Intense automorphisms of finite groups

Stanojkovski, M.

Author: Stanojkovski, M.
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Chapter 12

A characterization for high classes

Let $p > 3$ be a prime number and let G be a finite p-group. We recall that, for each positive integer i, the *i-th width* of G is $\mathrm{wt}_G(i) = \log_p |G_i : G_{i+1}|$. The group G is a *p-obelisk* if it is non-abelian, satisfying $G_3 = G^p$ and $|G : G_3| = p^3$. A p-obelisk G is *framed* if, for each maximal subgroup M of G, one has $\Phi(M) = G_3$. For more information about p-obelisks, we refer to Chapter 10.

In this chapter we prove the following result.

Theorem 375. *Let p be a prime number and let G be a finite p-group with $\mathrm{wt}_G(5) = 2$. Then the following are equivalent.*

1. *One has $\mathrm{int}(G) > 1$.*
2. *One has $p > 3$, the group G is a framed p-obelisk, and there exists an automorphism α of G of order 2 that induces the inversion map on G/G_2.*

We would like to stress that, from the combination of Lemma 322 with Theorem 375, it follows that each finite p-group G of class at least 6 with $\mathrm{int}(G) > 1$ is a framed p-obelisk.

12.1 A special case

The main result of this section is the following.

Proposition 376. *Let $p > 3$ be a prime number and let G be a p-obelisk. Write $C = \mathrm{C}_G(G_4)$. Assume that $\mathrm{wt}_G(5) = 1$ and $\mathrm{int}(G) > 1$. Then one has $\Phi(C) = G_3$.*

The goal of Section 12.1 is proving Proposition 376, so all assumption that we will make throughout the text (right now and right after Lemma 380) will hold until the end of Section 12.1.

Let $p > 3$ be a prime number and let G be a p-obelisk. Let $(G_i)_{i\geq 1}$ denote the lower central series of G. Assume that $\mathrm{wt}_G(5) = 1$ so, thanks to Proposition 321(2), the class of G is equal to 5. Write $C = \mathrm{C}_G(G_4)$.

Lemma 377. *The subgroup C is maximal in G.*

Proof. The commutator map induces a bilinear map $G/G_2 \times G_4/G_5 \to G_5$ whose image generates G_5, thanks to Lemma 24, and whose left kernel is C/G_2. As a consequence of Lemma 323, all quotients G_i/G_{i+1} are $\mathbb{F}_p$-vector spaces and, by assumption, $\mathrm{wt}_G(5) = 1$. By Lemma 322(2), the dimension of G_4/G_5 is equal to 1 and so Lemma 2 yields $|G : C| = p$. In other words, C is a maximal subgroup of G. ∎

Lemma 378. *One has $G_4 \subseteq C^p \subseteq G_3$ and $|G_3 : C^p| = |C^p : G_4| = p$.*

Proof. The subgroup C is maximal, by Lemma 377, and it is thus normal of index p in G. It follows that C^p is normal in G and, as a consequence of Corollary 329, the number 3 is a 1-dimensional jump of C^p in G. Lemma 327(2) yields $G_4 \subseteq C^p \subseteq G_3$ and thus, thanks to Lemma 322(2), we get $|G_3 : C^p| = |C^p : G_4| = p$. ∎

Lemma 379. *The subgroup C^p centralizes G_2.*

Proof. Each p-obelisk is regular, by Lemma 320, so, as a consequence of Lemma 54, the subgroups $[C, G_2^p]$ and $[C^p, G_2]$ are the same. Now, G_2^p is equal to G_4, by Lemma 323, and $[C, G_4] = \{1\}$, by definition of C. It follows that C^p centralizes G_2. ∎

Lemma 380. *The group C^p is contained in $\mathrm{Z}(C)$.*

Proof. The subgroup C^p is contained in G_3, by Lemma 378, and the commutator map $C \times C^p \to G_4$ is bilinear by Lemma 22. Such commutator map factors as $\gamma : C/G_2 \times C^p/G_4 \to G_4$, as a consequence of Lemma 379 and of the definition of C. Moreover, thanks to Corollary 329, if $C = \langle \{x\} \cup G_2 \rangle$ then $C^p = \langle \{x^p\} \cup G_4 \rangle$. The map γ being alternating, it follows that γ is the trivial map and so C^p centralizes C. ∎

