Intense automorphisms of finite groups

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Chapter 11

The most intense chapter

Let $p > 3$ be a prime number. We recall that a *p-obelisk* is a finite p-group G of class at least 2 that satisfies $G_3 = G^p$ and $|G : G_3| = p^3$. A p-obelisk G is *framed* if, for each maximal subgroup M of G, one has $\Phi(M) = G_3$. Some theory about p-obelisks is developed in Chapter 10.

The main results of this chapter are summarized in Theorems 343 and 344, which are proven in Section 11.4.

Theorem 343. *Let $p > 3$ be a prime number and let G be a finite p-group of class 4. Let α be an automorphism of order 2 of G. Then the following conditions are equivalent.*

1. *The group G is a p-obelisk and the automorphism $G/G_2 \to G/G_2$ that is induced by α is equal to the inversion map $\overline{x} \mapsto \overline{x}^{-1}$.*

2. *The automorphism α is intense.*

An analogue of Theorem 343 for higher nilpotency classes is proven in Chapter 12: the next theorem gives an essential contribution to its proof.

Theorem 344. *Let $p > 3$ be a prime number and let G be a framed p-obelisk. Let α be an automorphism of order 2 of G and assume that the automorphism $G/G_2 \to G/G_2$ that is induced by α is equal to the inversion map $\overline{x} \mapsto \overline{x}^{-1}$. Then α is intense.*

We remark that the structure of Chapter 11 is quite rigid and is meant to ease the understanding of the strategy behind the proof of Theorem 344. We will prove Theorem 344 by induction on the nilpotency class c of the group G and we will separate the cases according to the parity of c. Propositions 345, 358, and 369 will be the building blocks of the whole theory and will be verified respectively in Sections 11.1, 11.2, and 11.3. We will use several results from Section 10.4 to understand the structure of the subgroups of G, according to the size of their intersection with G_c. Moreover, the arguments that we will apply will heavily depend from the knowledge of the jumps of each subgroup in G. For more detailed information about jumps, we refer to Section 2.3.

11.1 The even case

The next proposition is proven for any p-obelisk, where p is a prime number greater than 3. We want to stress that, on the contrary, in Propositions 358 and 369 we ask for the p-obelisk to be framed.

Proposition 345. *Let $p > 3$ be a prime number and let G be a p-obelisk of class c. Assume that c is even. Let moreover α be an automorphism of G of order 2 and assume that the map $\alpha_c : G/G_c \to G/G_c$ that is induced by α is intense. Then α is intense.*

We give the proof of Proposition 345 in Section 11.1.2, after some preparation.

11.1.1 Some lemmas

We will work under the hypotheses of Proposition 345 until the end of Section 11.1.1. The class c of G being even, Lemma 322(4) yields that G_c has order p. We denote moreover $A = \langle \alpha \rangle$ and we recall that a subgroup H of G is said to be A-stable if the action of A on G induces an action of A on H.

Lemma 346. *Let H be a subgroup of G containing G_c. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. The automorphism α_c is intense so, by Lemma 93, there exists $g \in G$ such that $(gHg^{-1})/G_c$ is $\langle \alpha_c \rangle$-stable. It follows from the definition of α_c that gHg^{-1} is A-stable. ∎

Lemma 347. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Then all jumps of H in G are odd.*

Proof. The subgroup H has trivial intersection with G_c and c is even. It follows from Corollary 329 that H cannot have even jumps in G. ∎

Lemma 348. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Define $T = HG_c$ and assume that $\alpha(T) = T$. Then, for each subgroup K of T, one has $\alpha(KG_c) = KG_c$. Moreover, for each $x \in H$, there exists $\gamma \in G_c$ such that $\alpha(x) = x^{-1}\gamma$ and $\alpha(\gamma) = \gamma$.*

Proof. We denote $\overline{G} = G/G_c$ and we use the bar notation for its subgroups. By Lemma 347, all jumps of H in G are odd and so all jumps of $\overline{T}$ in $\overline{G}$ are odd. The subgroup $\overline{T}$ is $\langle\alpha_c\rangle$-stable so, as a consequence of Lemma 85, each element of $\overline{T}$ is sent to its inverse by α_c. Every subgroup of $\overline{T}$ is thus $\langle\alpha_c\rangle$-stable and, in particular, so is $\overline{K}$. It follows from the definition of α_c that KG_c is A-stable. Moreover, every element of H is inverted, modulo G_c, by α and the restriction of α to G_c is the identity map, thanks to Lemma 62. ∎

