Intense automorphisms of finite groups

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Chapter 6

Some structural restrictions

In this chapter we will see how the structure of finite p-groups whose intensity is greater than 1 starts becoming more and more rigid, as soon as the class is at least 4. We recall that, if $(G_i)_{i\geq 1}$ denotes the lower central series of G, then the class of G is the number of indices i for which $G_i \neq 1$. Recall moreover that, for each positive integer i, the i-th width of G is $\mathrm{wt}_G(i) = \log_p |G_i : G_{i+1}|$ (see Section 2.3). The main results from Chapter 6 are the following.

Theorem 164. *Let p be a prime number and let G be a finite p-group of class at least 4. For all $i \in \{1,2,3,4\}$, write $w_i = \mathrm{wt}_G(i)$. If $\mathrm{int}(G) > 1$, then $(w_1, w_2, w_3, w_4) = (2,1,2,1)$.*

Theorem 165. *Let p be a prime number and let G be a finite p-group of class at least 3. For all $i \in \mathbb{Z}_{\geq 1}$, write $w_i = \mathrm{wt}_G(i)$. Assume that $\mathrm{int}(G) > 1$. Then, for all $i \in \mathbb{Z}_{\geq 1}$, one has $w_i w_{i+1} \leq 2$.*

6.1 Normal subgroups

We devote Section 6.1 to understanding the normal subgroup structure of a finite p-group of intensity greater than 1. We prove the following result.

Proposition 166. *Let p be a prime number and let G be a finite p-group with $\mathrm{int}(G) > 1$. Let N be a subgroup of G. Then N is normal if and only if there exists $i \in \mathbb{Z}_{\geq 1}$ such that $G_{i+1} \subseteq N \subseteq G_i$.*

The following assumptions will be satisfied until the end of Section 6.1. Let p be a prime number and let G be a finite p-group of intensity greater than 1. It follows that p is odd and that G is non-trivial (see Section 3.2). Denote by $(G_i)_{i\geq 1}$ the lower central series of G and, for each positive integer i, write $w_i = \mathrm{wt}_G(i)$ for the i-th width of G. Let α be intense of order 2 and write $A = \langle \alpha \rangle$. Denote

$\chi = \chi_{G|A}$, the restriction of the intense character of G to A (once again, we refer to Section 3.2). In concordance with the notation from Section 2.2, let $G^+ = \{x \in G \mid \alpha(x) = x\}$ and $G^- = \{x \in G \mid \alpha(x) = x^{-1}\}$. For a subgroup H of G we will write $H^+ = H \cap G^+$ and $H^- = H \cap G^-$.

Lemma 167. *Let H be an A-stable subgroup of G. If H is cyclic, then $H \subseteq G^+$ or $H \subseteq G^-$.*

Proof. Assume that H is cyclic. As a consequence of Corollary 76, the subgroup H decomposes as $H = H^+ \oplus H^-$. Then H is cyclic if and only if one of H^+ and H^- is trivial. This concludes the proof. ∎

We recall that, as defined in Section 2.3, if x is a non-trivial element of G, then the depth $\mathrm{dpt}_G(x)$ of x is the unique positive integer d for which $x \in G_d \setminus G_{d+1}$.

Lemma 168. *Let $x \in G \setminus \{1\}$. Then the following hold.*

1. *The depth of x is even if and only if there exists $g \in G$ such that gxg^{-1} belongs to G^+.*
2. *The depth of x is odd if and only if there exists $g \in G$ such that gxg^{-1} belongs to G^-.*

Proof. The automorphism α being intense, it follows from Lemma 93 that there exists $g \in G$ such that $\langle gxg^{-1} \rangle$ is A-stable. Write $d = \mathrm{dpt}_G(x) = \mathrm{dpt}_G(gxg^{-1})$ and $H = \langle gxg^{-1} \rangle$. By Lemma 167, the subgroup H is contained either in G^+ or in G^-. By Lemma 104, the action of A on $(HG_{d+1})/G_{d+1}$ is through χ^d and the choice between G^+ and G^- only depends from the parity of d. If d is even, then $\chi^d = 1$ and H is contained in G^+. Otherwise, H is contained in G^-. ∎

We recall that, if H is a subgroup of G, then a jump of H in G is a positive integer j such that $H \cap G_j \neq H \cap G_{j+1}$.

