Intense automorphisms of finite groups

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Chapter 5

Intensity of groups of class 3

The purpose of this chapter is giving a complete overview of the case in which the class is 3. We will prove the following theorems.

Theorem 124. *Let p be a prime number and let G be a finite p-group of class 3. Then the following are equivalent.*

1. *One has* $\mathrm{int}(G) > 1$.
2. *The prime p is odd and* $\mathrm{int}(G) = 2$.
3. *The prime p is odd and* $|G : G_2| = p^2$.

We remind the reader that, if G is a finite p-group and j is a positive integer, then $\mathrm{wt}_G(j) = \log_p |G_j : G_{j+1}|$. For more detail, see Section 2.3.

Theorem 125. *Let p be a prime number and let G be a finite p-group of class at least 3. Assume that* $\mathrm{int}(G) > 1$. *For each positive integer j, set moreover $w_j = \mathrm{wt}_G(j)$. Then the following hold.*

1. *One has* $\mathrm{int}(G) = 2$.
2. *One has* $(w_1, w_2, w_3) = (2, 1, f)$, *where* $f \in \{1, 2\}$.

5.1 Low intensity

In Section 5.1 we derive some restrictions on the structure of finite p-groups of class at least 3 and intensity greater than 1. We will prove the following main result.

Proposition 126. *Let p be a prime number and let G be a finite p-group of class at least 3. Assume that* $\mathrm{int}(G) > 1$. *Then the following hold.*

1. *The prime p is odd.*
2. *One has* $\mathrm{int}(G) = 2$.
3. *One has* $|G : G_2| = p^2$.

Our main goal for this section being the proof of Proposition 126, we will work under the following assumptions until the end of Section 5.1. Let p be prime number and let G be a finite p-group of class at least 3. Assume that $\mathrm{int}(G) > 1$ and let α be intense of order $\mathrm{int}(G)$. Write $A = \langle \alpha \rangle$ and $\chi = \chi_{G|A}$, where χ_G denotes the intense character of G (see Section 3.2). For the rest of the notation we refer to the List of Symbols. We remark that, $\mathrm{int}(G)$ being greater than 1, the prime p is odd and G is non-trivial. For more detail see Chapter 3.

Lemma 127. *Assume that G has class 3. Then the following hold.*

1. *One has* $G^p \subseteq G_3$.
2. *One has* $|G_2 : G_3| = p$.
3. *One has* $\mathrm{Z}(G) = G_3$.

Proof. The subgroup G_3 is contained in $\mathrm{Z}(G)$ because G has class 3. By Lemma 101 the intensity of G/G_3 is greater than 1, and thus, by Theorem 105, the group G/G_3 is extraspecial of exponent p. It follows that G_2/G_3 has size p and that G^p is contained in G_3. Moreover, one has $\mathrm{Z}(G/G_3) = G_2/G_3$ and, since $\mathrm{Z}(G)/G_3$ is contained in $\mathrm{Z}(G/G_3)$, we get $G_3 \subseteq \mathrm{Z}(G) \subseteq G_2$. As the class of G is 3, the subgroup $\mathrm{Z}(G)$ is different from G_2, and so $\mathrm{Z}(G) = G_3$. ■

Lemma 128. *Assume G has class 3. Then the following hold.*

1. *The group G_2 is elementary abelian.*
2. *The group $\mathrm{C}_G(G_2)$ is abelian and A-stable.*

Proof. (1) The group G_4 is trivial and, as a consequence of Lemma 20, the subgroup G_2 is abelian. The exponent of G_2 is equal to p, by Proposition 123. This proves (1). We now prove (2). To lighten the notation, let $C = \mathrm{C}_G(G_2)$. The subgroup G_2 is central in C, by definition of C, and, as a consequence of Lemma 26, the commutator map induces a bilinear map $\phi : C/G_2 \times C/G_2 \to [C, C]$. The subgroups C and $[C, C]$ are characteristic in G and thus they are A-stable. Thanks to Lemma 104, the group A acts on C/G_2 through χ, and, by Lemma 61, it acts on $[C, C]$ through χ^2. By Lemma 104, the action of A on G_3 is through χ^3. The character χ not being trivial, one has $\chi^2 \neq \chi^3$, and Lemma 66 yields $[C, C] \cap G_3 = \{1\}$. By Lemma 127(3), the group G_3 is equal to $\mathrm{Z}(G)$ so the group $[C, C]$ is a normal subgroup of G that trivially intersects $\mathrm{Z}(G)$. It follows from Lemma 29 that $[C, C] = \{1\}$. ■

Lemma 129. *Assume that G_3 has order p. Then the following hold.*

1. *One has $|G : \mathrm{C}_G(G_2)| = p$.*
2. *One has $|G : G_2| = p^2$.*
3. *One has $|\mathrm{C}_G(G_2)| = p^3$.*