Let now α be an intense automorphism of G of order 2 and write $A = \langle \alpha \rangle$. Set $G^+ = \{x \in G \,:\, \alpha(x) = x\}$ and $G^- = \{x \in G \,:\, \alpha(x) = x^{-1}\}$ and, for each subgroup H of G, denote $H^+ = H \cap G^+$ and $H^- = H \cap G^-$. We will prove Proposition 376 *by contradiction* and, to this end, we assume that $\Phi(C) \neq G_3$. Let X be the collection of subgroups H of C of the form $H = \langle x, y \rangle$, where $x \in C \setminus G_2$ and $y \in G_4 \setminus G_5$. Then A acts on X in a natural way. Let X^+ be the collection of fixed points of X under A.

Lemma 381. *The exponent of C divides p^2.*

Proof. By definition, the subgroup C^{p^2} is contained in $(C^p)^p$. By Lemma 335, we have $[C,C]=C^p$ so it follows from Lemma 54 that $(C^p)^p=[C,C]^p=[C,C^p]$. As a consequence of Lemma 380, the subgroup $[C,C^p]$ is trivial, and thus $(C^p)^p=\{1\}$. In particular, the exponent of C divides p^2. ∎

Lemma 382. *Let x be an element of $C\setminus G_2$. Then x has order p^2.*

Proof. As a consequence of Corollary 329, the element x^p is non-trivial so the order of x is divisible by p^2. We conclude by Lemma 381. ∎

Lemma 383. *Let $H\in X$. Then H is abelian and $H\cap G_5=\{1\}$. Moreover, if $x,y\in H$ satisfy $\operatorname{dpt}_G(x)=1$ and $\operatorname{dpt}_G(y)=4$, then $H=\langle x\rangle\oplus\langle y\rangle$.*

Proof. Let $(x,y)\in(C\setminus G_2)\times(G_4\setminus G_5)$ be such that $H=\langle x,y\rangle$. Then $y\in\mathrm{Z}(C)$ and the group H is commutative. Moreover, as a consequence of Lemma 169, the subgroups $\langle x\rangle$ and $\langle y\rangle$ have respectively only odd and only even jumps. In particular, $\langle x\rangle\cap\langle y\rangle=\{1\}$ and $H=\langle x\rangle\oplus\langle y\rangle$. In addition, it follows from Lemma 328(1) that 5 is a jump of H in G if and only if $x^{p^2}\neq 1$. Lemma 382 yields $H\cap G_5=\{1\}$. ∎

Lemma 384. *Let $H\in X$ and, for each $i\in\mathbb{Z}_{\geq 1}$, denote $u_i=\operatorname{wt}_H^G(i)$. Then $(u_1,u_2,u_3,u_4,u_5)=(1,0,1,1,0)$ and H has order p^3.*

Proof. For each $i\in\mathbb{Z}_{\geq 1}$, write $w_i=\operatorname{wt}_G(i)$. Thanks to Lemma 322, we have $(w_1,w_2,w_3,w_4,w_5)=(2,1,2,1,1)$. Let x,y be as in Lemma 383: then $u_1,u_4\geq 1$ and $u_5=0$. Since, for each $i\geq 1$, one has $u_i\leq w_i$, we get $u_4=1$. Moreover, Lemma 328(1) ensures that $u_3\geq 1$. Let now $N=\langle y\rangle G_5$, which is a normal subgroup of G thanks to Lemma 327. Then $N\cap H=\langle y\rangle$ and, the quotient $H/\langle y\rangle$ being cyclic, so is HN/N. Thanks to Lemma 338, all jumps of HN/N have the same dimension and width 1 in G/N. As a result, 2 is not a jump of HN/N in G/N and, since $\langle y\rangle$ is contained in G_4, we have $u_2=0$ and $u_1=u_3=1$. The group H has order p^3, by Lemma 84. ∎

