Intense automorphisms of finite groups
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Chapter 3

Intense automorphisms

Let G be a group. An automorphism α of G is *intense* if for every subgroup H of G there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. We denote by $\mathrm{Int}(G)$ the collection of all intense automorphisms of G, which is easily seen to be a normal subgroup of $\mathrm{Aut}(G)$.

In this chapter we will prove some basic properties of intense automorphisms and formulate the main research question of this thesis. Among others, we will prove the following result.

Theorem 86. *Let p be a prime number and let G be a finite p-group. Then $\mathrm{Int}(G)$ is isomorphic to $S \rtimes C$, where S is a Sylow p-subgroup of $\mathrm{Int}(G)$ and C is a subgroup of $\mathbb{F}_p^*$. Moreover, if G is non-trivial abelian, then $C = \mathbb{F}_p^*$.*

3.1 Basic properties

Section 3.1 is devoted to basic properties of intense automorphisms. Most of the notation used appears in the List of Symbols, at the beginning of this thesis.

Proposition 87. *Let G be a group. Then $\mathrm{Inn}(G) \subseteq \mathrm{Int}(G) \subseteq \mathrm{Aut}(G)$ and $\mathrm{Int}(G)$ is normal in $\mathrm{Aut}(G)$.*

Proof. Straightforward application of the definitions. ∎

We recall that, if A is a group acting on a set X, a subset Y of X is A-stable if the action of A on X restricts to an action of A on Y (see Section 2.1).

Lemma 88. *Let G be a group and let N be a normal subgroup of G. Then the following hold.*

1. *The subgroup N is $\mathrm{Int}(G)$-stable.*

2. *The natural projection* $G \to G/N$ *induces a well-defined homomorphism* $\mathrm{Int}(G) \to \mathrm{Int}(G/N)$*, by means of* $\alpha \mapsto (xN \mapsto \alpha(x)N)$*.*

3. *Assume* N *is contained in* $\mathrm{Z}(G)$*. Then the image of the homomorphism* $\mathrm{Int}(G) \to \mathrm{Aut}(N)$*, sending* α *to* $\alpha_{|N}$*, is contained in* $\mathrm{Int}(N)$*.*

Proof. (1) Intense automorphisms send every subgroup to a conjugate and therefore each normal subgroup to itself. (2) The map is well-defined as a consequence of (1) and it is a homomorphism by construction. (3) This follows from the fact that conjugation in G restricts to the trivial map on each subgroup of $\mathrm{Z}(G)$. ∎

In the following proposition, let $\omega : \mathbb{F}_p^* \to \mathbb{Z}_p^*$ be the Teichmüller character at p as defined in Section 2.1.

Lemma 89. *Let* p *be a prime number and let* V *be a vector space over* $\mathbb{F}_p$*. Then there exists a unique injective homomorphism* $\lambda : \mathrm{Int}(V) \to \mathbb{Z}_p^*$ *such that the following hold.*

1. *The group* $\mathrm{Int}(V)$ *acts on* V *through* λ*.*

2. *If* $V \neq 0$*, then* $\lambda(\mathrm{Int}(V)) = \omega(\mathbb{F}_p^*)$*.*

Proof. If $V = 0$, define $\lambda : \mathrm{id}_V \mapsto 1$. Assume $V \neq 0$. Since V is abelian, every one-dimensional subspace of V is stable under the action of $\mathrm{Int}(V)$. It follows that, for all $v \in V \setminus \{0\}$ and $\alpha \in \mathrm{Int}(V)$, there exists (a unique) $\mu(\alpha, v) \in \mathbb{F}_p^*$ such that $\alpha(v) = \mu(\alpha, v)v$. We will show that $\mu(\alpha, v)$ is independent of the choice of v. To this end, let $\alpha \in \mathrm{Int}(V)$ and let v and w be elements of $V \setminus \{0\}$. If v and w are linearly dependent, then $\mu(\alpha, v) = \mu(\alpha, w)$. We assume that v and w are linearly independent. From the linearity of α, it follows that $\mu(\alpha, v+w)(v+w) = \mu(\alpha, v)v + \mu(\alpha, w)w$. The vectors v and w are linearly independent, so $\mu(\alpha, v) = \mu(\alpha, v+w) = \mu(\alpha, w)$, as required. We fix $v \in V \setminus \{0\}$ and define $\mu : \mathrm{Int}(V) \to \mathbb{F}_p^*$ by $\alpha \mapsto \mu(\alpha, v)$. The map μ is well-defined and it is an injective homomorphism of groups by construction. Moreover, μ is surjective, because scalar multiplication by any element of $\mathbb{F}_p^*$ is an intense automorphism of V. We define $\lambda = \omega \circ \mu$. Then $\mathrm{Int}(V)$ acts on V through λ and the image of λ is equal to $\omega(\mathbb{F}_p^*)$ by construction. The uniqueness of λ follows from Lemma 66. ∎

