Intense automorphisms of finite groups
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Chapter 9

The special case of 3-groups

Let $R = \mathbb{F}_3[\epsilon]$ be of cardinality 9, with $\epsilon^2 = 0$. Denote by $\mathbb{A}$ the quaternion algebra

$$\mathbb{A} = R + R\mathrm{i} + R\mathrm{j} + R\mathrm{k}$$

with defining relations $\mathrm{i}^2 = \mathrm{j}^2 = \epsilon$ and $\mathrm{k} = \mathrm{ji} = -\mathrm{ij}$. Let the *bar map* on $\mathbb{A}$ be defined by

$$x = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \ \mapsto\ \overline{x} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}.$$

We write $\mathfrak{m} = \mathbb{A}\mathrm{i} + \mathbb{A}\mathrm{j}$, which is a 2-sided nilpotent ideal of $\mathbb{A}$, and we define $\mathrm{MC}(3)$ to be the subgroup of $1 + \mathfrak{m}$ consisting of those elements x satisfying $\overline{x} = x^{-1}$. The main result of this chapter is the following.

Theorem 231. *Let G be a finite 3-group. Then the following are equivalent.*

1. *The group G has class at least 4 and $\mathrm{int}(G) > 1$.*
2. *The group G has class 4, order 729, and $\mathrm{int}(G) = 2$.*
3. *The group G is isomorphic to $\mathrm{MC}(3)$.*

A considerable part of the present chapter is devoted to the proof of Theorem 231, which is given in Section 9.7. An essential contribution to it is given by the theory of "κ-groups" we develop.

Definition 232. *A κ-group is a finite 3-group G such that $|G : G_2| = 9$ and with the property that the cubing map on G induces a bijective map $\kappa : G/G_2 \to G_3/G_4$.*

Our interest in κ-groups arises from Lemma 206(1), which asserts that, if p is an odd prime number and G is a finite p-group of class at least 4 with $\mathrm{int}(G) > 1$, then the map $x \mapsto x^p$ induces a bijection $\overline{\rho} : G/G_2 \to G_3/G_4$. As a consequence of Theorem 125, each finite 3-group of class at least 4 and intensity greater than 1 is a κ-group, where κ coincides with $\overline{\rho}$. The reason why, in this chapter, we work exclusively with 3-groups is that they are more "difficult to deal with": several techniques that apply to the case in which p is a prime larger than 3 do not apply to the case of 3-groups of higher class, as the results from Chapter 8 suggest. For example, it is not difficult to show, using results from Section 1.5, that, whenever $p > 3$ and G is a finite p-group, the map $\overline{\rho} : G/G_2 \to G_3/G_4$ from Lemma 206 is an isomorphism of groups, while, if G is a κ-group, then, given any two elements $x, y \in G/G_2$, one has

$$\kappa(xy) \equiv \kappa(x)\kappa(y)[xy^{-1}, [x, y]] \bmod G_4,$$

as we show in Lemma 237. What plays in our favour is that a finite 3-group G is a κ-group if and only if G/G_4 is a κ-group: to detect κ-groups it is thus sufficient to be able to detect κ-groups among the finite 3-groups of class 3. We will prove the following result.

Theorem 233. *Let G be a finite 3-group of class 3. Then G is a κ-group if and only if G is isomorphic to* $\mathrm{MC}(3) / \mathrm{MC}(3)_4$.

In Section 9.4, we prove Theorem 233 by building κ-groups as quotients of a free group: we give a sketch of the proof here. Let F be the free group on 2 generators and let $(F_i)_{i\geq 1}$ be defined recursively by $F_1 = F$ and $F_{i+1} = [F, F_i]F_i^3$. Then $V = F/F_2$ is a vector space over $\mathbb{F}_3$ of dimension 2. Let moreover $L = F_3F^3$ and set $\overline{F} = F/([F, L]F_2^3)$; we use the bar notation for the subgroups of $\overline{F}$. We will show that the cubing map on F induces a map $V \to \overline{L}$, which we denote by c, and we will construct, in Sections 9.3 and 9.4, isomorphisms of the following $\mathrm{Aut}(F)$-sets, all having cardinality 3.

$$\mathcal{I}_V = \{k \subseteq \mathrm{End}(V) \text{ subfield} : |k| = 9\}$$

$$\downarrow$$

$$\mathcal{K}_V = \{\kappa : V \to V \otimes \textstyle\bigwedge^2(V) \text{ bijective} : \text{for all } x, y \in V, \text{ one has}$$
$$\kappa(x + y) = \kappa(x) + \kappa(y) + (x - y) \otimes (x \wedge y)\}$$

$$\downarrow$$

$$\mathcal{P} = \{\pi \in \mathrm{Hom}(\overline{L}, \overline{F_3}) : \pi \circ c \text{ is bijective}, \pi_{|\bar{F_3}} = \mathrm{id}_{\bar{F_3}}\}$$

$$\downarrow$$

$$\mathcal{N}_3 = \{N \subseteq F \text{ normal subgroup} : F/N \text{ is a } \kappa\text{-group of class } 3\}.$$

We will then prove that the natural action of $\mathrm{Aut}(F)$ on $\mathcal{I}_V$ is transitive and so it will follow that $\mathrm{Aut}(F)$ acts transitively on $\mathcal{N}_3$, leading to the fact that all κ-groups of class 3 are isomorphic to the κ-group $\mathrm{MC}(3) / \mathrm{MC}(3)_4$. To extend the investigation of κ-groups to class 4, we consider the "smallest possible case" and look at extensions of $\mathrm{MC}(3) / \mathrm{MC}(3)_4$ by a group of order 3. In Section 9.5, we prove the following result.

Theorem 234. *Let G be a κ-group such that G_4 has order 3. Then G_2 is elementary abelian.*

It would be interesting to explore the world of κ-groups more extensively, however Theorems 233 and 234 provide us with sufficient information on the structure of κ-groups to be able to go into the proof of Theorem 231. Let G be a finite 3-group of class at least 4. We have seen that a necessary condition for $\mathrm{int}(G)$ to be greater than 1 is that of being a κ-group, however we can only hope to construct an intense automorphism of G of order 2 if

($*$) there exists an automorphism of G of order 2 that inverts all elements of G modulo G_2.

We proved in Section 5.4 that such an automorphism can always be constructed for G/G_4, so we want to understand which conditions we need to impose on the structure of G to be able to lift such an automorphism from G/G_4 to G. For this purpose, we define

$$\mathcal{N}_4 = \{N \subseteq F \text{ normal subgroup} : F/N \text{ is a } \kappa\text{-group of class 4 with } \mathrm{wt}_{F/N}(4) = 1 \text{ and satisfying } (*)\}.$$

Via building a bijection $\mathcal{N}_4 \to \mathcal{N}_3$, we will be able to prove that the natural action of $\mathrm{Aut}(F)$ on $\mathcal{N}_4$ is transitive and so that, given M and N in $\mathcal{N}_4$, the quotients F/M and F/N are isomorphic. The group $\mathrm{MC}(3)$ being a κ-group of class 4 with $\mathrm{wt}_{\mathrm{MC}(3)}(4) = 1$ and ($*$), it follows that each quotient F/N, with $N \in \mathcal{N}_4$, is isomorphic to $\mathrm{MC}(3)$. Since $\mathrm{MC}(3)$ has an elementary abelian commutator subgroup, Proposition 211 yields that each finite 3-group of intensity greater than 1 has class at most 4.

9.1 The cubing map

In this section we prove some structural properties about κ-groups of class 4. We remind the reader that, if G is a finite 3-group and i is a positive integer, then the i-th width of G is defined to be $\mathrm{wt}_G(i) = \log_3 |G_i : G_{i+1}|$ (see Section 2.3). We warn the reader that we will make a set of assumptions, which will hold until the end of Section 9.1, right after Lemma 239.

Lemma 235. *Let G be a group of order 81 and class 3. Then the exponent of G is different from 3.*

Proof. The class of G is 3 so, thanks to Lemma 31, the quotient G/G_2 is non-cyclic. The order of G being 81, it follows that $(\mathrm{wt}_G(1), \mathrm{wt}_G(2), \mathrm{wt}_G(3)) = (2,1,1)$. Let now $C = \mathrm{C}_G(G_2)$. Then, by Lemma 137(1), the subgroup C contains G_2 with index 3 and, by Lemma 140, the centre of G is equal to G_3. Let $(a,b) \in G \times C$ be such that $\{a,b\}$ generates G and define $c = [a,b]$, which is an element of $G_2 \setminus G_3$. Let moreover d be a generator for G_3. Assume by contradiction that the exponent of G is 3. As a consequence of Lemma 138, the subgroup C is elementary abelian and, in particular, $G_2 = \langle c \rangle \oplus \langle d \rangle$. Since G_2 is central modulo G_3 and d generates G_3, there exists an integer k such that $aca^{-1} = cd^k$. The element d^k is not equal to the identity element, because a and c do not centralize each other. Keeping in mind that C is abelian, we compute

$$1 = (ba)^3 = bababa = b(cba)(cba)a = b^2cacba^2 =$$

$$b^2c^2d^kaba^2 = b^2c^2d^kcba^3 = b^3c^3d^k = d^k.$$

Contradiction. ■

We recall here that, if G is a group and n is a positive integer, then G^n is defined to be $G^n = \langle x^n : x \in G \rangle$.

Lemma 236. *Let G be a finite 3-group of class 3 such that $|G : G_2| = 9$. Then $G_3 = G^3$.*

Proof. The class of G is equal to 3 and so G_3 is central in G. By Lemma 135(1), the index $|G_2 : G_3|$ is equal to 3 and, by Lemma 135(2), the order of G_3 is either 3 or 9. As a consequence of Lemma 139, moreover, the subgroup G^3 is contained in G_3. Assume by contradiction that $G_3 \neq G^3$. Then, by Lemma 35, there exists a normal subgroup M of G such that $G^3 \subseteq M \subseteq G_3$ and $|G_3 : M| = 3$. Fix such M. Then the quotient G/M has class 3 and order 81. Moreover, the exponent of G/M is equal to 3. Contradiction to Lemma 235. ■

Lemma 237. *Let G be a group of class at most 3 and assume that G_2 has exponent dividing 3. Then, for all $x, y \in G$, one has $(xy)^3 = x^3y^3[xy^{-1}, [x,y]]$.*

Proof. The class of G is at most 3, so the subgroup G_3 is central. Fix x and y in

G. Then we have

$$\begin{aligned}
(xy)^3 &= xyxyxy \\
&= xyx[y,x]xy^2 \\
&= xyx[[y,x],x]x[y,x]y^2 \\
&= xyxx[y,x]y^2[[y,x],x] \\
&= x[y,x]xyx[y,x]y^2[[y,x],x] \\
&= x[[y,x],x]x[y,x]yx[y,x]y^2[[y,x],x] \\
&= x^2[y,x]yx[y,x]y^2[[y,x],x]^2 \\
&= x^2[y,x]^2xy[y,x]y^2[[y,x],x]^2 \\
&= x^2[y,x]^2[xy,[y,x]][y,x]xy^3[[y,x],x]^2 \\
&= x^2[y,x]^3xy^3[[y,x],x]^2[xy,[y,x]].
\end{aligned}$$

The commutator subgroup of G being annihilated by 3, the element $[y,x]^3$ is trivial. Moreover, thanks to Lemma 24, the commutator map induces a bilinear map $G/G_2 \times G_2/G_3 \to G_3$. It follows that

$$\begin{aligned}
(xy)^3 &= x^3y^3[[y,x],x]^2[xy,[y,x]] \\
&= x^3y^3[[y,x],x^2][[y,x],(xy)^{-1}] \\
&= x^3y^3[[y,x],x^2y^{-1}x^{-1}] \\
&= x^3y^3[[y,x],xy^{-1}] \\
&= x^3y^3[xy^{-1},[x,y]].
\end{aligned}$$

The proof is now complete. ∎

Lemma 238. *Let G be a finite 3-group of class at least 3 and assume that $|G:G_2| = 9$. Then the cubing map induces a map $\kappa : G/G_2 \to G_3/G_4$.*

Proof. We assume without loss of generality that $G_4 = \{1\}$. As a consequence of Lemma 236, the image of the cubing map is contained in G_3 and, by Lemma 141, the commutator subgroup of G has exponent 3. We now prove that the map $\kappa : G/G_2 \to G_3$, given by $\kappa(xG_2) = x^3$, is well-defined. To this end, let $(x,y) \in G \times G_2$. Then $y^3 = 1$ and $[y,x]$ belongs to G_3, a central subgroup. From Lemma 237, we get

$$(xy)^3 = x^3y^3[[y,x],xy^{-1}] = x^3y^3 = x^3$$

so every element of xG_2 has the same cube x^3 in G, as claimed. ∎

We remark that, in concordance with Definition 232, the real requirement for a 3-group G satisfying $|G:G_2| = 9$ to be a κ-group is that the map from Lemma 238 is a bijection. The reason why we are interested in κ-groups is given by the following lemma.

Lemma 239. *Let G be a finite 3-group of class 4 and denote by $(G_i)_{i\geq 1}$ the lower central series of G. Assume that* $\mathrm{int}(G) > 1$*. Then G is a κ-group.*

Proof. Take $p = 3$ in Lemma 206(1). ∎

In the remaining part of this section, we will prove some structural results about κ-groups. Until the end of Section 9.1, let thus G be a finite 3-group of class 4. Let $(G_i)_{i\geq 1}$ denote the lower central series of G and, for each $i \in \mathbb{Z}_{\geq 1}$, denote $w_i = \mathrm{wt}_G(i)$. Assume that $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$ and, to conclude, let $\kappa : G/G_2 \to G_3/G_4$ be the map from Lemma 238.

Lemma 240. *The group G_2 is abelian.*

Proof. The quotient G_2/G_3 being cyclic, it follows from Lemma 28 that $[G_2, G_2] = [G_2, G_3]$, so, thanks to Lemma 20, we get $[G_2, G_3] \subseteq G_5$. The class of G is 4, so $G_5 = \{1\}$ and G_2 is abelian. ∎

Lemma 241. *The commutator map $G \times G_2 \to G_3$ induces an isomorphism $G/G_2 \otimes G_2/G_3 \to G_3/G_4$.*

Proof. By Lemma 25, the commutator map induces a surjective homomorphism $G/G_2 \otimes G_2/G_3 \to G_3/G_4$. The induced map is bijective because $|G/G_2 \otimes G_2/G_3| = 3^{w_1 w_2} = 9 = 3^{w_3} = |G_3 : G_4|$. ∎

We recall that, if C is a group and n is a positive integer, then C^n and $\mu_n(C)$ are respectively defined as $C^n = \langle x^n : x \in C\rangle$ and $\mu_n(C) = \langle x \in C : x^n = 1\rangle$.