Lemma 349. *Let H be a non-trivial subgroup of G such that $H \cap G_c = \{1\}$. Let l denote the least jump of H and assume that all jumps of H in G have the same width. Assume moreover that $\alpha(HG_c) = HG_c$. Then there exists $g \in G_{c-l}$ such that gHg^{-1} is A-stable.*

Proof. Define $T = HG_c$. All jumps of H in G are odd, by Lemma 347, and H is abelian, by Lemma 341(1). The subgroup G_c being central, the group T is in fact equal to $H \oplus G_c$. Moreover, the subgroup G_c being characteristic, $T = \alpha(T) = \alpha(H) \oplus G_c$ and $\alpha(H)$ is a complement of G_c in T. By Lemma 341(2), the Frattini subgroup of H is equal to $H \cap G_{l+1}$ so it follows from Lemma 342 that there exists $t \in G_{c-l}$ such that $\alpha(H) = tHt^{-1}$. Thanks to Lemma 92, there exists $g \in G_{c-l}$ such that gHg^{-1} is A-stable. ∎

Lemma 350. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Assume that all jumps of H in G have the same width. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. Denote $T = HG_c$. Thanks to Lemma 346, there exists $a \in G$ such that aTa^{-1} is A-stable. Write $T' = aTa^{-1}$ and $H' = aHa^{-1}$. Then $T' = H'G_c$ and, thanks to Lemma 349, there exists $b \in G$ such that $bH'b^{-1}$ is A-stable. To conclude, define $g = ba$. ∎

Lemma 351. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

We devote the remaining part of this section to the proof of Lemma 351. We warn the reader that the following assumptions will be valid until the end of Section 11.1.1.

Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Without loss of generality we assume that H is non-trivial and, in view of Lemma 350, that the jumps of H in G do not all have the same width. As a consequence of Proposition 321(1), each jump of H in G will have width 1 or 2. Let l and j denote respectively the least jump of width 1 and the least jump of width 2 of H in G. Write $T = HG_c$.

Lemma 352. *Let i and h be jumps of H. Assume that $\mathrm{wt}_H^G(i) = 1$ and that $\mathrm{wt}_H^G(h) = 2$. Then $i < h$.*

Proof. By Lemma 347, both i and h are odd so, the class c being even, Corollary 329 yields $i < h$. ∎

Lemma 353. *The following hold.*

1. *One has $l < j$.*

2. *One has $j + l > c$.*

3. *The subgroup H is abelian.*

Proof. Part (1) follows directly from Lemma 352. We prove (2) and (3) together. As a consequence of Lemma 339, the group $H/(H \cap G_j)$ is cyclic and, thanks to Lemma 28, one gets $[H, H] = [H, H \cap G_j]$. The number $l + j$ being even, it follows from Lemma 347 that $l + j$ is not a jump of H in G. Lemma 330 yields $l + j > c$ and, as a result, $[H, H] \subseteq G_{c+1} = \{1\}$ so H is abelian. ∎

Lemma 354. *There exist cyclic subgroups J and L of H such that $H = J \oplus L$ and j and l are respectively the least jump of J and the least jump of L in G.*

Proof. The subgroup H is abelian by Lemma 353(3) and, as a consequence of Lemma 352, the subgroup $H \cap G_j$ has only jumps of width 2. The smallest jump of H in G is l so, thanks to Lemma 82, there is an element z in H with $\mathrm{dpt}_G(z) = l$. Define $L = \langle z \rangle$. Then L is a subgroup of H and l is the least jump of L in G. Moreover, thanks to Lemma 338, all jumps of L are odd and of width 1 in G. Now, l is smaller than j, by Lemma 353(1), and j is a jump of L in G as a consequence of Corollary 329. However, j is a jump of width 2 of H, and thus there exists an element x in $H \setminus L$ such that $\mathrm{dpt}_G(x) = j$. Define $J = \langle x \rangle$. The group H being abelian, Corollary 329 yields $L \cap J = \{1\}$. Now, every jump $l \leq i < j$ of L in G is also a jump of H and it has width 1 by definition of j. Moreover, each jump $j \leq i < c$ of $J \oplus L$ is a jump of width 2 of H. Corollary 329 ensures that all odd integers $l \leq i < c$ are jumps of H in G, so Lemma 84 yields

$$|J \oplus L| = \prod_{i=l}^{c-1} p^{\dim^G_{J \oplus L}(i)} = \prod_{i=l}^{c-1} p^{\dim^G_H(i)} = |H|.$$