We recall that, an action of a group C on a group B is a homomorphism $C \to \mathrm{Aut}(B)$ (see Section 2.1).

Definition 90. *Let* C *be a finite group acting on a finite group* B*. Assume moreover that both* B *and* C *act on a set* X*. The actions are said to be* compatible *if for all* $x \in X$*,* $b \in B$*, and* $c \in C$*, one has* $c(bx) = (cb)(cx)$*.*

Lemma 91 (Glauberman's lemma)**.** *Let G and A be finite groups of coprime orders. Assume that at least one of A and G is solvable. Assume A acts on G and that each of them acts on some set X, where the action of G is transitive. Finally, assume the three actions are compatible. Then there exists an A-stable element in X.*

Proof. See [Isa08, Lemma 3.24]. ∎

Lemma 92 (An equivalent condition)**.** *Let G be a finite group and let $\alpha \in \mathrm{Aut}(G)$ be of order coprime to the order of G. Let H and N be subgroups of G and assume that $\alpha(N) = N$. Then the following are equivalent.*

1. *There exists $a \in N$ such that $\alpha(H) = aHa^{-1}$.*
2. *There exists $b \in N$ such that bHb^{-1} is $\langle\alpha\rangle$-stable.*

Proof. (2) $\Rightarrow$ (1) By assumption there exists an element $b \in N$ such that $bHb^{-1} = \alpha(bHb^{-1}) = \alpha(b)\alpha(H)\alpha(b)^{-1}$. Define $a = \alpha(b)^{-1}b$. (1) $\Rightarrow$ (2) Write $X = \{gHg^{-1} : g \in N\}$. Then N acts on X by conjugation and $\langle\alpha\rangle$ acts on X by assumption. The actions are compatible and the action of N is transitive. By Lemma 91, there exists an element of X that is fixed by α. ∎

Lemma 93. *Let G be a finite group and let α be an automorphism of G of order coprime to $|G|$. Then $\alpha \in \mathrm{Int}(G)$ if and only if each subgroup of G has an $\langle\alpha\rangle$-stable conjugate.*

Proof. Take $N = G$ in Lemma 92. ∎

Lemma 94. *Let G be a finite group and let $\alpha \in \mathrm{Int}(G)$ be of order coprime to the order of G. Let X be a collection of subgroups of G on which G acts by conjugation and let $X^+ = \{H \in X : \alpha(H) = H\}$. Then*

$$|X| \leq \sum_{H \in X^+} |G : \mathrm{N}_G(H)|.$$

Equality holds if and only if the elements of X^+ are pairwise non-conjugate in G.

Proof. Let $\mathcal{C}$ be the collection of orbits of X under G. By Lemma 93, there exists a subset $\mathcal{R}$ of X^+ whose elements are representatives for the elements of $\mathcal{C}$. It follows that

$$|X| = \sum_{C \in \mathcal{C}} |C| = \sum_{H \in \mathcal{R}} |G : \mathrm{N}_G(H)| \leq \sum_{H \in X^+} |G : \mathrm{N}_G(H)|.$$

Equality holds if and only if $\mathcal{R} = X^+$. ∎

3.2 The main question

Let p be a prime number and let G be a finite p-group. We recall that $\Phi(G) = [G,G]G^p$ and so $G/\Phi(G)$ is a vector space over $\mathbb{F}_p$ (see Section 1.3). In Section 3.2 we build the foundation for our theory and we give the dictionary that we will use throughout the whole thesis. We will also prove the following result.

