proposition 8.4.1) is an analogue of the map $\bar{\zeta}$ in [Wor13, Proposition 6.7]. The fact that ω is a bijection seems to be implied by the work in [Vie14] (or combined with [Wor13, Proposition 6.7]), but this is not completely clear to us. Due to the difference of characters in question (our $\mu : \mathbb{G}_{m,\kappa} \to G_{\kappa}$ is not minuscule), the proof of [Wor13, Proposition 6.7] cannot be adapted directly to our setting.