Lemma 169. *All jumps of a cyclic subgroup of G have the same parity.*

Proof. Let H be a cyclic subgroup of G. The automorphism α being intense, it follows from Lemma 93 that there exists $g \in G$ such that gHg^{-1} is A-stable. By Lemma 167, the subgroup gHg^{-1} is contained in G^+ or in G^-. We conclude by applying Lemma 168. ∎

Lemma 170. *Let $c \in \mathbb{Z}_{\geq 1}$ denote the class of G. Then the following hold.*

1. *The induced action of A on $\mathrm{Z}(G)$ is through χ^c.*
2. *If c is even, then $\mathrm{Z}(G) \subseteq G^+$.*
3. *If c is odd, then $\mathrm{Z}(G) \subseteq G^-$.*

Proof. The subgroup G_c is contained in $\mathrm{Z}(G)$ and, by Lemma 104, the group A acts on G_c through χ^c. From the combination of Corollary 103 with Lemma 66, it follows that A acts on $\mathrm{Z}(G)$ through χ^c. If c is even, then $\chi^c = 1$ and $\mathrm{Z}(G)$ is contained in G^+. Otherwise, $\chi^c = \chi$ and $\mathrm{Z}(G)$ is a subset of G^-. ∎

Lemma 171. *Let $c \in \mathbb{Z}_{\geq 1}$ be the class of G. Then, for all $i \in \{1, \ldots, c\}$, if H is a quotient of G of class i, then $\mathrm{Z}(H) = H_i$.*

Proof. If $i = 1$ the result is clear; we assume that i is at least 2 and that the result holds for $i - 1$. Let H be a quotient of G of class i, which has, thanks to Lemma 101, intensity greater than 1. Let β be intense of order 2 and let $B = \langle\beta\rangle$ and $\psi = \chi_{H|B}$. The subgroup H_i is central in H, so $\mathrm{Z}(H)/H_i$ is isomorphic to a subgroup of $\mathrm{Z}(H/H_i)$. By the induction hypothesis $\mathrm{Z}(H/H_i) = H_{i-1}/H_i$ and it follows that $H_i \subseteq \mathrm{Z}(H) \subseteq H_{i-1}$. By Lemma 104, the group B acts on H_{i-1}/H_i and H_i, respectively through ψ^{i-1} and ψ^i, which are distinct characters since $\psi \neq 1$. Moreover, the induced action of B on $\mathrm{Z}(H)$ is through ψ^i, by Lemma 170(1). Lemma 66 yields $\mathrm{Z}(H) = H_i$. ∎

We remark that, to prove Proposition 166, it now suffices to combine Lemma 171 with Lemma 30.

6.2 About the third width

Let p be a prime number and let G be a finite p-group. If i is a positive integer, we recall that the i-th width of G is defined to be $\mathrm{wt}_G(i) = \log_p |G_i : G_{i+1}|$, where $(G_i)_{i\geq 1}$ denotes the lower central series of G. Thanks to Theorem 125(2), we know that, if G has class at least 3 and $\mathrm{int}(G) > 1$, then $(\mathrm{wt}_G(1), \mathrm{wt}_G(2)) = (2, 1)$ and $\mathrm{wt}_G(3)$ is either 1 or 2. In the case in which the class of G equals 3, then both situations $\mathrm{wt}_G(3) = 1$ and $\mathrm{wt}_G(3) = 2$ occur. What about higher nilpotency classes? We prove the following.

