Intense automorphisms of finite groups

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Chapter 7

Higher nilpotency classes

The aim of this chapter is to gain better control of the p-power map on finite p-groups of intensity greater than 1. We remind the reader that, if n is a positive integer and G is a group, then G^n is equal to the subgroup of G that is generated by the n-th powers of the elements of G, i.e. $G^n = \langle x^n : x \in G \rangle$ (see List of Symbols). One of the most important results we achieve in Chapter 7 is the following.

Theorem 189. *Let p be a prime number and let G be a finite p-group. Assume that the class of G is at least 4 and that $\mathrm{int}(G) > 1$. Then $G^p = G_3$.*

We remark that, whenever p is larger than 3, Theorem 189 cannot be extended to groups of class 3. There are indeed examples, for $p > 3$, of finite p-groups of class 3, intensity greater than 1, and exponent p. We deal extensively with the case of 3-groups in Chapter 9.

7.1 Groups of class 4

In Section 7.1 we start the preparation for the proof of Theorem 189, which will be given in Section 7.3. This very section will thus just consist of structural lemmas, which will be later of use.

The following assumptions will be valid until the end of Section 7.1. Let p be a prime number. Let moreover G be a finite p-group of class 4 and denote by $(G_i)_{i\geq 1}$ the lower central series of G. For $i \in \{1,2,3,4\}$, we define w_i to be $\mathrm{wt}_G(i) = \log_p |G_i : G_{i+1}|$ (see Section 2.3).

Lemma 190. *Assume that $(w_1, w_2, w_3, w_4) = (2,1,2,1)$ and $\mathrm{Z}(G) = G_4$. Then the commutator map induces a non-degenerate map $G/G_2 \times G_3/G_4 \to G_4$.*

Proof. The commutator map induces a bilinear map $\gamma : G/G_2 \times G_3/G_4 \to G_4$ whose image generates G_4, by Lemma 24. The subgroup G_4 has dimension 1 as an $\mathbb{F}_p$-vector space, while $w_1 = w_3 = 2$. The quotient G/G_2 has exponent p, thanks to Lemma 36, and the exponent of G_3/G_4 is equal to p, as a consequence of Lemma 25 (the property of being elementary abelian is preserved by surjective homomorphisms and tensor products). Moreover, the centre of G being G_4, the right kernel of γ is trivial. It follows from Lemma 2 that the left kernel of γ has also dimension 0, so γ is non-degenerate. ∎

Lemma 191. *Assume that* $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. *Let* C *be a maximal subgroup of* G. *Then* C *contains* G_2 *and* $|G : C| = |C : G_2| = p$.

Proof. The subgroup C being maximal, it has index p in G and it contains the Frattini subgroup of G. Since G_2 is contained in $\Phi(G)$ and $|G : C| = p$, we get $|C : G_2| = p$. ∎

Lemma 192. *Assume that* $\mathrm{int}(G) > 1$. *Then* $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$ *and* G *has order* p^6. *Moreover, one has* $\mathrm{Z}(G) = G_4$.

Proof. The quadruple (w_1, w_2, w_3, w_4) is equal to $(2, 1, 2, 1)$ by Theorem 164 and the order of G is equal to p^6, as a consequence of Lemma 84. The centre of G is equal to G_4 thanks to Lemma 171. ∎

Lemma 193. *Let* C *be a maximal subgroup of* G *and assume that* $\mathrm{int}(G) > 1$. *Then* $G_4 \subseteq \mathrm{Z}(C) \subseteq G_3$ *and* $|G_3 : \mathrm{Z}(C)| = |\,\mathrm{Z}(C) : G_4| = p$.

Proof. We assume that $\mathrm{int}(G) > 1$. By Lemma 192, the subgroups G_4 and $\mathrm{Z}(G)$ are the same and $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. Moreover, by Lemma 190, the commutator map induces a non-degenerate map $G/G_2 \times G_3/G_4 \to G_4$. It follows from Lemma 2 that $G_3 \cap \mathrm{Z}(C)$ has index p in G_3. Now the subgroup $\mathrm{Z}(C)$ is normal in G, because it is characteristic in the normal subgroup C, and therefore Proposition 166 yields $\mathrm{Z}(C) \subseteq G_3$. We get that $|G_3 : \mathrm{Z}(C)| = |\,\mathrm{Z}(C) : G_4| = p$. ∎

Lemma 194. *Let* M *be a normal subgroup of* G *that contains* G_4 *with index* p. *If* $\mathrm{int}(G) > 1$, *then* $\mathrm{C}_G(M)$ *is a maximal subgroup of* G.