Proof. The group G_3 having order p, it follows from Lemma 29 that G_3 is central, and so G has class 3. To lighten the notation, let $C = \mathrm{C}_G(G_2)$. Let moreover $V = G/G_2$, $Z = G_2/G_3$, and $T = C/G_2$. The groups V, Z, and T are vector spaces over $\mathbb{F}_p$, as a consequence of Lemma 110. We prove (1). By Lemma 24, the commutator map induces a bilinear map $\psi : V \times Z \to G_3$ whose left kernel is T. The centre of G is equal to G_3, by Lemma 127(3), so the right kernel of ψ is trivial. The map $\psi_C : V/T \times Z \to G_3$ that is induced from ψ is thus non-degenerate. The dimension of Z is equal to 1, by Lemma 127(2), and Lemma 2 yields $\dim V/T = 1$. This proves (1). We prove (2) and (3) together. Let $\phi : V \times V \to Z$ be the bilinear map from Lemma 24. The map ϕ is induced from the commutator map and, by Lemma 128(2), the group C is abelian. It follows that T is isotropic. As a consequence of (1) the space T has codimension 1 in V and T is maximal isotropic, because ϕ is not the zero map. From Corollary 117, it follows that $1 = \dim(V/T) = \dim \mathrm{Hom}(T, Z) = \dim T$ and so $\dim V = 2$. To conclude, we compute $|G : G_2| = p^{\dim V} = p^2$ and $|C| = |C : G_2||G_2 : G_3||G_3| = p^{\dim T}p^{\dim Z}p = p^3$. ■

Lemma 130. *Assume that $\chi^2 \neq 1$ and that G has class 3. Then $\mathrm{C}_G(G_2)$ is elementary abelian.*

Proof. Let $C = \mathrm{C}_G(G_2)$. The group C is abelian and A-stable by Lemma 128(2). We will show that C has exponent p. By Lemma 128(1) the group G_2 is elementary abelian and $G_2 \subseteq C$. The group A acts on C/G_2 through χ, as a consequence of Lemma 104, and, by Lemma 63, it acts on C^p also through χ. It follows from Lemma 127(1) that $C^p \subseteq G_3$. The action of A on G_3 is through χ^3, by Lemma 128(2), and thus A acts on C^p both through χ and χ^3. Since $\chi^2 \neq 1$, the characters χ and χ^3 are distinct and, as a consequence of Lemma 66, the group C has exponent p. ■

Lemma 131. *Assume that $\chi^2 \neq 1$ and that G_3 has order p. Then $\mathrm{C}_G(G_2)$ is a vector space over $\mathbb{F}_p$ and there exist unique A-stable subspaces C_1 and C_2 of $\mathrm{C}_G(G_2)$, of dimension 1, such that $\mathrm{C}_G(G_2) = C_1 \oplus C_2 \oplus G_3$.*

Proof. To lighten the notation, let $C = \mathrm{C}_G(G_2)$. Since G_3 has order p, it follows from Lemma 29 that G_3 is central so G has class 3. As a consequence of Lemmas 129(3) and 130, the group C is a vector space over $\mathbb{F}_p$ of dimension 3. The group C is A-stable, by Lemma 128(2), and, by Lemma 104, the action of A on C/G_2,

G_2/G_3, and G_3 is respectively through χ, χ^2, and χ^3. The three characters are pairwise distinct because $\chi^2 \neq 1$. We first apply Theorem 68 to C/G_3. Then there exists a unique A-stable complement D_1/G_3 of G_2/G_3. It follows that $D_1 \cap G_2 = G_3$. We now apply Theorem 68 to both D_1 and G_2 to get unique A-stable subspaces C_1 and C_2 of C satisfying $D_1 = C_1 \oplus G_3$ and $G_2 = C_2 \oplus G_3$. As a consequence of Lemma 127(2), the subspace G_2 has dimension 2, so both C_1 and C_2 have dimension 1. Moreover, the intersection of D_1 with G_2 being equal to G_3, it follows that $C = C_1 \oplus C_2 \oplus G_3$. ∎

Lemma 132. *Assume that G_3 has order p. Then* $\text{int}(G) = 2$.

Proof. If $\chi^2 = 1$, then $1 < \text{int}(G) \leq 2$ and we are done. We assume now that $\chi^2 \neq 1$ and we will derive a contradiction. Let now C_1 and C_2 be as in Lemma 131 and denote by X be the collection of subspaces of dimension 1 of C. Since C is normal, the group G acts on X by conjugation. By Lemma 129(1), the index of C in G is equal to p and the size of each orbit of X under G is at most p. Moreover, the elements of X that are stable under the action of A are precisely C_1, C_2, and G_3. Lemma 94 yields

$$p^2 + p + 1 = |X| \leq |G : \mathrm{N}_G(C_1)| + |G : \mathrm{N}_G(C_2)| + |G : \mathrm{N}_G(G_3)| \leq 3p,$$

which is satisfied if and only if $(p-1)^2 \leq 0$. Contradiction. ∎

We can finally give the proof of Proposition 126. The prime p is odd as a consequence of Proposition 95. Since G has class at least 3, the group G_3 is non-trivial, so, by Lemma 35, there exists a normal subgroup M of G that is contained in G_3 with index p. By Lemma 101, the group G/M has intensity greater than 1 and, as a consequence of Lemma 132, the intensity of G/M is equal to 2. From Lemma 101 it follows that $1 < \text{int}(G) \leq \text{int}(G/M) = 2$. Moreover, Lemma 129(2) yields

$$|G : G_2| = |G/M : [G/M, G/M]| = p^2.$$

We remark that Proposition 126 gives (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) in Theorem 124 and Theorem 125(1). We give the full proof of Theorem 124 in Section 5.4 and the full proof of Theorem 125 in Section 5.2.