Proposition 172. *Let p be a prime number and let G be a finite p-group of class at least 4. For each positive integer i, denote $w_i = \mathrm{wt}_G(i)$. Assume that $\mathrm{int}(G) > 1$. Then $(w_1, w_2, w_3) = (2, 1, 2)$.*

Until the end of Section 6.2, the following assumptions will hold. Let p be a prime number and let G be a finite p-group. Let $(G_i)_{i\geq 1}$ denote the lower central series of G and, for each positive integer i, denote $w_i = \mathrm{wt}_G(i)$. We assume that G has class 4 and that $(w_1, w_2, w_3, w_4) = (2, 1, 1, 1)$. We will show that $\mathrm{int}(G) = 1$.

Lemma 173. *Assume that p is odd. Then $\mathrm{Z}(G) = G_4$.*

Proof. The class of G being 4, the subgroup G_4 is contained in $\mathrm{Z}(G)$. Now, the group $\mathrm{Z}(G)/G_4$ is contained in $\mathrm{Z}(G/G_4)$ and, as a consequence of Lemma 140, the centre of G/G_4 is equal to G_3/G_4. It follows that $G_4 \subseteq \mathrm{Z}(G) \subseteq G_3$. The groups $\mathrm{Z}(G)$ and G_3 are distinct, because the class of G is 4, so, the index $|G_3 : G_4|$ being p, we get that $G_4 = \mathrm{Z}(G)$. ■

Lemma 174. *The subgroup G_2 is abelian.*

Proof. The group G_2/G_3 is cyclic, because $w_2 = 1$, so $[G_2, G_2] = [G_2, G_3]$, by Lemma 28. It follows from Lemma 20 that $[G_2, G_2] \subseteq G_5 = \{1\}$, and thus G_2 is abelian. ■

Lemma 175. *Assume that p is odd. Then the following hold.*

1. *The subgroup $[\mathrm{C}_G(G_3), G_2]$ is contained in G_4.*

2. *One has $|\,\mathrm{C}_G(G_3) : G_2| = p$.*

Proof. To lighten the notation, write $C = \mathrm{C}_G(G_3)$. We first prove (1). The group $[G_2, [G, C]]$ is contained in $[G_2, G_2]$ so it follows from Lemma 174 that $[G_2, [G, C]] = \{1\}$. Moreover, $[C, [G_2, G]] = [C, G_3] = \{1\}$, by definition of C. Thanks to Lemma 19, the subgroup $[G, [C, G_2]]$ is trivial, and thus $[C, G_2]$ is contained in $\mathrm{Z}(G)$. The centre of G is equal to G_4, by Lemma 173, and (1) is proven. We prove (2). Let D be the unique subgroup of G containing G_4 such that $D/G_4 = \mathrm{C}_{G/G_4}(G_2/G_4)$. Then one has $[[G, D], G_2] \subseteq [G_2, G_2] = \{1\}$ and $[[D, G_2], G] \subseteq [G_4, G] = \{1\}$. Lemma 19 yields that $[D, G_3] = [D, [G, G_2]] = \{1\}$ and therefore D is contained in C. It follows from Lemma 137(1) that

$$p \leq |C : G_2| \leq |G : G_2| = p^2.$$

The group G_3 is not central, and so $|C : G_2| = p$. ■

Lemma 176. *If* $\mathrm{int}(G) > 1$, *then* $\mathrm{C}_G(G_3)$ *is abelian.*

Proof. Assume that $\mathrm{int}(G) > 1$. As a consequence of Proposition 95, the prime p is odd. Let α be an intense automorphism of G of order 2 and write $A = \langle \alpha \rangle$ and $\chi = \chi_{G|A}$. To lighten the notation, write $C = \mathrm{C}_G(G_3)$. By Lemma 175(2), the index of G_2 in C is p, so it follows from Lemma 28 that $[C, C] = [C, G_2]$. Moreover, $[C, G_2]$ is contained in G_4, by Lemma 175(1), and $G_4 = \mathrm{Z}(G)$ by Lemma 173. By Lemma 22, the commutator map $C \times G_2 \to G_4$ is bilinear and, as a consequence of Lemma 174, it factors as $\phi : C/G_2 \times G_2/G_3 \to G_4$. By Lemma 104, the group A acts on C/G_2 and G_2/G_3 respectively through χ and χ^2, so, as a consequence of Lemma 61, the group A acts on $[C, G_2]$ through $\chi^3 = \chi$. By Lemma 104, the group A acts on G_4 through $\chi^4 = 1$. Since $\chi \neq 1$, it follows from Lemma 66 that $[C, C]$ is trivial, and therefore C is abelian. ■