It follows that H and $J \oplus L$ coincide. ∎

Lemma 355. *Let J and L be as in Lemma 354. Assume that $\alpha(T) = T$. Then there exists $g \in G_{c-l}$ such that the following hold.*

1. *The group gLg^{-1} is A-stable.*

2. *One has $gTg^{-1} = T$ and $gJg^{-1} = J$.*

Proof. We define $R = LG_c$. By Lemma 348, the group R is A-stable. The subgroup L is cyclic so, by Lemma 338, all its jumps in G are odd and of width 1. With L in the role of H, it follows from Lemma 349 that there exists $g \in G_{c-l}$ such that gLg^{-1} is A-stable. We fix such an element g and prove that g normalizes both J and T. The least jump of J is j and therefore $[g, J] = \{[g, x] : x \in J\}$ is contained in $[G_{c-l}, G_j]$. As a consequence of Lemma 20, the set $[g, L]$ is contained in G_{c-l+j}, which is itself contained in G_{c+1} thanks to Lemma 353(1). It follows that g centralizes J and $gJg^{-1} = J$. To conclude, we show that $gTg^{-1} = T$. The subgroup T is contained in G_l, so $[g, T]$ is contained in $[G_{c-l}, G_l]$. Again applying Lemma 20, we get that $[g, T]$ is contained in G_c. The subgroup G_c being contained in T, we are done by Lemma 13. ∎

Lemma 356. *Let J and L be as in Lemma 354. Assume that $\alpha(T) = T$ and $\alpha(L) = L$. Then there exists $g \in G_{c-j}$ such that gHg^{-1} is A-stable.*

Proof. We will construct g. Let $x, z \in H$ be such that $J = \langle x \rangle$ and $L = \langle z \rangle$. As a consequence of Lemma 354, one has $\mathrm{dpt}_G(x) = j$ and $\mathrm{dpt}_G(z) = l$. Let moreover $\gamma \in G_c$ be such that $\alpha(x) = x^{-1}\gamma$ and $\alpha(\gamma) = \gamma$; the existence of γ is confirmed by Lemma 348. Define $m = (j + l - c)/2$, which is a positive integer thanks to Lemma 353(2). By Lemma 323, there exists a of depth $c - j$ in G such that $\rho^m(a) = z$. We fix a and remark that a belongs to $\mathrm{C}_G(L)$, because $[a, z] = [a, \rho^m(a)] = 1$ and z generates L. Now, x does not belong to L, but, as a consequence of Corollary 329, the jump j of J is also a jump of L in G. Since $\mathrm{wt}_G(j) = 2$, it follows from Lemma 330 that there exists $s \in \mathbb{Z}$ such that $[a^s, x] = \gamma^{\frac{p-1}{2}}$. We define $g = a^s$ and claim that gHg^{-1} is A-stable. We recall that γ belongs to the central subgroup G_c and that, because of Lemma 323, the exponent of G_c is p. We compute

$$\alpha(gxg^{-1}) = \alpha([g, x]x) = \alpha(\gamma^{\frac{p-1}{2}}x) = \alpha(\gamma^{\frac{p-1}{2}})\alpha(x) = \gamma^{\frac{p-1}{2}}x^{-1}\gamma =$$

$$\gamma^{\frac{p+1}{2}}x^{-1} = (\gamma^{\frac{p-1}{2}}x)^{-1} = ([g, x]x)^{-1} = (gxg^{-1})^{-1}$$

and so gJg^{-1} is A-stable. Moreover, g centralizes L and therefore $gHg^{-1} = gJg^{-1} \oplus L$. As a consequence, gHg^{-1} is itself A-stable. ∎

Lemma 357. *Let J and L be as in Lemma 354. Assume that $\alpha(T) = T$. Then there exists $g \in G_{c-j}$ such that gHg^{-1} is A-stable.*

Proof. As a consequence of Lemma 355, there exists $a \in G_{c-l}$ such that aLa^{-1} is A-stable, $aTa^{-1} = T$, and $aHa^{-1} = J \oplus aLa^{-1}$. We fix such a and we take $h \in G_{c-j}$ making $h(aHa^{-1})h^{-1}$ stable under the action of A. With H replaced by aHa^{-1}, Lemma 356 guarantees the existence of h. We define $g = ha$ and we claim that $g \in G_{c-j}$. The jump j is larger than the jump l, by Lemma 353(1), and therefore $G_{c-l} \subseteq G_{c-j}$. It follows that the product ah belongs to G_{c-l}. ∎

To conclude the proof of Lemma 351, we construct $g \in G$ such that gHg^{-1} is A-stable. Let $b \in G$ be such that bTb^{-1} is A-stable and observe that such element b exists by Lemma 346. Lemma 357, with H replaced by bHb^{-1}, provides an element $a \in G$ such that $a(bHb^{-1})a^{-1}$ is A-stable. We define $g = ab$ and the proof of Lemma 351 is complete.