Proposition 133. *Let p be a prime number and let G be a finite p-group. Then the following are equivalent.*

1. *One has* $\text{int}(G) > 1$.

2. *The prime p is odd and* $\text{int}(G)$ *is even.*

Proof. The implication (2) $\Rightarrow$ (1) is clear. Assume now (1). Then G is non-trivial and p is odd, by Proposition 95. Moreover, if G has class at least 3, then

Proposition 126 yields $\mathrm{int}(G) = 2$. On the other hand, if G has class at most 2, then we know from Theorems 86 and 105 that $\mathrm{int}(G) = p - 1$, which is even because p is odd. ■

Proposition 134. *Let p be a prime number and let G be a finite p-group of class at least 3. Let α be an intense automorphism of G of order $\mathrm{int}(G)$ and assume that $\mathrm{int}(G) > 1$. Then α has order 2 and, for all $i \geq 1$, it induces scalar multiplication by $(-1)^i$ on G_i/G_{i+1}.*

Proof. Let χ denote the restriction of χ_G to $\langle\alpha\rangle$. By Proposition 126, the intensity of G is 2 and p is odd. In particular, $\chi(\alpha)$ has order 2 in $\omega(\mathbb{F}_p^*)$, so $\chi(\alpha) = -1$. Lemma 104 draws the conclusion. ■

5.2 Groups of class 3

This section is devoted to understanding the structure of p-groups G of class 3 with the property that $|G : G_2| = p^2$. We will see in the next section that all these groups have intensity 2. We also give the proof of Theorem 125.

Lemma 135. *Let p be a prime number and let G be a finite p-group of class 3. Assume that $|G : G_2| = p^2$. Then the following hold.*

1. *One has $|G_2 : G_3| = p$.*
2. *One has $|G_3| \in \{p, p^2\}$.*
3. *The subgroup G_3 is elementary abelian.*

Proof. The group G is non-abelian and, as a consequence of Lemma 36, the group G/G_2 is a vector space over $\mathbb{F}_p$ of dimension 2. By Lemma 25, the commutator map induces a surjective homomorphism $G/G_2 \otimes G/G_2 \to G_2/G_3$ which factors as a surjective homomorphism $\bigwedge^2(G/G_2) \to G_2/G_3$ by the universal property of wedge products. The space $\bigwedge^2(G/G_2)$ has dimension 1 and so $|G_2 : G_3| = p$. Again applying Lemma 25, we derive that G_3 is isomorphic to a quotient of $G/G_2 \otimes G_2/G_3$. In particular, one has

$$1 < |G_3| \leq |G/G_2 \otimes G_2/G_3| = p^2$$

and G_3 is elementary abelian. ■

Thanks to Lemma 135, we can finally give the proof of Theorem 125. Indeed, if p is a prime number and G is a finite p-group of class at least 3 with $\mathrm{int}(G) > 1$, then Proposition 126 yields $\mathrm{int}(G) = 2$ and $w_1 = 2$. We now apply Lemma 135, with G/G_4 in place of G, to get $w_2 = 1$ and $w_3 \in \{1, 2\}$. The proof of Theorem 125 is now complete.

Lemma 136. *Let p be an prime number and let G be a finite p-group of class* 3. *Assume that $|G : G_2| = p^2$. Then the following hold.*

1. *One has $G_2 \subseteq \mathrm{C}_G(G_2)$.*

2. *The commutator map induces an isomorphism $G/\mathrm{C}_G(G_2) \otimes G_2/G_3 \to G_3$.*

3. *One has $|G : \mathrm{C}_G(G_2)| = |G_3|$.*

Proof. (1) As a consequence of Lemma 20, the group $[G_2, G_2]$ is contained in $G_4 = \{1\}$ and G_2 centralizes itself. This proves (1). We now prove (2) and (3) together. The subgroup G_3 is central, because the class of G is 3, so, by Lemma 22, the commutator map $\gamma : G \times G_2 \to G_3$ is bilinear. The right kernel of γ is equal to $G_2 \cap \mathrm{Z}(G)$, which is equal to G_3 as a consequence of Lemma 135(1). The left kernel of γ coincides with $\mathrm{C}_G(G_2)$ and, in particular, γ induces a non-degenerate map $\gamma_1 : G/\mathrm{C}_G(G_2) \times G_2/G_3 \to G_3$ whose image generates G_3. Thanks to (1) the quotient $G/\mathrm{C}_G(G_2)$ is abelian and, thanks to Lemma 36, it has exponent p. By the universal property of tensor products, γ_1 induces a surjective homomorphism $G/\mathrm{C}_G(G_2) \otimes G_2/G_3 \to G_3$, which is also an isomorphism since the index $|G_2 : G_3|$ is equal to p. Moreover, we have $|G : \mathrm{C}_G(G_2)| = |G_3|$. ■

Lemma 137. *Let p be a prime number and let G be a finite p-group of class* 3. *Assume that $|G : G_2| = p^2$. Then the following hold.*

1. *One has $|G_3| = p$ if and only if $|\mathrm{C}_G(G_2) : G_2| = p$.*

2. *One has $|G_3| = p^2$ if and only if $\mathrm{C}_G(G_2) = G_2$.*

Proof. By Lemma 136(3), we have $|G : \mathrm{C}_G(G_2)| = |G_3|$. By Lemma 136(1), the subgroup G_2 is contained in $\mathrm{C}_G(G_2)$, so (1) and (2) follow from the fact that $|G : G_2| = p^2$. ■

