Wave propagation in mechanical metamaterials

Zhou, Y.; Zhou Y.

Chapter 6

Elastic waves in flexible strings

In this chapter we study the elastic waves induced by an abrupt impact at a point. The impact has a constant velocity, and the string is initially straight and tensionless.

6.1 Introduction

The theory of flexible strings has been intensively developed in classic studies by Navier, Poisson, Stokes, Rayleigh and Kelvin, to name but a few [86]. The conceptual success of this classical field theory not only provides the general principle for the description of the mechanical behavior of strings, but also makes it possible to perform rational analysis in many physics and engineering problems [87–96]. Typically the linear theory only considers infinitesimal displacements and omits the coupling between transverse and longitudinal displacements. This approximation is, however, no longer accurate when the effect of nonlinearities becomes dominant, e.g., when the motion of the string has a large amplitude. Such studies of nonlinear dynamics of elastic waves in flexible strings are motivated by engineering challenges such as the deformation of yarns in weaving machines, the strength of ropes of parachutes or cables in mechanical structures like cranes and bridges [97–100].

In this chapter, we consider the case of a flexible string which is initially straight and tensionless. In this case, no linear transverse waves propagate. In general, such media are said to be in "sonic vacuum"[101], meaning that the velocity of linear waves vanishes. As a consequence, any small disturbance will generate a strongly nonlinear effect and dominate the dynamics. We consider a constant impact velocity, in which case nonlinear shock waves are generated. We study the nonlinear dynamics and obtain the shock velocity which scales with the impact velocity to a fractional exponent. This result has been obtained in literature as a special case [93, 102], but our interpretation articulates its mechanism, which hopefully can help explain similar types of shear shocks in mechanical models for solids in higher dimensions. Furthermore, we perform simulations that demonstrate this phenomenon in a simple model of wide applicability.

6.2 Lattice model and simulation

We start by considering the classical model of the 1D lattice of identical masses m confined in a plane (see Fig. 6.1a). The masses are coupled with their nearest neighbours by identical Hookean springs with the spring constant k and the rest length a. In linear theory the model supports elastic waves along both the longitudinal and the transverse direction and the two waves are decoupled. In the long-wave limit, the velocity of linear transverse waves is [103]

$$v_h = \sqrt{\tau/\rho}. \tag{6.1}$$

where τ is the constant tension in the springs and ρ the linear density. The nonlinear effect of the infinitesimal perturbation of longitudinal and transverse waves on τ is omitted. For the special case in which the spring rest length equals the equilibrium lattice spacing, τ goes to 0. In this case, the linear transverse wave has a vanishing speed and the effect of nonlinearities becomes dominant.

The perturbation that we choose to study in this chapter is an abrupt impact upon the mass at origin with constant velocity v_E along the transverse direction at time 0. This impact will generate a longitudinal front with velocity

$$v_l = a\sqrt{k/m} \tag{6.2}$$

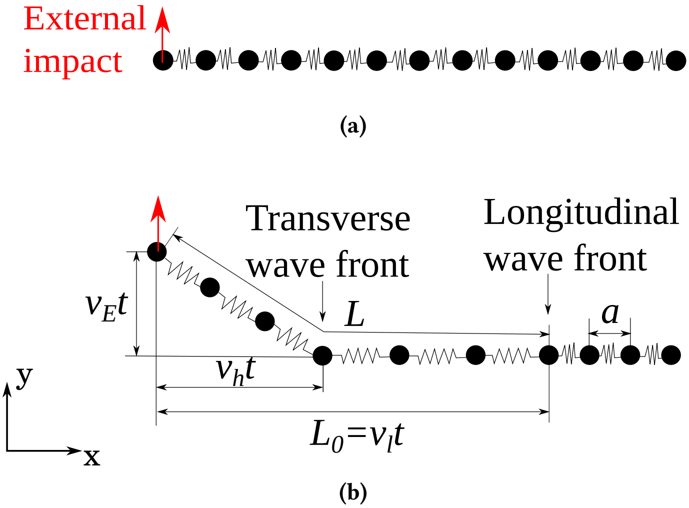

Figure 6.1. (**a**) The 1D lattice model within a plane, subject to an abrupt transverse impact of constant velocity v_E at time 0. (**b**) The impact results in two shocks: a transverse shock of speed v_h and a longitudinal shock of speed v_l. The lattice has the shape of a kink at the transverse shock.