Proof. We assume that $\mathrm{int}(G) > 1$. Then, by Lemma 192, we have that $\mathrm{Z}(G) = G_4$ and $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. Let now M be a normal subgroup of G that contains G_4 with index p. As a consequence of Proposition 166, the subgroup M is contained in G_3. The commutator map from Lemma 190 being non-degenerate, it follows from Lemma 2 that $\mathrm{C}_G(M)$ is maximal in G. ∎

Lemma 195. *Assume that* $\mathrm{int}(G) > 1$. *Let* $\mathcal{M}$ *be the collection of maximal subgroups of* G *and let* $\mathcal{N}$ *be the collection of normal subgroups of* G *that contain* G_4 *with index* p. *Let* $f : \mathcal{M} \to \mathcal{N}$ *be defined by* $N \mapsto \mathrm{Z}(N)$. *Then* f *is a bijection with inverse* $f^{-1} : M \mapsto \mathrm{C}_G(M)$.

Proof. The map f is well-defined, as a consequence of Lemma 193. Also the map $g : \mathcal{N} \to \mathcal{M}$, sending M to $\mathrm{C}_G(M)$, is well-defined thanks to Lemma 194. It is now easy to show that f and g are inverses to each other. ∎

7.2 Class 4 and p-th powers

The goal of this section is to prove some technical lemmas regarding the p-th powering on finite p-groups of intensity greater than 1. We will use such lemmas in Section 7.3, where Theorem 189 is proven.

Through the whole of Section 7.2, the following assumptions will be valid. Let p be a prime number. Let moreover G be a finite p-group of class 4 and denote by $(G_i)_{i\geq 1}$ the lower central series of G. Let $\rho : G \to G$ be defined by $x \mapsto x^p$; this is the first time we introduce this notation, which can be also found in the List of Symbols. For each $i \in \{1, 2, 3, 4\}$, we will write $w_i = \mathrm{wt}_G(i)$ for the i-th width of G (see Section 2.3). Assume that $\mathrm{int}(G) > 1$. It follows from Proposition 95 that G is non-trivial and p is odd.

Lemma 196. *The map ρ induces a map $\overline{\rho} : G/G_2 \to G_3/G_4$.*

Proof. For each index $i \in \mathbb{Z}_{\geq 1}$, the subset $\rho(G_i)$ is contained in G_{i+2}, as a consequence of Proposition 123. In particular, $\rho(G_2)$ is contained in G_4. By Lemma 192, the subgroup $\mathrm{Z}(G)$ is equal to G_4 and $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$. Let x be an element of G and define $C = \langle x, G_2\rangle$; denote by $(C_i)_{i\geq 1}$ the lower central series of C. The quotient C/G_2 is cyclic, so, thanks to Lemma 28, the subgroups C_2 and $[C, G_2]$ are equal. It follows that $C_3 = [C, C_2]$ is contained in G_4, and, the prime p being odd, we get that $C_2^p C_p$ is contained in G_4. Let now $y \in G_2$. By Lemma 48, we have that $\rho(xy) \equiv \rho(x)\rho(y) \bmod C_2^p C_p$, and therefore $\rho(xy) \equiv \rho(x)\rho(y) \bmod G_4$. Since the element $\rho(y)$ belongs to G_4 and the choices of x and y were arbitrary, the map $\overline{\rho}$ is well-defined. ∎

Lemma 197. *Let C be a maximal subgroup of G and assume that $\rho(C \setminus G_2) \cap G_4$ is not empty. Then $\rho(C) \subseteq G_4$.*

Proof. By Lemma 196, the map ρ induces a function $\overline{\rho} : G/G_2 \to G_3/G_4$, which then becomes a homomorphism whenever we restrict it to a cyclic subgroup of G/G_2. Since $\rho^{-1}(G_4) \cap (C \setminus G_2)$ is non-empty, it generates C modulo G_2, and thus $\overline{\rho}(C/G_2) \subseteq G_4$. It follows that $\rho(C) \subseteq G_4$. ∎