Lemma 138. *Let p be a prime number and let G be a finite p-group of class* 3. *Assume that $|G : G_2| = p^2$. Then $\mathrm{C}_G(G_2)$ is abelian.*

Proof. Write $C = \mathrm{C}_G(G_2)$. As a consequence of Lemma 137, the group C/G_2 is cyclic. It follows from Lemma 28 that $[C, C] = [C, G_2] = \{1\}$. ■

Lemma 139. *Let p be an odd prime number and let G be a finite p-group of class* 3. *Assume that $|G : G_2| = p^2$. Then G/G_3 is extraspecial of exponent p.*

Proof. We write $\overline{G} = G/G_3$ and we use the bar notation for the subgroups of $\overline{G}$. The group $\overline{G}$ has class 2 and $\overline{G_2}$ is contained in $\mathrm{Z}(\overline{G})$. By Lemma 135(1), the order of $\overline{G_2}$ is equal to p and, as a consequence of Lemma 27, the groups $\overline{G_2}$ and $\mathrm{Z}(\overline{G})$ coincide. In particular, $\overline{G}$ is extraspecial. We now show that $\overline{G}$ has exponent p. Define $C = \mathrm{C}_G(G_2)$ and $D = \{x \in G \ : \ x^p \in G_3\}$. Then $C \neq G$, because G_2

is not central, and D is a group, thanks to Corollary 48. Let now $x \in G \setminus C$. As a consequence of Lemma 36, the element x^p belongs to G_2. Moreover, by Lemma 136(2), the commutator map induces an isomorphism $G/C \otimes G_2/G_3 \to G_3$, so, since x is not in the centralizer of G_2, the element x^p belongs to G_3. It follows that $x \in D$ and, in particular, we have proven that $G = C \cup D$. The group C is different from G, thus the groups D and G are the same. It follows that $\overline{G} = \overline{D}$ and so $\overline{G}$ has exponent p. ■

Lemma 140. *Let p be an odd prime number and let G be a finite p-group of class 3. Assume that $|G : G_2| = p^2$. Then $G_3 = \mathrm{Z}(G)$.*

Proof. The subgroup G_3 is contained in $\mathrm{Z}(G)$, since G has class 3 and, as a consequence of Lemma 139, the centre of G/G_3 is equal to G_2/G_3. It follows that $\mathrm{Z}(G)/G_3 \subseteq G_2/G_3$ and $\mathrm{Z}(G) \subseteq G_2$. Moreover, the group $\mathrm{Z}(G)$ does not contain G_2, because G has class 3. The group G_2/G_3 having order p, one gets $G_3 = \mathrm{Z}(G)$. ■

Lemma 141. *Let p be an odd prime number and let G be a finite p-group of class 3. Assume that $|G : G_2| = p^2$. Then G_2 is elementary abelian.*

Proof. The group G_2 is abelian as a consequence of Lemma 136(1). We prove that it has exponent p. Let M be a maximal subgroup of G_3; then M has index p in G_3 and it is normal, because G_3 is central. We write $\overline{G} = G/M$ and use the bar notation for the subgroups of $\overline{G}$. The subgroup $\overline{G_3}$ has order p and $|\overline{G} : \overline{G_2}| = |G : G_2| = p^2$. It follows from Lemma 138 that $\mathrm{C}_{\overline{G}}(\overline{G_2})$ is abelian and, from Lemma 137(1), that it contains $\overline{G_2}$ with index p. Write $\overline{C} = \mathrm{C}_{\overline{G}}(\overline{G_2})$. As a consequence of Lemma 139, the subgroup $\overline{C}^p$ is contained in $\overline{G_3}$, so $\mu_p(\overline{C})$ is a normal subgroup of $\overline{G}$ of order at least p^2. Moreover, $\overline{G_3}$ is contained in $\mu_p(\overline{C})$, so $\mu_p(\overline{C})/\overline{G_3}$ is a non-trivial normal subgroup of G/G_3. The quotient G/G_3 is extraspecial, by Lemma 139, so G_2/G_3 is equal to $\mathrm{Z}(G/G_3)$. As a consequence of Lemma 135(1), the quotient G_2/G_3 has order p, so Lemma 29 yields $\overline{G_2} \subseteq \mu_p(\overline{C})$. In particular, one has $G_2^p \subseteq M$. If $M = \{1\}$ we are done, otherwise let N be another maximal subgroup of G_3. In this case, G_3 is elementary abelian of order p^2, by Lemma 135(2-3), and G_2^p is contained in $N \cap M = \{1\}$, by the previous arguments. The exponent of G_2 is thus p. ■

5.3 Intensity given the automorphism

We recall that, for any group G, the lower central series of G is denoted $(G_i)_{i \geq 1}$ and it consists of characteristic subgroups of G. For more detail see Section 1.2. The main result of this section is the following.

Proposition 142. *Let p be an odd prime number and let G be a finite p-group of class 3 such that $|G : G_2| = p^2$. Let moreover α be an automorphism of G of order 2 that induces the inversion map $x \mapsto x^{-1}$ on G/G_2. Then α is intense and* $\mathrm{int}(G) = 2$.