as well as a nonlinear transverse shock whose velocity v_h is what we aim to derive. Because v_h is zero to linear order, we assume that the actual v_h is also smaller than v_l when the external impact $v_E \ll v_l$. In other words, the impact generates a fast longitudinal stress wave as well as a slow transverse displacement wave. The abrupt stimulus causes abrupt responses, i.e., the deformed structure takes the shape of a kink. This kink is a sharp transition in the direction of the spatial arrangement of the masses, which happens right at the transverse wave front. [1]

Now let us relate the shape and the propagation speed of the kink. We make use of the result that transverse waves do not influence longitudinal strain [93, 96–98]. In other words, the spring tensions in front of and behind the kink are the same. This suggests that transverse shocks propagate due to the tension induced by the longitudinal shock. We assume that behind

[1] The longitudinal wave also has a sharp wave front, but since the displacement is along the same direction of propagation there is no change of shape.

the longitudinal shock, all of the springs experience approximately the same tension. From Eq. (6.1), this implies that v_h is also nonzero and constant. This in turn suggests that the kink shape consists of two rectilinear parts, as shown in Fig. 6.1b.

Next, we derive v_h. At time t, the longitudinal wave propagates a distance of $L_0 = v_l t$. The number of springs in this region is L_0/a. The contour length of the lattice behind the front of longitudinal wave is L. Then, the uniform tension along the contour is

$$\tau = \frac{k(L - L_0)}{L_0/a}. \tag{6.3}$$

The mass at the origin subject to the impact moves a distance $v_E t$ in the y-direction and the transverse shock propagates a distance $v_h t$ in the x-direction. Using the geometry of the configuration, we find

$$L = \sqrt{(v_E t)^2 + (v_h t)^2} + L_0 - v_h t. \tag{6.4}$$

Combining Eq. (6.4) and Eq. (6.3), we find

$$\tau = ka\left(\sqrt{\left(\frac{v_E}{v_l}\right)^2 + \left(\frac{v_h}{v_l}\right)^2} - \frac{v_h}{v_l}\right). \tag{6.5}$$

Furthermore, combining Eq. (6.1), Eq. (6.2), Eq. (6.5) and using $\rho = m/a$, we obtain the desired relation

$$\left(\frac{v_E}{v_l}\right)^2 = \left(\frac{v_h}{v_l}\right)^4 + 2\left(\frac{v_h}{v_l}\right)^3. \tag{6.6}$$

If the external impact velocity v_E vanishes, v_h vanishes as well. Then, the last term in Eq. (6.6) dominates the right-hand side. Thus, we obtain the result

$$\frac{v_h}{v_l} = \frac{1}{2^{1/3}}\left(\frac{v_E}{v_l}\right)^{2/3}. \tag{6.7}$$

This power-law relation between v_E and v_h with a fractional exponent is a remarkable result of nonlinear dependence. This result has been obtained in Ref. [93, 102], and supported by experimental data [104].

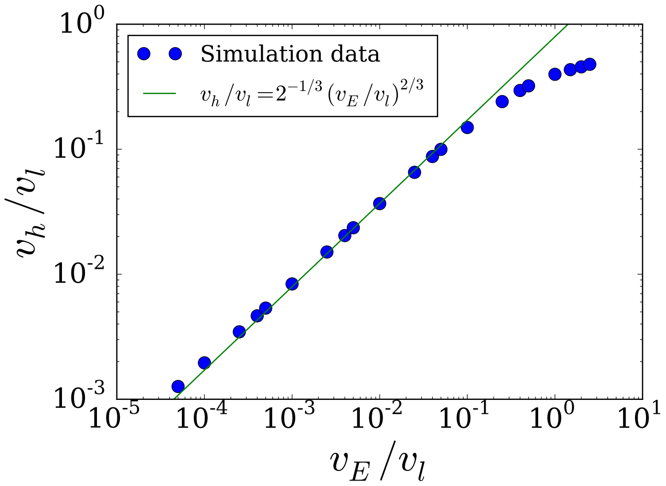

Figure 6.2. The velocity of the nonlinear transverse wave v_h vs the impact velocity v_E, with both velocities rescaled by the speed of sound v_l.