Lemma 385. *The cardinality of X is p^4.*

Proof. Thanks to Lemma 383, the set X consists of subgroups of the form $\langle x\rangle\oplus\langle y\rangle$, with $x\in C\setminus G_2$ and $y\in G_4\setminus G_5$. The cardinality of X will be thus equal to the quotient $\frac{n}{m}$, where n is the cardinality of $(C\setminus G_2)\times(G_4\setminus G_5)$ and m denotes the number of elements of $(C\setminus G_2)\times(G_4\setminus G_5)$ that generate the same subgroup. Let H be in X and let x and y be generators of H, as described before. Then, as a consequence of Lemma 384, the orders of x and y are respectively p^2 and p. It follows that $m=(p^3-p^2)(p-1)$ so, in view of Lemmas 384 and 319, we get

$$|X|=\frac{n}{m}=\frac{(p^6-p^5)(p^2-p)}{(p^3-p^2)(p-1)}=p^4.$$

∎

Lemma 386. *Let $H \in X$. Then the following are equivalent.*

1. *The subgroup H is A-stable.*

2. *There exists $x \in C^- \setminus G_2$ such that $H = \langle x \rangle \oplus G_4^+$.*

Proof. To prove that (2) implies (1) is an easy exercise; we prove the other implication. Assume (1). The group H is abelian, by Lemma 383, and it is A-stable. By Corollary 76, it decomposes as $H = H^+ \oplus H^-$. In view of Lemmas 384 and 85(1), we have that $H^+ = G_4^+$ and that $H \cap G_4 = G_4^+$. It follows from Lemma 383 that there exists a cyclic subgroup Q of H such that $H = Q(H \cap G_4)$, and thus H^- is cyclic. The proof is now complete. ∎

Lemma 387. *The cardinality of X^+ is p^2.*

Proof. Let $\mathcal{C}$ denote the collection of subgroups $\langle x \rangle$ of C, where x is an element of $C^- \setminus G_2$. Thanks to Lemma 386, one can define the map $\mathcal{C} \to X^+$, by $Q \mapsto Q \oplus G_4^+$, which is easily shown to be a bijection. In particular, the cardinality of X^+ is equal to that of $\mathcal{C}$. Now, the group C is normal in G, as a consequence of Lemma 377, and therefore it is A-stable. By Lemma 382, each element of $C^- \setminus G_2$ has order p^2 and, as a consequence of Proposition 134, the set $C^- \setminus G_2$ is equal to $C^- \setminus G_3^-$. It follows from Lemma 85 that

$$|X^+| = \frac{|C^-| - |G_3^-|}{p^2 - p} = \frac{p^4 - p^3}{p^2 - p} = p^2.$$

∎

Lemma 388. *For each subgroup L of C^-, the commutator map induces a bilinear map $L \times G_3 \to G_4^+$.*

Proof. The subgroup G_4 is central in C so, by Lemma 22, the commutator map $L \times G_3 \to G_4$ is bilinear. Since L is contained in $C = \mathrm{C}_G(G_4)$, the commutator map induces a bilinear map $L \times G_3/G_4 \to G_4$. Now, thanks to Proposition 134, the map α induces the inversion map on G_3/G_4 and so, thanks to Lemma 61, we get $[L, G_3] = [L, G_3]^+$. In particular, $[L, G_3]$ is contained in G_4^+ and the proof is complete. ∎

Lemma 389. *Let $H \in X^+$. Then $G_3 \subseteq \mathrm{N}_G(H)$.*

Proof. By Lemma 386, the subgroup H is of the form $\langle x \rangle \oplus G_4^+$, for some element $x \in C^- \setminus G_2$. As a consequence of Lemma 20, the subgroup $[G_3, G_4^+]$ is trivial so, from Lemma 18(1), it follows that $[G_3, H] = [G_3, \langle x \rangle]$. Lemma 388 yields that $[G_3, \langle x \rangle]$ is contained in G_4^+, a subgroup of H, and so, by Lemma 13, one has $G_3 \subseteq \mathrm{N}_G(H)$. ∎

We will now prove Proposition 376 by building a contradiction. We remind the reader that we have assumed that $\Phi(C) \neq G_3$.