The following assumptions will be valid until the end of Section 5.3. Let p be an odd prime number and let G be a finite p-group of class 3 such that $|G : G_2| = p^2$. Let α be an automorphism of G of order 2 and write $A = \langle \alpha \rangle$. Let moreover $\chi : A \to \{\pm 1\}$ be an isomorphism of groups and assume that the induced action of A on G/G_2 is through χ. We will prove that α is intense.

Lemma 143. *Every subgroup of G that contains G_3 has an A-stable conjugate in G.*

Proof. Let H be a subgroup of G that contains G_3. By Lemma 139, the group G/G_3 is extraspecial of exponent p and by assumption A acts on G/G_2 through χ. As a consequence of Lemmas 121 and Lemma 93, there exists $g \in G$ such that $\alpha(gHg^{-1})/G_3 = (gHg^{-1})/G_3$ and, G_3 being A-stable, $\alpha(gHg^{-1}) = gHg^{-1}$. ∎

We remind the reader that, if H is a subgroup of G, then a positive integer j is a jump of H in G if $H \cap G_j \neq H \cap G_{j+1}$. The j-th width of H in G is $\mathrm{wt}_H^G(j) = \log_p |H \cap G_j : H \cap G_{j+1}|$. For more information about jumps and width see Section 2.3.

Lemma 144. *Let H be a subgroup of G that trivially intersects G_3. Then the following hold.*

1. *If 1 is a jump of H in G, then* $\mathrm{wt}_H^G(1) = 1$.
2. *If 2 is a jump of H in G, then* $H \subseteq \mathrm{C}_G(G_2)$.

Proof. (1) Assume that 1 is a jump of H in G. By Lemma 36, the Frattini subgroup of G is equal to G_2. The subgroup H does not contain G_3 and thus $H \neq G$. By Lemma 33, we have $H\Phi(G) \neq G$ so $H\Phi(G)/\Phi(G) = HG_2/G_2$ has order p. Since HG_2/G_2 is isomorphic to $(H \cap G_1)/(H \cap G_2)$, this proves (1). We now prove (2). Assume that 2 is a jump of H in G. Then by Lemma 82 there exists an element $x \in (H \cap G_2) \setminus G_3$. Fix x. As a consequence of Lemma 135(1), the group G_2 is equal to $\langle x, G_3 \rangle$. The group G_3 being central, it follows that $[H, G_2] = [H, \langle x \rangle]$. The subgroup $[H, \langle x \rangle]$ is contained in $H \cap [G, G_2] = H \cap G_3$, which is trivial by assumption. In particular, H centralizes G_2. ∎

Lemma 145. *Let H be a subgroup of G that trivially intersects G_2. Then H has an A-stable conjugate in G.*

Proof. The group H is abelian, because $[H,H] \subseteq H \cap [G,G] = \{1\}$. By Lemma 140, the groups G_3 and $\mathrm{Z}(G)$ are equal, so the group $T = H \oplus G_3$ is abelian. By Lemma 143, there exists $g \in G$ such that gTg^{-1} is A-stable and, the group G_3 being characteristic, $gTg^{-1} = gHg^{-1} \oplus G_3$. We fix such element g and note that $gTg^{-1} \cap G_2 = G_3$. It follows from Lemma 63 that the induced action of A on gTg^{-1}/G_3 is through χ. Moreover, by Lemma 62, the group A acts on G_3 through $\chi^3 = \chi$. From Lemma 77, it follows that α sends each element of gTg^{-1} to its inverse, so each subgroup of gTg^{-1} is A-stable. In particular, gHg^{-1} is A-stable. ∎

Lemma 146. *Let H be a subgroup of G such that $G_2 = H \oplus G_3$. Then H has an A-stable conjugate in G.*

Proof. By Lemma 135(1), the index $|G_2 : G_3|$ is equal to p, so H has order p. By Lemma 62, the induced action of A on G_2/G_3 and G_3 is respectively through χ^2 and $\chi^3 = \chi$. By assumption, the characters χ and χ^2 are distinct. Moreover, by Lemma 136(1), the group G_2 is abelian and so, by Theorem 68, there exists a unique A-stable complement K of G_3 in G_2. We want to show that H and K are conjugate in G. The groups G_3 and $\mathrm{Z}(G)$ coincide, by Lemma 140, thus $\mathrm{C}_G(H) = \mathrm{C}_G(G_2)$. Moreover, we have that $H \cap [H,G] \subseteq H \cap G_3 = \{1\}$, so $\mathrm{C}_G(H) = \mathrm{N}_G(H)$. Let X be the collection of complements of G_3 in G_2. Then K and all conjugates of H in G are in X. By Lemma 136(3), we have $|G : \mathrm{C}_G(G_2)| = |G_3|$. By Lemma 135(3), the subgroup G_3 is elementary abelian and, by Lemma 114, the cardinality of X is equal to the cardinality of $\mathrm{Hom}(H, G_3)$, which coincides with $|G_3|$ because H has order p. It follows that $|X| = |G : \mathrm{C}_G(G_2)| = |G : \mathrm{N}_G(H)|$ and, every conjugate of H being in X, every complement of G_3 in G_2 is conjugate to H. In particular, K and H are conjugate in G. ∎