Lemma 198. *Let C be a maximal subgroup of G and assume that $\rho(C \setminus G_2) \cap G_4$ is not empty. Then $\rho(C \setminus G_2) = \{1\}$.*

Proof. Let α be an intense automorphism of G of order 2 and set $A = \langle \alpha\rangle$. Let H be a cyclic subgroup of C, not contained in G_2, and such that $\rho(H) \subseteq G_4$. Without loss of generality we assume that H is A-stable (otherwise we can take a conjugate of H that is A-stable, thanks to Lemma 93). As a consequence of Proposition 134, the automorphism α induces scalar multiplication by -1 on $H/(H\cap G_2)$, so, thanks to Lemma 63, the restriction of α to H^p coincides with scalar multiplication by -1. However, the subgroup H^p being contained in G_4, it follows from Proposition 134 that α coincides with the identity map on H^p. Lemma 66 yields $H^p = \{1\}$, and, the choiche of H being arbitrary, we get $\rho(C \setminus G_2) = \{1\}$. ∎

7.3 Class 4 and intensity

The unique purpose of Section 7.3 is to give the proof of the following proposition.

Proposition 199. *Let p be a prime number and let G be a finite p-group of class at least 4. Denote by $(G_i)_{i\geq 1}$ the lower central series of G. If $\mathrm{int}(G) > 1$, then $G^p = G_3$.*

Until the end of Section 7.3, the following assumptions will hold. Let p be a prime number and let G be a finite p-group of class 4. Let $\rho : G \to G$ be defined by $x \mapsto x^p$ (see also the List of Symbols). Assume that $\mathrm{int}(G) > 1$. It follows that p is odd and the group G is non-trivial (see Section 3.2). Let α denote an intense automorphism of G of order 2 and write $A = \langle\alpha\rangle$. Set $G^+ = \{g \in G : \alpha(g) = g\}$ and $G^- = \{g \in G : \alpha(g) = g^{-1}\}$. For each maximal subgroup C of G, define moreover Y_C to be the collection of abelian subgroups of G that can be written as $\langle x\rangle \oplus \langle y\rangle$, with $x \in C \setminus G_2$ and $y \in \mathrm{Z}(C) \setminus G_4$. We will call Y_C^+ the set consisting of the A-stable elements of Y_C.

Lemma 200. *Let C be a maximal subgroup of G and assume that $\rho(C\setminus G_2)\cap G_4$ is not empty. Let H be an element of Y_C. Then H has exponent p and $H\cap G_4 = \{1\}$.*

Proof. Let $H = \langle x\rangle\oplus\langle y\rangle$ be an element of Y_C, where $x \in C\setminus G_2$ and $y \in \mathrm{Z}(C)\setminus G_4$. The group $\mathrm{Z}(C)$ is normal in G, because $\mathrm{Z}(C)$ is characteristic in the normal subgroup C, and, by Lemma 193, the group G_3 contains $\mathrm{Z}(C)$. From Proposition 123, it follows that $\mathrm{Z}(C)$ has exponent p, and thus $y^p = 1$. The element x^p is 1, by Lemma 198, and so $H^p = \{1\}$. To conclude, assume that $x^a y^b \in H \cap G_4$. Then $x^a = (x^a y^b)y^{-b}$ belongs to $H \cap G_3$, so $a \equiv 0 \bmod p$. From the fact that $\langle y\rangle \cap G_4 = \{1\}$, we conclude that $H \cap G_4 = \{1\}$. ∎

Lemma 201. *Let C be a maximal subgroup of G and assume that $\rho(C \setminus G_2)\cap G_4$ is not empty. If $H \in Y_C^+$, then $H \subseteq G^-$.*

Proof. Let $H = \langle x\rangle\oplus\langle y\rangle$ be an element of Y_C^+, where $x \in C\setminus G_2$ and $y \in \mathrm{Z}(C)\setminus G_4$. By Lemma 200, the group H has exponent p and so the order of H is p^2. The element x has depth 1 and the depth of y is 3, as a consequence of Lemma 193. It follows from Lemma 85 that

$$|H| \geq |H \cap G^-| = p^{\mathrm{wt}_H^G(1)} p^{\mathrm{wt}_H^G(3)} \geq p^2 = |H|.$$