Proposition 95. *Let G be a finite 2-group. Then $\operatorname{Int}(G)$ is a finite 2-group.*

Definition 96. *Let p be a prime number and let G be a finite p-group. The* intense character *of G is the homomorphism $\chi_G : \operatorname{Int}(G) \to \mathbb{Z}_p^*$ that is gotten from composition of the following.*

- *The homomorphism $\operatorname{Int}(G) \to \operatorname{Int}(G/\Phi(G))$ from Proposition 88(2).*
- *The homomorphism $\lambda : \operatorname{Int}(G/\Phi(G)) \to \mathbb{Z}_p^*$ from Proposition 89.*

Lemma 97. *Let p be a prime number and let G be a finite p-group. Let moreover $\chi_G : \operatorname{Int}(G) \to \mathbb{Z}_p^*$ be the intense character of G. Then the group $\ker \chi_G$ is the unique Sylow p-subgroup of $\operatorname{Int}(G)$.*

Proof. If G is the trivial group, then $\operatorname{Int}(G) = \ker \chi_G = \{1\}$ and $\{1\}$ is a Sylow p-subgroup of $\operatorname{Int}(G)$. Assume now G is non-trivial and set $V = G/\Phi(G)$. By Lemma 89, the map $\lambda : V \to \operatorname{Int}(V)$ is injective and so, with the notation of Lemma 34, the subgroup $\ker \chi_G$ is equal to $\operatorname{Int}(G) \cap \ker \phi$. As a consequence of Lemma 34, the kernel of χ_G is a normal Sylow p-subgroup of $\operatorname{Int}(G)$. Since $\operatorname{Int}(G)$ acts on the collection of its Sylow p-subgroup in a transitive manner, $\operatorname{Int}(G)$ has a unique Sylow p-subgroup, namely $\ker \chi_G$. ∎

Definition 98. *Let p be a prime number and let G be a finite p-group. Let $\chi_G : \operatorname{Int}(G) \to \mathbb{Z}_p^*$ be the intense character of G. The* intensity *of G is the value $|\operatorname{Int}(G) : \ker \chi_G|$ and it is denoted by $\operatorname{int}(G)$.*

We remark that the intensity of a p-group G is equal to the size of the image of the intense character χ_G inside $\omega(\mathbb{F}_p^*)$. In particular, if G is a 2-group, then its intensity is always 1.

Lemma 99. *Let p be a prime number and let G be a finite p-group. Let moreover $\chi_G : \operatorname{Int}(G) \to \mathbb{Z}_p^*$ be the intense character of G. Then $\operatorname{int}(G)$ divides $p-1$ and $\ker \chi_G$ has a cyclic complement in $\operatorname{Int}(G)$ of order $\operatorname{int}(G)$.*

Proof. The image of χ_G is a subgroup of $\omega(\mathbb{F}_p^*)$, which has order $p-1$. It follows that $\operatorname{int}(G)$ divides $p-1$. Now, by Proposition 97, the kernel of χ_G is the unique Sylow p-subgroup of G and it is therefore normal. The group $\ker \chi_G$ has a complement C in $\operatorname{Int}(G)$, by the Schur-Zassenhaus theorem, and C is cyclic because it is isomorphic to a subgroup of $\mathbb{F}_p^*$. ∎

Proposition 95 now follows from Lemmas 97 and 99.

The major goal of this thesis if classifying all pairs (p, G) where p is a prime number and G is a finite p-group with $\mathrm{int}(G) > 1$. Starting from the next chapter, we will therefore often be working with odd primes. Explicit assumptions will be made at the beginning of each section.

3.3 The abelian case

The main result of Section 3.3 is the following.

Proposition 100. *Let p be a prime number and let G be a finite non-trivial abelian p-group. Then* $\mathrm{int}(G) = p - 1$.