Lemma 147. *Let H be a subgroup of G that is not contained in $\mathrm{C}_G(G_2)$ and that has trivial intersection with G_3. Then H has a conjugate that is A-stable.*

Proof. As a consequence of Lemma 144(2), the subgroup H has trivial intersection with G_2. We now apply Lemma 145. ∎

Lemma 148. *Let H be a subgroup of $\mathrm{C}_G(G_2)$ of order p that has trivial intersection with G_3. Then H has a conjugate that is A-stable.*

Proof. Let us call $T = H \oplus G_3$. If $T = G_2$, then H has an A-stable conjugate by Lemma 146. Assume now that $T \cap G_2 = G_3$. Then $H \cap G_2 = H \cap T \cap G_2 = H \cap G_3 = \{1\}$, so we conclude applying Lemma 145. ∎

We denote $G^+ = \{x \in G : \alpha(x) = x\}$ and $G^- = \{x \in G : \alpha(x) = x^{-1}\}$, in concordance with the notation from Section 2.2. In the context of Section 5.3, we will use this notation in Lemmas 149 and 150.

Lemma 149. *Let H be a subgroup of $\mathrm{C}_G(G_2)$ such that $H \cap G_3 = \{1\}$. Then the following hold.*

1. *The subgroup H is elementary abelian.*
2. *One has $G^+ \mathrm{N}_G(H) = \mathrm{N}_G(H)$.*

Proof. (1) The subgroup $\mathrm{C}_G(G_2)$ is abelian, by Lemma 138, and therefore H is abelian. Moreover, as a consequence of Lemma 139, the subgroup H^p is contained in $H \cap G_3 = \{1\}$, so H is elementary abelian. This proves (1). We now prove (2). The subgroup G^+ is contained in G_2, thanks to Lemma 85, and G_2 centralizes C, by definition of C. It follows that $G^+ \mathrm{N}_G(H) \subseteq G_2 \mathrm{N}_G(H) = \mathrm{N}_G(H)$. Since $\mathrm{N}_G(H)$ is contained in $G^+ \mathrm{N}_G(H)$, the proof is complete. ■

Lemma 150. *Let H be a subgroup of G such that $\mathrm{C}_G(G_2) = H \oplus G_3$. Then H has a conjugate that is A-stable.*

Proof. To lighten the notation, write $C = \mathrm{C}_G(G_2)$. If $C = G_2$, then we are done by Lemma 146. Assume now that $C \neq G_2$. As a consequence of Lemma 137(1), the group C contains G_2 with index p and G_3 has order p. We define X to be the collection of subgroups K of G such that $C = K \oplus G_3$ and denote $X^+ = \{K \in X \mid \alpha(K) = K\}$. The centre of G is equal to G_3, by Lemma 140, and, as a consequence of Lemma 29, all elements of X are non-normal subgroups of G. In particular, for any $K \in X$, one has $|G : \mathrm{N}_G(K)| \geq p$. Now, by Lemma 149(1), the subgroup H is elementary abelian, and, G_3 being central of order p, it follows that C is naturally an $\mathbb{F}_p$-vector space. Combining Lemmas 135(1) and 137(1), we get that $\dim C = 3$. Write $C^+ = \{x \in C : \alpha(x) = x\}$ and $C^- = \{x \in C : \alpha(x) = x^{-1}\}$. Then $C = C^+ \oplus C^-$, thanks to Corollary 76 and, as a consequence of Lemma 85, one has $|C^-| = p^2$ and $|C^+| = p$. Moreover, C being abelian, both C^+ and C^- are linear subspaces of C. It is not difficult to show at this point that

$$X^+ = \{C^+ \oplus \ell : \ell \subseteq G^-, \ell \cap G_3 = \{1\}, \dim(\ell) = 1\}.$$

It follows that X^+ has cardinality p, while the cardinality of X is p^2. Moreover, the combination of Lemmas 81 and 149(2), ensures that no two elements of X^+ are conjugate in G. It follows from Lemma 94 that

$$p^2 = |X| \geq \sum_{K \in X^+} |G : \mathrm{N}_G(K)| \geq \sum_{K \in X^+} p = |X^+| p = p^2,$$

and therefore every element of X is conjugate to an element of X^+. In particular, H has an A-stable conjugate. ■

Lemma 151. *Every subgroup of G that trivially intersects G_3 has an A-stable conjugate in G.*

Proof. Let H be a subgroup of G such that $H \cap G_3 = \{1\}$. If H is contained in $\mathrm{C}_G(G_2)$ and has order p, then we are done by Lemma 148. Assume now that H is contained in $\mathrm{C}_G(G_2)$ and that H has order p^2. The group $\mathrm{C}_G(G_2)$ is abelian, by Lemma 138, and thus, as a consequence of Lemmas 135(1) and 137, one has $\mathrm{C}_G(G_2) = H \oplus G_3$. The group H has an A-stable conjugate by Lemma 150. We conclude by Lemma 147, in case H is not contained in $\mathrm{C}_G(G_2)$. ■

Lemma 152. *Let H be a subgroup of G such that $H \cap G_3 \neq \{1\}$. Then H has a conjugate that is A-stable.*

Proof. Lemma 143 covers the case in which H contains G_3. Assume now that the group $H \cap G_3$ is different from both $\{1\}$ and G_3. As a consequence of Lemma 135(2), the cardinality of G_3 is p^2, so $H \cap G_3$ has order p. We write $\overline{G} = G/(H \cap G_3)$ and use the bar notation for the subgroups of $\overline{G}$. The group $\overline{G}$ has class 3 and $|\overline{G} : \overline{G_2}| = p^2$. Moreover, $\overline{H} \cap \overline{G_3} = \{1\}$. Thanks to Lemma 151, the subgroup $\overline{H}$ has an A-stable conjugate, and therefore so does H. ■

Lemma 153. *The automorphism α is intense and* $\mathrm{int}(G) = 2$.