All inequalities are in fact equalities and $H \cap G^- = H$. ∎

Lemma 202. *Let C be a maximal subgroup of G and assume that $\rho(C \setminus G_2) \cap G_4$ is not empty. Then the cardinality of Y_C^+ is equal to p.*

Proof. Write $C^- = C \cap G^-$ and $\mathrm{Z}(C)^- = \mathrm{Z}(C) \cap G^-$. Thanks to Lemma 201 we are reduced to count the subgroups of the form $\langle x \rangle \oplus \langle y \rangle$, with $x \in C^- \setminus G_2$ and $y \in \mathrm{Z}(C)^- \setminus G_4$. By Lemma 193, the subgroup $\mathrm{Z}(C)$ contains G_4. By Lemma 192, the quadruple $(\mathrm{wt}_G(1), \mathrm{wt}_G(2), \mathrm{wt}_G(3), \mathrm{wt}_G(4))$ is equal to $(2, 1, 2, 1)$ so, as a consequence of Lemma 85, the cardinalities of $C^- \setminus G_2$ and $\mathrm{Z}(C)^- \setminus G_4$ are respectively $p^3 - p^2$ and $p - 1$. Fix now a basis (x, y) for a subgroup H, where $x \in C^- \setminus G_2$ and $y \in \mathrm{Z}(C)^- \setminus G_4$. Thanks to Lemma 200, the set of equivalent bases for H is $B = \{(x^a y^b, y^c) : a, c \in \mathbb{F}_p^*,\, b \in \mathbb{F}_p\}$, and thus $|B| = p(p-1)^2$. The cardinality of Y_C^+ is

$$|Y_C^+| = \frac{|C^- \setminus G_2|\,|\mathrm{Z}(C^-) \setminus G_4|}{|B|} = \frac{(p^3 - p^2)(p-1)}{p(p-1)^2} = p.$$

∎

Lemma 203. *Let C be a maximal subgroup of G and assume that $\rho(C \setminus G_2) \cap G_4$ is not empty. Then the cardinality of Y_C is equal to p^4.*

Proof. We want to count the subgroups of the form $\langle x \rangle \oplus \langle y \rangle$, with $x \in C \setminus G_2$ and $y \in \mathrm{Z}(C) \setminus G_4$. The quadruples $(\mathrm{wt}_G(1), \mathrm{wt}_G(2), \mathrm{wt}_G(3), \mathrm{wt}_G(4))$ and $(2, 1, 2, 1)$ are the same, by Lemma 192, and so $|C| - |G_2| = p^5 - p^4$. Moreover, thanks to Lemma 193, the set $\mathrm{Z}(C) \setminus G_4$ has cardinality $p^2 - p$. Fix now (x, y) a basis for an element $H \in Y_C$, such that $x \in C \setminus G_2$ and $y \in \mathrm{Z}(C) \setminus G_4$. As a consequence of Lemma 200, the set of equivalent bases for H is $B = \{(x^a y^b, y^c) : a, c \in \mathbb{F}_p^*,\, b \in \mathbb{F}_p\}$, and so B has cardinality $p(p-1)^2$. It follows that

$$|Y_C| = \frac{|C \setminus G_2|\,|\mathrm{Z}(C) \setminus G_4|}{|B|} = \frac{(p^5 - p^4)(p^2 - p)}{p(p-1)^2} = p^4.$$

∎

Lemma 204. *Let C be a maximal subgroup of G and assume that $\rho(C \setminus G_2) \cap G_4$ is not empty. Let H be an element of Y_C^+. Then one has $\mathrm{N}_G(H) = HG_4$ and $|G : \mathrm{N}_G(H)| \leq p^3$.*

Proof. Let H be an arbitrary element of Y_C^+. By Lemma 192, the subgroup G_4 is central of order p so, as a consequence of Lemma 200, the cardinality of HG_4 is at least p^3. Moreover, by Lemma 192, the cardinality of G is equal to p^6. The subgroup HG_4 is contained in the normalizer of H and, in particular, $|G : \mathrm{N}_G(H)| \leq |G : HG_4| \leq p^3$. Assume by contradiction that there exists $K \in Y_C^+$ such that $\mathrm{N}_G(K) \neq KG_4$. Then $|G : \mathrm{N}_G(K)| < p^3$, and thus it follows from Lemma 94 that

$$|Y_C| \leq \sum_{H \in Y_C^+} |G : \mathrm{N}_G(H)| < |Y_C^+|\, p^3.$$

By Lemma 202, the cardinality of Y_C^+ is equal to p, so we get a contradiction to Lemma 203. ■

