Intense automorphisms of finite groups

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Chapter 8

A disparity between the primes

The main result of Chapter 8 is Theorem 218. We recall that, if G is a finite p-group and i is a positive integer, then $\mathrm{wt}_G(i) = \log_p |G_i : G_{i+1}|$, where $(G_i)_{i\geq 1}$ denotes the lower central series of G.

Theorem 218. *Let $p > 3$ be a prime number and let G be a finite p-group with $\mathrm{int}(G) > 1$. Let c denote the class of G and assume that $c \geq 3$. If i is a positive integer such that $\mathrm{wt}_G(i)\,\mathrm{wt}_G(i+1) = 1$, then $i = c - 1$.*

An equivalent way of formulating Theorem 218 is that of saying that, if G satisfies the assumptions of Theorem 218 and we writw $w_i = \mathrm{wt}_G(i)$, then

$$(w_i)_{i\geq 1} = (2, 1, 2, 1, \ldots, 2, 1, f, 0, 0, \ldots) \quad \text{where} \quad f \in \{0, 1, 2\}.$$

The restriction to primes greater than 3 in Theorem 218 is superfluous; it is however not worth the effort proving the result in general, since, as we will see in the next chapter, 3-groups of intensity greater than 1 have class at most 4 and we know from Theorems 125(2) and 164 that Theorem 218 is valid when c is 3 or 4.

8.1 Regularity

In Section 8.1 we make a distinction, for the first time, among the odd primes: namely we separate the cases $p = 3$ and $p > 3$. The main result of this section is Proposition 219. We refer to Section 1.5, for an overview of regular p-groups.

Proposition 219. *Let p be a prime number and let G be a finite p-group. Assume that $\mathrm{int}(G) > 1$. Then the following are equivalent.*

1. *The group G is not regular.*
2. *The class of G is larger than 2 and $p = 3$.*

We will give the proof of Proposition 219 at the end of Section 8.1.

Lemma 220. *Let $p > 3$ be a prime number and let G be a finite p-group. Assume that* $\mathrm{int}(G) > 1$*. Then the following hold.*

1. *The group G is regular.*

2. *If the class of G is at least* 4*, then $G_3 = \rho(G)$.*

Proof. If the class of G is at most 4, the group G is regular by Lemma 50. We assume that G has class at least 4. It follows from Lemma 101 that $\mathrm{int}(G/G_5)$ is larger than 1, and so, thanks to Lemma 192, the index $|G : G_3|$ is equal to p^3. From Theorem 189, we get that $G^p = G_3$, and therefore $|G : G^p| < p^p$. The group G is regular, by Lemma 53, so, thanks to Lemma 52, the subgroup G^p coincides with $\rho(G)$. The proof is now complete. ∎

We would like to stress that, for $p > 3$, Proposition 220(2) is a stronger version of Theorem 189. In fact, not only $G_3 = G^p = \langle\rho(G)\rangle$ but G_3 coincides with the set of p-th powers of elements of G.

Lemma 221. *Let G be a finite* 3*-group with* $\mathrm{int}(G) > 1$*. Then G is regular if and only if G has class at most* 2*.*

Proof. If G has class at most 2, then G is regular by Lemma 50. Assume by contradiction that G is regular of class at least 3. As a consequence of Theorem 125(2), the group G is 2-generated, and so, by Lemma 55, the subgroup G_2 is cyclic. Proposition 123 yields that $G_3 = \{1\}$. Contradiction. ∎

We now give the proof of Proposition 219. To this end, let p be a prime number and let G be a finite p-group with $\mathrm{int}(G) > 1$. The intensity of G being greater than 1, it follows from Proposition 95 that p is odd. The implication (2) $\Rightarrow$ (1) is given by Lemma 221. We prove (1) $\Rightarrow$ (2). Assume that G is not regular. Then Lemma 220 yields $p = 3$. Moreover, the class of G is larger than 2, as a consequence of Lemma 50. The proof of Proposition 219 is complete.

8.2 Rank

The *rank* of a finite group G is the smallest integer r with the property that each subgroup of G can be generated by r elements. We denote the rank of G by $\mathrm{rk}(G)$. We will prove the following.

Proposition 222. *Let $p > 3$ be a prime number and let G be a finite p-group of class at least* 4*. If* $\mathrm{int}(G) > 1$*, then* $\mathrm{rk}(G) = 3$*.*

We recall that, if G is a group and n is a positive integer, then the subgroup $\mu_n(G)$ is defined to be $\langle x \in G : x^n = 1 \rangle$.