Proof. We will show that $\alpha \in \mathrm{Int}(G)$. Thanks to Lemma 93, it suffices to show that every subgroup of G has an A-stable conjugate. Let H be a subgroup of G. If $H \cap G_3 = \{1\}$, we are done by Lemma 151, otherwise apply Lemma 152. ■

Thanks to Lemma 153, Proposition 142 is proven.

5.4 Constructing intense automorphisms

The aim of Section 5.4 is giving the proof of Theorem 124. We will prove the following essential result.

Proposition 154. *Let p be an odd prime number and let G be a finite p-group of class 3 such that $|G : G_2| = p^2$. Then there exists an automorphism α of G of order 2 that induces the inversion map $x \mapsto x^{-1}$ on G/G_2.*

In order to prove Proposition 154, let p be an odd prime number and let G be a finite p-group of class 3. Let moreover $(G_i)_{i \geq 1}$ denote the lower central series of G and assume that $|G : G_2| = p^2$. We will keep these assumptions and notation until the end of Section 5.4. We will work to construct an automorphism α of G and an isomorphism $\chi : \langle \alpha \rangle \to \{\pm 1\}$ in order to apply the results achieved in the previous section.

Let F be the free group on the set $S = \{a, b\}$ and let $\iota : S \to G$ be a map such that $G = \langle \iota(S) \rangle$. By the universal property of free groups, there exists a unique homomorphism $\theta : F \to G$ such that $\theta(a) = \iota(a)$ and $\theta(b) = \iota(b)$. In particular, the map θ is surjective. We denote by $(F_i)_{i\geq 1}$ the p-central series of F, which is recursively defined as

$$F_1 = F \quad \text{and} \quad F_{i+1} = [F, F_i]F_i^p.$$

We want to stress the fact that the notation we use here for the p-central series of F clashes with the notation we have adopted so far (see the section "Exceptions" from the List of Symbols). Define additionally

$$L = F_3F^p \;\text{ and }\; E = [F, L]F_2^p.$$

The notation we just introduced will be valid until the end of Section 5.4. We will introduce some extra notation between Lemma 158 and Lemma 159. We refer to the diagram given at the end of the present section for a visualization of the proof of Proposition 154.

Lemma 155. *One has $\theta^{-1}(G_2) = F_2$.*

Proof. Since θ is a surjective homomorphism, one has $\theta(F_2) = \theta([F, F]F^p) = G_2G^p = \Phi(G)$ and so, as a consequence of Lemma 36, we get $\theta(F_2) = G_2$. In other words, $F_2 \subseteq \theta^{-1}(G_2)$. The group F being 2-generated, we have that $|F : F_2| = p^2$, and so $F_2 = \theta^{-1}(G_2)$. ∎

Lemma 156. *The commutator map induces an alternating map $F/F_2 \times F/F_2 \to F_2/L$ whose image generates F_2/L. Furthermore, the index $|F_2 : L|$ is at most p.*

Proof. We write $\overline{F} = F/L$ and we will use the bar notation for the subgroups of $\overline{F}$. The subgroup $[F, [F, F]]$ is contained in F_3 and $[\overline{F}, \overline{F}]$ is central. Moreover, $\overline{F}$ being annihilated by p, the subgroup $\overline{F_2}$ coincides with $[\overline{F}, \overline{F}]$. As a consequence of Lemma 22, the commutator map induces a bilinear map $\phi : \overline{F}/\overline{F_2} \times \overline{F}/\overline{F_2} \to \overline{F_2}$ whose image generates $\overline{F_2} = [\overline{F}, \overline{F}]$. The map ϕ is alternating because every element of a group commutes with itself. By the universal property of the exterior square, ϕ factors as a surjective homomorphism $\bigwedge^2(\overline{F}/\overline{F_2}) \to \overline{F_2}$. As a consequence of Lemma 36, the quotient $\overline{F}/\overline{F_2}$ is a 2-dimensional vector space over $\mathbb{F}_p$ and $\bigwedge^2(\overline{F}/\overline{F_2})$ has dimension 1. It follows that $\overline{F_2}$ has order at most p. ∎

Lemma 157. *One has $\theta^{-1}(G_3) = L$ and $|F_2 : L| = p$.*

Proof. As a consequence of Lemma 139, the group G_3 contains G^p, from which it follows that $\theta(L) = G_3$. In particular, the subgroup L is contained in $\theta^{-1}(G_3)$. As a consequence of Lemma 155, the subgroup $\theta^{-1}(G_3)$ is contained in F_2 and $\theta^{-1}(G_3) \neq F_2$, because $G_3 \neq G_2$. In particular, F_2 is different from L. By Lemma 156, the index $|F_2 : L|$ is at most p, and so we get that $|F_2 : L| = p$ and $\theta^{-1}(G_3) = L$. ∎

Lemma 158. *One has* $E \subseteq \ker\theta \cap F_3$.