Lemma 242. *Let C be a maximal subgroup of G. Then $G_4 C^3 \subseteq \mathrm{Z}(C)$.*

Proof. The subgroup G_4 is central in G, because the class of G is 4, so G_4 is contained in $\mathrm{Z}(C)$. By Lemma 25, the commutator map induces a homomorphism $\gamma : G/G_2 \otimes G_3/G_4 \to G_4$ and, C/G_2 being cyclic, the subgroup $\gamma(C/G_2 \otimes \kappa(C/G_2))$ is trivial. The quotient $G_4 C^3/G_4$ being equal to $\kappa(C/G_2)$, it follows that $G_4 C^3$ is contained in the centre of C. ∎

Lemma 243. *There exists at most one maximal subgroup C of G such that $G_3 \subseteq \mathrm{Z}(C)$.*

Proof. Let C and D be maximal subgroups of G such that G_3 is contained in $\mathrm{Z}(C) \cap \mathrm{Z}(D)$. Then CD centralizes G_3 and, the class of G being equal to 4, the subgroup CD is different from G. It follows that $C = D$. ∎

Lemma 244. *Assume that G is a κ-group. Then* $\mathrm{Z}(G) = G_4$*.*

Proof. We first claim that $G_4 \subseteq \mathrm{Z}(G) \subsetneq G_3$. The subgroup G_4 is contained in $\mathrm{Z}(G)$ and, as a consequence of Lemma 140, one has $\mathrm{Z}(G)/G_4 \subseteq \mathrm{Z}(G/G_4) = G_3/G_4$. Since the class of G is 4, the inclusion $\mathrm{Z}(G) \subseteq G_3$ is not an equality so the claim is proven. Now, by Lemma 36, the subgroup G_2 is equal to $\Phi(G)$ and so, the dimension w_1 being equal to 2, the group G has precisely 4 maximal subgroups. Thanks to Lemma 243, there exist two distinct maximal subgroups C and D of G such that both $\mathrm{Z}(C)$ and $\mathrm{Z}(D)$ do not contain G_3. Fix such C and D. Since κ is a bijection and $w_3 = 2$, Lemma 242 yields $\mathrm{Z}(C) \cap G_3 = C^3G_4$ and $\mathrm{Z}(D) \cap G_3 = D^3G_4$. Now, the subgroup $\mathrm{Z}(G)$ contains G_4 and is contained in $\mathrm{Z}(C) \cap \mathrm{Z}(D) \cap G_3 = C^3G_4 \cap D^3G_4$. The map κ being a bijection, the subgroup $C^3G_4 \cap D^3G_4$ is equal to G_4 and therefore $\mathrm{Z}(G) = G_4$. ∎

Lemma 245. *Let C be a maximal subgroup of G. Assume moreover that G is a κ-group and that G_2 has exponent* 3*. Then $[C, C] \cap \mathrm{Z}(C) = G_4$.*

Proof. The quotient C/G_2 is cyclic of order 3 so, by Lemma 28, the subgroups $[C, C]$ and $[C, G_2]$ are equal. It follows that $[C, C]$ is contained in G_3 and, from Lemma 241, that the index of $([C, C]G_4)/G_4$ in G_3/G_4 is equal to $|G : C| = 3$. In particular, $[C, C]$ is non-trivial. Now, the subgroup $[C, C]$ is characteristic in the normal subgroup C and therefore it is itself normal in G; Lemma 29 yields $[C, C] \cap \mathrm{Z}(G) \neq 1$. By Lemma 244, the centre of G is equal to G_4 and thus $[C, C]$ contains G_4. As a result, $[C, C]$ is equal to $[C, C]G_4$ and it has thus cardinality 9. In an analogous way, since G is a κ-group, the normal subgroup C^3 is non-trivial and it contains therefore G_4. However, C^3 is different from G_4 because G is a κ-group. We have proven that G_4 is contained in $[C, C] \cap \mathrm{Z}(C)$. We assume now by contradiction that $[C, C] \cap \mathrm{Z}(C)$ is different from G_4. It follows that $[C, C] \cap \mathrm{Z}(C)$ has cardinality at least 9, which is the same as the cardinality of $[C, C]$. We get that $[C, C]$ is contained in $\mathrm{Z}(C)$ and so, as a consequence of Lemmas 50 and 51, the cubing map is an endomorphism of C. By assumption, the exponent of G_2 is 3, and so it follows that

$$|C^3| = |C : \mu_3(C)| \leq |C : G_2| = 3.$$

Since C^3 contains G_4, we get that $C^3 = G_4$. Contradiction. ∎

Lemma 246. *Assume that G is a κ-group. Then G_3 has exponent* 3*.*

Proof. The subgroup G_2 is abelian, by Lemma 240, and G_2^3 is contained in G_4, as a consequence of Lemma 238. It follows that $\mu_3(G_2)$ has cardinality at least $|G_2 : G_4| = 27$. Set $N = \mu_3(G_2) \cap G_3$. We denote $\overline{G} = G/N$ and use the bar notation for the subgroups of $\overline{G}$. If $\overline{G_3} = \{1\}$, then G_3 is contained in $\mu_3(G_2)$ and we are done. Assume by contradiction that $\overline{G_3}$ is non-trivial. Then $\overline{G_3}$ has cardinality at least 3 so, $\mu_3(G_2)$ consisting of at least 27 elements, it follows that

$\overline{\mu_3(G_2)}$ is non-trivial. However, $\overline{\mu_3(G_2)}$ has trivial intersection with $\overline{G_3}$, which is equal to $\mathrm{Z}(\overline{G})$, thanks to Lemma 140. Contradiction to Lemma 29. ■

9.2 A specific example

This section is entirely devoted to understanding the structure of the group MC(3), which is defined at the beginning of the present chapter. The name MC(3) refers to the fact that MC(3) turns out to be an example of maximal class among the finite 3-groups of intensity greater than 1. Moreover, as stated in Theorem 231, given any finite 3-group G of class at least 4, either $\mathrm{int}(G) = 1$ or G is isomorphic to MC(3). We recall the definition of MC(3).

Let $R = \mathbb{F}_3[\epsilon]$ be of cardinality 9, with $\epsilon^2 = 0$, and let $\mathbb{A}$ denote the quaternion algebra $\left(\frac{\epsilon,\epsilon}{R}\right)$. In other words, $\mathbb{A}$ is given by

$$\mathbb{A} = R + R\mathrm{i} + R\mathrm{j} + R\mathrm{k}$$

with defining relations $\mathrm{i}^2 = \mathrm{j}^2 = \epsilon$ and $\mathrm{k} = \mathrm{ji} = -\mathrm{ij}$. The ring $\mathbb{A}$ has a unique (left/right/2-sided) maximal ideal $\mathfrak{m} = \mathbb{A}\mathrm{i} + \mathbb{A}\mathrm{j}$ and the residue field $k = \mathbb{A}/\mathfrak{m}$ is equal to $\mathbb{F}_3$. The algebra $\mathbb{A}$ is also equipped with a natural anti-automorphism of order 2, which is defined by

$$x = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{k} \;\mapsto\; \overline{x} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{k}.$$

We define MC(3) to be the subgroup of $1 + \mathfrak{m}$ consisting of those elements x satisfying $\overline{x} = x^{-1}$. We denote by $(\mathrm{MC}(3)_i)_{i\geq 1}$ the lower central series of MC(3) and, for each $i \in \mathbb{Z}_{\geq 1}$, we define $M_i = (1 + \mathfrak{m}^i) \cap G$. One easily shows that $(M_i)_{i\geq 1}$ is central and that, for each $i \geq 1$, the commutator map induces a map $M_1/M_2 \times M_i/M_{i+1} \to M_{i+1}/M_{i+2}$ whose image generates M_{i+1}/M_{i+2}. For each $i \geq 1$, it follows that $M_{i+1} = [M_1, M_i]$ and, since $M_1 = G$, we have that

$$\mathrm{MC}(3)_i = \mathrm{MC}(3) \cap (1 + \mathfrak{m}^i).$$

The rest of the present section is devoted to the proof of some technical Lemmas that we will use in the proof of Theorem 231.

Lemma 247. *The group* MC(3) *has class* 4 *and order* 729.

Proof. We start by proving that MC(3) has order 729. The cardinality of R is equal to 9 and therefore the cardinality of $\mathbb{A}$ is 9^4. Since $\mathbb{A}/\mathfrak{m}$ is isomorphic to $\mathbb{F}_3$, the cardinality of $\mathfrak{m}$ is equal to $(9^4/3) = 3^7$ and therefore also $1+\mathfrak{m}$ has cardinality 3^7. Now, asking for an element $x \in 1 + \mathfrak{m}$ to satisfy $x\overline{x} = 1$ lowers our freedom in the choice of coordinates of x by 1 and therefore G has cardinality $3^6 = 729$. To conclude the proof, we note that $\mathrm{MC}(3)_5$ is trivial, because $\mathfrak{m}^5 = \{0\}$, while $1 + \epsilon\mathrm{k}$ is a non-trivial element of $\mathrm{MC}(3)_4$. It follows that MC(3) has class 4. ■

Lemma 248. *Set $G = \mathrm{MC}(3)$ and, for each $i \in \mathbb{Z}_{\geq 1}$, denote $w_i = \mathrm{wt}_G(i)$. Then the following hold.*

1. *One has $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$.*
2. *There exist generators a and b of G such that $a^3 \equiv [b, [a, b]]^{-1} \bmod G_4$ and $b^3 \equiv [a, [a, b]] \bmod G_4$.*

Proof. (1) Let $i \in \{1, 2, 3, 4\}$. Then the function $G \to \mathfrak{m}$ that is defined by $x \mapsto x - 1$ induces an injective homomorphism $d_i : G_i/G_{i+1} \to \mathfrak{m}^i/\mathfrak{m}^{i+1}$, which commutes with the bar map of $\mathbb{A}$. Now, for each element $x \in G_i$, one has that $\overline{x - 1} + (x - 1)$ belongs to $\mathfrak{m}^{i+1}$ and therefore the image of d_i is contained in $D_i = \{y + \mathfrak{m}^{i+1} : y \in \mathfrak{m}^i, \overline{y} + y \in \mathfrak{m}^{i+1}\}$. With an easy computation, one shows that D_i coincides with the image of d_i and, consequently, that $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. To prove (2), define

$$a = 1 - \epsilon + \mathrm{i} \quad \text{and} \quad b = 1 - \epsilon + \mathrm{j}$$

and note that a and b belong to G. Since $w_1 = 2$ and a and b are linearly independent modulo $G_2 = G \cap (1 + \mathfrak{m}^2)$, the group G is generated by a and b. Using the defining properties of $\mathbb{A}$, we compute $a^3 = 1 + \epsilon\mathrm{i}$ and $b^3 = 1 + \epsilon\mathrm{j}$. Define $c = [a, b]$, $d = [a, c]$, and $e = [b, c]$. Then, working modulo G_3, we get

$$\begin{aligned} c = ab\overline{a}\overline{b} &\equiv (1 - \epsilon + \mathrm{i})(1 - \epsilon + \mathrm{j})(1 - \epsilon - \mathrm{i})(1 - \epsilon - \mathrm{j}) \\ &\equiv (1 + \epsilon + \mathrm{i} + \mathrm{j} + \mathrm{k})(1 + \epsilon - \mathrm{i} - \mathrm{j} + \mathrm{k}) \\ &\equiv 1 - \mathrm{k} \bmod G_3. \end{aligned}$$

Thanks to Lemma 24, one has $d \equiv [a, 1+\mathrm{k}] \bmod G_4$ and $e \equiv [b, 1+\mathrm{k}] \bmod G_4$ and it is now easy to compute $d \equiv 1 + \epsilon\mathrm{j} \bmod G_4$ and $e \equiv 1 - \epsilon\mathrm{i} \bmod G_4$. It follows that both ea^3 and $d^{-1}b^3$ belong to G_4 and so the proof is complete. ∎

We remind the reader that, in concordance with Definition 232, a κ-group is a finite 3-group G such that $|G : G_2| = 9$ and such that the cubing map on G induces a bijection $G/G_2 \to G_3/G_4$.

Lemma 249. *The group* $\mathrm{MC}(3)$ *is a κ-group.*

Proof. Write $G = \mathrm{MC}(3)$ and, for each $i \in \mathbb{Z}_{\geq 1}$, denote $w_i = \mathrm{wt}_G(i)$. By Lemma 247, the group G has class 4 and, by Lemma 248(1), one has $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. Let $\kappa : G/G_2 \to G_3/G_4$ be as in Lemma 238; we want to show that κ is a bijection. Let a and b be as in Lemma 248(2) and define $d = [a, [a, b]]$ and $e = [b, [a, b]]$. Then $\kappa(a) \equiv e^{-1} \bmod G_4$ and $\kappa(b) \equiv d \bmod G_4$. Moreover, since $w_2 = 1$, it follows from Lemma 241 that d and e generate G_3 modulo G_4. We claim that κ is surjective. Let r, s be integers and let $y = d^s e^r$. If $r = 0$ or $s = 0$, then

$\kappa(b^s) \equiv y \bmod G_4$ or $\kappa(a^{-r}) \equiv e^r \bmod G_4$. The quotient G_3/G_4 being elementary abelian, we may now assume that r and s are both non-zero modulo 3 and they satisfy therefore $r^2 \equiv s^2 \equiv 1 \bmod 3$. Define $x = a^r b^{-s}$. Working modulo G_4, we get from Lemma 237 that

$$\begin{aligned}\kappa(x) &\equiv a^{3r}b^{-3s}[a^r b^s, [a^r, b^{-s}]] \\ &\equiv e^{-r}d^{-s}[a,[a,b]]^{-r^2 s}[b,[a,b]]^{-rs^2} \\ &\equiv e^{-r-rs^2}d^{-s-r^2 s} \\ &\equiv e^{-2r}d^{-2s} \\ &\equiv y \bmod G_4.\end{aligned}$$

We have proven that κ is surjective so, the widths w_1 and w_3 being the same, it follows that κ is a bijection. ■

Lemma 250. *Define* $\alpha : \mathrm{MC}(3) \to \mathrm{MC}(3)$ *by*

$$x = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{k} \;\mapsto\; \alpha(x) = s - t\mathrm{i} - u\mathrm{j} + v\mathrm{k}.$$

Then α *is an automorphism of order* 2 *of* $\mathrm{MC}(3)$. *Moreover,* α *induces the inversion map on* $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_2$.

Proof. Set $G = \mathrm{MC}(3)$. It is easy to check that α is an automorphism of order 2 of G, so we prove that α induces the inversion map on G/G_2. The subgroup G_2 is equal to $G \cap (1+\mathfrak{m}^2)$ and, thanks to Lemma 248(1), the order of G/G_2 is 9. It follows from Lemma 36(2) that G/G_2 is elementary abelian. We define $a = 1 - \epsilon + \mathrm{i}$ and $b = 1 - \epsilon + \mathrm{j}$. Then a and b span G modulo G_2 and

$$\alpha(a) = \bar{a} = a^{-1} \;\text{ and }\; \alpha(b) = \bar{b} = b^{-1}.$$

The quotient G/G_2 being commutative, the map $G/G_2 \to G/G_2$ that is induced by α is equal to the inversion map $x \mapsto x^{-1}$. ■

We conclude Section 9.2 by remarking that another characterization of MC(3) has been provided by Derek Holt and Frieder Ladisch; this characterization was found using computer algebra systems. The group MC(3) turns out to be isomorphic to a Sylow 3-subgroup of the Schur cover $3.\mathrm{J}_3$ of the simple Janko-3 group J_3. If S is a Sylow 3-subgroup of $3.\mathrm{J}_3$ and N denotes the normalizer of S in $3.\mathrm{J}_3$, then conjugation under any element of order 2 of N restricts to an automorphism of order 2 of S that induces the inversion map on the abelianization. The isomorphism class of MC(3) is denoted by $[729, 57]$ in the GAP system.