Lemma 101. *Let p be a prime number and let G be a finite p-group. Let N be a normal subgroup of G. If $N \neq G$, then $\mathrm{int}(G)$ divides $\mathrm{int}(G/N)$.*

Proof. Assume $N \neq G$; then G is non-trivial. Let $\phi : \mathrm{Int}(G) \to \mathrm{Int}(G/N)$ be as in Lemma 88(2). The subgroup $N\Phi(G)$ is different from G, thanks to Lemma 33, and therefore $\Phi(G)N/N \neq G/N$. The groups $(G/N)/\Phi(G/N)$ and $G/(\Phi(G)N)$ being isomorphic (and non-trivial!), it follows that $\chi_G = \chi_{G/N} \circ \phi$. The image of χ_G is a subgroup of the image of $\chi_{G/N}$ and thus $\mathrm{int}(G)$ divides $\mathrm{int}(G/N)$. ∎

We recall that a group A acts through a character on a finite abelian p-group G if there exists a homomorphism $\chi : A \to \mathbb{Z}_p^*$ such that, for all $a \in A$ and $x \in G$, one has $ax = \chi(a)x$. For more details, see Section 2.1.

Lemma 102. *Let p be a prime number and let G be a finite abelian p-group. Let α be intense of order dividing $\mathrm{int}(G)$ and write $\chi = \chi_{G|\langle\alpha\rangle}$. Then $\langle\alpha\rangle$ acts on G through χ and, if G is non-trivial, then $\mathrm{int}(G) = p - 1$.*

Proof. Write $A = \langle\alpha\rangle$. If G is the trivial group, then the only automorphism of G is the identity, which is intense. Assume now G is non-trivial. The group $\omega(\mathbb{F}_p^*)$ acts on G (as described at the beginning of Section 2.1) via intense automorphisms and it induces scalar multiplication by elements of $\mathbb{F}_p^*$ on $G/\Phi(G)$. The image of the intense character of G is thus $\omega(\mathbb{F}_p^*)$, and so, $\mathrm{int}(G) = p-1$. Let now Ω denote the image of $\omega(\mathbb{F}_p^*) \to \mathrm{Int}(G)$ and write $\Omega = \langle\beta\rangle$. Then $\mathrm{Int}(G) = \ker \chi_G \rtimes \Omega$, and, as a consequence of Schur-Zassenhaus, there exist $m \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \ker \chi_G$ such that $\alpha = \gamma\beta^m\gamma^{-1}$. We get

$$\chi(\alpha) = \chi_G(\alpha) = \chi_G(\gamma\beta^m\gamma^{-1}) = \chi_G(\beta^m).$$

Since each homomorphism of abelian groups is $\mathbb{Z}_p$-linear and Ω acts on G through $\chi_{G|\Omega}$, the group A acts on G through χ. ∎

We remark that Proposition 100 is a special case of Lemma 102. Moreover, Theorem 86 is proven by combining Lemmas 97, 99, and Proposition 100.

Corollary 103. *Let p be a prime number and let G be a finite p-group. Let moreover α be an intense automorphism of G of order dividing* $\mathrm{int}(G)$*. Then $\langle\alpha\rangle$ acts on the centre of G through a character $\langle\alpha\rangle \to \mathbb{Z}_p^*$.*

Proof. Let $\zeta : \mathrm{Int}(G) \to \mathrm{Int}(\mathrm{Z}(G))$ be the map from Lemma 88(3) and define $\sigma = \chi_{\mathrm{Z}(G)|\langle\zeta(\alpha)\rangle} \circ \zeta_{|\langle\alpha\rangle}$. Lemma 97 yields that $\langle\alpha\rangle$ acts on $\mathrm{Z}(G)$ through σ. ∎

Lemma 104. *Let p be a prime number and let G be a finite p-group. Let α be intense of order dividing* $\mathrm{int}(G)$ *and write $\chi = \chi_{G|\langle\alpha\rangle}$. Denote by $(G_i)_{i\geq 1}$ the lower central series of G. Then, for all $i \in \mathbb{Z}_{\geq 1}$, the induced action of $\langle\alpha\rangle$ on G_i/G_{i+1} is through χ^i.*

Proof. Denote $A = \langle\alpha\rangle$. As a consequence of Proposition 88(2), the action of A on G induces an action of A on G/G_2. By Proposition 102, the action of A on G/G_2 is through χ. We now apply Lemma 62. ∎