11.1.2 The induction step

Under the hypotheses of Proposition 345, we want to show that α is an intense automorphism of G. In view of this, let H be a subgroup of G. If H contains G_c, then H has an A-stable conjugate in G by Lemma 346. We assume that $H \cap G_c \neq G_c$. The class c being even, it follows from Lemma 322(4) that $|G_c| = p$ and so H intersects G_c trivially. By Lemma 351, there exists an element $g \in G$ such that gHg^{-1} is A-stable. The automorphism α is intense as a consequence of Lemma 93 and the fact that H was chosen arbitrarily. Proposition 345 is proven.

11.2 The odd case, part I

In Proposition 358 an additional assumption compared to Proposition 345 is made: that G be a *framed* p-obelisk. We recall that, if p is a prime number, then a p-obelisk G is framed if, for each maximal subgroup M of G, one has $\Phi(M) = G_3$. We refer to Section 10.3 for useful facts related to framed p-obelisks.

Proposition 358. *Let $p > 3$ be a prime number and let G be a framed p-obelisk of class c. Assume that c is odd and that G_c has order p. Let α be an automorphism of G of order 2 and assume that the map $\alpha_c : G/G_c \to G/G_c$ that is induced by α is intense. Then α is intense.*

The proof of Proposition 358 is given in Section 11.2.2.

11.2.1 Some lemmas

The goal of this section is to give all ingredients for the proof of Proposition 358 so we will keep the following assumptions until the end of Section 11.2.1. Let $p > 3$ be a prime number and let G be a p-obelisk of class c. Assume that c is odd and that G_c has order p. Let moreover α be an automorphism of G of order 2 and assume that the map $\alpha_c : G/G_c \to G/G_c$ that is induced by α is intense. Set $A = \langle \alpha \rangle$ and, in concordance with Section 2.2, write $G^+ = \{x \in G \ : \ \alpha(x) = x\}$ and $G^- = \{x \in G \ : \ \alpha(x) = x^{-1}\}$. For a subgroup H of G, we denote $H^+ = H \cap G^+$ and $H^- = H \cap G^-$ and we use the same “plus-minus" notation for any subgroup of G/G_c with respect to α_c.

We have intentionally not yet asked for G to be framed: we will make such assumption right after stating Lemma 363.

Lemma 359. *Let H be a subgroup of G containing G_c. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. By assumption, the automorphism α_c is intense and, by Lemma 93, there exists $g \in G$ such that $(gHg^{-1})/G_c$ is $\langle \alpha_c \rangle$-stable. It follows from the definition of α_c that gHg^{-1} is A-stable. ∎

Lemma 360. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Then all jumps of H in G have width* 1.

Proof. As a consequence of Proposition 321(1), every jump of H in G has width at most 2. Assume by contradiction that l is a jump of H in G of width 2. The jump l is odd, thanks to Lemma 322(1), and $G_l/G_{l+1} = (H \cap G_l)G_{l+1}/G_{l+1}$. Looking at $\rho_l^{(c-l)/2} : G_l/G_{l+1} \to G_c$, it follows from Lemma 328(1) that $H \cap G_c \neq \{1\}$. Contradiction. ∎

Lemma 361. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Assume that all jumps of H in G are even. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. All jumps of H in G are even so, by Lemma 360, they also all have width 1. Let now l be the least jump of H in G. Then, by Lemma 340, the subgroup H is cyclic and, by Lemma 341, the subgroups $\Phi(H)$ and $H \cap G_{l+1}$ are the same. Let $T = H \oplus G_c$. Assume first that $\alpha(T) = T$. Then $T = \alpha(H) \oplus G_c$ and, by Lemma 342, there exists $t \in G$ such that $\alpha(H) = tHt^{-1}$. Thanks to Lemma 92, there exists thus $t \in G$ such that tHt^{-1} is A-stable. In general, by Lemma 359, there exists $a \in G$ such that aTa^{-1} is A-stable. There now exists $t \in G$ such that $t(aHa^{-1})t^{-1}$ is A-stable, so we conclude by defining $g = ta$. ∎