To check the relation (6.7) numerically, we performed Newtonian dynamics simulations on the lattice model to confirm the theoretical result, see Fig. 6.2. The theory fits well for small v_E. At higher ratios v_E/v_l the effect of higher-order nonlinearities cause deviations from this power law.

6.3 Continuum theory

In this section we derive v_h in the rectilinear flexible string – the continuum counterpart of the 1D lattice. We obtain the equations of motion and analyse them directly. Let l and h be the continuum fields of the longitudinal and transverse displacements and x the Lagrangian coordinate along the string. From the geometry shown in Fig. 6.3, we obtain the strain $\frac{\partial s}{\partial x}$, which to the lowest order is given by:

$$\frac{\partial s}{\partial x} = \frac{\partial l}{\partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 . \tag{6.8}$$

The flexible string has only stretching potential energy $V = \frac{\kappa}{2}(\frac{\partial s}{\partial x})^2$ is quadratic in $\frac{\partial s}{\partial x}$. Using this, we write down the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho\left(\dot{h}^2 + \dot{l}^2\right) - \frac{\kappa}{2}\left(\frac{\partial l}{\partial x} + \frac{1}{2}\left[\frac{\partial h}{\partial x}\right]^2\right)^2 \tag{6.9}$$

where κ is the elastic modulus of the string. The Euler-Lagrange equations are

$$\rho\ddot{l} - \kappa\frac{\partial}{\partial x}\left(\frac{\partial l}{\partial x} + \frac{1}{2}\left[\frac{\partial h}{\partial x}\right]^2\right) = 0, \tag{6.10}$$

$$\rho\ddot{h} - \kappa\frac{\partial}{\partial x}\left\{\left(\frac{\partial l}{\partial x} + \frac{1}{2}\left[\frac{\partial h}{\partial x}\right]^2\right)\frac{\partial h}{\partial x}\right\} = 0. \tag{6.11}$$

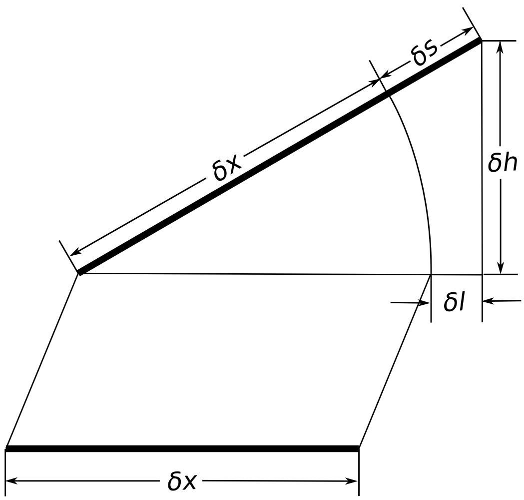

Figure 6.3. An infinitesimal element δx (thick line) subject to longitudinal and transverse displacements. This geometry lets us calculate $\frac{\partial s}{\partial x}$ in terms of $\frac{\partial l}{\partial x}$ and $\frac{\partial h}{\partial x}$, Eq. (6.8).

The external impact acts on the string transversely with constant velocity v_E at the origin starting at time 0. Eq. (6.10) is a linear wave equation for the longitudinal wave with a “source” term of second order in the transverse field

h. We seek steady-state solutions in which the displacement fields change at constant speed, i.e., $\ddot{l} = 0$ and $\ddot{h} = 0$. The condition $\ddot{l} = 0$ together with Eq. (6.10) requires that the tension $\frac{\partial l}{\partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2$ should be constant elsewhere except for the longitudinal wave front where there is a sharp jump of l. Then, Eq. (6.11) turns into the form of the linear wave equation in the tensioned region behind the longitudinal wave front:

$$\rho\ddot{h} - \kappa\frac{\partial s}{\partial x}\frac{\partial^2 h}{\partial x^2} = 0, \text{for } x < v_l t. \tag{6.12}$$

To obtain the tension $\frac{\partial s}{\partial x}$, we integrate the strain at time t along the string and use the boundary conditions to get the total deformation of the string in the tensioned region $[0, v_l t]$. The strain is this deformation divided by $v_l t$.