We recall here that, if $A = \langle\alpha\rangle$ is a multiplicative group of order 2 acting on a finite group B of odd order, then one defines $B^+ = \{x \in B : \alpha(x) = x\}$ and $B^- = \{x \in B : \alpha(x) = x^{-1}\}$. (See Section 2.2.)

Lemma 177. *Assume that* $\mathrm{int}(G) > 1$ *and let* α *be an intense automorphism of* G *of order* 2. *Write* $C = \mathrm{C}_G(G_3)$. *Then* $C = C^+ \oplus C^-$ *and* $|C^+| = |C^-| = p^2$.

Proof. The group C is A-stable and it is abelian by Lemma 176. From Corollary 76, it follows that $C = C^+ \oplus C^-$. The cardinalities of C^+ and C^- are both equal to p^2, as a consequence of Lemma 85. ∎

Lemma 178. *Assume that* $\mathrm{int}(G) > 1$ *and let* α *be an intense automorphism of* G *of order* 2. *Write* $C = \mathrm{C}_G(G_3)$. *Then* C^+ *is cyclic if and only if* C^- *is cyclic.*

Proof. The subgroup C is A-stable and $C = C^+ \oplus C^-$, by Lemma 177. Moreover, both C^+ and C^- have cardinality p^2. The subgroup C^p is characteristic in G, so C^p is A-stable. The group G has class 4 and, as a consequence of Lemma 170(2), the subgroup $\mathrm{Z}(G)$ is contained in G^+. We first prove the implication from right to left. Assume that C^- is cyclic. Then C^p is a non-trivial subgroup of the p-group G. From Lemma 29, it follows that C^p has non-trivial intersection with the centre of G and, in particular, $C^p \cap G^+ \neq \{1\}$. The group C^+ is cyclic, because it has order p^2 and exponent different from p. To prove the implication from left to right, assume that C^+ is cyclic. As a consequence of Lemma 85, the group C^+ is contained in G_2. We claim that there exists an element $x \in C \setminus G_2$ of order p^2. If not, then it means that C is equal to the union of two of its proper subgroups, namely $C \cap G_2$ with $\mu_p(C)$, which is impossible. It follows that there exists $x \in C$ of order p^2, with $\mathrm{dpt}_G(x) = 1$. Fix such x. By Lemma 168, there exists $g \in G$ such that gxg^{-1} belongs to G^-. Since both C^- and $\langle x\rangle$ have order p^2, the group C^- is cyclic. ∎

Lemma 179. *Let* H *be a subgroup of* $\mathrm{C}_G(G_3)$. *If* $\mathrm{int}(G) > 1$, *then* H *has at most* p *conjugates in* G.

Proof. Assume that $\mathrm{int}(G) > 1$. The group $\mathrm{C}_G(G_3)$ is abelian, by Lemma 176, and therefore $\mathrm{C}_G(G_3)$ normalizes H. It follows that $|G : \mathrm{N}_G(H)|$ is at most $|G : \mathrm{C}_G(G_3)|$. By Lemma 175(2) the index $|G : \mathrm{C}_G(G_3)|$ is equal to p, and thus H has at most p conjugates in G. ∎

Lemma 180. *Assume that* $\mathrm{int}(G) > 1$ *and let* α *be an intense automorphism of* G *of order* 2. *Write* $C = \mathrm{C}_G(G_3)$. *Then* C^+ *is cyclic.*