Let H be an element of X^+ with the property that $|G : \mathrm{N}_G(H)|$ is maximal. Let moreover $\mathcal{J}$ denote the collection of jumps of $\mathrm{N}_G(H)$ in G. As a consequence of Lemma 389, the normalizer of H contains HG_3. It follows from Lemma 384 that $\{1, 3, 4, 5\}$ is contained in $\mathcal{J}$ and, thanks also to Lemma 319(2), that $|G : \mathrm{N}_G(H)| \leq |G : HG_3| = p^2$. Now, by Lemmas 385 and 387, the cardinalities of X and X^+ are respectively p^4 and p^2. It follows from Lemma 94 that

$$p^4 = |X| \leq \sum_{K \in X^+} |G : \mathrm{N}_G(K)| \leq |X^+||G : \mathrm{N}_G(H)| \leq p^2 p^2 = p^4,$$

and therefore $|G : \mathrm{N}_G(H)| = p^2$. In particular, we get $\mathrm{N}_G(H) = HG_3$ and $\mathcal{J} = \{1, 3, 4, 5\}$. Moreover, again by Lemma 94, no two elements of X^+ are conjugate in G. As a consequence of Lemma 81, the subgroup G^+ is contained in $\mathrm{N}_G(H) = HG_3$ and so, thanks to Lemma 85(1), the number 2 is a jump of $\mathrm{N}_G(H)$ in G. Contradiction.

12.2 The last exotic case

The aim of Section 12.2 is that of exploring the last exotic case for what concerns the structure of finite p-groups of intensity greater than 1. As a consequence of Theorem 375, the finite p-groups of "high class" and intensity greater than 1 all need to be framed obelisks. Theorem 390 is the last result we present that still allows some "structural freedom" to p-obelisks.

Theorem 390. *Let p be a prime number and let G be a finite p-group with* $\mathrm{wt}_G(5) = 1$. *Write $C = \mathrm{C}_G(G_4)$. Then the following are equivalent.*

1. *One has* $\mathrm{int}(G) > 1$.
2. *One has $p > 3$, the group G is a p-obelisk, and $\Phi(C) = G_3$. Moreover, there exists an automorphism α of G of order 2 that induces the inversion map on G/G_2.*

The remaining part of Section 12.2 will be devoted to the proof of Theorem 390 and we will thus work under the hypotheses of such theorem.

Assume first (1). As a consequence of Proposition 95 and Corollary 284, the prime p is larger than 3 and so, thanks to Proposition 318, the group G is a p-obelisk. Thanks to Theorem 125(1), there exists an intense automorphism α of order 2 of G, which induces the inversion map on G/G_2 by Proposition 134. Proposition 376 yields $\Phi(C) = G_3$.

Assume now that $p > 3$, that G is a p-obelisk, and that $\Phi(C) = G_3$. Let moreover α be an automorphism of order 2 of G that induces the inversion map on G/G_2. We will prove (1). Set $A = \langle\alpha\rangle$ and, for each $i \in \mathbb{Z}_{\geq 1}$, denote $w_i = \mathrm{wt}_G(i)$. Thanks to Lemma 322, we have $(w_1, w_2, w_3, w_4, w_5) = (2, 1, 2, 1, 1)$ and so, thanks to Proposition 321(2), the class of G is equal to 5. We remind the reader that, for each $k \in \mathbb{Z}_{\geq 0}$, the map $G \to G$ sending x to x^{p^k} is denoted by ρ^k. Furthermore, by Lemma 320, the group G is regular and so, given any subgroup K of G, Lemma 52 yields that $\rho^k(K) = K^{p^k}$.

Lemma 391. *One has* $\rho^2(C) = G_5$.

Proof. The group $\Phi(C)$ is equal to $C^p[C, C]$ and $\Phi(C) = G_3$, by assumption. It follows from Lemma 328(1) that $\Phi(C)^p = G_5$. Now, the group C is maximal in G, by Lemma 377, and so, as a consequence of Lemma 319(2), the quotient C/G_2 is cyclic. Lemma 28 yields $[C, C] = [C, G_2]$. Now, by Lemma 54, one has $[C, G_2]^p = [C, G_2^p]$ and so, thanks to Lemma 323, one gets $[C, G_2]^p = [C, G_4] = \{1\}$. It follows that $\Phi(C)^p$ is equal to C^{p^2} and therefore $\rho^2(C) = G_5$. ■