9.3 Structures on vector spaces

Until the end of Section 9.3, the following notation will be adopted. Let V be a 2-dimensional vector space over $\mathbb{F}_3$. A *κ-structure* on V is a bijective map $\kappa : V \to V \otimes \bigwedge^2 V$ such that, for each $x, y \in V$, one has

$$\kappa(x+y) = \kappa(x) + \kappa(y) + (x-y) \otimes (x \wedge y). \tag{A1}$$

We denote by $\mathcal{K}_V$ the collection of κ-structures of V and by $\mathcal{I}_V$ the collection of subfields of $\mathrm{End}(V)$ of cardinality 9. We remark that, for each element k of $\mathcal{I}_V$, there exists $i \in \mathrm{End}(V)$ such that $i^2 = -1$ and $k = \mathbb{F}_3[i]$. Moreover, V is naturally a vector space of dimension 1 over each of the elements of $\mathcal{I}_V$. The rest of Section 9.3 will be devoted to the proof of the following result. Until the end of Section 9.3, all tensor and wedge products will be defined over $\mathbb{F}_3$.

Proposition 251. *Let V be a 2-dimensional vector space over $\mathbb{F}_3$ and let the map $s_V : \mathcal{I}_V \longrightarrow \mathcal{K}_V$ be defined by*

$$k = \mathbb{F}_3[i] \ \mapsto \ (x \mapsto ix \otimes (ix \wedge x)).$$

Then s_V is a bijection. Moreover, the cardinality of $\mathcal{K}_V$ is equal to 3.

As the goal of this section is to prove Proposition 251, we will respect the notation of the very same proposition until the end of Section 9.3.

We put a field structure on V, via an $\mathbb{F}_3$-linear isomorphism with $\mathbb{F}_9$. We define then Λ to be the collection of bijective maps $\lambda : V \to V$ such that, for all $x, y \in V$, one has

$$\lambda(x+y) = \lambda(x) + \lambda(y) + (x-y)(xy^3 - x^3y). \tag{A2}$$

We let moreover $\sigma_V : \mathcal{I}_V \to \Lambda_V$ be defined by

$$k = \mathbb{F}_3[i] \ \mapsto \ (x \mapsto ix((ix)x^3 - (ix)^3x)).$$

Lemma 252. *The map $V \to V$, defined by $x \mapsto x^5$, is an element of Λ.*

Proof. The group of units of V has order 8 and, since 8 and 5 are coprime, the map $x \mapsto x^5$ is a bijection $V^* \to V^*$ which extends to a bijection $V \to V$. Let now x and y be elements of V. Keeping in mind that V has characteristic 3, one computes

$$\begin{aligned}(x+y)^5 &= \sum_{k=0}^{5} \binom{5}{k} x^k y^{5-k} = x^5 - xy^4 + x^2y^3 + x^3y^2 - x^4y + y^5 \\ &= x^5 + y^5 + (x-y)(xy^3 - x^3y)\end{aligned}$$

and therefore $x \mapsto x^5$ satisfies (A2). ∎

Lemma 253. *The map σ_V is well-defined.*

Proof. Let $k = \mathbb{F}_3[i]$ be an element of $\mathcal{I}_V$. The group k^* is cyclic of order 8 and there are therefore exactly two square roots of -1 in k, namely i and $-i$. Now, for each element x of V, we have

$$ix((ix)x^3 - (ix)^3x)) = -ix((-ix)x^3 - (-ix)^3x))$$

and thus k gives a map $V \to V$. Let now $k \to V$ denote an isomorphism of fields and identify i with its image in V. Then, for each $x \in V$, we have

$$ix((ix)x^3 - (ix)^3x) = x(ix)^2(x^2 - (ix)^2) = -x^3(x^2 + x^2) = x^5$$

and so, as a consequence of Lemma 252, the map σ_V is well-defined. ∎

Lemma 254. *Let $\mathbb{P}V$ denote the collection of 1-dimensional subspaces of V. Then the natural homomorphism $\mathrm{Aut}(V) \to \mathrm{Sym}(\mathbb{P}V)$ induces an isomorphism $\mathrm{Aut}(V)/\mathbb{F}_3^* \to \mathrm{Sym}(\mathbb{P}V)$.*

Proof. The natural homomorphism $\mathrm{Aut}(V) \to \mathrm{Sym}(\mathbb{P}V)$ factors as an injective homomorphism $\mathrm{Aut}(V)/\mathbb{F}_3^* \to \mathrm{Sym}(\mathbb{P}V)$, which is in fact also surjective, because $|\,\mathrm{Aut}(V) : \mathbb{F}_3^*| = 48/2 = 24 = |\,\mathrm{S}_4\,| = |\,\mathrm{Sym}(\mathbb{P}V)|$. ∎

Lemma 255. *The set $\mathcal{I}_V$ has cardinality 3. Moreover, the action by conjugation of $\mathrm{Aut}(V)$ on $\mathcal{I}_V$ is transitive.*

Proof. Let $f : \mathcal{I}_V \to \mathrm{Aut}(V)/\mathbb{F}_3^*$ be defined by $k = \mathbb{F}_3[i] \mapsto i\mathbb{F}_3^*$ and observe that, since $\mathbb{F}_3[i] = \mathbb{F}_3[-i]$, the map f is well-defined. Moreover, since each element of $\mathcal{I}_V$ is uniquely determined, modulo $\mathbb{F}_3^*$, by a square root of -1, the map f is injective. Let $\mathbb{P}V$ denote the collection of 1-dimensional subspaces of V and let $\epsilon : \mathrm{Aut}(V)/\mathbb{F}_3{*} \to \mathrm{S}_4$ be the composition of the isomorphism $\mathrm{Aut}(V)/\mathbb{F}_3^* \to \mathrm{Sym}(\mathbb{P}V)$ from Lemma 254 with a given isomorphism $\mathrm{Sym}(\mathbb{P}V) \to \mathrm{S}_4$. Then $(\epsilon \circ f)(\mathcal{I}_V)$ consists of elements of order 2. Now, each element k of $\mathcal{I}_V$ can be written as $k = \mathbb{F}_3[i]$, with $i^2 = -1$, and this suffices to show that $(\epsilon \circ f)(\mathcal{I}_V)$ is in fact contained in the Klein subgroup V_4 of S_4. The set $V_4 \setminus \{1\}$ forms a unique conjugacy class in S_4 and thus the elements of $\mathcal{I}_V$ form a unique orbit under the action by conjugation of $\mathrm{Aut}(V)$. Since the set $V_4 \setminus \{1\}$ has cardinality 3, the cardinality of $\mathcal{I}_V$ is also equal to 3. ∎

Lemma 256. *Write $V = \mathbb{F}_3[i]$, with $i^2 = -1$. Then the map $\bigwedge^2 V \to \mathbb{F}_3 i$ that is defined by $x \wedge y \mapsto xy^3 - x^3y$ is an isomorphism of vector spaces.*

Proof. Let $\phi : V \times V \to V$ be defined by $(x, y) \mapsto xy^3 - x^3y$. It is easy to show that ϕ is alternating and that $\phi(V \times V)$ is contained in $\mathbb{F}_3 i$, the eigenspace of the Frobenius homomorphism that is associated to -1. Moreover, the map ϕ is non-zero. It follows that ϕ induces a linear homomorphism $\phi' : \bigwedge^2 V \to \mathbb{F}_3 i$ that is non-trivial. Since both $\bigwedge^2 V$ and $\mathbb{F}_3 i$ have dimension 1 over $\mathbb{F}_3$, the map ϕ' is an isomorphism. ∎

Lemma 257. *Write $V = \mathbb{F}_3[i]$, with $i^2 = -1$. Then the map $\mu : V \otimes \mathbb{F}_3 i \to V$ that is defined by $x \otimes y \mapsto xy$ is an isomorphism of vector spaces.*

Proof. The map $V \times \mathbb{F}_3 i \to V$ that is defined by $(x, y) \mapsto xy$ is a bilinear surjective map. By the universal property of tensor products, it factors as the surjective homomorphism $\mu : V \otimes \mathbb{F}_3 i \to V$. The dimensions of $V \otimes \mathbb{F}_3 i$ and V being the same, μ is an isomorphism. ∎

Write $V = \mathbb{F}_3[i]$ with $i^2 = -1$. Define $\theta : V \otimes \bigwedge^2 V \to V \otimes \mathbb{F}_3 i$ by

$$\theta(a \otimes (x \wedge y)) = a \otimes (xy^3 - x^3 y)$$

and note that θ is an isomorphism of vector spaces, as a consequence of Lemma 256. Let μ be as in Lemma 257. We keep this notation until the end of Section 9.3.

Lemma 258. *The map $l_V : \mathcal{K}_V \to \Lambda$ that is defined by $\kappa \mapsto \mu \circ \theta \circ \kappa$ is bijective.*

Proof. Let κ be an element of $\mathcal{K}_V$. Then $l_V(\kappa)$ is bijective, because it is the composition of bijective maps, and, for each $x, y \in V$, one has

$$\begin{aligned}
l_V(\kappa)(x + y) &= \mu \circ \theta \circ \kappa(x + y) \\
&= \mu \circ \theta(\kappa(x) + \kappa(y) + (x - y) \otimes (x \wedge y)) \\
&= l_V(\kappa)(x) + l_V(\kappa)(y) + \mu \circ \theta((x - y) \otimes (x \wedge y)) \\
&= l_V(\kappa)(x) + l_V(\kappa)(y) + (x - y)(xy^3 - x^3 y).
\end{aligned}$$

We have proven that $l_V(\kappa)$ belongs to Λ and so l_V is well-defined. Moreover, l_V is bijective, because μ and θ are bijective. ∎

Lemma 259. *Let l_V be as in Lemma 258. Then $\sigma_V = l_V \circ s_V$ and s_V is well-defined.*

Proof. Let $k = \mathbb{F}_3[i]$ be an element of $\mathcal{I}_V$. Let moreover κ and λ respectively denote $s_V(k)$ and $\sigma_V(k)$. Then one has

$$\begin{aligned}
l_V(\kappa)(x) &= \mu \circ \theta(ix \otimes ix \wedge x) \\
&= \mu(ix \otimes ((ix)x^3 - (ix)^3 x)) \\
&= ix((ix)x^3 - (ix)^3 x) \\
&= \lambda(x)
\end{aligned}$$

and so, the choices of k and x being arbitrary, $\sigma_V = l_V \circ s_V$. As a consequence, the map s_V is well-defined. ∎

Lemma 260. *The map s_V is injective.*

Proof. Let k and k' be elements of $\mathcal{I}_V$ and let $i, j \in \mathrm{End}(V)$ be such that $k = \mathbb{F}_3[i]$, $k' = \mathbb{F}_3[j]$, and $i^2 = j^2 = -1$. Assume moreover that $s_V(k) = s_V(k')$. For each $x \in V$, we have $\mathbb{F}_3 x + \mathbb{F}_3 ix = V = \mathbb{F}_3 x + \mathbb{F}_3 jx$ and therefore there exists $\omega_x \in \{\pm 1\}$ such that $ix \equiv \omega_x jx \bmod \mathbb{F}_3 x$. For each $x \in V$, it then follows that

$$jx \otimes (jx \wedge x) = ix \otimes (ix \wedge x) = ix \otimes ((\omega_x jx) \wedge x) = \omega_x ix \otimes (jx \wedge x)$$

and, $\mu \circ \theta$ being bijective, the elements jx and $\omega_x ix$ are the same. The choice of x being arbitrary, we get

$$V = \{x \in V : ix = jx\} \cup \{x \in V : ix = -jx\}$$

and so, V being equal to the union of two subgroups, either $i = j$ or $i = -j$. In either case, i and j are linearly dependent over $\mathbb{F}_3$ and so $k = k'$. ∎

Lemma 261. *Let ϕ be an $\mathbb{F}_3$-linear endomorphism of V. Then there exist unique $a, b \in V$ such that, for each $x \in V$, one has $\phi(x) = ax^3 + bx$.*

Proof. The characteristic of V being 3, for each pair (a, b) in V^2, the map $x \mapsto ax^3 + bx$ is an $\mathbb{F}_3$-linear endomorphism of V. The order of $\mathrm{End}(V)$ being equal to the order of V^2, it follows that each element ψ of $\mathrm{End}(V)$ is of the form $x \mapsto ax^3 + bx$, where $a, b \in V$ are uniquely determined by ψ. In particular, this holds for ϕ. ∎

Lemma 262. *Let $\lambda \in \Lambda$. Then there exist $a, b \in V$ such that, for each $x \in V$, one has $\lambda(x) = x^5 + ax^3 + bx$.*

Proof. Because of (A2), the difference of any two elements of Λ belongs to $\mathrm{End}(V)$, so, thanks to Lemma 252, we have $\lambda \in (x \mapsto x^5) + \mathrm{End}(V)$. It now follows from Lemma 261 that there exist $a, b \in V$ such that, for each $x \in V$, we have $\lambda(x) = x^5 + ax^3 + bx$. ∎

Lemma 263. *Let m be a positive integer and let q be a prime power. Then*

$$\sum_{x \in \mathbb{F}_q} x^m = \begin{cases} -1 & \textit{when} \quad (q-1) | m \\ 0 & \textit{otherwise} \end{cases}$$

Proof. This is Lemma 2.5.1 from [Coh07]. ∎

Lemma 264. *Let $\lambda \in \Lambda$. Then there exists $b \in V$ such that, for each $x \in V$, one has $\lambda(x) = x^5 + bx$.*

Proof. Let $a, b \in V$ be as in Lemma 262. By definition of Λ, the map λ is bijective so each element of V belongs to the image of λ. With x replaced by $\lambda(x)$, Lemma 263 yields

$$0 = \sum_{x \in V} \lambda(x)^2 = \sum_{x \in V} (x^5 + ax^3 + bx)^2 = \sum_{x \in V} 2ax^8 = -2a.$$

It follows that $a = 0$ and therefore, for each $x \in V$, one has $\lambda(x) = x^5 + bx$. ∎

Lemma 265. *The cardinality of* Λ *is at most* 3.

Proof. Let $\lambda \in \Lambda$ and let $b \in V$ be as in Lemma 264. The map λ is bijective and so, with x replaced by $\lambda(x)$, Lemma 263 gives

$$\begin{aligned} 0 = \sum_{x \in V} \lambda(x)^4 &= \sum_{x \in V} (x^5 + bx)^4 \\ &= \sum_{x \in V} (x^{10} - bx^6 + b^2x^2)^2 \\ &= \sum_{x \in V} (bx^{16} + b^3x^8) \\ &= -b(1 + b^2). \end{aligned}$$

It follows that there are at most 3 choices for b in V and thus Λ has cardinality at most 3. ∎

We conclude Section 9.3 by giving the proof of Proposition 251. The function $s_V : \mathcal{I}_V \to \mathcal{K}_V$ is injective by Lemma 260 and, by Lemma 255, the cardinality of $\mathcal{I}_V$ is equal to 3. It follows that $\mathcal{K}_V$ has at least 3 elements. Now, as a consequence of Lemma 258, the set Λ has the same cardinality as $\mathcal{K}_V$ and thus, as a consequence of Lemma 265, the cardinality of $\mathcal{K}_V$ is equal to 3. From its injectivity, it now follows that s_V is bijective. The proof of Proposition 251 is complete.