Proof. The group E is contained in $[F, F_2]F_2^p = F_3$, by definition of the p-central series of F. As a consequence of Lemma 139, the image of L under θ is equal to G_3 and, as a consequence of Lemma 141, the subgroup F_2^p is contained in the kernel of θ. It follows that $\theta(E) = \theta([F, L]F_2^p) = [G, G_3] = G_4 = \{1\}$. ∎

Let β be the endomorphism of F sending a to a^{-1} and b to b^{-1}, and note that β exists by the universal property of free pro-p-groups. Then β^2 is equal to id_F, and thus β is an automorphism of F. Write $B = \langle\beta\rangle$ and define the homomorphism $\sigma : B \to \{\pm 1\}$ by $\beta \mapsto -1$. We will respect this notation until the end of Section 5.4.

Lemma 159. *The induced action of B on F/F_2 and F_2/L is respectively through σ and σ^2.*

Proof. By definition of β, every element of F is inverted by β modulo F_2; in other words, the action of B on F/F_2 is through σ. By Lemma 156, the commutator map induces a bilinear map $\phi : F/F_2 \times F/F_2 \to F_2/L$ whose image generates F_2/L. The group B acts on F/F_2 through σ and, by Lemma 61, the action of B on F_2/L is through σ^2. ∎

Lemma 160. *The induced action of B on L/F_3 is through σ.*

Proof. We write $\overline{F} = F/F_3$ and we use the bar notation for its subgroups. Then $\overline{L}$ is equal to $\overline{F}^p$ and, since $[F, [F, F]]$ is contained in F_3, the group $\overline{F}$ has class at most 2. Moreover, we have that $[F, F]^p \subseteq F_2^p \subseteq F_3$, so $[\overline{F}, \overline{F}]$ is annihilated by p. It follows from Lemma 48 that the p-power map is an endomorphism of $\overline{F}$, and therefore $\overline{L}$ is an epimorphic image of $\overline{F}/\overline{F_2}$. By Lemma 63, the induced action of B on $\overline{L}$ is through σ. ∎

Lemma 161. *The induced action of B on F_3/E is through σ.*

Proof. The group F_2/L is cyclic, thanks to Lemma 157, so Lemma 28 yields $[F_2, F_2] = [F_2, L]$. It follows that $[F_2, F_2]$ is contained in E. The group $[F, F_3]$ is also contained in E and, as a consequence of Lemma 22, the commutator map induces a bilinear map $\phi : F/F_2 \times F_2/L \to F_3/E$. By definition of F_3, the image of ϕ generates F_3/E. By Lemma 159, the induced actions of B on F/F_2 and F_2/L are respectively through σ and σ^2 and, by Lemma 61, the action of B on F_3/E is through $\sigma^3 = \sigma$. ∎

Lemma 162. *The induced action of B on L/E is through σ. Moreover, the kernel of θ is B-stable.*

Proof. As a consequence of Lemmas 160 and 161, the induced actions of B on L/F_3 and F_3/E are both through σ. It follows from Lemma 77 that the action of B on L/E is through σ. As a consequence of Lemmas 157 and 158, one has $E \subseteq \ker\theta \subseteq L$, and, in particular, the action of B on L/E restricts to an action of B on $\ker\theta/E$. It follows that $\ker\theta$ is B-stable and the proof is complete. ∎

Lemma 163. *Given any two generators x and y of G, there exists an intense automorphism of G such that $\alpha(x) = x^{-1}$ and $\alpha(y) = y^{-1}$.*

Proof. Let x and y be generators of G. Without loss of generality, we assume that $\iota(a) = x$ and $\iota(b) = y$. Let moreover $\bar{\theta} : F/\ker\theta \to G$ be the isomorphism that is induced from θ. By Lemma 162, the subgroup $\ker\theta$ of F is B-stable, and therefore β induces an automorphism $\bar{\beta}$ of $F/\ker\theta$. Define $\alpha : G \to G$ by $\alpha = \bar{\theta} \circ \bar{\beta} \circ \bar{\theta}^{-1}$. Then α is an automorphism G of order 2 that inverts the generators x and y. Proposition 142 yields that α is intense. ∎

We remark that Proposition 154 follows directly from Lemma 163. Moreover, we are also finally ready to give the proof of Theorem 124. Proposition 126 gives (1) ⇔ (2) and (1) ⇒ (3). On the other hand, the implication (3) ⇒ (2) is given by the combination of Lemma 163 and Proposition 142. The proof of Theorem 124 is finally complete.

$$\begin{tikzcd}
& F \arrow[rr, dashed] \arrow[d, dash, "-"'] & & G \arrow[d, dash, "-"] \\
& F_2 \arrow[rr, dashed] \arrow[d, dash, "+"'] & & G_2 \arrow[d, dash, "+"] \\
& L = F_3F^p \arrow[rr, dashed] \arrow[dl, dash, "-"'] \arrow[dr, dash, "-"] & & G_3 \arrow[d, dash, "-"] \\
F_3 \arrow[dr, dash, "-"'] & & \ker\theta \arrow[r, dashed] \arrow[dl, dash, "-"] & 1 \\
& E = [F, L]F_2^p & &
\end{tikzcd}$$