Lemma 362. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Assume that all jumps of H in G are odd. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. Let $T = HG_c$. The class of G being odd, it follows from the assumptions that all jumps of T in G are odd. By Lemma 359, there exists $g \in G$ such that gTg^{-1} is A-stable. By Lemma 83, the subgroups gTg^{-1} and T have the same jumps in G so, as a consequence of Lemma 85, we get that $gTg^{-1} = (gTg^{-1})^-$. In particular, $gHg^{-1} = (gHg^{-1})^-$ and gHg^{-1} is A-stable. ∎

Lemma 363. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Assume moreover that G is framed. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

The remaining part of Section 11.2.1 will be entirely dedicated to the proof of Lemma 363. For this purpose, all assumptions that we now make will hold until the end of the very same section.

Assume that G is a framed p-obelisk. Let moreover H be a subgroup of G that trivially intersects G_c. If all jumps of H in G have the same parity, then we are done by Lemmas 361 and 362. We assume that H has jumps of each parity and we define i and j respectively to be the least odd jump and the least even jump of H in G. Write $T = HG_c$. We recall that, the class of G being c, the subgroup G_c is central in G.

Lemma 364. *The following hold.*

1. *One has $i + j > c$.*

2. *The subgroups H and T are abelian.*

Proof. The numbers i and j having different parities, their sum $m = i + j$ is odd. Let $k = \max\{i, j\}$. Then, as a consequence of Lemma 360, all jumps of H in G that are smaller than k have width 1 and so, by Lemma 339, the group $H/(H \cap G_k)$ is cyclic. From Lemma 28, we get $[H, H] = [H, H \cap G_k]$. By Lemma 20, the subgroup $[H, H]$ is contained in G_m. If $m > c$, then $G_m \subseteq G_{c+1} = \{1\}$, and thus (1) and (2) are proven. Assume by contradiction that $m \leq c$. Let y and x be elements of H respectively of depth i and j in G. Then the image of $\langle y \rangle$ under the natural projection $G \to G/G_{i+1}$ is a 1-dimensional subspace of G_i/G_{i+1}. Thanks to Proposition 336(3), with $h = i$ and $k = j$, the elements $y^{p^{j/2}}$ and $[y, x]$ of H span G_m/G_{m+1}. It follows from Lemma 360 that m is a jump of H of width 1 in G so, from Lemma 322(1), we derive $m = c$. Contradiction to H trivially intersecting G_c. ∎

Lemma 365. *Let $\pi : G \to G/G_c$ denote the natural projection. Assume that $\alpha(T) = T$. Then $\pi(H)$ is $\langle \alpha_c \rangle$-stable and $\pi(H) = \pi(H)^+ \oplus \pi(H)^-$. Moreover, both $\pi(H)^+$ and $\pi(H)^-$ are cyclic.*

Proof. To lighten the notation, we will denote $\overline{G} = \pi(G)$ and we will use the bar notation for the subgroups of $\overline{G}$. By assumption, $\alpha(T) = T$ and thus $\alpha_c(\overline{T}) = \overline{T}$. Moreover, $\overline{H}$ is equal to $\overline{T}$, so $\overline{H}$ is itself $\langle \alpha_c \rangle$-stable. As a consequence of Lemma 364(2), the group $\overline{H}$ is abelian so, by Corollary 76, it decomposes as $\overline{H} = \overline{H}^+ \oplus \overline{H}^-$. It follows from Lemma 85 that $\overline{H}^+$ and $\overline{H}^-$ have respectively only even jumps and only odd jumps in $\overline{G}$. Moreover, thanks to Lemma 360, all jumps of $\overline{H}$, and thus of its subgroups, in $\overline{G}$ have width 1. Lemma 339 yields that both $\overline{H}^+$ and $\overline{H}^-$ are cyclic. ∎

Lemma 366. *Assume that $\alpha(T) = T$. Then there exist cyclic subgroups I and J of H, with least jumps in G respectively equal to i and j, such that the following hold.*