9.4 Structures and free groups

We recall that a κ-group is a finite 3-group G such that $|G : G_2| = 9$ and such that the cubing map on G induces a bijection $G/G_2 \to G_3/G_4$. In the present section, we consider κ-groups of class 3 and we prove the following main result.

Proposition 266. *Let* G *be a* κ*-group of class* 3. *Then* G *is isomorphic to* $\mathrm{MC}(3) \,/\, \mathrm{MC}(3)_4$.

As a consequence of Lemma 36(2), each κ-group is 2-generated. Our strategy, for proving Proposition 266, will be that of constructing all κ-groups of class 3 as quotients of a free group. To this end, the following assumptions will be valid until the end of Section 9.4. Let F be the free group on two generators and let $(F_i)_{i\geq 1}$ denote the lower 3-series of F, which we recall from Section 5.4 to be defined by

$$F_1 = F \quad \text{and} \quad F_{i+1} = [F, F_i]F_i^3.$$

We remark that the notation we use for the lower 3-series is not concordant with our usual notation (see Exceptions in List of Symbols). We denote

$$V = F/F_2, \; L = F_3F^3, \quad \text{and} \quad E = [F, L]F_2^3.$$

The group V is a vector space of dimension 2 over $\mathbb{F}_3$, by construction, so we let $\mathcal{K}_V$ be defined as in Section 9.3. We write moreover $\overline{F} = F/E$ and we use the bar notation for the subsets of $\overline{F}$. We define additionally $\mathcal{N}_3$ to be the collection of normal subgroups N of F with the property that F/N is a κ-group of class 3.

Lemma 267. *The map $c_3 : F \to L/F_3$, defined by $x \mapsto x^3F_3$, is surjective. Moreover, c_3 induces an isomorphism $V \to L/F_3$ and $|L : F_3| = 9$.*

Proof. The map c_3 is well-defined, by definition of L, and $L/F_3 = (F^3F_3)/F_3$. As a consequence of Lemma 48, the map c_3 is a surjective homomorphism, which, F_2^3 being contained in F_3, factors as a surjective homomorphism $c_2 : F/F_2 \to L/F_3$. Since $V = F/F_2$ has order 9, the order of L/F_3 is at most 9. Let now $A = \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ and let $\psi : F \to A$ be a surjective homomorphism. The group A being abelian, we have that F_3 is contained in $\ker\psi$. Moreover, since $L = F^3F_3$, the group $\psi(L)$ is equal to $3\mathbb{Z}/9\mathbb{Z} \times 3\mathbb{Z}/9\mathbb{Z}$, which has order 9. As a consequence, L/F_3 has cardinality at least 9 and so $|L : F_3| = 9$. In addition, the map c_2 is an isomorphism of groups. ∎

Lemma 268. *One has $|F_3 : E| \geq 9$.*

Proof. Thanks to Lemma 248, we have $|\operatorname{MC}(3) : \operatorname{MC}(3)_2| = 9$ and therefore, as a consequence of Lemma 36(2), the group $\operatorname{MC}(3)$ is 2-generated. We fix a surjective homomorphism $\phi : F \to \operatorname{MC}(3)$ and we denote by π the canonical projection $\pi : \operatorname{MC}(3) \to \operatorname{MC}(3) / \operatorname{MC}(3)_4$. The homomorphism $\pi \circ \phi : F \to \operatorname{MC}(3) / \operatorname{MC}(3)_4$ is surjective and so, from Lemma 157, we get $L = (\pi\circ\phi)^{-1}(\operatorname{MC}(3)_3 / \operatorname{MC}(3)_4)$. As a consequence, L is equal to $\phi^{-1}(\operatorname{MC}(3)_3)$ and thus $\phi(L) = \operatorname{MC}(3)_3$. Moreover, thanks to Lemma 141, we know that $\operatorname{MC}(3)_2^3$ is contained in $\operatorname{MC}(3)_4$ and therefore $\phi(F_2^3) \subseteq \operatorname{MC}(3)_4$. It follows that

$$\phi(F_3) = \phi([F, F_2])\phi(F_2^3) = [\operatorname{MC}(3), \operatorname{MC}(3)_2] \operatorname{MC}(3)_2^3 = \operatorname{MC}(3)_3$$

and also that

$$\phi(E) = \phi([F, L])\phi(F_2^3) = [\operatorname{MC}(3), \operatorname{MC}(3)_3] \operatorname{MC}(3)_2^3 = \operatorname{MC}(3)_4 .$$

As a result, the index $|F_3 : E|$ is at least $|\operatorname{MC}(3)_3 : \operatorname{MC}(3)_4|$, which is, by Lemma 248(1), equal to 9. ∎

$$
\begin{tikzcd}
\mathbb{Z}/9\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z} \arrow[d, no head, "9"'] & & F \arrow[ll, dashed] \arrow[r, dashed] \arrow[d, no head, "9"'] & \mathrm{MC}(3) \arrow[d, no head, "9"] \\
3\mathbb{Z}/9\mathbb{Z}\times 3\mathbb{Z}/9\mathbb{Z} \arrow[d, no head, "9"'] & & F_2 \arrow[r, dashed] \arrow[d, no head, "3"'] & \mathrm{MC}(3)_2 \arrow[d, no head, "3"] \\
0 & & L = F_3F^3 \arrow[llu, dashed] \arrow[r, dashed] \arrow[dl, no head, "9"'] & \mathrm{MC}(3)_3 \arrow[d, no head, "9"] \\
& F_3 \arrow[lu, dashed] \arrow[rru, dashed] \arrow[dr, no head, "9"'] & & \mathrm{MC}(3)_4 \\
& & E = [F,L]F_2^3 \arrow[ru, dashed] &
\end{tikzcd}
$$

Lemma 269. *The commutator map $F \times F_2 \to F_3$ induces an isomorphism of groups $\delta : F/F_2 \otimes F_2/L \to \overline{F_3}$. Moreover, $|F_3 : E| = 9$.*

Proof. The subgroup $\overline{F_3}$ is central in $\overline{F}$, by definition of E, and so, by Lemma 22, the commutator map $F \times F_2 \to \overline{F_3}$ is bilinear. Moreover, the quotient F_2/L is cyclic of order 3, thanks to Lemma 157, and so, by Lemma 28, the subgroup $[F_2, F_2]$ is equal to $[F_2, L]$. The commutator map factors thus as a surjective homomorphism $\delta : F/F_2 \otimes F_2/L \to \overline{F_3}$ and therefore $|F_3 : E| \leq |F/F_2 \otimes F_2/L| = 9$. Now, the group $\overline{F_3}$ has order at least 9, by Lemma 268, and therefore $|F_3 : E| = 9$ and δ is an isomorphism. ∎

Lemma 270. *The group $\overline{L}$ is an $\mathbb{F}_3$-vector space of dimension 4.*

Proof. By the definition of E, the group $\overline{L}$ is central in $\overline{F}$ and so it is abelian. Moreover L^3 is contained in F_2^3, which is itself contained in E. It follows that $\overline{L}$ is naturally a vector space over $\mathbb{F}_3$. The dimension of $\overline{L}$ is equal to 4, thanks to the combination of Lemmas 267 and 269. ∎

Lemma 271. *The commutator map $F \times F_2 \to F_3$ induces an isomorphism of groups $\gamma : V \otimes \bigwedge^2 V \to \overline{F_3}$.*

Proof. The subgroup F_2 is central modulo L so, thanks to Lemma 22, the commutator map $F \times F \to F_2/L$ is bilinear. Since $[F, F_2]$ is contained in L, we get a bilinear map $V \times V \to F_2/L$, which is also alternating. By the universal property of wedge products, the last map factors as a homomorphism $\theta : \bigwedge^2 V \to F_2/L$ mapping $x \wedge y$ to $[x, y]$. By Lemma 157, the cardinality of F_2/L is equal to 3, which is the same as the cardinality of $\bigwedge^2 V$ and so, θ being non-trivial, it is an isomorphism of groups. We conclude by defining $\gamma = \delta \circ (1 \otimes \theta)$, where δ is as in Lemma 269. ∎

Lemma 272. *Let γ be as in Lemma 271 and use the additive notation for the vector spaces V and $\overline{L}$. Then the cubing map on $\overline{F}$ induces a map $c : V \to \overline{L}$ such that, for each $x, y \in V$, one has*

$$c(x + y) = c(x) + c(y) + \gamma((x - y) \otimes (x \wedge y)).$$

Proof. The subgroup $[F, [F, F]]$ is contained in L, which is central modulo E, so $\overline{F}$ has class at most 3. Moreover, $[F, F]^3$ is contained in F_2^3 and so $[\overline{F}, \overline{F}]$ has exponent at most 3. By Lemma 237, given any two elements x, y of $\overline{F}$, one has $(xy)^3 = x^3y^3[xy^{-1}, [x, y]]$. Since both F_2^3 and $[F, [F, F_2]]$ are contained in E, cubing on $\overline{F}$ induces a map $c : V \to \overline{L}$. Using the additive notation for the vector spaces V and $\overline{L}$, it follows from the definition of γ that, for each $x, y \in V$, one has $c(x + y) = c(x) + c(y) + \gamma((x - y) \otimes (x \wedge y))$. ∎

Lemma 273. *Let $0 \to A \xrightarrow{\iota} B \xrightarrow{\sigma} C \to 0$ be a short exact sequence of abelian groups. Let moreover $s : C \to B$ be a function such that $\sigma \circ s = \mathrm{id}_C$. Write $\mathcal{R} = \{f \in \mathrm{Hom}(B, A) : f \circ \iota = \mathrm{id}_A\}$ and let $\mathcal{H}$ be the collection of maps $g : C \to A$ such that, for all $u, v \in C$, one has*

$$\iota(g(u + v) - g(u) - g(v)) = s(u + v) - s(u) - s(v).$$

Then the function $\mathcal{R} \to \mathcal{H}$ that is defined by $f \mapsto f \circ s$ is bijective.

Proof. Let $\nu : \mathcal{R} \to \mathcal{H}$ be defined by $f \mapsto f \circ s$. We first prove that ν is well-defined. To this end, let $f \in \mathcal{R}$ and let $u, v \in C$. Since $\sigma \circ s = \mathrm{id}_C$, the element $s(u + v) - s(u) - s(v)$ belongs to $\ker \sigma = \iota(A)$. Since $f \circ \iota = \mathrm{id}_A$, we get that $\iota \circ f_{|\iota(A)} = \mathrm{id}_{|\iota(A)}$ and therefore

$$\iota((f \circ s)(u + v) - (f \circ s)(u) - (f \circ s)(v)) = \iota(f(s(u + v) - s(u) - s(v))) =$$

$$s(u + v) - s(u) - s(v).$$

We have proven that ν is well-defined. We now prove that ν is injective. Let $f, h \in \mathcal{R}$ be such that $\nu(f) = \nu(h)$. Since $f \circ \iota = h \circ \iota = \mathrm{id}_A$, the group $\iota(A)$ is contained in $\ker(f - h)$ and thus $f - h$ induces a homomorphism $B/\iota(A) \to A$.

Now, $B/\iota(A) = \{s(c) + \iota(A) : c \in C\}$ and, the maps $f \circ s$ and $h \circ s$ being the same, we get $f - h = 0$. The maps f and g are the same and ν is injective. To conclude, we prove that ν is surjective. Let $g \in \mathcal{H}$. Since each element x of B can be written uniquely as $x = \iota(a) + s(u)$, with $a \in A$ and $u \in C$, we define $f : B \to A$ by

$$x = \iota(a) + s(u) \mapsto f(x) = a + g(u).$$

For each $u \in C$, we have then $f \circ s(u) = g(u)$. We prove now that f is a homomorphism. Let $x, y \in B$ and let $a, b \in A$ and $u, v \in C$ be such that $x = \iota(a) + s(u)$ and $y = \iota(b) + s(v)$. Keeping in mind that g belongs to $\mathcal{H}$, we compute

$$\begin{aligned} f(x+y) - f(x) - f(y) &= f(\iota(a) + s(u) + \iota(b) + s(v)) - f(\iota(a) + s(u)) + \\ &\quad - f(\iota(b) + s(v)) \\ &= f(\iota(a) + \iota(b) - g(u+v) + g(u) + g(v) + s(u+v)) + \\ &\quad - f(\iota(a) + s(u)) - f(\iota(b) + s(v)) \end{aligned}$$

and, since $g(C)$ is contained in A, we get

$$\begin{aligned} f(x+y) - f(x) - f(y) &= a + b - g(u+v) + g(u) + g(v) + g(u+v) + \\ &\quad - f(a + s(u)) - f(b + s(v)) \\ &= a + b + g(u) + g(v) - a - g(u) - b - g(v) \\ &= 0. \end{aligned}$$

We have proven that f is a homomorphism and so ν is surjective. ∎

Proposition 274. *Let c be as in Lemma 272 and let γ be as in Lemma 271. Set $\mathcal{P} = \{\pi \in \mathrm{Hom}(\overline{L}, \overline{F_3}) : \pi_{|\bar{F_3}} = \mathrm{id}_{\bar{F_3}}, \pi \circ c$ bijective$\}$ and let $t_V : \mathcal{P} \to \mathcal{K}_V$ be defined by $\pi \mapsto \gamma^{-1} \circ \pi \circ c$. Then t_V is a bijection and $\mathcal{P}$ has cardinality 3.*

Proof. Let $c_2 : V \to L/F_3$ be the isomorphism from Lemma 267. Composing the canonical projection $\overline{L} \to L/F_3$ with c_2^{-1}, we get the short exact sequence of abelian groups $0 \to \overline{F_3} \to \overline{L} \to V \to 0$. With $A = \overline{F_3}$, $B = \overline{L}$, $C = V$, and $s = c$, Lemma 273 applies. Let thus $\mathcal{R} = \{\pi \in \mathrm{Hom}(\overline{L}, \overline{F_3}) : \pi_{|\bar{F_3}} = \mathrm{id}_{\bar{F_3}}\}$ and let $\mathcal{H}$ be the collection of maps $g : V \to \overline{F_3}$ such that, for all $x, y \in V$, one has $g(x+y) - g(x) - g(y) = c(x+y) - c(x) - c(y)$. Then, thanks to Lemma 273, each element of $\mathcal{H}$ is of the form $\pi \circ c$, where π belongs to $\mathcal{R}$. In particular, the subset $\mathcal{P}$ of $\mathcal{R}$ is sent bijectively to the subset $\mathcal{H}_{\mathrm{bij}}$ of bijective elements of $\mathcal{H}$. Now, by Lemma 272, given any two elements $x, y \in V$, we have $c(x+y) - c(x) - c(y) = \gamma((x-y) \otimes (x \wedge y))$ and therefore each element $\kappa = \gamma^{-1} \circ \pi \circ c$, with $\pi \in \mathcal{P}$, belongs to $\mathcal{K}_V$. The map γ being an isomorphism, t_V is injective. Moreover, since γ is bijective, Lemma 272 yields a well-defined injection $\mathcal{K}_V \to \mathcal{H}_{\mathrm{bij}}$, given by $\kappa \mapsto \gamma \circ \kappa$. It follows that $|\mathcal{P}| \leq |\mathcal{K}_V| \leq |\mathcal{H}_{\mathrm{bij}}| = |\mathcal{P}|$ and therefore t_V is a bijection. Thanks to Proposition 251, the cardinality of $\mathcal{P}$ is 3. ∎

We remind the reader that $\mathcal{N}_3$ has been defined to be the collection of normal subgroups N of F such that F/N is a κ-group of class 3.