Proof. Assume the contrary. Then, as a consequence of Lemma 178, both C^+ and C^- are elementary abelian. From Lemma 177 it follows that C is an $\mathbb{F}_p$-vector space of dimension 4. Let X be the collection of 1-dimensional subspaces of C; then we have

$$|X| = \frac{p^4-1}{p-1} = p^3+p^2+p+1.$$

Let moreover $X^+ = \{H \in X \,:\, \alpha(H) = H\}$. As a consequence of Lemma 167, the set X^+ consists of the 1-dimensional subspaces of C that are contained in $C^+ \cup C^-$. Then $|X^+| = 2(p+1)$. By Lemma 179, each element of X^+ has at most p conjugates in G, so it follows from Lemma 94 that

$$2p(p+1) = p|X^+| \geq \sum_{H \in X^+} |G : \mathrm{N}_G(H)| \geq |X| = p^3+p^2+p+1.$$

Contradiction. ∎

Lemma 181. *The intensity of G is equal to* 1.

Proof. Assume by contradiction that $\mathrm{int}(G) > 1$ and let α be an intense automorphism of G of order 2. Write $C = \mathrm{C}_G(G_3)$. The group C is abelian, by Lemma 176, and $C = C^+ \oplus C^-$, by Lemma 177. Moreover, C^+ and C^- have both cardinality p^2. By Lemma 180, the subgroup C^+ is cyclic so, by Lemma 178, the subgroup C^- is also cyclic. Let X be the collection of cyclic subgroups of C of order p^2 and let X^+ be the subset of X consisting of the A-stable ones. It follows from Lemma 167 that $X^+ = \{C^+, C^-\}$ and the cardinality of X^+ is thus 2. On the other hand, the cardinality of X is equal to

$$|X| = \frac{p^4-p^2}{p^2-p} = p(p+1).$$

By Lemma 179, each element of X^+ has at most p conjugates, so it follows from Lemma 94 that

$$2p \geq p|X^+| \geq |X| = p^2+p.$$

Contradiction. ∎

We conclude Section 6.2 by giving the proof of Proposition 172. Let Q be a finite p-group of class at least 4 with $\mathrm{int}(Q) > 1$. The class of Q being 4, the subgroup Q_4 is non-trivial and, by Lemma 35, there exists a normal subgroup M of Q that is contained in Q_4 with index p. Fix M and denote $\overline{Q} = Q/M$. Thanks to Lemma 101, the intensity of $\overline{Q}$ is greater than 1, so it follows from Theorem 125(2) that $(\mathrm{wt}_{\overline{Q}}(1), \mathrm{wt}_{\overline{Q}}(2), \mathrm{wt}_{\overline{Q}}(3), \mathrm{wt}_{\overline{Q}}(4)) = (2,1,f,1)$, where $f \in \{1,2\}$. Lemma 181 yields $f = 2$ and the proof of Proposition 172 is complete.

6.3 A bound on the width

From Section 2.3, we recall that, given a finite p-group G and a positive integer i, the i-th width of G is defined to be $\mathrm{wt}_G(i) = \log_p |G_i : G_{i+1}|$. The unique purpose of Section 6.3 is to prove the following result.

Proposition 182. *Let p be a prime number and let G be a finite p-group. Let c denote the class of G. Assume $c \geq 3$ and $\mathrm{int}(G) > 1$. Then, for each i in $\{1, 2, \ldots, c-1\}$, one has $\mathrm{wt}_G(i)\,\mathrm{wt}_G(i+1) \leq 2$.*

We devote the remaining part of this section to the proof of Proposition 182. Until the end of Section 6.3 we work thus under the assumptions of Proposition 182. As a consequence of Proposition 95, the prime p is odd.

