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Intense automorphisms of finite groups

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Chapter 4

Intensity of groups of class 2

The main goal of this thesis, as stated in Section 3.2, is to classify all finite p -groups whose group of intense automorphisms is not itself a p -group. We will proceed to a classification by separating the cases according to the class of the p -groups. We remind the reader that a finite p -group is always nilpotent and that its (nilpotency) class is defined to be the number of non-trivial successive quotients of the lower central series (see Section 1.2). If the class is 0, the group is trivial and the intensity is 1. For the case in which the class is 1 (non-trivial abelian case) we refer to Chapter 3. In this chapter we study the case in which the class is equal to 2. We prove the following main result.

Theorem 105. *Let p be a prime number and let G be a finite p -group of class 2. Then the following are equivalent.*

1. *One has $\text{int}(G) \neq 1$.*
2. *The group G is extraspecial of exponent p .*
3. *The prime p is odd and $\text{int}(G) = p - 1$.*

4.1 Small commutator subgroup

Let p be a prime number. We recall that a group A acts on a finite abelian p -group G through a character if there exists a homomorphism $\chi : A \rightarrow \mathbb{Z}_p^*$ such that, for all $x \in G$, $a \in A$, one has $ax = \chi(a)x$. For more detail about actions through characters see Section 2.1.

Until the end of Section 4.1, the following assumptions will be valid. Let p be a prime number and let G denote a finite p -group of nilpotency class 2 (see Section 1.2). Let moreover α be intense of order $\text{int}(G)$. Write $A = \langle \alpha \rangle$ and $\chi = \chi_{G|A}$.

Assume that the intensity of G is greater than 1. It follows that G is non-trivial and p is odd (see Sections 3.2 and 3.3). We will keep this notation until the end of this section, together with the one from the List of Symbols.

Lemma 106. *Assume G_2 has exponent p . Then $\Phi(G)$ is central and A acts on G_2 is through χ^2 .*

Proof. The Frattini subgroup of G is central by Lemma 40 and A acts on G_2 through χ^2 by Lemma 104. ■

Lemma 107. *The homomorphisms $\chi, \chi^2 : A \rightarrow \mathbb{Z}_p^*$ are distinct.*

Proof. Assume $\chi = \chi^2$. Then $\chi(\alpha) = \chi(\alpha)^2$ and $\chi(\alpha) = 1$. It follows that the intensity of G is equal to 1. Contradiction. ■

Lemma 108. *Assume G_2 has exponent p . Then $Z(G) = \Phi(G) = G_2$ and A acts on $Z(G)$ through χ^2 .*

Proof. The group G_2 is a non-trivial subgroup of $Z(G)$ and, by Lemma 106, the group A acts on G_2 through χ^2 . By Corollary 103, the group A acts on $Z(G)$ through a character and, as a consequence of Lemma 66, the action of A on the centre is through χ^2 . On the other hand, by Lemma 104, the induced action of A on G/G_2 is through χ . The group A acts hence on $Z(G)/G_2$ both through χ and χ^2 . The characters χ and χ^2 being distinct, Lemma 66 yields $Z(G) = G_2$. By Lemma 106 the subgroup $\Phi(G)$ is central, and thus $G_2 = \Phi(G) = Z(G)$. ■

Lemma 109. *Assume G_2 has order p . Then G is an extraspecial group of exponent p .*

Proof. Thanks to Lemma 108 we are only left with showing that G has exponent p . Assume by contradiction there exists $g \in G$ of order p^2 and write $H = \langle g \rangle$. Then H^p has order p . Now, H^p is contained in $\Phi(G)$ and, as a consequence of Lemma 108, the Frattini subgroup of G has itself order p . It follows that $H^p = \Phi(G)$ and, in particular, H contains $\Phi(G)$. The group G_2 is equal to $\Phi(G)$, by Lemma 108, so the group H is normal. By Lemma 88(1), the subgroup H is A -stable. As a consequence of Lemma 104, the actions of A on H/G_2 and G_2 are respectively through χ and χ^2 and, by Lemma 107, the characters χ and χ^2 are distinct. From Theorem 68 it follows that the groups H and $(H/G_2) \oplus G_2$ are isomorphic. Contradiction. ■

Lemma 110. *Let Q be a finite p -group of both class and intensity greater than 1. Denote by $(Q_i)_{i \geq 1}$ the lower central series of Q . Then, for all $i \in \mathbb{Z}_{\geq 1}$, the exponent of Q_i/Q_{i+1} divides p .*