Lemma 392. *Let* $\alpha_5 : G/G_5 \to G/G_5$ *denote the automorphism that is induced by* α. *Then* α_5 *is intense.*

Proof. Let $\overline{G}$ denote G/G_5 and use the bar notation for the subgroups of $\overline{G}$. The automorphism α_5 induces the inversion map on $\overline{G}/\overline{G_2}$, because α does so on G/G_2. Moreover, thanks to Lemma 326, the group $\overline{G}$ is a p-obelisk of class 4. We conclude by applying Theorem 343. ■

Let H be a subgroup of G and, for each $i \in \mathbb{Z}_{\geq 1}$, write $u_i = \mathrm{wt}_H^G(i)$. We will show that H has an A-stable conjugate in G. We assume, without loss of generality, that H is non-trivial. As a consequence of Lemma 392, the automorphism that α induces on G/G_5 is intense. If G_5 is contained in H, then, thanks to Lemma 359, there exists $g \in G$ such that gHg^{-1} is A-stable. Since G_5 has order p, we now assume that $H \cap G_5 = \{1\}$. By Lemma 360, all jumps of H in G have dimension 1 and, if they all have the same parity, Lemmas 361 and 362 yield that H has an A-stable conjugate. We assume now that H has jumps of both parities and we denote by i and j respectively the least odd and the least even jump of H in G.

Lemma 393. *One has* $u_4 = 1$.

Proof. The group G having class 5, we have $j \in \{2, 4\}$. It follows from Corollary 329 that $0 \neq u_4 \leq w_4$ and therefore $u_4 = 1$. ■

Lemma 394. *One has* $i = 3$.

Proof. Since $u_5 = 0$, the index i is different from 5 and so $i \in \{1, 3\}$. Assume by contradiction that $i = 1$. As a consequence of Lemma 393, the subgroups G_4 and $(H \cap G_4)G_5$ are equal. The group G_5 being central, it follows from Lemma 18(1) that

$$G_5 = [G, G_4] = [G, (H \cap G_4)G_5] = [G, H \cap G_4].$$

Thanks to Lemma 20, the group $[G_2, G_4]$ is trivial ans so Lemma 18(2) yields

$$[HG_2, G_4] = [H, G_4] = [H, H \cap G_4] \subseteq H \cap G_5 = \{1\}.$$

In particular, H is contained in C and so, as a consequence of Lemma 328(1), we get $\rho^2(H) = \rho^2(C)$. It follows from Lemma 391 that H contains G_5. Contradiction to $H \cap G_5 = \{1\}$. ∎

Let D be a maximal subgroup of G with the property that $(H \cap G_3)G_4 = D^p G_4$ and note that, thanks to Corollary 329, the subgroup D is uniquely determined by H. Since D^p is characteristic in the normal subgroup D, Lemma 327 yields $D^p = D^p G_4$ and therefore, from Corollary 329, one gets $|D^p : G_4| = p$.

Lemma 395. *One has $\rho^2(D) = \{1\}$.*

Proof. From the definition of D together with Lemma 328(1), it follows that $\rho^2(D) = \rho(D^p) = \rho(H \cap G_3)$. As a consequence of Lemma 323, the subgroup $\rho(H \cap G_3)$ is contained in $H \cap G_5 = \{1\}$ and thus $\rho^2(D) = \{1\}$. ∎

Lemma 396. *One has $D \neq C$ and $[D, G_4] = G_5$.*

Proof. The subgroups D and C are both maximal in G and so, as a consequence of Lemmas 391 and 395, they are distinct. Moreover, the class of G being 5, the subgroup $[D, G_4]$ is non-trivial. Lemma 20 gives that $[D, G_4]$ is contained in G_5 and, since $w_5 = 1$, we get $[D, G_4] = G_5$. ∎

Lemma 397. *One has $[G_2, D^p] = G_5$.*

Proof. The group G is regular, by Lemma 320, and therefore, by Lemma 54, the subgroups $[G_2, D^p]$ and $[G_2^p, D]$ are equal. By Lemma 323, we have $G_2^p = G_4$ and so, from Lemma 396, we derive $[G_2, D^p] = G_5$. ∎