Lemma 205. *One has $\rho^{-1}(G_4) \subseteq G_2$.*

Proof. Assume by contradiction that there exists a maximal subgroup C of G such that $\rho(C \setminus G_2) \cap G_4$ is not empty. Thanks to Lemma 204, the normalizer of each element H of Y_C^+ is equal to HG_4. It follows from the definition of Y_C that, given any $H \in Y_C^+$, the A-stable subgroup $\mathrm{N}_G(H)$ does not contain G_2. As a consequence of Lemma 85, the subgroup G^+ is not contained in $\mathrm{N}_G(H)$. From the combination of Lemmas 81 and 94, we get that $|Y_C| < \sum_{H \in Y_C^+} |G : \mathrm{N}_G(H)|$. By Lemma 204, the normalizer of each element of Y_C^+ has index at most p^3 in G, so, together with Lemmas 202 and 203, we obtain $p^4 = |Y_C| < |Y_C^+|\, p^3 = p^4$. Contradiction. ■

Lemma 206. *Let $\overline{\rho}$ be as in Lemma 196. Let moreover C be a maximal subgroup of G. Then the following hold.*

1. *The map $\overline{\rho}$ is a bijection.*
2. *One has $\mathrm{Z}(C) = C^p$.*

Proof. The restriction of $\overline{\rho}$ to any cyclic subgroup of G/G_2 is a homomorphism, in particular the restriction to C/G_2. As a consequence of Lemma 205, the subgroup $\overline{\rho}(C/G_2)$ has size p, and so C^p is not contained in G_4. The subgroup C^p is characteristic in the normal subgroup C, and therefore C^p is normal in G. It follows from Lemma 166 that C^p contains G_4, and so, if $x \in C \setminus G_2$, then $C^p = \langle x^p, G_4 \rangle$. By Lemma 190, the commutator map induces a non-degenerate map $\gamma : G/G_2 \times G_3/G_4 \to G_4$ and, if $x \in C$, then $\gamma(xG_2, x^pG_4) = 1$. It follows that $\gamma(C/G_2, \overline{\rho}(C/G_2)) = 1$ and so $C^p \subseteq \mathrm{Z}(C)$. Since γ is non-degenerate, we get $C^p = \mathrm{Z}(C)$, and thus (2) is proven. We now prove (1). Denote by $\mathcal{M}$ the collection of maximal subgroups of G. As a consequence of Lemma 195, the quotient G_3/G_4 is equal to $\bigcup_{N \in \mathcal{M}} \mathrm{Z}(N)/G_4 = \bigcup_{N \in \mathcal{M}} \overline{\rho}(N/G_2)$ and so $\overline{\rho}$ is surjective. By Lemma 192, the indices $|G_1 : G_2|$ and $|G_3 : G_4|$ are equal, so the map $\overline{\rho}$ is a bijection. ■

We remark that Theorem 189 is the same as Proposition 199, which we now prove. Let Q be a finite p-group of class at least 4. Assume that $\mathrm{int}(G) > 1$. As a consequence of Lemma 101, the group Q/Q_5 has intensity greater than 1, so Lemma 206 yields $Q_3 = Q^p Q_5$. The subgroup Q^p being normal in Q, it follows from Lemma 166 that $Q^p = Q_3$. The proof of Proposition 199 is now complete.

7.4 Groups of class 5

In analogy with Sections 7.1 and 7.3, this section serves as foundations for the results in Section 7.5.

Until the end of Section 7.4, the following assumptions will hold. Let p be a prime number and let G be a finite p-group. Let $\rho : G \to G$ denote the p-the powering on G, i.e. the map $x \mapsto x^p$. Denote by $(G_i)_{i\geq 1}$ the lower central series of G and, for each positive integer i, write $w_i = \mathrm{wt}_G(i)$. Assume that $|G_5| = p$, so G has class 5. Furthermore, assume that $\mathrm{int}(G) > 1$, so p is odd. Let α be an intense automorphism of G of order 2 and write $A = \langle \alpha \rangle$.