Lemma 275. *There exists M in $\mathcal{N}_3$ such that the quotient F/M is isomorphic to* $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_4$*. Moreover, the set $\mathcal{N}_3$ is non-empty.*

Proof. The group $\mathrm{MC}(3)$ is a κ-group, by Lemma 249, and it has class 4, by Lemma 247. It follows that there exists M in $\mathcal{N}_3$ such that F/M is isomorphic to $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_4$ and, in particular, $\mathcal{N}_3$ is non-empty. ∎

Lemma 276. *Let $\mathcal{P}$ be as in Proposition 274 and denote, for each $\pi \in \mathcal{P}$, by K_π the unique normal subgroup of F containing E such that $\overline{K_\pi} = \ker \pi$. Then the map $r : \mathcal{P} \to \mathcal{N}_3$ that is defined by $\pi \mapsto K_\pi$ is a bijection. Moreover, for each $N \in \mathcal{N}_3$, one has $|L : N| = 9$.*

Proof. We first show that r is well-defined. To this end, let π be an element of $\mathcal{P}$ and set $G = F/K_\pi$. Since $|F : F_2| = 9$, Lemma 36 yields that G_2 is equal to F_2/K_π. Moreover, it easily follows from the definition of $\mathcal{P}$ that $\overline{L}$ decomposes as $\ker \pi \oplus \overline{F_3} = \overline{K_\pi} \oplus \overline{F_3}$. In particular, L/K_π and $\overline{F_3}$ are naturally isomorphic and so, as a consequence of Lemma 269, the subgroup G_3 coincides with L/K_π. The class of G is equal to 3, because L is central modulo E. Now, the map $\pi \circ c$ being bijective, it follows that the cubing map induces a bijection $F/F_2 \to \overline{F_3}$ and so, via the natural isomorphism $\overline{F_3} \to L/K_\pi$, the cubing map induces a bijection $G/G_2 \to G_3$. As a result, we have that $|L : K_\pi| = |G_3| = |G : G_2| = |F : F_2| = 9$ and G is a κ-group. The choice of π being arbitrary, we have proven that r is well-defined. It is now easy to show that r is bijective. From the surjectivity of r one deduces that, for all $N \in \mathcal{N}_3$, the index $|L : N|$ is equal to 9. ∎

Proposition 277. *The set $\mathcal{N}_3$ has cardinality* 3 *and the natural action of* $\mathrm{Aut}(F)$ *on $\mathcal{N}_3$ is transitive.*

Proof. Let $\mathcal{I}_V$ be defined as in Section 9.3. Define moreover $\psi : \mathcal{I}_V \to \mathcal{N}_3$ to be $\psi = r \circ t_V^{-1} \circ s_V$, where s_V, t_V, and r are as in Propositions 251 and 274 and Lemma 276. The combination of the just-mentioned results yields that ψ is a bijection and, from its definition, it is easy to check that it respects the action of $\mathrm{Aut}(F)$. Now, by Lemma 255, the set $\mathcal{I}_V$ has cardinality 3 and so $\mathcal{N}_3$ has cardinality 3. Again by Lemma 255, the action of $\mathrm{Aut}(V)$ on $\mathcal{I}_V$ is transitive and thus the action of $\mathrm{Aut}(F)$ on $\mathcal{I}_V$ is transitive. Since the map ψ is an isomorphism of $\mathrm{Aut}(F)$-sets, the action of $\mathrm{Aut}(F)$ on $\mathcal{N}_3$ is transitive. ∎

We are finally ready to give the proof of Proposition 266. Let G be a κ-group of class 3. As a consequence of Proposition 275, there exist N and M normal subgroups of F such that F/N and F/M are respectively isomorphic to G and $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_4$. Fix such M and N. Then, thanks to Lemma 277, there exists an automorphism of F mapping M to N, which thus induces an isomorphism between G and $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_4$. The choice of G being arbitrary, the proof of Proposition 266 is complete.

We conclude the present section by giving the proof of Theorem 233. If G is a κ-group of class 3, then, by Proposition 266, the group G is isomorphic to $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_4$. On the other hand, the group $\mathrm{MC}(3)$ has class 4, by Lemma 247, and it is a κ-group, by Lemma 249. It follows that $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_4$ is a κ-group of class 3. This proves Theorem 233.

9.5 Extensions of κ-groups

We recall here that a κ-group is a finite 3-group G such that $|G : G_2| = 9$ and such that cubing in G induces a bijection $G/G_2 \to G_3/G_4$. We remind the reader that we investigate κ-groups because we aim at classifying 3-groups of class at least 4 and intensity greater than 1: those groups are all κ-groups, as a consequence of Lemma 239. The main purpose of the present section is that of proving the following proposition, which is the same as Theorem 234.

Proposition 278. *Let G be a κ-group such that G_4 has order 3. Then the subgroup G_2 is elementary abelian.*

Until the end of Section 9.5, we will work under the assumptions of Proposition 278. For each $i \in \mathbb{Z}_{\geq 1}$, we set $w_i = \mathrm{wt}_G(i)$. It follows from the assumptions, together with Lemma 135(1), that $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. Moreover, the group G/G_4 being a κ-group of class 3, Proposition 266 yields that G/G_4 is isomorphic to $\mathrm{MC}(3)\,/\,\mathrm{MC}(3)_4$. It follows from Lemma 248(2) that there exist generators a, b of G satisfying $a^3 \equiv [b, [a, b]]^{-1} \bmod G_4$ and $b^3 \equiv [a, [a, b]] \bmod G_4$. Call $c = [a, b]$, $d = [a, c]$, and $e = [b, c]$. Let moreover $f = [a, d]$. Then we have $a^3 \equiv e^{-1} \bmod G_4$ and $b^3 \equiv d \bmod G_4$.

Lemma 279. *The elements d and e generate G_3 modulo G_4.*

Proof. The index $|G_2 : G_3|$ is equal to 3, so c generates G_2 modulo G_3. By Lemma 241, the commutator map induces an isomorphism $G/G_2 \otimes G_2/G_3 \to G_3/G_4$, and so d and e span G_3 modulo G_4. ∎

Lemma 280. *One has $G_4 = \langle f \rangle = \langle [b, e] \rangle$.*

Proof. By Lemma 244, the centre of G is equal to G_4 and, by Lemma 190, the commutator map $G/G_2 \times G_3/G_4 \to G_4$ is non-degenerate. The elements d and e generate G_3 modulo G_4, thanks to Lemma 279, and, by the choice of a and b, we also have $a^3 \equiv e^{-1} \bmod G_4$ and $b^3 \equiv d \bmod G_4$. From the non-degeneracy of the commutator map, it follows that both f and $[b, e]$ are non-trivial elements of G_4, which, being cyclic of order 3, then satisfies $G_4 = \langle f \rangle = \langle [b, e] \rangle$. ■

Lemma 281. *There exists a pair (u, t) in $\{\pm 1\} \times \mathbb{Z}$ such that $[b, e] = f^u$ and $c^3 = f^t$. Moreover, there exist $r, s \in \mathbb{Z}$ such that $a^3 = e^{-1} f^r$ and $b^3 = df^s$.*

Proof. By assumption, the order of G_4 is 3 and, by Lemma 279, both elements f and $[b, e]$ generate G_4. There exists thus $u \in \{\pm 1\}$ such that $[b, e] = f^u$. Moreover, by the choice of a and b, we know that $a^3 \equiv e^{-1} \bmod G_4$ and $b^3 \equiv d \bmod G_4$. There exist hence integers r and s such that $a^3 = e^{-1} f^r$ and $b^3 = df^s$. To conclude, thanks to Lemma 141, the subgroup G_2^3 is contained in G_4 so there exists $t \in \mathbb{Z}$ such that $c^3 = f^t$. ■

We are now ready to give the proof of Proposition 278. To this end, let u, t, r, s be as in Lemma 281. By Lemma 240, the subgroup G_2 is abelian and, by Lemma 246, the exponent of G_3 is equal to 3. It follows that

$$\begin{aligned} ab^3 &= cbab^2 \\ &= cbcbab \\ &= cbcbcba \\ &= cecbecb^2a \\ &= ec^2 f^u ebcb^2 a \\ &= f^u e^2 c^2 ecb^3 a \\ &= f^u e^3 c^3 b^3 a \\ &= f^u f^t b^3 a \\ &= f^{u+t} b^3 a \end{aligned}$$

from which we derive

$$fdaf^s = adf^s = ab^3 = f^{u+t} b^3 a = f^{u+t} df^s a.$$

The subgroup G_4 is central, thanks to Lemma 244, and so one gets

$$fda = f^{u+t} da.$$

Since the exponent of G_3 is equal to 3, we have $u + t \equiv 1 \bmod 3$ and so

$$(u, t) \equiv (1, 0) \bmod 3 \quad \text{or} \quad (u, t) \equiv (-1, -1) \bmod 3.$$

If $(u, t) \equiv (1, 0) \bmod 3$, then we are done. We assume by contradiction that

$(u,t) \equiv (-1,-1) \bmod 3$. Then $c^3 = f^{-1}$ and we compute

$$\begin{aligned} a^3b &= a^2cba \\ &= adcaba \\ &= fdac^2ba^2 \\ &= fd^2cacba^2 \\ &= fd^2cdcaba^2 \\ &= fd^3c^3ba^3 \\ &= ba^3. \end{aligned}$$

We have shown that a^3 centralizes b in G. Call $C = \langle \{b\} \cup G_2 \rangle$. Then a^3 belongs to $\mathrm{Z}(C)$, which then, thanks to Lemma 242, contains $\{a^3, b^3\} \cup G_4$. The group G being a κ-group, it follows that $\mathrm{Z}(C)$ contains G_3, and so $[b,e]=1$. Contradiction to Lemma 280. The proof of Proposition 278, and thus that of Theorem 234, is now complete.

Corollary 282. *The subgroup* $\mathrm{MC}(3)_2$ *of* $\mathrm{MC}(3)$ *is elementary abelian.*

Proof. The group $\mathrm{MC}(3)$ is a κ-group by Lemma 249 and, thanks to Lemma 248(1), the subgroup $\mathrm{MC}(3)_4$ has order 3. It follows from Proposition 278 that $\mathrm{MC}(3)_2$ is elementary abelian. ∎

Corollary 283. *Let Q be a finite 3-group of class 4 and let $(Q_i)_{i \geq 1}$ denote the lower central series of Q. If $\mathrm{int}(Q) > 1$, then Q_2 is elementary abelian.*

Proof. By Lemma 239, the group Q is a κ-group. Moreover, thanks to Theorem 164, the subgroup Q_4 has order 3. It follows from Proposition 278 that Q_2 is elementary abelian. ∎

Corollary 284. *Let Q be a finite 3-group with $\mathrm{int}(Q) > 1$. Then Q has nilpotency class at most 4.*

Proof. Assume that Q has class at least 4. Thanks to Lemma 101, the intensity of Q/Q_5 is greater than 1, and so, as a consequence of Corollary 283, the subgroup Q_2^3 is contained in Q_5. However, because of Proposition 211, each finite 3-group H of class at least 5 with $\mathrm{int}(H) > 1$ satisfies $H_2^3 = H_4$ and so it follows that Q has class at most 4. ∎

9.6 Constructing automorphisms

In this section we aim at understanding the structure of finite 3-groups of class 4 and intensity greater than 1. We recall that a κ-group is a finite 3-group G such that $|G:G_2|=9$ and the cubing map on G induces a bijection $G/G_2 \to G_3/G_4$ (see Section 9.1 for a closer look at κ-groups). The reason why κ-groups are so special for us is Lemma 239, which asserts that any finite 3-group of class 4 and intensity greater than 1 is a κ-group. Moreover, we know from Proposition 134, that if we hope to construct a 3-group G of large class and intensity greater than 1, then we need as well to construct an automorphism of order 2 of G that induces the inversion map on the abelianization of G. We will devote the present section to the proof of the following result.

Proposition 285. *Let G be a κ-group such that G_4 has order* 3*. Assume that G possesses an automorphism of order* 2 *that induces the inversion map on G/G_2. Then G is isomorphic to* MC(3)*.*

We will prove Proposition 285 at the end of the present section and so the following assumptions will hold until the end of Section 9.6. Let G be a κ-group such that G_4 has order 3. Then the group G has class 4 and $(\mathrm{wt}_G(i))_{i=1}^4=(2,1,2,1)$. Let F be the free group on the set $S=\{a,b\}$ and let $\iota : S \to G$ be such that $G=\langle \iota(S)\rangle$. By the universal property of free groups, there exists a unique homomorphism $\phi : F \to G$ such that $\phi(a)=\iota(a)$ and $\phi(b)=\iota(b)$. As a consequence of its definition, the map ϕ is surjective. Let $(F_i)_{i\geq 1}$ denote the lower 3-series of F, which is defined recursively as

$$F_1 = F \quad \text{and} \quad F_{i+1}=[F,F_i]F_i^3.$$

and, in addition, let

$$L = F^3F_3 \quad \text{and} \quad E=[F,L]F_2^3.$$

All F_i's, L, and E are stabilized by any endomorphism of F. For a visualization of such groups we refer to the end of Section 5.4 or to the diagram before Lemma 303. Let β be the endomorphism of F sending a to a^{-1} and b to b^{-1}. Since $\beta^2=\mathrm{id}_F$, the map β is an automorphism of F. We remind the reader that we have already worked with such an automorphism β in Section 5.4 and we will thus, in this section, often apply results achieved in Section 5.4. We conclude by defining two specific sets, consisting of normal subgroups of F. Let $\mathcal{N}_3$ denote the collection of normal subgroups N of F such that F/N is a κ-group of class 3, as defined in Section 9.4. For each element N of $\mathcal{N}_3$, we set

$$D_N=[F,N]F_2^3[F_2,F_2].$$

We define moreover $\mathcal{N}_4$ to be the collection of normal subgroups M of F such that F/M is a κ-group of class 4 with $\mathrm{wt}_{F/M}(4)=1$ and such that F/M possesses an automorphism of order 2 that induces the inversion map on the abelianization $(F/M)/(F/M)_2$ of F/M. We will keep this notation until the end of Section 9.6.