Lemma 223. *Let p be a prime number and let G be a non-trivial finite p-group. Then* $\mathrm{rk}(G) \leq \log_p |\mu_p(G)|$.

Proof. This is is Corollary 2 from [Laf73]. ∎

Lemma 224. *Let $p > 3$ be a prime number and let G be a finite p-group of class at least* 4. *If* $\mathrm{int}(G) > 1$, *then* $\mathrm{rk}(G) \leq 3$.

Proof. Assume that $\mathrm{int}(G) > 1$. By Theorem 189, the subgroup G^p is equal to G_3 so, by Lemma 52(3), the order of $\mu_p(G)$ is equal to $|G : G^p| = |G : G_3|$. As a consequence of Theorem 164, the index $|G : G_3|$ is equal to p^3, and thus Lemma 223 yields $\mathrm{rk}(G) \leq \log_p |G : G_3| = 3$. ∎

We can now finally prove Proposition 222. In order to do this, let p be a prime number and let G be a finite p-group of class at least 4, with $\mathrm{int}(G) > 1$. Thanks to Lemma 224, it suffices to present a subgroup of G whose minimum number of generators is at least 3. The group G/G_5 has class 4 and, thanks to Lemma 101, it has intensity greater than 1. As a consequence of Theorem 164, the index $|G_2 : G_4|$ is equal to p^3 and, thanks to Proposition 123, the quotient G_2/G_4 is elementary abelian. It follows that $\Phi(G_2) \subseteq G_4$ and the minimum number of generators for G_2 is at least $\log_p(|G_2 : G_4|) = 3$. Proposition 222 is now proven.

We would like to remark that, if $p = 3$, then Proposition 222 is not valid. We will see indeed in the next chapter that finite 3-groups of class 4 and intensity larger than 1 have a commutator subgroup that is elementary abelian of order p^4, so the rank of such groups is at least 4.

8.3 A sharper bound on the width

The aim of Section 8.3 is to give the proof of Proposition 225, which is the same as Theorem 218.

Proposition 225. *Let $p > 3$ be a prime number and let G be a finite p-group with* $\mathrm{int}(G) > 1$. *Let c denote the class of G and assume that that $c \geq 3$. If i is a positive integer such that* $\mathrm{wt}_G(i)\,\mathrm{wt}_G(i+1) = 1$, *then* $i = c - 1$.

We list here a number of assumptions that will hold until the end of Section 8.3. Let $p > 3$ be a prime number and let G be a finite p-group with lower central series $(G_i)_{i \geq 1}$. Let c denote the class of G and, for each positive integer i, write $w_i = \mathrm{wt}_G(i)$. Assume that $\mathrm{int}(G) > 1$. Then, as a consequence of Proposition 95, the prime p is odd and G is non-trivial. Let α be an intense automorphism of G of order 2 and write $A = \langle \alpha \rangle$.

Lemma 226. *Let $i \in \mathbb{Z}_{\geq 1}$ be such that $w_i w_{i+1} = 1$. Then $i > 1$.*

Proof. The subgroup G_{i+1} being non-trivial, Lemma 31 yields $i > 1$. ∎

Lemma 227. *Assume that $w_2 w_3 = 1$. Then $c = 3$.*

Proof. The widths w_2 and w_3 are both equal to 1, so Theorem 164 yields $c = 3$. ∎

Lemma 228. *Let $i \in \mathbb{Z}_{\geq 1}$ be such that $w_i w_{i+1} = 1$. If $c > 3$, then $i \geq 4$.*

Proof. Assume that the class of G is at least 4. Then, by Lemma 101, the group G/G_5 has intensity greater than 1. Theorem 164 yields $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$ and therefore $i \geq 4$. ∎

Lemma 229. *Let $i \in \mathbb{Z}_{\geq 1}$ be minimal such that $w_i w_{i+1} = 1$. If $c > 3$, then i is even and $w_{i-1} = 2$.*

Proof. Assume $c > 3$. Then Lemma 228 yields $i-1 > 1$. The width w_{i-1} is at most 2, as a consequence of Theorem 165, and thus, i being minimal with the property that $w_i w_{i+1} = 1$, it follows that $w_{i-1} = 2$. Another consequence of the minimality of i is that i is even. Indeed, thanks to Theorem 165 and the minimality of i, whenever $j < i$, the product $w_j w_{j+1}$ is equal to 2. Moreover, by Theorem 125(2), we have that $w_1 = 2$, so i is even. ∎