Lemma 398. *The subgroup H is abelian.*

Proof. As a consequence of Lemma 394, the subgroup H is contained in G_2 and, since $w_2 = 1$, the quotient $H/(H \cap G_3)$ is cyclic. It follows from Lemma 28 that $[H, H] = [H, H \cap G_3]$ and so, thanks to Lemma 20, one gets $[H, H] \subseteq H \cap G_5 = \{1\}$. In particular, H is abelian. ∎

Lemma 399. *One has $(i, j) = (3, 4)$.*

Proof. By Lemma 394, the jump i is equal to 3 and, by definition of D, we have $D^p = (H \cap G_3)G_4$. Moreover, since G has class 5, the jump j belongs to $\{2,4\}$. Assume by contradiction that $j = 2$. Then one has $u_2 = w_2 = 1$ and so $G_2 = HG_3$. By Lemma 397, the subgroups $[G_2, D^p]$ and G_5 coincide and so, the group G_5 being central, Lemma 22 ensures that the commutator map $G_2 \times D^p \to G_5$ is bilinear and differs from the trivial map. Now, the induced map $G_2/G_3 \times D^p/G_4 \to G_5$, derived from Lemma 20, is non-trivial and so $[H, H \cap G_3] \neq 1$. Contradiction to Lemma 398. ∎

Lemma 400. *Let x and y be elements of H, respectively belonging to $D^p \setminus G_4$ and $G_4 \setminus G_5$. Then $H = \langle x\rangle \oplus \langle y\rangle$ and $(u_1, u_2, u_3, u_4, u_5) = (0, 0, 1, 1, 0)$.*

Proof. Thanks to Lemma 399, we have $(u_1, u_2, u_3, u_4, u_5) = (0, 0, 1, 1, 0)$. The subgroup H is thus contained in G_3 and so, by Lemma 323, one has $H^p \subseteq G_5 \cap H = \{1\}$. It follows from Lemma 398 that H is elementary abelian. Given any two elements x and y of H, satisfying $x \in D^p \setminus G_4$ and $y \in G_4 \setminus G_5$, Lemma 82 now yields $H = \langle x\rangle \oplus \langle y\rangle$. ∎

We define X to be the collection of all subgroups of G of the form $\langle x\rangle \oplus \langle y\rangle$, where (x, y) belongs to $(D^p \setminus G_4) \times (G_4 \setminus G_5)$. Thanks to Lemmas 395 and 398, each such subgroup is elementary abelian and thus X is well defined. We remark that, the group D^p being normal in G, the group G acts naturally on X by conjugation. Write $X^+ = \{K \in X : \alpha(K) = K\}$.

Lemma 401. *The cardinality of X is p^2.*

Proof. Let K be an element of X. Then there exist elements x and y of order p, respectively of depth 3 and 4 in G, such that $x \in D^p$ and $K = \langle x\rangle \oplus \langle y\rangle$. Since $|D^p : G_4| = p$ and $(w_4, w_5) = (1, 1)$, we get

$$|X| = \frac{(p^3 - p^2)(p^2 - p)}{(p-1)p(p-1)} = p^2.$$

∎

Lemma 402. *One has $\mathrm{N}_G(H) \cap D = \mathrm{N}_G(H) \cap G_2$.*

Proof. Assume by contradiction that $\mathrm{N}_G(H) \cap D \neq \mathrm{N}_G(H) \cap G_2$. As a consequence of Lemma 393, we have that $(H \cap G_4)G_5 = G_4$ and, the group G_5 being central, it follows from Lemma 18(1) that $[D, G_4] = [D, H \cap G_4]$. Lemma 13 yields $[D, G_4] \subseteq H$ and thus, by Lemma 396, the subgroup G_5 is contained in H. Contradiction. ∎

Lemma 403. *One has $\mathrm{N}_G(H) \cap G_2 = G_3$.*

Proof. As a consequence of Lemma 399, the subgroup H is contained in G_3 and, thanks to Lemma 20, one has $[G_3, G_3] \subseteq G_6 = \{1\}$. In particular, G_3 normalizes H. Assume by contradiction that 2 is a jump of $\mathrm{N}_G(H)$ in G. Thanks to Lemma 20, the group G_2 centralizes G_4 and, by definition of D, we have $(H \cap G_3)G_4 = D^p$. It follows from Lemma 18(1) that $[G_2, D^p] = [G_2, H \cap G_3]$ and so, thanks to Lemma 13, the subgroup $[G_2, D^p]$ is contained in H. Lemma 397 yields $G_5 \subseteq H$. Contradiction. ∎