1. *One has $H = I \oplus J$.*

2. *The group I is A-stable and $I = I^-$.*

3. *The group $S = J \oplus G_c$ is A-stable and $S = S^+ \oplus G_c$.*

Proof. We denote $\overline{G} = G/G_c$ and we will use the bar notation for the subgroups of $\overline{G}$. By Lemma 365, the subgroup $\overline{H}$ is $\langle \alpha_c \rangle$-stable and it decomposes as $\overline{H} = \overline{H}^+ \oplus \overline{H}^-$, where both $\overline{H}^+$ and $\overline{H}^-$ are cyclic. Let R and S be subgroups of G, containing G_c, such that $\overline{S} = \overline{H}^+$ and $\overline{R} = \overline{H}^-$. Because of their definitions, both R and S are A-stable. The subgroup G_c is contained in G^- as a consequence of Lemma 62, so it follows from Lemma 77 that $R = R^-$. Moreover, by Corollary 76, one has $S = S^+ \oplus S^-$. However, as $\overline{S} = \overline{S}^+$, the subgroups S^- and G_c are equal, and hence $S = S^+ \oplus G_c$. We define $I = H \cap R$ and $J = H \cap S$. The subgroup I, being contained in $R = R^-$, is itself A-stable and $I = I^-$. Moreover, with G and N respectively replaced by T and H, Lemma 325 yields $JG_c = (H \cap S)G_c = S \cap (HG_c) = S$. Since $H \cap G_c = \{1\}$, we get $S = J \oplus G_c$. In the same way, we have $R = I \oplus G_c$. It follows that J and I are respectively isomorphic to $\overline{H}^+$ and $\overline{H}^-$, and therefore they are cyclic. What is left to show is that indeed $H = I \oplus J$. The subgroup H is abelian by Lemma 364(2) and $I \cap J = \{1\}$, since $R \cap S = G_c$. The subgroup $I \oplus J$ is contained in H and

$$\overline{I \oplus J} = \overline{I} \oplus \overline{J} = \overline{R} \oplus \overline{S} = \overline{H}^+ \oplus \overline{H}^- = \overline{H},$$

so we derive $H = I \oplus J$. ∎

Lemma 367. *Let $\gamma \in G_c$ and let x, y be elements of G be such that* $\mathrm{dpt}_G(x) = j$ *and* $\mathrm{dpt}_G(y) = i$. *Then there exist $n \in \mathbb{Z}$ and $d \in \mathrm{C}_G(y) \cap G_{c-j}$ such that $\gamma = y^n[d, x]$.*

Proof. By Lemma 364 the sum $i + j$ is larger than c and so $i > c - j$. Define

$$r = \frac{i - (c - j)}{2} \quad \text{and} \quad s = \frac{j}{2} - r.$$

Let now $a \in G_{c-j} \setminus G_{c-j+1}$ be such that $\rho^r(a) = y$; the existence of a is granted by Lemma 323. As a consequence of Proposition 336, the subgroup G_c is generated by $\rho^{\frac{j}{2}}(a)$ and $[a, x]$. There exist thus $A, B \in \mathbb{Z}$ such that

$$\gamma = \rho^{\frac{j}{2}}(a)^A [a, x]^B.$$

We recall that, for any $k \in \mathbb{Z}_{\geq 0}$, the map ρ^k is given by $z \mapsto z^{p^k}$, hence

$$\rho^{\frac{j}{2}}(a) = \rho^{s+r}(a) = \rho^s(\rho^r(a)) = \rho^s(y) = y^{p^s}.$$

Now, the group G_c being central, Lemma 22 implies that the commutator map $G_{c-j} \times G_j \to G_c$ is bilinear so $[a, x]^B = [a^B, x]$. We define

$$n = Ap^s \text{ and } d = a^B$$

and get $\gamma = y^n[d, x]$. To conclude, the element d belongs to $\mathrm{C}_G(y)$, because d and y belong to $\langle a \rangle$. ∎

Lemma 368. *Assume that $\alpha(T) = T$. Let I and J be as in Lemma 366. Then there exists $g \in \mathrm{C}_G(I)$ such that $\alpha(gJg^{-1}) \subseteq gHg^{-1}$.*