Lemma 230. *Let $i \in \mathbb{Z}_{\geq 1}$ be minimal such that $w_i w_{i+1} = 1$. Assume that $c > 3$ and that $w_{i+2} = 1$. Then G_{i-1}/G_{i+3} has exponent p.*

Proof. We write $\overline{G} = G/G_{i+3}$ and we will use the bar notation for the subgroups of $\overline{G}$. The intensity of $\overline{G}$ is larger than 1 thanks to Lemma 101. The group $[G_{i-1}, G_{i-1}]$ is contained in G_{2i-2}, by Lemma 20, and, by Lemma 228, the index i is larger than 3. It follows that $[G_{i-1}, G_{i-1}] \subseteq G_{2i-2} \subseteq G_{i+2}$, and therefore $\overline{G}_{i-1}$ has class at most 2 and $[\overline{G}_{i-1}, \overline{G}_{i-1}]^p = \{1\}$. As a consequence of Corollary 48, the p-power map is an endomorphism of $\overline{G}_{i-1}$. Now, thanks to Proposition 123, the subgroup $\overline{G}_{i-1}^p$ is contained in $\overline{G}_{i+1}$ and, from Lemmas 52(3) and 229, it follows that $|\mu_p(\overline{G}_{i-1})| = |\overline{G}_{i-1} : \overline{G}_{i-1}^p| \geq |\overline{G}_{i-1} : \overline{G}_{i+1}| = p^3$. Also the order of $\overline{G}_i$ is equal to p^3 and, as a consequence of Proposition 166, the subgroup $\overline{G}_i$ is contained in $\mu_p(\overline{G}_{i-1})$. Hence the p-power map factors thus as a homomorphism $\overline{G}_{i-1}/\overline{G}_i \to \overline{G}_{i+1}$. By Lemma 229, the index i is even, and so, by Proposition 134, the automorphism of $\overline{G}_{i-1}/\overline{G}_i$ that is induced by α is equal to the inversion map. It follows from Lemma 63 that α restricts to the inversion map on $\overline{G}_{i-1}^p$. Moreover, again by Proposition 134, the action of A on $\overline{G}_{i+2}$ is trivial. It follows from Lemma 66 that $\overline{G}_{i-1}^p \cap \overline{G}_{i+2} = \{1\}$. The subgroup $\overline{G}_{i-1}^p$ is clearly characteristic in $\overline{G}$, while the subgroup $\overline{G}_{i+2}$ is equal to the centre of $\overline{G}$, by Lemma 171. Lemma 29 yields $\overline{G}_{i-1}^p = \{1\}$. ∎

We conclude Section 8.3 with the proof of Proposition 225. By Lemma 226, the integer i is larger than 1. If $i = 2$, then Lemma 227 yields $c = 3 = i + 1$. We assume that i is greater than 2, so $c > 3$, and, without loss of generality, that i is minimal with the property that $w_i w_{i+1} = 1$. In particular, the subgroup G_{i+1} is non-trivial. If $G_{i+2} = \{1\}$, then the class of G is equal to $i + 1$, and so $i = c - 1$. Assume now by contradiction that G_{i+2} is non-trivial. By Lemma 35, there exists a normal subgroup N of G that is contained in G_{i+2} with index p. We fix N and denote the quotient G/N by $\overline{G}$. Lemma 101 yields $\mathrm{int}(\overline{G}) > 1$. By Lemma 229, the width $\mathrm{wt}_{\overline{G}}(i-1) = w_{i-1}$ is equal to 2 so, by Lemma 84, the order of $\overline{G}_{i-1}$ is equal to p^5. By Lemma 228, the index i is at least 4, and thus, as a consequence of Lemma 20, the subgroup $[\overline{G}_{i-1}, \overline{G}_i]$ is contained in $\overline{G}_{i+3} = \{1\}$. It follows that $\overline{G}_{i-1}$ and $\overline{G}_i$ centralize each other. Let now M be a maximal subgroup of $\overline{G}_{i-1}$ that contains $\overline{G}_i$. The index $|M : \overline{G}_i|$ is equal to p, because $w_{i-1} = 2$, and so Lemma 28 gives $[M, M] = [M, \overline{G}_i] = \{1\}$. Moreover, the order of M is equal to p^4 and M has exponent p, because of Lemma 230. In particular, M is a 4-dimensional vector space over $\mathbb{F}_p$. Contradiction to proposition 222.