We claim that the action of G on X is transitive. As a consequence of Lemma 401, we have that $p^2 = |X| \geq |G : \mathrm{N}_G(H)|$ and therefore, applying Lemmas 402 and 403, we get

$$p^2 \geq |G : \mathrm{N}_G(H)| \geq |D : G_2||G_2 : G_3| = |D : G_3|.$$

As a consequence of Lemma 319(2), the index $|D : G_3|$ is equal to p^2 and therefore the number of conjugates of H in G is equal to p^2. This proves the claim. To conclude, we remark that $\alpha(H)$ is an element of X and therefore $\alpha(H)$ and H are conjugate. The choice of H being arbitrary, Lemma 93 yields that α is intense and so $\mathrm{int}(G) > 1$. The proof of Theorem 390 is now complete.

12.3 Proving the main theorem

In this section we prove Proposition 404 and Theorem 375. We remind the reader that a p-obelisk G is framed if, for each maximal subgroup M of G, one has $\Phi(M) = G_3$.

Proposition 404. *Let $p > 3$ be a prime number and let G be a finite p-group of class at least 5. Assume that $\mathrm{int}(G) > 1$. Then G is a p-obelisk and one of the following holds.*

1. *One has $\mathrm{wt}_G(5) = 1$ and G has class 5.*

2. *One has $\mathrm{wt}_G(5) = 2$ and G is framed.*

Proof. By Proposition 318, the group G is a p-obelisk so, thanks to Proposition 321(1), the width $\mathrm{wt}_G(5)$ is either 1 or 2. The 4-th width of G is 1, thanks to Lemma 322(2), so, if $\mathrm{wt}_G(5) = 1$, then Proposition 321(2) yields that G has class 5. Assume now that $\mathrm{wt}_G(5) = 2$. We will show that, for each maximal subgroup M of G, one has $\Phi(M) = G_3$. To this end, let M be a maximal subgroup of G. By Lemma 322, the widths $\mathrm{wt}_G(1)$ and $\mathrm{wt}_G(4)$ are respectively 2 and 1 so, the index $|G : M|$ being p, it follows from Lemma 330, that 5 is a jump of $[M, G_4]$ of width 1 in G. Moreover, 5 is the smallest jump of $[M, G_4]$ in G, and so Lemma 327 yields $G_6 \subseteq [M, G_4]$. We denote $\overline{G} = G/[M, G_4]$ and use the bar notation for the subgroups and the elements of $\overline{G}$. We remark that, by construction, we have $\overline{M} \subseteq \mathrm{C}_{\overline{G}}(\overline{G_4})$ and $\mathrm{wt}_{\overline{G}}(5) = 1$. The class of $\overline{G}$ being 5, we have in fact that $\overline{M} = \mathrm{C}_{\overline{G}}(\overline{G_4})$ and so Proposition 376 yields $\Phi(\overline{M}) = \overline{G_3}$. The subgroup $\Phi(M)$ being normal in G, it follows from Lemma 327 that $\Phi(M) = \{x \in G : \overline{x} \in \Phi(\overline{M})\}$ and therefore $\Phi(M) = G_3$. The choice of M being arbitrary, the proof is complete. ■

We are finally ready to prove Theorem 375. Let p be a prime number and let G be a finite p-group with $\mathrm{wt}_G(5) = 2$. The implication (2) $\Rightarrow$ (1) is given by Theorem 344. Assume now (1). Since $\mathrm{wt}_G(5) \neq 1$, the class of G is at least 5. Moreover, thanks to Proposition 95 and Corollary 284, the prime p is larger than 3. Proposition 404 yields that G is a framed p-obelisk. As a consequence of Theorem 125, the intensity of G is equal to 2 and so, thanks to the Schur-Zassenhaus theorem, G has an intense automorphism of order 2 that, by Proposition 134, induces the inversion map on G/G_2. The proof of Theorem 375 is complete.