Proof. We work by induction on i and we start by assuming $i = 1$. Let M be a normal subgroup of Q that is contained in Q_2 with index p ; the group M exists by Lemma 35. Thanks to the isomorphism theorems, the groups Q/Q_2 and $(Q/M)/(Q_2/M)$ are isomorphic. We write $\overline{Q} = Q/M$ and use the bar notation for the subgroups of $\overline{Q}$. Then $\overline{Q}_2 = [\overline{Q}, \overline{Q}]$ has order p and $\overline{Q}$ has intensity greater than 1, by Lemma 101. From Lemma 109, it follows that $\overline{Q}/\overline{Q}_2$ is elementary abelian and therefore so is Q/Q_2 . Assume now that i is greater than 1 and that the result holds for all indices smaller than i . The property of being annihilated by p is preserved by tensor products and surjective homomorphisms so, as a consequence of Lemma 25, the exponent of Q_i/Q_{i+1} divides p . ■

Corollary 111. *Let Q be a finite p -group of nilpotency class 2. If $\text{int}(Q) > 1$, then $Z(Q) = Q_2$.*

Proof. By Lemma 110, the commutator subgroup of Q has exponent p . To conclude, apply Lemma 108. ■

4.2 More general setting

Throughout this whole section (Section 4.2), let p be a prime number and let G be a finite p -group of class 2 and intensity greater than 1. It follows from the work done in Sections 3.2 and 3.3 that G is not trivial and p is odd. Let α be intense of order $\text{int}(G)$ and write $A = \langle \alpha \rangle$ and $\chi = \chi_{G|A}$. We denote by V and Z respectively G/G_2 and G_2 and by π the canonical projection $G \rightarrow V$. From Lemma 110 it follows that both V and Z are vector spaces over $\mathbb{F}_p$. By Corollary 111, the non-trivial subgroup Z is equal to $Z(G)$ and, as a consequence of Lemma 26, the map $\phi : V \times V \rightarrow Z$ that is induced by the commutator map is alternating.

Lemma 112. *Let H be a linear subspace of Z of codimension 1. Then the map $\phi_H : V \times V \rightarrow Z/H$, defined by $(x, y) \mapsto \phi(x, y) + H$, is non-degenerate.*

Proof. The subgroup H is contained in the centre Z and is therefore a normal subgroup of G . It follows from Lemma 101 that $\text{int}(G/H) > 1$. As a consequence of Lemma 109, the group G/H is extraspecial, and so, thanks to Lemma 26, the map $\phi_H : V \times V \rightarrow Z/H = [G/H, G/H]$ is non-degenerate. ■

Corollary 113. *There exists $n \in \mathbb{Z}_{>0}$ such that $\dim V = 2n$.*

Proof. Let H be a linear subspace of Z of codimension 1 and let ϕ_H be as in Lemma 112. Then ϕ_H is non-degenerate, and so, by Lemma 10, the dimension of V is even. The dimension is positive, because G has class 2. ■

Lemma 114. *Let G be a group, let N be a central subgroup, and let H be a complement of N in G . Let moreover $\mathcal{C}_N$ be the collection of complements of N in G and, for all $f \in \text{Hom}(H, N)$, call $\mathcal{G}_f = \{f(h)h : h \in H\}$. Then the map $\text{Hom}(H, N) \rightarrow \mathcal{C}_N$, given by $f \mapsto \mathcal{G}_f$, is well-defined and bijective.*

Proof. Straightforward. ■

We recall that, as defined in Section 1.1, an isotropic subspace of V is a linear subspace T of V such that $\phi(T \times T) = 0$.

Lemma 115. *Let T be a linear subspace of V . Then T is isotropic if and only if $\pi^{-1}(T)$ is abelian.*

Proof. The subspace T is isotropic if and only if $\phi(T \times T) = 0$, which happens if and only if $[\pi^{-1}(T), \pi^{-1}(T)] = 1$. ■

In the next lemma, we use the same notation as in Section 1.1. The map ϕ_T is given in Definition 6.

Lemma 116. *Let T be an isotropic subspace of V . Then the map $\phi_T : V/T \rightarrow \text{Hom}(T, Z)$, defined by $v + T \mapsto (t \mapsto \phi(v, t))$, is surjective.*