Lemma 207. *One has $(w_1, w_2, w_3, w_4, w_5) = (2, 1, 2, 1, 1)$ and the order of G is p^7. Moreover, one has $\mathrm{Z}(G) = G_5$.*

Proof. As a consequence of Lemma 101, the intensity of G/G_5 is greater than 1. The group G/G_5 has class 4, so from Lemma 192 it follows that $(w_1, w_2, w_3, w_4) = (2, 1, 2, 1)$ and that the order of G/G_5 is p^6. Since G_5 has order p, the order of G is equal to p^7. The centre of G is equal to G_5 by Lemma 171. ∎

Lemma 208. *The subgroup G_3 is abelian and $G_2 \subseteq \mathrm{C}_G(G_4)$.*

Proof. The group G_6 being trivial, both claims follow from Lemma 20. ∎

Lemma 209. *One has $|G_3 : \mathrm{C}_{G_3}(G_2)| \leq p$.*

Proof. To lighten the notation, let $C = \mathrm{C}_{G_3}(G_2)$. By Lemma 23, the commutator map induces a bilinear map $\gamma : G_2/G_3 \times G_3/G_4 \to G_5$ whose right kernel is equal to C/G_4. It follows that γ induces an injective homomorphism $G_3/C \to \mathrm{Hom}(G_2/G_3, G_5)$. By Lemma 207, both w_2 and w_5 are equal to 1, so $\mathrm{Hom}(G_2/G_3, G_5)$ has order p. In particular, we get $|G_3 : C| \leq p$. ∎

Lemma 210. *The restriction of ρ to G_2 is an endomorphism of G_2.*

Proof. Thanks to Lemma 207, the group G_2/G_3 is cyclic and so $[G_2, G_2] = [G_2, G_3]$. From Lemma 20, it follows that $[G_2, G_2]$ is contained in G_5, which is equal to the centre of G by Lemma 207. In particular, the class of G_2 is at most 2 and, p being odd, Lemma 50 yields that G_2 is regular. Now the commutator subgroup of G_2 is contained in G_5, whose order is p, so, by Lemma 51, the restriction of ρ to G_2 is an endomorphism of G_2. ∎

7.5 Class 5 and intensity

We recall that, if G is a finite group, we denote by $(G_i)_{i\geq 1}$ the lower central series of G (see List of Symbols). In this section, we prove the following result.

Proposition 211. *Let p be a prime number and let G be a finite p-group of class at least* 5*. If* $\mathrm{int}(G) > 1$*, then $G_2^p = G_4$.*

We will keep the following assumptions until the end of Section 7.5. Let p be a prime number and let G be a finite p-group. For any positive integer i, write $w_i = \mathrm{wt}_G(i)$ and assume that $w_5 = 1$. Then the class of G is 5. Assume moreover that $\mathrm{int}(G) > 1$, so, thanks to Proposition 95, the prime p is odd. Let α be an intense automorphism of G of order 2 and write $A = \langle \alpha \rangle$. In concordance with the notation from Section 2.2, write $G^+ = \{x \in G : \alpha(x) = x\}$. In conclusion, define X to be the collection of all subgroups of G whose jumps in G are exactly 2 and 4 and denote $X^+ = \{H \in X : \alpha(H) = H\}$. In this section, the List of Symbols will be fully respected.

Lemma 212. *The elements of X have order p^2.*

Proof. Let H be an element of X. As a consequence of Lemma 207, both widths $\mathrm{wt}_H^G(2)$ and $\mathrm{wt}_H^G(4)$ are equal to 1. Now apply Lemma 84. ■

Lemma 213. *Assume that G_2 has exponent p. Let H be a subgroup of G. Then $H \in X$ if and only if there exist $x \in G_2 \backslash G_3$ and $y \in G_4 \backslash G_5$ such that $H = \langle x \rangle \oplus \langle y \rangle$.*

Proof. If $H = \langle x \rangle \oplus \langle y \rangle$, with $x \in G_2 \setminus G_3$ and $y \in G_4 \setminus G_5$, then H belongs to X, thanks to Lemma 82. We prove the converse. The subgroup H has order p^2, by Lemma 212, and H cannot be cyclic, because it is contained in G_2. The jumps of H in G being 2 and 4, it follows from Lemma 82 that there exist elements x and y in H of depths respectively 2 and 4 in G. As a consequence of Lemma 208, the subgroup H decomposes as $H = \langle x \rangle \oplus \langle y \rangle$. ■