The boundary conditions of l are $l(x = 0, t) = 0$ and $l(x = v_l t, t) = 0$. The boundary conditions of h at the origin are $\dot{h}(x = 0, t) = v_E$ and $h(x = 0, t) = v_E t$. Because v_h is smaller than v_l, the boundary condition of h at the longitudinal wave front is $h(x = v_l t, t) = 0$. In addition, we assume that h has a traveling-wave solution of the form $h(x - v_h t)$, so $\frac{\partial h}{\partial x} = -\frac{1}{v_h}\frac{\partial h}{\partial t}$, and the steady-state solution requires $\frac{\partial^2 h}{\partial x^2} = \frac{1}{v_h^2}\frac{\partial^2 h}{\partial t^2} = 0$.

The total deformation along the string at time t is equal to the integral of the strain:

$$\int_0^{v_l t} \frac{\partial l}{\partial x} + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 \mathrm{d}x \tag{6.13}$$

$$= l\Big|_{x=0}^{v_l t} + \frac{1}{2}h\frac{\partial h}{\partial x}\Big|_{x=0}^{v_l t} - \int_0^{v_l t} \frac{1}{2}h\frac{\partial^2 h}{\partial x^2}\mathrm{d}x \tag{6.14}$$

$$= -\frac{1}{2}h\frac{\partial h}{\partial x}\Big|_{x=0} \tag{6.15}$$

$$= \frac{v_E^2 t}{2v_h}. \tag{6.16}$$

Therefore, the strain is

$$\frac{\partial s}{\partial x} = \frac{v_E^2 t}{2v_h}/(v_l t) = \frac{v_E^2}{2v_h v_l}. \tag{6.17}$$

In addition, the linear transverse-wave equation Eq. (6.12), with traveling-wave solution $h(x - v_h t)$, gives

$$\rho v_h^2 - \kappa \frac{\partial s}{\partial x} = 0. \tag{6.18}$$

From Eq. (6.17) and (6.18), we obtain

$$\frac{v_h}{v_l} = \frac{1}{2^{1/3}} \left(\frac{v_E}{v_l} \right)^{2/3}. \tag{6.19}$$

In principle, the method used in this section provides insight into the governing differential equations of the string dynamics.

6.4 Outlook

In this chapter, we have demonstrated the propagation mechanism of the transverse shock waves in a tensionless flexible string under a constant impact. One may expect to discover nonlinear waves of the same nature in many classical mechanical models for solids, e.g., marginally-rigid random spring networks [105] and some isostatic lattice networks like kagome. For untwisted kagome lattices, the shear moduli vanish along special directions, along which there are zero-frequency transverse modes of all wave numbers across the Brillouin zone [53]. This situation is analogous to a tensionless string, where even infinitesimal perturbations will generate nonlinear responses. But for such two-dimensional lattices with more complex unit cells, the coupling between degrees of freedom in transverse and longitudinal directions will not be as simple as that in strings. Besides, we have only studied the response of a string to a perturbation of a constant impact at a single point. So, extending the analysis from this chapter to general cases will require further investigations.

Moreover, for some inhomogeneous structures in toplogical mechanical lattices, e.g., dislocations [13], and interfaces between different phases [25], there are topological zero modes associated. How do these zero modes behave in the context of nonlinear motions? Do any of them also propagate in forms of solitons like vortices or skyrmions, as high-dimensional counterpart of kinks in topological rotor chains. If so, how does the boundary condition influence the behavior of such nonlinear objects. Or do they all remain localized and oscillate with large amplitudes? These are the open questions that we hope to get answers in future works.