Lemma 183. *One has $\mathrm{wt}_G(1)\,\mathrm{wt}_G(2) = 2$.*

Proof. This follows directly from Theorem 125(2). ∎

Lemma 184. *Let $\phi : G/G_2 \to \mathrm{Hom}(G_{c-1}/G_c, G_c)$ be the function defined by $xG_2 \mapsto (aG_c \mapsto [x, a])$. Then ϕ is a homomorphism.*

Proof. The map ϕ is induced from the surjective homomorphism from Lemma 25 and ϕ is thus itself a homomorphism of groups. ∎

Lemma 185. *The group G_{c-1} is abelian.*

Proof. By Lemma 20, the group $[G_{c-1}, G_{c-1}]$ is contained in G_{2c-2}, which is itself contained in G_{c+1} because $c \geq 3$. Since $G_{c+1} = \{1\}$, the group G_{c-1} is abelian. ∎

Lemma 186. *Let α be an intense automorphism of G of order $\mathrm{int}(G)$ and write $A = \langle \alpha \rangle$. Then G_c has a unique A-stable complement in G_{c-1}.*

Proof. Denote by χ the restriction of the intense character of G to A (see Section 3.2). The group G_{c-1} is abelian, by Lemma 185, and, by Lemma 104, the group A acts on G_{c-1}/G_c and G_c respectively through χ^{c-1} and χ^c. The characters χ^{c-1} and χ^c are distinct, because $\chi \neq 1$, so, by Theorem 68, the subgroup G_c has a unique A-stable complement in G_{c-1}. ∎

Lemma 187. *The homomorphism ϕ from Lemma 184 is surjective.*

Proof. Let α be an intense automorphism of G of order $\mathrm{int}(G)$ and write $A = \langle \alpha \rangle$. By Lemma 185, the group G_{c-1} is abelian and, by Lemma 186, there exists a unique A-stable complement M of G_c in G_{c-1}. Now the group A acts in a natural way on the set of complements of G_c in G_{c-1} and, the automorphism α being intense, it follows from Lemma 93 that all complements of G_c in G_{c-1} are conjugate to M in G. On the other hand, by Lemma 114, the set of complements of G_c in G_{c-1} consists of the elements $\{mf(m) : m \in M\}$ as f varies in $\mathrm{Hom}(M, G_c)$. It follows that, for each $f \in \mathrm{Hom}(M, G_c)$, there exists $x \in G$, such that $\{mf(m) : m \in M\} = xMx^{-1}$. Fix the pair (f, x). Then, for all $m \in M$, there exists $n \in M$ such that $mf(m) = xnx^{-1} = [x, n]n$. It follows that $n^{-1}m = [x, n]f(m)^{-1}$ belongs to $M \cap G_c = \{1\}$, so $m = n$. We have proven that $f(m) = [x, m]$. Now, the groups $\mathrm{Hom}(G_{c-1}/G_c, G_c)$ and $\mathrm{Hom}(M, G_c)$ are isomorphic and, the choice of f being arbitrary, each homomorphism $f : G_{c-1}/G_c \to G_c$ is of the form $mG_c \mapsto [x, m]$, for some $x \in G$. In other words, we have proven surjectivity of ϕ. ■

Lemma 188. *One has* $\mathrm{wt}_G(c-1)\,\mathrm{wt}_G(c) \leq 2$.

Proof. The groups G_{c-1}/G_c and G_c are $\mathbb{F}_p$-vector spaces, as a consequence of Lemma 110. It follows that the dimension of $\mathrm{Hom}(G_{c-1}/G_c, G_c)$ is equal to $\mathrm{wt}_G(c-1)\,\mathrm{wt}_G(c)$. Thanks to Lemma 187, the product $\mathrm{wt}_G(c-1)\,\mathrm{wt}_G(c)$ is at most $\mathrm{wt}_G(1)$, which is equal to 2, by Theorem 125(2). We get thus that $\mathrm{wt}_G(c-1)\,\mathrm{wt}_G(c) \leq 2$. ■

The proof of Proposition 182 is now an easy exercise, which we spell out here. If $i = 1$, then we are done by Lemma 183. Assume that $i > 1$. For all indices $j \leq i$, the quotient G/G_{j+1} has class j so, without loss of generality, we assume that $c = i + 1$. We conclude by applying Lemma 188.

We remark that Theorem 165 is the same as Proposition 182. Moreover, Theorem 164 is given by the combination of Propositions 172 and 182.