Lemma 214. *Assume that G_2 has exponent p. Then $|X| = p^4$.*

Proof. Thanks to Lemma 213, all elements H of X are of the form $H = \langle x \rangle \oplus \langle y \rangle$, with $x \in G_2 \setminus G_3$ and $y \in G_4 \setminus G_5$. Let $(x, y) \in (G_2 \setminus G_3) \times (G_4 \setminus G_5)$ and let H be the $\mathbb{F}_p$-vector space that is spanned by x and y. The collection of equivalent bases for H is $B = \{(x^a y^b, y^c) \ : \ a, c \in \mathbb{F}_p^*, \, b \in \mathbb{F}_p\}$ and so B has cardinality $p(p-1)^2$. From Lemma 207 it follows that the cardinalities of $G_2 \setminus G_3$ and $G_4 \setminus G_5$ are respectively $p^5 - p^4$ and $p^2 - p$. We conclude by computing

$$|X| = \frac{|G_2 \setminus G_3| \, |G_4 \setminus G_5|}{|B|} = \frac{(p^5 - p^4)(p^2 - p)}{p(p-1)^2} = p^4.$$

■

Lemma 215. *One has $X^+ = \{G^+\}$.*

Proof. From Lemmas 85 and 207, it follows that G^+ has order p^2. Let now H be an element of X^+. It follows from Lemma 85 that $H \cap G^+$ has cardinality p^2 and, thanks to Lemma 212, the subgroups H and G^+ are the same. In particular, the only element of X^+ is G^+. ∎

Lemma 216. *The exponent of G_2 is different from p.*

Proof. We work by contradiction, assuming that the exponent of G_2 is p. To lighten the notation, write $C = \mathrm{C}_{G_3}(G_2)$ and $N = CG^+$. The group C is characteristic in G, so N is a subgroup of G. Moreover, G^+ is contained in G_2, thanks to Lemma 85, so N is a subgroup of $\mathrm{N}_G(G^+)$. By Lemma 209, the group C is contained in G_3 with index at most p and, by Lemma 207, the quadruple (w_2, w_3, w_4, w_5) is equal to $(1, 2, 1, 1)$. It follows from Lemma 84 that the order of N is at least p^4. The order of G is p^7, by Lemma 207, and thus $|G : N| \leq p^3$. By Lemma 215, the set X^+ has only one element, namely G^+, so Lemma 94 yields

$$|X| \leq |G : \mathrm{N}_G(G^+)| \leq |G : N| \leq p^3.$$

Contradiction to Lemma 214. ∎

Lemma 217. *One has $\rho(G_2) = G_4$.*

Proof. As a consequence of Lemma 210, the set $\rho(G_2)$ is a characteristic subgroup of G and, by Lemma 216, it is non-trivial. By Lemma 207, the centre of G is equal to G_5 so, as a consequence of Lemma 29, the intersection $G_5 \cap \rho(G_2)$ is non-trivial. The order of G_5 being p, the subgroup $\rho(G_2)$ contains G_5. Thanks to Proposition 123, the quotient G_2/G_4 is elementary abelian and so $G_5 \subseteq \rho(G_2) \subseteq G_4$. By Lemma 207, the dimension of G_4/G_5 is 1 and therefore there are only two possibilities: either $\rho(G_2) = G_4$ or $\rho(G_2) = G_5$. In the first case we are done, so assume by contradiction the second. Then, by Lemma 169, each element of $G_2 \setminus G_3$ has order p. It follows that G_2 is equal to the union of two proper subgroups, namely $\ker \rho_{|G_2}$ and G_3. Contradiction. ∎

We are finally ready to prove Proposition 211. To this end, let Q be a finite p-group of class at least 5 with $\mathrm{int}(Q) > 1$. Then the group Q/Q_6 has class 5 and, as a consequence of Lemma 101, the intensity of Q/Q_6 is greater than 1. By Lemma 217, the subgroups $(Q_2/Q_6)^p$ and Q_4/Q_6 are equal, and so $Q_2^p Q_6 = Q_4$. The subgroup Q_2^p being normal in G, it follows from Lemma 166 that $Q_2^p = Q_4$. This concludes the proof of Proposition 211.

We remark that Proposition 211 can be easily derived, when p is greater than 3, from Theorem 189. We will show a way of doing so in Section 8.1.