Proof. Let y be a generator of I and let x be a generator of J. Then one has $\mathrm{dpt}_G(y) = i$ and $\mathrm{dpt}_G(x) = j$. As a consequence of Lemma 366(3), there exists $\gamma \in G_c$ such that $\alpha(x) = x\gamma$. We fix γ and we want to construct g. Let $n \in \mathbb{Z}$ and $d \in \mathrm{C}_G(y) \cap G_{c-j}$ be such that $\gamma = y^n[d, x]$; the existence of n and d is given by Lemma 367. We define $g = d^{\frac{p+1}{2}}$ and we claim that $\alpha(gxg^{-1})$ belongs to gHg^{-1}. We will use some properties of G_c that we list here. The group G_c is central and annihilated by p, by hypothesis. Moreover, as a consequence of Lemma 62, the restriction of α to G_c coincides with the map $z \mapsto z^{-1}$. To conclude, the commutator map $G_{c-j} \times G_j \to G_c$ is bilinear by Lemma 22. We compute

$$\alpha(gxg^{-1}) = \alpha([g, x]x) = \alpha([g, x])\alpha(x) = [g, x]^{-1}x\gamma = [g^{-1}, x]x\gamma =$$

$$[g^{-1}, x]xy^n[d, x] = [g^{-1}, x][d, x]xy^n = [g^{-1}d, x]xy^n = [d^{\frac{p-1}{2}}d, x]xy^n =$$
$$[d^{\frac{p+1}{2}}, x]xy^n = [g, x]xy^n = (gxg^{-1})y^n.$$

The element g centralizes y, because d does, so $\alpha(gxg^{-1}) = g(xy^n)g^{-1}$ belongs to gHg^{-1}. In particular, $\alpha(gJg^{-1}) \subseteq gHg^{-1}$. ∎

We conclude the proof of Lemma 363. By Lemma 359 there exists $a \in G$ such that aTa^{-1} is A-stable. We fix a and write $aHa^{-1} = I \oplus J$, with I and J as in Lemma 366 and H replaced by aHa^{-1}. By Lemma 368, there exists an element $b \in G$ such that $bIb^{-1} = I$ and $\alpha(bJb^{-1})$ is contained in $baHa^{-1}b^{-1}$. We select such an element b and define $g = ba$. Then I is contained in gHg^{-1} and

$$\alpha(gHg^{-1}) = \alpha(baHa^{-1}b^{-1}) = \alpha(bIb^{-1} \oplus bJb^{-1}) = \alpha(I \oplus bJb^{-1}) =$$

$$\alpha(I) \oplus \alpha(bJb^{-1}) = I \oplus \alpha(bJb^{-1}) \subseteq gHg^{-1}.$$

It follows that $\alpha(gHg^{-1}) = gHg^{-1}$ and gHg^{-1} is itself A-stable. The proof of Lemma 363 is now complete.

11.2.2 The induction step

In this paragraph we give the proof of Proposition 358 and we work thus under the assumptions of the very same proposition. Let H be a subgroup of G; we will show that H has an A-stable conjugate. If H contains G_c, then, by Lemma 359, there exists $g \in G$ such that gHg^{-1} is A-stable. Assume now that G_c is not contained in H, i.e. $H \cap G_c = \{1\}$. Thanks to Lemma 363, the subgroup H has an A-stable conjugate. In other words, by Lemma 93, the subgroups H and $\alpha(H)$ are conjugate in G. As the choice of H was arbitrary, α is intense and the proof of Proposition 358 is complete.

11.3 The odd case, part II

Proposition 369. *Let $p > 3$ be a prime number and let G be a framed p-obelisk of class c. Assume that c is odd and that G_c has order p^2. Let α be an automorphism of G of order 2 and assume that the map $\alpha_c : G/G_c \to G/G_c$ that is induced by α is intense. Then α is intense.*

The proof of Proposition 369 is given in Section 11.3.2.

11.3.1 Some lemmas

The purpose of this section is laying the ground for the proof of Proposition 369. We will therefore, until the end of Section 11.3.1, work under the assumptions of Proposition 369. Denote $A = \langle \alpha \rangle$.

Lemma 370. *Let H be a subgroup of G containing G_c. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. We write $\overline{G} = G/G_c$ and we use the bar notation for the subgroups of $\overline{G}$. By hypothesis, the automorphism α_c is intense so, by Lemma 93, there exists $g \in G$ such that $\alpha_c(\overline{gHg^{-1}}) = \overline{gHg^{-1}}$. The map α_c being induced from α, it follows that $\alpha(gHg^{-1}) = gHg^{-1}$ and gHg^{-1} is A-stable. ∎

Lemma 371. *Let H be a subgroup of G such that $H \cap G_c \neq \{1\}$. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. Let $N = H \cap G_c$. If $N = G_c$, then, by Lemma 370, there exists $g \in G$ such that gHg^{-1} is A-stable. We assume that $N \neq G_c$. The group N being non-trivial, it follows from Proposition 321(1) that G_c and N have orders respectively p^2 and p. Moreover, the group G_c being central, N is normal in G. It follows from Lemma 88(2) that the action of A on G induces an action of A on $\overline{G} = G/N$. Moreover, $\overline{G}$ has class c and the subgroup $\overline{H} = H/N$ has trivial intersection with $\overline{G_c} = G_c/N$.