Lemma 286. *Let $N \in \mathcal{N}_3$. Then D_N is contained in E.*

Proof. As a consequence of Lemma 157, the subgroup L contains N. Again by Lemma 157, the index $|F_2 : L|$ is equal to 3 and so, thanks to Lemma 28, one has $[F_2, F_2] = [F_2, L]$. We get

$$[F, N]F_2^3[F_2, F_2] \subseteq [F, L]F_2^3[F_2, L] = [F, L]F_2^3 = E$$

and therefore D_N is contained in E. ∎

Lemma 287. *For each $k \in \mathbb{Z}_{\geq 5}$, one has $\phi(F_k) = \{1\}$.*

Proof. Let $k \in \mathbb{Z}_{\geq 5}$ and recall that $F_k = [F, F_{k-1}]F_{k-1}^3$. By definition of E, one has $\phi([F, F_{k-1}]) \subseteq \phi([F, F_4]) \subseteq \phi([F, E]) = [\phi(F), \phi(E)]$ and so, as a consequence of Lemma 158, we get $\phi([F, F_{k-1}]) \subseteq [G, G_4] = \{1\}$. It follows from Lemma 36 that $\phi(F_k) = \phi(F_{k-1}^3) \subseteq \phi(F_2^3) \subseteq \Phi(G)^3 = G_2^3$ and so Proposition 278 yields $\phi(F_k) = \{1\}$. ∎

Lemma 288. *Let α be an automorphism of order 2 of G that induces the inversion map on G/G_2. Then there exist generators x and y of G such that $\alpha(x) = x^{-1}$ and $\alpha(y) = y^{-1}$.*

Proof. Write $G^- = \{g \in G : \alpha(g) = g^{-1}\}$. Since $(\mathrm{wt}_G(i))_{i=1}^4 = (2, 1, 2, 1)$, Lemma 85 yields that the map $G^- \to G/G_2$, defined by $g \mapsto gG_2$, is surjective. Thanks to Lemma 36, the subgroups G_2 and $\Phi(G)$ coincide and therefore there exist two elements x and y of G^- that generate G. ∎

Proposition 289. *Let α be an automorphism of order 2 of G that induces the inversion map on G/G_2. Let moreover $k \in \mathbb{Z}_{\geq 5}$ and let $\phi_k : F/F_k \to G$ be the map induced by ϕ. Then there exists $\epsilon \in \mathrm{Aut}(F/F_k)$ of order 2 such that $\alpha\phi_k = \phi_k\epsilon$.*

Proof. For each $k \in \mathbb{Z}_{\geq 5}$, the map $\phi_k : F/F_k \to G$ is well-defined, thanks to Lemma 287. Let now x and y be as in Lemma 288 and let c and d be elements of F such that $\phi(c) = x$ and $\phi(d) = y$, which exist because ϕ is surjective. As a consequence of Lemma 155, the map ϕ induces an isomorphism $F/F_2 \to G/G_2$ and therefore c and d generate F modulo F_2. Let now $\psi : F \to F$ be the endomorphism of F sending $a \mapsto c$ and $b \mapsto d$; such ψ exists thanks to the universal property of free groups. Fix $k \in \mathbb{Z}_{\geq 5}$. The subgroup F_k being being stabilized by any endomorphism of F, the map ψ induces an endomorphism $\overline{\psi}$ of the 3-group $\overline{F} = F/F_k$. However, since $\Phi(\overline{F}) = \overline{F_2}$, the map $\overline{\psi}$ induces an automorphism of $\overline{F}/\Phi(\overline{F})$ and so $\overline{\psi}$ is in fact an automorphism of $\overline{F}$. Let $\overline{\beta}$ be the automorphism of $\overline{F}$ that is induced by β and define $\epsilon = \overline{\psi}\overline{\beta}\overline{\psi^{-1}}$. By construction, the following diagram is commutative.

$$\begin{array}{ccc} F/F_k & \xrightarrow{\phi_k} & G \\ \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\alpha} \\ F/F_k & \xrightarrow{\phi_k} & G \end{array}$$

Moreover, ϵ has order 2, because it is conjugate in $\mathrm{Aut}(\overline{F})$ to $\overline{\beta}$. ∎

Lemma 290. *Let M be an element of $\mathcal{N}_4$. Then $N = ME$ belongs to $\mathcal{N}_3$ and D_N is contained in M.*

Proof. Let $H = F/M$ and let $\pi : F \to H$ be the canonical projection. Then $\pi(N) = \pi(ME) = \pi(E)$ and so, as a consequence of Lemma 158, we get $\pi(N) \subseteq H_4$. The order of H_4 being 3, either $\pi(N) = H_4$ or $N \subseteq M$. Assume first that $\pi(N) = H_4$. Then we have $M \subseteq N \subseteq \pi^{-1}(H_4)$ and $M \neq N$. On the other hand, we know $|\pi^{-1}(H_4) : M| = |H_4| = 3$ and therefore $N = \pi^{-1}(H_4)$. As a result, F/N is isomorphic to H/H_4 and so N belongs to $\mathcal{N}_3$. We prove that $\pi(D_N)$ is trivial. The image of F_2 under π is equal to H_2, thanks to Lemma 155. Moreover, the subgroup H_4 is central in H, because H has class 4, and the commutator subgroup of H is elementary abelian, thanks to Proposition 278. We compute

$$\pi(D_N) = \pi([F, N])\pi(F_2^3)\pi([F_2, F_2]) = [H, H_4]H_2^3[H_2, H_2] = \{1\},$$

and so D_N is contained in M. We now prove that $\pi(N) = H_4$. We work by contradiction, assuming that $N \subseteq M$. Since $N = ME$, we get that E is contained in M. As a consequence, the group F/E has class at least 4. However, one has

$$[F, [F, [F, F]]] \subseteq [F, [F, F_2]] \subseteq [F, F_3] \subseteq [F, L] \subseteq E$$

and therefore F/E has class at most 3. Contradiction. ∎

Lemma 291. *Let $N \in \mathcal{N}_3$. Denote moreover $H = \mathrm{MC}(3)$. Then there exists a surjective homomorphism $\varphi : F \to G$ such that $N = \varphi^{-1}(G_4)$. Moreover, φ induces isomorphisms $\varphi_1 : F/F_2 \to H/H_2$ and $\varphi_3 : L/N \to H_3/H_4$ and a surjective homomorphism $\varphi_4 : E/D_N \to H_4$.*

Proof. Let $\psi : F \to H$ be a surjective homomorphism, which exists thanks to the universal property of free groups. Set $K = \psi^{-1}(H_4)$. Then K belongs to $\mathcal{N}_3$, because H is a κ-group, and so, thanks to Lemma 277, there exists an automorphism r of F such that $r(N) = K$. Define $\varphi = \psi \circ r$. Then φ is a surjective homomorphism $F \to H$ such that $\varphi^{-1}(H_4) = N$. Moreover, φ induces isomorphisms $\varphi_1 : F/F_2 \to H/H_2$ and $\varphi_3 : L/N \to H_3/H_4$ as a consequence of Lemmas 155 and 157. We conclude by showing that φ induces a surjective homomorphism

$E/D_N \to H_4$. Thanks to Lemma 290, the subgroup D_N is contained in the kernel of φ. Moreover, since $\varphi(F_2) = H_2$ and $\varphi(L) = H_3$, we get

$$\varphi(E) = \varphi([F, L]F_2^3) = [H, H_3]H_2^3 = H_4H_2^3.$$

Now, the group H is a κ-group and hence $H_2^3 \subseteq H_4$. It follows that $\varphi(E) = H_4$ and therefore φ induces a surjective homomorphism $E/D_N \to H_4$. ∎

Lemma 292. *Let $N \in \mathcal{N}_3$. Then the commutator map induces a non-degenerate map $F/F_2 \times L/N \to E/D_N$ whose image generates E/D_N. In addition, one has $E \neq D_N$.*

Proof. Write $\overline{F} = F/D_N$ and use the bar notation for the subgroups of $\overline{F}$. From the definition of E, one sees that $\overline{E} = [\overline{F}, \overline{L}]$. Moreover, by Lemma 158, the subgroup E is contained in N and so $[F, E] \subseteq [F, N] \subseteq D_N$. In particular, $\overline{E}$ is central in $\overline{F}$ and so it follows from Lemma 22 that the commutator map $F \times L \to \overline{E}$ is bilinear. Since $[F_2, L]$ and $[F, N]$ are both contained in D_N, the last map factors as a bilinear map $\gamma : F/F_2 \times L/N \to \overline{E}$ whose image generates $\overline{E}$. Set now $H = \mathrm{MC}(3)$. Then, as a consequence of Lemmas 249 and 244, the centre of H is equal to H_4 and thus Lemma 190 yields that the commutator map induces a non-degenerate map $\nu : H/H_2 \times H_3/H_4 \to H_4$. With the notation from Lemma 291, the following diagram is commutative.

$$\begin{tikzcd}
F/F_2 \times L/N \arrow[r, "\gamma"] \arrow[d, shift right, "\varphi_1"'] \arrow[d, shift left, "\varphi_3"] & E/D_N \arrow[d, "\varphi_4"] \\
H/H_2 \times H_3/H_4 \arrow[r, "\nu"] & H_4
\end{tikzcd}$$

Since the map ν is non-degenerate and both φ_1 and φ_3 are isomorphisms, the map γ is non-degenerate. It follows in particular that $E \neq D_N$. ∎

Lemma 293. *Let V and W be 2-dimensional vector spaces over $\mathbb{F}_3$ and let $\eta : V \to W$ be a bijective map with the property that, for each $\lambda \in \mathbb{F}_3$ and $v \in V$, one has $\eta(\lambda v) = \lambda\eta(v)$. Define $K = \langle v \otimes \eta(v) : v \in V\rangle$. Then the quotient $(V \otimes W)/K$ has dimension 1 as a vector space over $\mathbb{F}_3$.*

Proof. Without loss of generality we assume that $V = W$. Assume first that η is an automorphism of V and define the automorphism σ of $V \otimes V$ by $x \otimes y \mapsto x \otimes \eta(y)$. Then the subspace $\Delta = \langle v \otimes v : v \in V\rangle$ is mapped isomorphically to K via σ. It follows that $(V \otimes V)/K$ has the same dimension as $(V \otimes V)/\Delta = \bigwedge^2 V$ and so $(V \otimes V)/K$ has dimension 1. Let now η be any map satisfying the hypotheses of Lemma 293. Then η induces a bijective map $\overline{\eta} : \mathbb{P}V \to \mathbb{P}V$, where $\mathbb{P}V$ denotes the collection of 1-dimensional subspaces of V. As a consequence of Lemma 254, there exists an automorphism τ of V such that $\overline{\tau} = \overline{\eta}$ and, for each $v \in V$, one has $\mathbb{F}_3\tau(v) = \mathbb{F}_3\eta(v)$. As a consequence, we get $K = \langle v \otimes \tau(v) : v \in V\rangle$ and therefore $(V \otimes V)/K$ has dimension 1 over $\mathbb{F}_3$. ∎

Lemma 294. *Let $N \in \mathcal{N}_3$. Then $|E : D_N| = 3$.*

Proof. The quotient F/F_2 is a 2-dimensional vector space over $\mathbb{F}_3$, by definition of F_2, while L/E is a 4-dimensional vector space over $\mathbb{F}_3$, thanks to Lemma 270. Moreover, by Lemma 158, the subgroup N contains E and, as a consequence of Lemma 276, the quotient L/N is a vector space of dimension 2 over $\mathbb{F}_3$. Let $\gamma : F/F_2 \otimes L/N \to E/D_N$ be the surjective homomorphism induced from the non-degenerate map of Lemma 292. Let moreover $c : F/F_2 \to L/E$ be the map from Lemma 272 and let π denote the canonical projection $L/E \to L/N$. Denote $c_N = \pi \circ c$ and note that, as a consequence of Lemma 276, the map $c_N : F/F_2 \to L/N$ is a bijection between vector spaces of dimension 2 over $\mathbb{F}_3$. From Lemma 272, it is clear that c commutes with scalar multiplication by elements of $\mathbb{F}_3$. Define $K = \langle x \otimes c_N(x) : x \in F/F_2\rangle$. As a consequence of the definition of c, each element $x \otimes c_N(x)$, with $x \in F/F_2$, belongs to the kernel of γ, and therefore K is contained in $\ker\gamma$. It follows from Lemma 293 that $(F/F_2 \otimes L/N)/K$ has dimension 1 and therefore E/D_N has dimension at most 1 as a vector space over $\mathbb{F}_3$. By Lemma 292, the quotient E/D_N is non-trivial and so γ is not the trivial map. It follows that $\ker\gamma$ has dimension 3 and thus E/D_N has cardinality 3. ∎

The following lemmas pave the way to proving Proposition 300.

Lemma 295. *Let η be an automorphism of F/L of order 2 that induces the inversion map on F/F_2. Then there exists $\varphi_L \in \mathrm{Inn}(F/L)$ such that, for each $x \in F$, one has $\beta(x) \equiv (\varphi_L\eta\varphi_L^{-1})(x) \bmod L$.*

Proof. Write $H = F/L$. The group F being 2-generated, $|F : F_2| = 9$ and so, as a consequence of Lemma 157, the group H has order 27. Thanks to the definitions of F_2 and L, one easily sees that H is non-abelian of exponent 3 and that $H_2 = F_2/L$. Lemma 29 yields that H_2 is central in H. Applying Lemma 27, we get that $H_2 = \mathrm{Z}(H)$ and therefore H is extraspecial. Let now β_L be the automorphism of H that is induced by β. Then $\eta^{-1}\beta_L$ induces the identity on H/H_2 and so, thanks to Lemma 46, one gets $\eta^{-1}\beta_L \in \mathrm{Inn}(H)$. The group $\mathrm{Inn}(H)$ being a normal 3-subgroup of $\mathrm{Aut}(H)$, the Schur-Zassenhaus theorem applies to $\mathrm{Inn}(H) \rtimes \langle\eta\rangle$ and ensures that there exists $\varphi_L \in \mathrm{Inn}(H)$ with the property that $\beta_L = \varphi_L\eta\varphi_L^{-1}$. ∎

Lemma 296. *Let η be an automorphism of F/E of order 2 that induces the inversion map on F/F_2. Assume that β coincides with η modulo L. Then, for all $x \in F$, one has $\beta(x) \equiv \eta(x) \bmod E$.*

Proof. Let Δ denote the subgroup of $\mathrm{Aut}(F/E)$ consisting of all those automorphisms of F/E inducing the identity on both F/L and L/E. Let β_E be the automorphism that is induced on F/E by β. As a consequence of Lemma 162, the element $\psi = \eta^{-1}\beta_E$ belongs to Δ and so, thanks to Lemma 45, there exists a homomorphism $h : F/L \to L/E$ such that, for all $x \in F/E$, one has $\psi(x) = h(x)x$. The quotient L/E being elementary abelian, the groups $\mathrm{Hom}(F/L, L/E)$ and $\mathrm{Hom}(F/F_2, L/E)$ are naturally isomorphic and so h factors as a homomorphism $F/F_2 \to L/E$. Now η coincides with β on F/F_2 and so, thanks to Lemma 162, it induces the inversion map on L/E. For each $x \in F/F_2$, it follows that

$$(\eta h \eta^{-1})(x) = \eta(h(x^{-1})) = (h(x^{-1}))^{-1} = h(x).$$

However, the automorphisms η and β_E having order 2, one also has

$$\eta^2 = 1 = \beta_E^2 = \eta\psi\eta\psi$$

and therefore $\eta\psi\eta^{-1} = \psi^{-1}$. For all $x \in F/E$, we compute

$$\begin{aligned}
\psi(x)x^{-1} = h(x) &= (\eta h \eta^{-1})(x) \\
&= \eta(h(\eta^{-1}(x))) \\
&= \eta\big(\psi(\eta^{-1}(x))(\eta^{-1}(x))^{-1}\big) \\
&= (\eta\psi\eta^{-1})(x)x^{-1} \\
&= \psi^{-1}(x)x^{-1}
\end{aligned}$$

and therefore $\psi(x)^2 = 1$. The group F/E being a 3-group, it follows that ψ coincides with the trivial map and therefore η and β_E are equal. ∎