Proof. Let T be an isotropic subspace of V . The subgroup $\pi^{-1}(T)$ is abelian, by Lemma 115, and it contains Z . It follows that $\pi^{-1}(T)$ is normal, and so, by Proposition 88(1), it is A -stable. By Lemma 104, the actions of A on $\pi^{-1}(T)/Z$ and on Z are respectively through χ and χ^2 , which are distinct by Lemma 107. By Theorem 68 the subgroup Z has a unique A -stable complement H in $\pi^{-1}(T)$, which is isomorphic to T via π . We now show that ϕ_T is surjective. For this purpose, let $f \in \text{Hom}(T, Z)$ and note that $\text{Hom}(T, Z)$ and $\text{Hom}(H, Z)$ are naturally isomorphic. We identify f with its image in $\text{Hom}(H, Z)$. By Lemma 114, the set $L = \{f(t)t \mid t \in H\}$ is a complement of Z in $\pi^{-1}(T)$ and so, being H the unique A -stable complement of Z , Lemma 93 guarantees that there exists $g \in G$ such that $L = gHg^{-1}$. Fix such an element g . Then, for each $h \in H$, there exists $t \in H$ such that $[g, h]h = ghg^{-1} = f(t)t$. It follows that $ht^{-1} = [h, g]f(t)$ belongs to both H and Z , but H and Z intersect trivially, so we get $h = t$. We have proven that f is the map $t \mapsto [g, t]$. It follows from Definition 6 that ϕ_T is surjective. ■

Corollary 117. *Let T be an isotropic subspace of V . Then T is maximal isotropic if and only if the map $\phi_T : V/T \rightarrow \text{Hom}(T, Z)$, defined by $v + T \mapsto (t \mapsto \phi(v, t))$, is a bijection.*

Proof. The map ϕ_T is surjective by Lemma 116 and it is injective by Lemma 7. ■

Lemma 118. *The dimension of Z is different from 2.*

Proof. Assume by contradiction that Z has dimension 2. Let T be an isotropic subspace of V of maximal dimension t and let $d = \dim V$, which is positive. From Corollary 117, it follows that $d = 3t$ and in particular that $t > 0$. Let L be a subspace of T of codimension 1, which is itself isotropic. Let moreover $\phi_L : V/L \rightarrow \text{Hom}(L, Z)$ be defined by $v + L \mapsto (l \mapsto \phi(v, l))$. The linear map ϕ_L is surjective by Lemma 116. Let U be the kernel of ϕ_L and let $\phi_U : U \times U \rightarrow Z$ be induced by ϕ . Then $\dim U = d - 3(t - 1) = 3$ and ϕ_U is alternating. By the universal property of wedge products, there exists a unique linear map $\psi : \bigwedge^2 U \rightarrow Z$ that, composed with the canonical map $U \times U \rightarrow \bigwedge^2 U$, gives ϕ_U . The dimension of $\bigwedge^2 U$ being 3, the dimension of $\ker \psi$ is positive and, as a consequence of Lemma 4, there are linearly independent elements $s, r \in U$ such that $\psi(s \wedge r) = 0$. Set $R = L \oplus \mathbb{F}_p s \oplus \mathbb{F}_p r$. By construction, R is an isotropic subspace of V of dimension $t + 1$. Contradiction to the maximality of t . ■

Corollary 119. *The group G is extraspecial of exponent p .*

Proof. The commutator subgroup of G is non-trivial. If G_2 has order p , then G is extraspecial of exponent p , by Lemma 109. We claim that the order of G_2 is in fact p . Assume by contradiction that G_2 has order larger than p . Then, by Lemma 35, there exists a normal subgroup M of G that is contained in G_2 with index p^2 . The group G/M has class 2 and, by Lemma 101, its intensity is greater than 1. This is a contradiction to Lemma 118, with G_2/M in the role of Z . ■

We remark that Corollary 119 gives (1) $\Rightarrow$ (2) in Theorem 105. We complete the proof in the next section.

4.3 The extraspecial case

In Section 4.3 we will see how the structure of extraspecial groups of exponent p (see Section 1.4) is particularly suitable for explicit construction of intense automorphisms of order coprime to p . In this section, we conclude the proof of Theorem 105.

Lemma 120. *Let p be a prime number and let G be a non-abelian extraspecial group of exponent p . Let moreover H be a subgroup of G that trivially intersects G_2 . Then $|G : N_G(H)| = |\text{Hom}(H, G_2)|$.*

Proof. The group G being non-abelian, Lemma 39 yields that $Z(G) = G_2$ and G_2 has order p . Since $H \cap G_2$ is trivial, we have $N_G(H) = C_G(H)$ and H is abelian. By Lemma 22, the commutator map $G \times G \rightarrow G_2$ is bilinear, and moreover, since $H \cap Z(G)$ is trivial, it induces a non-degenerate map $G/C_G(H) \times H \rightarrow G_2$. Now,

both $G/C_G(H)$ and H are $\mathbb{F}_p$ -vector spaces and G_2 has order p . It follows from Lemma 2 that $|G : N_G(H)| = |G : C_G(H)| = |H| = |\text{Hom}(H, G_2)|$. ■