By Lemma 363, there exists $\bar{g} \in \overline{G}$ such that $\bar{g}\overline{H}\bar{g}^{-1}$ is A-stable, and so there exists $g \in G$ such that gHg^{-1} is A-stable. ∎

Lemma 372. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Then H has only even jumps.*

Proof. The index c is odd and $H \cap G_c = \{1\}$. It follows from Corollary 329 that H cannot have odd jumps. ∎

Lemma 373. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Let $T = HG_c$ and assume that $\alpha(T) = T$. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. Let l denote the least jump of H in G. By Lemma 372, all jumps of H in G are even so, as a consequence of Lemma 322(2), all jumps of H have width 1. It follows from Lemma 341 that H is abelian and $\Phi(H) = H \cap G_{l+1}$. The subgroup G_c being central, we get $T = H \oplus G_c$. Now, by Lemma 85, the subgroup $T^+ = \{t \in T : \alpha(t) = t\}$ has the same jumps as H, and it is therefore a complement of G_c in T. Thanks to Lemma 342, the subgroups H and T^+ are conjugate in G. In particular, H has an A-stable conjugate. ∎

Lemma 374. *Let H be a subgroup of G such that $H \cap G_c = \{1\}$. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. Define $S = HG_c$. By Lemma 370, there exists $a \in G$ such that aSa^{-1} is A-stable. Let now $T = aSa^{-1}$. Then $\alpha(T) = T$ and $T = a(HG_c)a^{-1} = aHa^{-1}G_c$. Moreover, the intersection $aHa^{-1} \cap G_c$ is trivial. Thanks to Lemma 373 (with aHa^{-1} in the place of H), there exists $b \in G$ such that $b(aHa^{-1})b^{-1}$ is A-stable. To conclude, we define $g = ba$. ∎

11.3.2 The last step

We give here the proof of Proposition 369 and we make thus all assumptions from Proposition 369 hold, until the end of Section 11.3.2. To show that α is intense, we will show that each subgroup of G has an A-stable conjugate. Let H be a subgroup of G. If H trivially intersects G_c, then, by Lemma 374, there exists $g \in G$ such that gHg^{-1} is A-stable. If, on the contrary, $H \cap G_c \neq \{1\}$, then, by Lemma 371, there exists a conjugate of H in G that is A-stable. We have proven that, in any case, H has an A-stable conjugate and, by Lemma 92, the subgroups H and $\alpha(H)$ are conjugate in G. The choice of H being arbitrary, the automorphism α is intense and we have proven Proposition 369.

11.4 Proving the main theorems

In Sections 11.4.1 and 11.4.2 we finally prove the two main results of this Chapter, which were stated at the beginning of it.

11.4.1 The proof of Theorem 343

We work under the assumptions of Theorem 343. The implication (2) $\Rightarrow$ (1) follows from the combination of Propositions 318 and 134. We now prove (1) $\Rightarrow$ (2). To this end, denote by $\overline{G}$ the quotient G/G_4 and by α_4 the automorphism of $\overline{G}$ that is induced by α. The map α induces the inversion map on G/G_2 and thus so does α_4 on $\overline{G}/\overline{G}_2$. It follows from Proposition 142 that α_4 is intense and consequently, from Proposition 345, that α is intense too. The proof of Theorem 343 is complete.

11.4.2 The proof of Theorem 344

Under the hypotheses of Theorem 344, we will work by induction on the class c of G. As a consequence of Lemma 319(1-3), the group G has class at least 2 and G/G_3 is extraspecial of exponent p. If $c = 2$, then Lemma 121 yields that α is intense. We assume that $c > 2$ and denote by $\overline{G}$ the quotient G/G_c. We denote moreover by α_c the automorphism of $\overline{G}$ that is induced by α and assume that α_c is intense. The group $\overline{G}$ is a framed obelisk, because $c > 2$, and α_c induces the inversion map on $\overline{G}/\overline{G}_2$, because α does. If c is even, then, by Proposition 345, the map α is intense. Suppose that c is odd. From Proposition 321(1) it follows that the cardinality of G_c is p or p^2. In the first case we apply Proposition 358, in the second Proposition 369. Theorem 344 is now proven.