Lemma 297. *Let $N = \phi^{-1}(G_4)$ and let Δ denote the subgroup of* $\mathrm{Aut}(F/D_N)$ *consisting of all those maps inducing the identity on both F/E and E/D_N. Then Δ is contained in* $\mathrm{Inn}(F/D_N)$*.*

Proof. The group N belongs to $\mathcal{N}_3$, because G/G_4 is a κ-group of class 3. As a consequence of Lemma 292, the commutator map induces an injective homomorphism $\varphi : L/N \to \mathrm{Hom}(F/F_2, E/D_N)$. Combining Lemmas 276 and 294, we get that the orders of $\mathrm{Hom}(F/F_2, E/D_N)$ and L/N are the same and therefore φ is also surjective. It follows that, for each element f of $\mathrm{Hom}(F/F_2, E/D_N)$, there exists $l \in L$ such that f equals $xF_2 \mapsto [l, x]D_N$. Set $\overline{F} = F/D_N$ and use the bar notation for the subgroups of $\overline{F}$. We now prove that Δ is contained in $\mathrm{Inn}(\overline{F})$. Let $\delta \in \Delta$. Then, as a consequence of Lemma 45, there exists a homomorphism $f : \overline{F} \to \overline{E}$ whose kernel contains $\overline{E}$ and such that, for each $x \in \overline{F}$, one has $\delta(x) = f(x)x$. Fix such f. The group $\overline{E}$ being elementary abelian, the kernel of f contains $\overline{F_2}$ and therefore f factors as a homomorphism $F/F_2 \to \overline{E}$. As a result, there exists $l \in \overline{L}$ such that, for each $x \in \overline{F}$, one has $f(x) = [l, x]$ and thus

$\delta(x) = [l, x]x = lxl^{-1}$. In particular, δ is an inner automorphism of $\overline{F}$ and, the choice of δ being arbitrary, Δ is contained in $\mathrm{Inn}(\overline{F})$. ∎

Lemma 298. *Let N be an element of $\mathcal{N}_3$. Then $\beta(D_N) = D_N$.*

Proof. As a consequence of Lemma 162, the group N is $\langle\beta\rangle$-stable and therefore so is D_N. ∎

Lemma 299. *Let $N = \phi^{-1}(G_4)$. Let $\eta \in \mathrm{Aut}(F/D_N)$ be of order 2 and assume that η induces the inversion map on F/F_2. Assume moreover that β and η induce the same automorphism of F/E. Then there exists $\psi_N \in \mathrm{Inn}(F/D_N)$ such that, for all $x \in F$, one has $\beta(x) \equiv (\psi_N \eta \psi_N^{-1})(x) \bmod D_N$.*

Proof. Set $\overline{F} = F/D_N$ and use the bar notation for the subgroups of $\overline{F}$. Thanks to Lemma 298, the map β induces an automorphism of $\overline{F}$, which we denote by $\overline{\beta}$. Let Δ denote the subgroup of $\mathrm{Aut}(\overline{F})$ consisting of all those elements δ such that δ induces the identity on both $\overline{E}$ and $\overline{F}/\overline{E}$. Then, as a consequence of Lemmas 292 and 61, the automorphism $\eta^{-1}\overline{\beta}$ belongs to Δ and thus, thanks to Lemma 297, we have $\eta^{-1}\overline{\beta} \in \mathrm{Inn}(\overline{F})$. Applying the Schur-Zassenhaus theorem to $\mathrm{Inn}(\overline{F}) \rtimes \langle\eta\rangle$, we get that there exists $\overline{\psi} \in \mathrm{Inn}(\overline{F})$ such that $\overline{\beta} = \overline{\psi}\eta\overline{\psi}^{-1}$. This concludes the proof. ∎

Proposition 300. *Let α be an automorphism of order 2 of G that induces the inversion map on G/G_2. Then there exists $\gamma \in \mathrm{Inn}(F)$ such that $\alpha\phi = \phi(\gamma\beta\gamma^{-1})$.*

Proof. Thanks to proposition 289, there exists an automorphism ϵ of F/F_5 of order 2 such that $\alpha\phi_5 = \phi_5\epsilon$. As a consequence, the map ϵ induces the inversion map on F/F_2. Let now $M = \ker\phi$ and let $N = ME$. Thanks to Lemma 290, the group N belongs to $\mathcal{N}_3$ and $D_N \subseteq M$. One easily shows that F_5 is contained in D_N. It follows that ϵ induces an automorphism η of order 2 of F/D_N. Let η_L be the automorphism that η induces on F/L. Then, via the choice of a representative, Lemma 295 ensures that there exists an inner automorphism φ_N of F/D_N such that β and $\varphi_N\eta\varphi_N^{-1}$ induce the same automorphism of F/L. Fix such φ_N and define $\eta_1 = \varphi_N\eta\varphi_N^{-1}$. Since η has order 2, the order of η_1 is equal to 2. Lemma 296 yields that in fact η_1 and β are the same modulo E. At last, let ψ_N be as in Lemma 299 and define $\eta_2 = \psi_N\eta_1\psi_N^{-1}$. As a consequence of Lemma 299, the maps η_2 and β induce the same map on F/D_N. Via the choice of a representative, the inner automorphism $\psi_N\varphi_N$ of F/D_N lifts to an inner automorphism γ of F with the property that η and $\gamma\beta\gamma^{-1}$ induce the same automorphism on F/D_N. To conclude, let $\phi_N : F/D_N \to G$ be the map induced by ϕ. Since $\alpha\phi_5 = \phi_5\epsilon$, one gets $\alpha\phi_N = \phi_N\eta$ and therefore $\alpha\phi = \phi(\gamma\beta\gamma^{-1})$. ∎

Lemma 301. *There exists $M \in \mathcal{N}_4$ such that F/M is isomorphic to* $\mathrm{MC}(3)$. *Moreover, $\mathcal{N}_4$ is non-empty.*

Proof. The group MC(3) is a κ-group, by Lemma 249, and $\mathrm{MC}(3)_4$ has cardinality 3, thanks to Lemma 248(1). By Lemma 247, the class of MC(3) is 4 and moreover, thanks to Lemma 250, the group MC(3) possesses an automorphism that induces the inversion map on the quotient $\mathrm{MC}(3) / \mathrm{MC}(3)_2$. By the universal property of free groups, there exists $M \in \mathcal{N}_4$ such that F/M is isomorphic to MC(3). In particular, $\mathcal{N}_4$ is non-empty. ∎

Lemma 302. *For each $M \in \mathcal{N}_4$, one has $\beta(M) = M$.*

Proof. Let $M \in \mathcal{N}_4$. Without loss of generality $G = F/M$ and so $M = \ker\phi$. Let moreover α be an automorphism of G of order 2 that induces the inversion map on G/G_2. Then, thanks to Proposition 300, there exists $\gamma \in \mathrm{Inn}(F)$ such that $\alpha\phi = \phi(\gamma\beta\gamma^{-1})$. It follows that

$$\{1\} = \alpha(\phi(M)) = \phi(\gamma\beta\gamma^{-1})(M) = \phi\beta(M)$$

and therefore $\beta(M)$ is contained in $\ker\phi = M$. Since β induces an automorphism of each quotient F/F_k and since, for large enough k one has $F_k \subseteq M$, we have in fact that $\beta(M) = M$. ∎

$$\begin{tikzcd}
& F \arrow[d, dash, "-"'] \arrow[rrr, dashed] & & & G \arrow[d, dash, "-"] \\
& F_2 \arrow[d, dash, "+"'] \arrow[rrr, dashed] & & & G_2 \arrow[d, dash, "+"] \\
& L \arrow[dl, dash, "-"'] \arrow[dr, dash, "-"] \arrow[rrr, dashed] & & & G_3 \arrow[d, dash, "-"] \\
F_3 \arrow[dr, dash, "-"'] & & N \arrow[dl, dash, "-"] \arrow[dr, dash, "+"] \arrow[rr, dashed] & & G_4 \arrow[d, dash, "+"] \\
& E \arrow[dr, dash, "+"] & & M \arrow[dl, dash, "-"'] \arrow[r, dashed] & 1 \\
& & D_N & &
\end{tikzcd}$$

Lemma 303. *Let N be an element of $\mathcal{N}_3$ and write $\overline{F} = F/D_N$. Set moreover $\overline{N} = N/D_N$ and $\overline{E} = E/D_N$. Define $\overline{\beta}$ to be the map that is induced by β on $\overline{F}$ and set*

$$\overline{N}^+ = \{\overline{x} \in \overline{N} : \overline{\beta}(\overline{x}) = \overline{x}\} \quad \text{and} \quad \overline{N}^- = \{\overline{x} \in \overline{N} : \overline{\beta}(\overline{x}) = \overline{x}^{-1}\}.$$

Then $\overline{N}^+ = \overline{E}$ and $\overline{N}^-$ is the unique $\langle\overline{\beta}\rangle$-stable complement of $\overline{E}$ in $\overline{N}$.

Proof. As a consequence of Lemma 162, the group $\overline{N}$ is $\langle\overline{\beta}\rangle$-stable and, being central in $\overline{F}$, it is also abelian. Write now $B = \langle\beta\rangle$ and let $\sigma : B \to \{\pm 1\}$ be the isomorphism mapping β to -1. By Lemma 159, the group B acts on F/F_2 through σ and, by Lemma 162, the induced action of B on L/E is through σ. As a consequence, the induced action of B on both L/N and N/E is through σ. It follows from Lemmas 292 and 61 that β induces the identity map on $\overline{E}$ and so, thanks to Theorem 68, the subgroup $\overline{E}$ has a unique $\langle\overline{\beta}\rangle$-stable complement in $\overline{N}$, which coincides with $\overline{N}^-$. ∎

Lemma 304. *The map $\mathcal{N}_4 \to \mathcal{N}_3$ that is defined by $M \mapsto ME$ is an injection respecting the natural actions of* $\mathrm{Aut}(F)$*.*

Proof. The map $\mathcal{N}_4 \to \mathcal{N}_3$ is well-defined, thanks to Lemma 290, and it is clear that it respects the action of $\mathrm{Aut}(F)$. We prove injectivity. To this end, let M_1 and M_2 be elements of $\mathcal{N}_4$ such that $M_1E = M_2E$ and set $N = M_1E = M_2E$. Since M_1 and M_2 belong to $\mathcal{N}_4$, Lemma 302 yields $\beta(M_1) = M_1$ and $\beta(M_2) = M_2$. It follows then from Lemma 303 that both $\overline{M_1}$ and $\overline{M_2}$ are the unique $\langle\overline{\beta}\rangle$-stable complement of $\overline{E}$ and so $M_1 = M_2$. ∎

Lemma 305. *The map $\mathcal{N}_4 \to \mathcal{N}_3$ that is defined by $M \mapsto ME$ is a bijection respecting the natural actions of* $\mathrm{Aut}(F)$*.*

Proof. The map $\mathcal{N}_4 \to \mathcal{N}_3$ is well-defined, injective, and respects the action of $\mathrm{Aut}(F)$ thanks to Lemma 304. We prove surjectivity. To this end, let N be an element of $\mathcal{N}_3$. Write $\overline{F} = F/D_N$ and use the bar notation for the subgroups of $\overline{F}$. Let moreover $\overline{N}^-$ be as in Lemma 303. As a consequence of the definition of D_N, the subgroup $\overline{N}$ is central in $\overline{F}$ and so $\overline{N}^-$ is normal in $\overline{F}$. Let M be the unique normal subgroup of F containing D_N such that $\overline{M} = \overline{N}^-$. Then, as a consequence of Lemma 303, one has $N = ME$. Write $H = F/M$ and denote by π the canonical projection $F \to H$. We will prove that $M \in \mathcal{N}_4$. Thanks to the isomorphism theorems, the groups $\pi(N)$ and $\overline{E}$ are naturally isomorphic and, by Lemma 294, the group $\overline{E}$ has order 3. It follows that $|N : M| = 3$. Moreover, the group N being an element of $\mathcal{N}_3$, the quotient F/N has class 3 and so $M \subseteq \pi^{-1}(H_4) \subseteq N$. Only two cases can occur: either $N = \pi^{-1}(H_4)$ or $M = \pi^{-1}(H_4)$. Assume by contradiction that $M = \pi^{-1}(H_4)$ and so that H has class 3. Since $H/\pi(N)$ is isomorphic to F/N, Lemma 157 yields that $\pi(L) = H_3\pi(N)$ and, since N is central modulo D_N, the subgroup $\pi(N)$ is central in H. It follows that

$$\pi([F,L]) = [\pi(F), \pi(L)] = [H, H_3\pi(N)] = [H, H_3] = H_4 = \{1\}$$

and therefore $[F, L]$ is contained in M. We get then

$$E = [F,L]F_2^3 \subseteq [F,L]F_2^3[F_2,F_2] \subseteq [F,L]D_N \subseteq M,$$

and thus $N = ME = M$, which is a contradiction. We have proven that $N = \pi^{-1}(H_4)$, from which it follows in particular that $|H_4| = |N : M| = 3$ and so H has class 4. Moreover, H is a κ-group, because F/N is. To prove that M belongs to $\mathcal{N}_4$, we are left with proving that H has an automorphism of order 2 that induces the inversion map on H/H_2 and in fact such an automorphism can be gotten, for example, by inducing β to H. We have proven that $M \in \mathcal{N}_4$ and so, the choice of N being arbitrary, the map $\mathcal{N}_3 \to \mathcal{N}_4$ is surjective. ■

Corollary 306. *The set $\mathcal{N}_4$ has 3 elements and the action of $\mathrm{Aut}(F)$ on $\mathcal{N}_4$ is transitive.*

Proof. Combine Proposition 277 and Lemma 305. ■

We are now ready to prove Proposition 285. By Lemma 301, there exists an element M of $\mathcal{N}_4$ with the property that F/M is isomorphic to $\mathrm{MC}(3)$. As a consequence of Corollary 306, the natural action of $\mathrm{Aut}(F)$ on $\mathcal{N}_4$ is transitive and therefore G and $\mathrm{MC}(3)$ are isomorphic. The proof of Proposition 285 is complete.

9.7 Intensity

In Section 9.5 we have proven Corollary 284, which asserts that finite 3-groups of intensity larger than 1 have class at most 4. We will prove in this section that the bound is best possible by showing that the group $\mathrm{MC}(3)$, introduced at the beginning of this chapter and whose structure we investigated in Section 9.2, has intensity 2. Thanks to results coming from the previous sections, we will, at the end of the current section, finally be able to give the proof of Theorem 231.

Proposition 307. *The group $\mathrm{MC}(3)$ has intensity 2.*

We will devote a big part of the present section to the proof of Proposition 307. To this end, let the following assumptions hold until the end of Section 9.7. Set $G = \mathrm{MC}(3)$ and denote by $(G_i)_{i\geq 1}$ its lower central series. For all $i \in \mathbb{Z}_{\geq 1}$, write $\mathrm{wt}_G(i) = w_i$. By Lemma 247, the group G has class 4 and order 729. Moreover, thanks to Lemmas 248(1) and 249, the group G is a κ-group satisfying

$(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. Let α be as in Lemma 250 and set $A = \langle \alpha \rangle$. In concordance with the notation from Section 2.2, we define $G^+ = \{x \in G \ : \ \alpha(x) = x\}$ and $G^- = \{x \in G \ : \ \alpha(x) = x^{-1}\}$. Moreover, for each subgroup H of G, we denote $H^+ = H \cap G^+$ and $H^- = H \cap G^-$.