Lemma 121. *Let p be a prime number and let G be a non-abelian extraspecial group of exponent p . Let α be an automorphism of G such that $\langle \alpha \rangle$ acts on G/G_2 through a character. Then $\alpha \in \text{Int}(G)$.*

Proof. Let H be a subgroup of G and write $A = \langle \alpha \rangle$. We want to show that H and $\alpha(H)$ are conjugate in G . As G is non-abelian, Lemma 39 yields that $G_2 = Z(G)$ and G_2 has order p . It follows that either H contains G_2 or the intersection of H with G_2 is trivial. In the first case, H/G_2 is a linear subspace of G/G_2 , and is therefore A -stable; in particular, also H is A -stable. We now consider the case in which $H \cap G_2 = \{1\}$. In this case, H is abelian and the group $T = H \oplus G_2$ is A -stable. The group G_2 being A -stable, $\alpha(H)$ is a complement of G_2 in T . Also each G -conjugate of H is a complement of G_2 in T , because G_2 and T are both normal. By Lemma 114, the number of complements of G_2 in T equals the cardinality of $\text{Hom}(H, G_2)$, which is equal to $|G : N_G(H)|$ by Lemma 120. It follows that the number of complements of G_2 in T is equal to the number of conjugates of H in G . As all conjugates of H are themselves complements of G_2 in T , we get that every complement of G_2 in T is conjugate to H in G . In particular, H and $\alpha(H)$ are conjugate in G . The choice of H being arbitrary, it follows that $\alpha \in \text{Int}(G)$. ■

Lemma 122. *Let p be a prime number and let G be a non-abelian extraspecial p -group of exponent p . Then p is odd and $\text{int}(G) = p - 1$.*

Proof. The prime p is odd, because all groups of exponent 2 are abelian. By Proposition 43, we can write G in the form $G(Z, Y, X, \theta)$, where X, Y , and Z are vector spaces over $\mathbb{F}_p$ and $\theta : X \times Y \rightarrow Z$ is bilinear. Now, the group $\mathbb{F}_p^*$ acts on X, Y , and Z , as described in Section 2.1, and so each $m \in \mathbb{F}_p^*$ gives rise to an automorphism of each of the three vector spaces. For each $m \in \mathbb{F}_p^*$, the following diagram is commutative because θ is bilinear.

$$\begin{array}{ccc} X \times Y & \xrightarrow{\theta} & Z \\ \downarrow m & & \downarrow m^2 \\ X \times Y & \xrightarrow{\theta} & Z \end{array}$$

By Proposition 44, for each $m \in \mathbb{F}_p^*$ there exists an automorphism a_m of G such that the maps induced by a_m respectively on $X \times Y$ and Z are scalar multiplications by m and m^2 . The set $A = \{a_m \mid m \in \mathbb{F}_p^*\}$ is a subgroup of $\text{Aut}(G)$ that is isomorphic to $\mathbb{F}_p^*$. Thanks to Lemma 121, the subgroup A is contained in $\text{Int}(G)$ and therefore $\text{int}(G) = p - 1$. ■

We remark that Lemma 122 is the same as $(2) \Rightarrow (3)$ in Theorem 105. Since the implication $(3) \Rightarrow (1)$ is clear and $(1) \Rightarrow (2)$ is given by Corollary 119, Theorem 105 is finally proven.

Proposition 123. *Let p be a prime number and let G be a finite p -group. Denote by $(G_i)_{i \geq 1}$ the lower central series of G . Assume both the class and the intensity of G are greater than 1. Then, for all $i \in \mathbb{Z}_{\geq 1}$, the exponent of G_i/G_{i+2} divides p .*

Proof. Let α be intense of order $\text{int}(G)$ and write $\chi = \chi_{G|\langle \alpha \rangle}$. Let moreover i be a positive integer. The case in which $i = 1$ is given by the combination of Lemma 101 and Theorem 105; we assume that $i > 1$. As a consequence of Lemma 20, the quotient G_i/G_{i+2} is abelian. By Lemma 104, the action of $\langle \alpha \rangle$ on G_i/G_{i+1} and G_{i+1}/G_{i+2} is respectively through χ^i and χ^{i+1} , which are distinct because $\text{int}(G) \neq 1$. It follows from Theorem 68 that the groups G_i/G_{i+2} and $G_i/G_{i+1} \oplus G_{i+1}/G_{i+2}$ are isomorphic. The exponent of G_i/G_{i+2} divides p as a consequence of Lemma 110. ■