Lemma 308. *Let H be a subgroup of G_2 and let g be an element of G. Then the following hold.*

1. *The group G_2 normalizes H.*
2. *If both H and gHg^{-1} are A-stable, then $gHg^{-1} = H$.*

Proof. The group G_2 is abelian, by Corollary 282, and in particular it normalizes each of its subgroups. As a consequence of Lemma 85(1), the subgroup G^+ is contained in G_2 and we conclude combining (1) with Lemma 81. ∎

Lemma 309. *Let H be a subgroup of G that contains G_4. Then there exists $g \in G$ such that gHg^{-1} is A-stable.*

Proof. We denote by α_4 the automorphism of G/G_4 that is induced by α. By Proposition 142, the automorphism α_4 is intense so, by Lemma 93, there exists $g \in G$ such that gHg^{-1}/G_4 is $\langle \alpha_4 \rangle$-stable. It follows from the definition of α_4 that gHg^{-1} is A-stable. ∎

We recall that a positive integer j is a jump of a subgroup H of G if and only if $H \cap G_j \neq H \cap G_{j+1}$. For the theory of jumps we refer to Section 2.3.

Lemma 310. *Let H be a subgroup of G such that $H \cap G_4 = \{1\}$. Assume moreover that H is not contained in G_2. Then there exists $x \in G \setminus G_2$ such that $H = \langle x \rangle$.*

Proof. The subgroup H is different from G, since $H \cap G_4 = \{1\}$, and it is therefore contained in a maximal subgroup C of G. Moreover, H not being contained in G_2, we have $\mathrm{wt}_H^G(1) = 1$. We first show that H is abelian. The subgroup $[H, H]$ is contained in $[C, C]$ and $[C, C] = [C, G_2]$, as a consequence of Lemma 28. Thanks to Lemma 20, the subgroup $[H, H]$ is contained in G_3. By Lemma 244, the centre of G is equal to G_4 and so, by Lemma 190, the map $\gamma : G/G_2 \times G_3/G_4 \to G_4$ that is induced from the commutator map is non-degenerate. Since $C = HG_2$, we get $[C, [H, H]] = [H, [H, H]] \subseteq H \cap G_4$ and so, since $H \cap G_4 = \{1\}$, the subgroup $[H, H]$ is contained in $\mathrm{Z}(C)$. It follows that $[H, H]$ is contained in $\mathrm{Z}(C) \cap [C, C] \cap H$ and so, thanks to Corollary 282 and Lemma 245, the commutator subgroup of H is trivial. The group H being abelian, it follows, from the non-degeneracy of γ, that $\mathrm{wt}_H^G(3) \leq 1$ and, from Lemma 241, that $\mathrm{wt}_H^G(2) = 0$. Let now x be an element of $H \setminus G_2$. Then 1 is a jump of $\langle x \rangle$ in G and, the group G being a κ-group, it follows that $x^3 \in G_3 \setminus G_4$. As a consequence of Lemma 84, we get

$$|\langle x \rangle| \geq 3^{\mathrm{wt}_{\langle x \rangle}^G(1)} 3^{\mathrm{wt}_{\langle x \rangle}^G(3)} \geq 9 \geq 3^{\mathrm{wt}_H^G(1)} 3^{\mathrm{wt}_H^G(3)} = \prod_{i=1}^{4} 3^{\mathrm{wt}_H^G(i)} = |H|$$

and therefore H is cyclic generated by x. ∎

Lemma 311. *Let H be a subgroup of G such that $H \cap G_4 = \{1\}$. Assume that H is not contained in G_2. Then H and $\alpha(H)$ are conjugate in G.*

Proof. By Lemma 310, the group H is cyclic. We define $T = H \oplus G_4$ so, by Lemma 309, there exists $g \in G$ such that gTg^{-1} is A-stable. We fix such g and denote $T' = gTg^{-1}$ and $H' = gHg^{-1}$. The subgroup G_4 being characteristic, it follows that $H' \oplus G_4 = T' = \alpha(H') \oplus G_4$. Let $\mathcal{C}$ denote the collection of complements of G_4 in T'. By Lemma 114, the elements of $\mathcal{C}$ are in bijection with the elements of $\mathrm{Hom}(H', G_4)$, which is naturally isomorphic to $\mathrm{Hom}(H'/\Phi(H'), G_4)$, because G_4 has order 3. The group H' is cyclic, so $\Phi(H') = (H')^3$ and, the group G being a κ-group, one gets $\Phi(H') = H' \cap G_2$. By Lemma 36, the quotient G/G_2 is elementary abelian and the restriction map $\mathrm{Hom}(G/G_2, G_4) \to \mathrm{Hom}(H'G_2/G_2, G_4)$ is thus surjective. Moreover, by Lemma 244, the subgroup G_4 coincides with $\mathrm{Z}(G)$ so, as a consequence of Lemma 190, the map $G_3/G_4 \to \mathrm{Hom}(G/G_2, G_4)$, defined by $xG_4 \mapsto (tG_2 \mapsto [x,t])$, is an isomorphism. It follows from Lemma 114 that, for each $K \in \mathcal{C}$, there exists $x \in G$ such that $K = \{[x,t]t = xtx^{-1} \mid t \in H'\}$. As a consequence, there is $x \in G$ such that $\alpha(H') = xH'x^{-1}$ and so, since $H' = gHg^{-1}$, also $\alpha(H)$ and H are conjugate in G. ∎

Lemma 312. *Let H be a subgroup of G_3 such that $H \cap G_4 = \{1\}$. Then H and $\alpha(H)$ are conjugate in G.*

Proof. Let $T = HG_4$. The group G_3 is elementary abelian, as a consequence of Corollary 282, and therefore so is $T = H \oplus G_4$. Let $g \in G$ be such that gTg^{-1} is A-stable, as in Lemma 309, and set $T' = gTg^{-1}$ and $H' = gHg^{-1}$. Let moreover $\mathcal{C}$ be the set of complements of G_4 in T' and note that, G_4 being characteristic, both H' and $\alpha(H')$ belong to $\mathcal{C}$. Thanks to Lemma 114, the elements of $\mathcal{C}$ are in bijection with the elements of $\mathrm{Hom}(H', G_4)$. By the isomorphism theorems, H' is isomorphic to $(H'G_4)/G_4$ and the restriction map induces a surjection $\mathrm{Hom}(G_3/G_4, G_4) \to \mathrm{Hom}(H', G_4)$. By Lemma 244, the subgroup G_4 coincides with $\mathrm{Z}(G)$ so, as a consequence of Lemma 190, the map $G/G_2 \to \mathrm{Hom}(G_3/G_4, G_4)$, defined by $xG_4 \mapsto (tG_2 \mapsto [x,t])$ is an isomorphism. It follows from Lemma 114 that each element of $\mathcal{C}$ is of the form $\{[x,t]t = xtx^{-1} \mid t \in H'\} = xH'x^{-1}$, for some $x \in G$. In particular, $\alpha(H')$ and H' are conjugate in G. The groups H' and H being conjugate in G, it follows that H and $\alpha(H)$ are conjugate, too. ∎

Lemma 313. *Let H be a subgroup of G such that $H \oplus G_4 = G_2$. Then H has an A-stable conjugate in G.*

Proof. We define X to be the collection of all subgroups K of G for which $G_2 = K \oplus G_4$. The group G_2 is elementary abelian, by Corollary 282, so the set X

is non-empty. Moreover, as a consequence of Lemma 114, the cardinality of X is equal to the cardinality of $\mathrm{Hom}(H, G_4)$, which is 27. We define $X^+ = \{K \in X : \alpha(K) = K\}$ and we will show, with a counting argument, that H is conjugate to an element of X^+. Let $K \in X^+$. Thanks to Corollary 76, we can write $K = K^+ \oplus K^-$ and, as a consequence of Lemma 85, the subgroup K^- is equal to G_2^-. Again by Lemma 85, the subgroup G_2^+ has order 9 and it contains G_4. It follows that $|X^+|$ is equal to the number of 1-dimensional subspaces of G_2^+ that are different from G_4, i.e. $|X^+| = 3$. By Lemma 308(1), the group G_2 normalizes K, but in fact $G_2 = \mathrm{N}_G(K)$, as a consequence of Lemma 190. It follows that the orbit of K in X has size $|G : G_2| = 9$ so, thanks to Lemma 308(2), the element K is the only element of X^+ belonging to its orbit under G/G_2. The number $|X|/|X^+|$ being equal to 9, it follows that each orbit of the action of G/G_2 on X has a representative in X^+. The same holds for the orbit of H. ■

Lemma 314. *Let H be a subgroup of G such that $H \oplus G_3 = G_2$. Then H has an A-stable conjugate in G.*

Proof. Let X be the collection of all complements of G_3 in G_2. The group G_2 is elementary abelian, by Corollary 282, and so, from Lemma 114 it follows that the cardinality of X is equal to $|\mathrm{Hom}(H, G_3)| = 27$. We define $X^+ = \{K \in X : \alpha(K) = K\}$. As a consequence of Lemma 85, if K is an element of X^+, then $K = K^+$. The elements of X^+ are thus exactly the one-dimensional subspaces of G_2^+ that are different from G_4 and so we have that $|X^+| = 3$. Fix $K \in X^+$. Then, as a consequence of Lemma 241, the commutator map induces an isomorphism $G/G_2 \otimes K \to G_3/G_4$. It follows that $\mathrm{N}_G(K)$ is contained in G_2 so, thanks to Lemma 308(1), one has $\mathrm{N}_G(K) = G_2$. Lemma 308(2) yields that K is the only element of X^+ belonging to its orbit under the action of G/G_2 on X. The number $|X|/|X^+|$ being equal to 9, it follows that each orbit of the action of G/G_2 on X has a representative in X^+ so, in particular, H has an A-stable conjugate in G. ■

Lemma 315. *Let $x \in G \setminus G_2$ and let $a \in G_2^+ \setminus G_3$. Then $[x, a]$ does not belong to G^-.*

Proof. Let $C = \mathrm{C}_G([x, a])$ and $D = \langle x, G_2 \rangle$. By Lemma 241, the element $[x, a]$ belongs to $G_3 \setminus G_4$ so, as a consequence of Lemmas 244 and 190, the index of C in G is equal to 3. In particular, both C and D are maximal subgroups of G. Assume now by contradiction that $[x, a] \in G^-$. Since x belongs to $G \setminus G_2$, there exists $\gamma \in G_2$ such that $\alpha(x) = x^{-1}\gamma$ and so we have

$$[x, a]^{-1} = \alpha([x, a]) = [x^{-1}\gamma, a].$$

The group G_2 is elementary abelian, by Corollary 282, and therefore, applying Lemma 18(2), one gets

$$\begin{aligned}[x,a]^{-1} &= [x^{-1}\gamma, a] \\ &= x^{-1}[\gamma, a]x[x^{-1}, a] \\ &= [x^{-1}, a] \\ &= x^{-1}axa^{-1} \\ &= x^{-1}[a,x]x \\ &= [x^{-1}, [a,x]][a,x] \\ &= [x^{-1}, [x,a]^{-1}][x,a]^{-1}.\end{aligned}$$

As a result, the element $[x^{-1}, [x,a]^{-1}]$ is trivial, and so $x \in C$. It follows that $C = D$, and thus $[x,a]$ belongs to $[C,C] \cap \mathrm{Z}(C)$. Lemma 245 yields $[x,a] \in G_4$. Contradiction. ∎

Lemma 316. *Let $x \in G_2 \setminus G_3$ and $y \in G_3 \setminus G_4$. Define $H = \langle x, y\rangle$. Then H has an A-stable conjugate.*

Proof. The group G_2 is elementary abelian, by Corollary 282, and therefore $H = \langle x\rangle \oplus \langle y\rangle$. Let X be the set consisting of all subgroups of G_2 of the form $\langle u\rangle \oplus \langle v\rangle$, where $u \in G_2 \setminus G_3$ and $v \in G_3 \setminus G_4$. The cardinality of X is then equal to 108. We define $X^+ = \{K \in X \ : \alpha(K) = K\}$ and we fix $K \in X^+$. By Corollary 76, the subgroup K decomposes as $K = K^+ \oplus K^-$ and, as a consequence of Lemma 85, there exists $a \in G_2^+$ such that $K = \langle a\rangle \oplus K^-$. Fix such a. Again thanks to Lemma 85, we get that $|X^+| = 12$. We want to count the conjugates of K. By Lemma 308(1), the subgroup G_2 is contained in $\mathrm{N}_G(K)$ and, if $x \in \mathrm{N}_G(K)$, then $[x,a] \in K \cap G_3$. The intersection $K \cap G_3$ being equal to K^-, it follows from Lemma 315 that $\mathrm{N}_G(K) = G_2$. As a consequence of Lemma 308(2), the element K is the only element of X^+ belonging to its orbit under G/G_2 so, from the equality $|X|/|X^+| = 9$, we can deduce that each orbit of the action of G/G_2 on X has a representative in X^+. The same holds for the orbit of H. ∎

Lemma 317. *The automorphism α is intense. Moreover, the intensity of G is equal to* 2.

Proof. Let H be a subgroup of G. If H contains G_4, then, by Lemma 309, there exists a conjugate of H that is A-stable. Assume that $H \cap G_4 = \{1\}$. If H is not contained in G_2, then H is conjugate to $\alpha(H)$, thanks to Lemma 311. Assume that H is contained in G_2. If 2 is not a jump of H in G, then, by Lemma 312, the subgroups H and $\alpha(H)$ are conjugate in G. We suppose that 2 is a jump of H in G. By Corollary 282, the group G_2 is elementary abelian and so H is a subspace of G_2, not contained in G_3, that trivially intersects G_4. The combination of Lemmas

313, 314, and 316 guarantees that H has an A-stable conjugate in G. The choice of H being arbitrary, it follows from Lemma 93 that α is intense. The intensity of G is at least 2, because α has order 2, but in fact $\text{int}(G) = 2$, as a consequence of Theorem 125(1). ■

We remark that, thanks to Lemma 317, the proof of Proposition 307 is complete. Moreover, we are now also able to prove Theorem 231. The implication (2) $\Rightarrow$ (1) is clear and the implication (3) $\Rightarrow$ (2) is given by the combination of Proposition 307 and Lemma 247. We now prove (1) $\Rightarrow$ (2). To this end, let Q be a finite 3-group of class at least 4 with $\text{int}(Q) > 1$. Because of Corollary 284, the class of Q is equal to 4 so, as a consequence of Theorem 164, the order of Q is equal to 729. The intensity of Q is equal to 2, thanks to Theorem 125(1). We have concluded the proof of (1) $\Rightarrow$ (2) and, to finish the proof of Theorem 231, we will next prove (2) $\Rightarrow$ (3). Let Q be a finite 3-group of class 4 and intensity 2. Then, by Lemma 239, the group Q is a κ-group and, as a consequence of Proposition 134, it possesses an automorphism of order 2 that induces the inversion map on Q/Q_2. By Theorem 164, the order of Q_4 is 3. Proposition 285 yields that Q is isomorphic to MC(3). The proof of Theorem 231 is now complete.