Intense automorphisms of finite groups

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Title: Intense automorphisms of finite groups
Issue Date: 2017-09-05

Chapter 2

Coprime actions

The aim of this chapter is to create tools for later use, giving them however their own chance to shine. In Section 2.1, we define *actions through characters* and prove a fundamental result, Theorem 68, in the context of intense automorphisms of groups. In Section 2.2, we prove some elementary, yet quite entertaining, results concerning involutions of groups of odd order. The results from Section 2.2 will spark throughout the thesis, starting with Chapter 5. The last section of this chapter, Section 2.3, is dedicated to the theory of *jumps*. In some sense, through jumps (and their width), we are able to recover structural information about subgroups of a given finite p-group. This theory will be heavily used when dealing with p-obelisks (from Chapter 10 onwards).

2.1 Actions through characters

Until the end of Section 2.1, let p be a prime number. Every finite abelian p-group G is naturally a $\mathbb{Z}_p$-module, with scalar multiplication $\mathbb{Z}_p \to \mathrm{End}(G)$ defined by

$$m \mapsto [x \mapsto (m \bmod |G|)\, x]\,.$$

It follows directly from this definition that every homomorphism between abelian p-groups is $\mathbb{Z}_p$-linear, a fact that we will make hidden use of in several proofs from Chapter 2. To conclude, we remark that we have here adopted the additive notation for the abelian group G, but this will sadly not be the case through the whole thesis. We will indeed often deal, instead of abelian groups, with abelian quotients of non-abelian groups (for which the multiplicative notation will be used). The first time we adopt the multiplicative notation in this context is in the proof of Lemma 77.

Definition 56. *Let A and G be groups. An* action of A on G *is a homomorphism $A \to \mathrm{Aut}(G)$.*

Definition 57. *Let A be a group acting on a set X. A subset Y of X is A-*stable *(or* stable under the action of A*) if the action of A on X restricts to an action of A on Y.*

Definition 58. *Let A be a group and let $\mathbb{Z}A$ denote its group ring over $\mathbb{Z}$. An* A-module *is a module over $\mathbb{Z}A$.*

With respect to the last definition, any finite abelian p-group is naturally a $\mathbb{Z}_p^*$-module. We stress that, if A is a group, then each A-module is, in particular, an abelian group.

Definition 59. *Let A be a group acting on two sets T and Z. A map $\phi : T \to Z$ is said to* respect the action of A *if, for all $t \in T$, $a \in A$, one has $\phi(at) = a\phi(t)$.*

Definition 60. *Let A be a group and let G be a finite p-group that is also an A-module. Let $\chi : A \to \mathbb{Z}_p^*$ be a homomorphism. Then* A acts on G through χ *if, for all $a \in A$ and $x \in G$, one has $ax = \chi(a)x$.*

We want to emphasize the fact that $\mathrm{Hom}(A, \mathbb{Z}_p^*)$ is a group under multiplication (induced by that in $\mathbb{Z}_p^*$). We will refer to the elements of $\mathrm{Hom}(A, \mathbb{Z}_p^*)$ as *characters* of A.

Lemma 61. *Let X, Y, and Z be finite abelian p-groups. Let A be a group acting on X, Y, and Z and let $\phi : X \times Y \to Z$ be a bilinear map respecting the action of A. Let moreover, χ and ψ be group homomorphisms $A \to \mathbb{Z}_p^*$ such that A acts on X and Y respectively through χ and ψ. Then A acts on $\langle \phi(X \times Y) \rangle$ through $\chi\psi$.*

Proof. Let $(x, y) \in X \times Y$ and $a \in A$. Then one has

$$a\phi(x, y) = \phi(ax, ay) = \phi(\chi(a)x, \psi(a)y) = \chi(a)\psi(a)\phi(x, y) = (\chi\psi)(a)\phi(x, y).$$

Since χ and ψ are homomorphisms, the action of A on $\phi(X \times Y)$ is through $\chi\psi$. ∎

Lemma 62. *Let p be a prime number and let G be a finite p-group. Let moreover A be a finite group acting on G and let $\chi : A \to \mathbb{Z}_p^*$ be a homomorphism. Denote by $(G_i)_{i \geq 1}$ the lower central series of G and assume that the induced action of A on G/G_2 is through χ. Then, for all $i \in \mathbb{Z}_{\geq 1}$, the induced action of A on G_i/G_{i+1} is through χ^i.*

The elements of the lower central series of a group are characteristic subgroups and, for each $i \in \mathbb{Z}_{\geq 1}$, the quotient G_i/G_{i+1} is abelian. Lemma 62 is thus well-stated.

Proof. We will work by induction on i. If $i = 1$, we are done by hypothesis. Suppose now that $i > 1$ and that the result holds for all indices smaller than i. By Lemma 24 the commutator map induces a bilinear map $G/G_2 \times G_{i-1}/G_i \to G_i/G_{i+1}$ whose image generates G_i/G_{i+1}. By the induction hypothesis, the induced action of A on G_{i-1}/G_i is through χ^{i-1} and, by Lemma 61, the group A acts on G_i/G_{i+1} through $\chi\chi^{i-1} = \chi^i$. ∎

Lemma 63. *Let A be a group and let G and H be finite p-groups that are also A-modules. Let moreover $\phi : G \to H$ and $\chi : A \to \mathbb{Z}_p^*$ be group homomorphisms. Assume that the action of A on G is through χ. If ϕ is surjective and ϕ respects the action of A, then A acts on H through χ.*

Proof. Let $a \in A$. If ϕ is surjective, then, for each $h \in H$ there exists $g \in G$ such that $\phi(g) = h$. If, moreover, the action of A is respected by ϕ, then $ah = a\phi(g) = \phi(ag) = \phi(\chi(a)g) = \chi(a)\phi(g) = \chi(a)h$. ∎

Lemma 64. *The short exact sequence of abelian groups*

$$1 \longrightarrow 1 + p\mathbb{Z}_p \longrightarrow \mathbb{Z}_p^* \longrightarrow \mathbb{F}_p^* \longrightarrow 1$$

has a unique section $\omega : \mathbb{F}_p^* \to \mathbb{Z}_p^*$.

The last is a classical result, which can be found for example in [Coh07, §4.3]. The homomorphism $\omega : \mathbb{F}_p^* \to \mathbb{Z}_p^*$ is called the *Teichmüller character* at p and its image is contained in the torsion subgroup of $\mathbb{Z}_p^*$. (For a reminder of the notation, see the List of Symbols.) Moreover, if p is odd, then $\omega(\mathbb{F}_p^*)$ is in fact equal to the torsion subgroup of $\mathbb{Z}_p^*$; for more information see for example Section 4.3 from [Coh07].

We remark that, if V is a vector space over $\mathbb{F}_p$, then, for each $v \in V$ and for each $a \in \mathbb{F}_p^*$, one has $av = \omega(a)v$. It follows that the natural action of $\mathbb{F}_p^*$ on a vector space over $\mathbb{F}_p$ is through the Teichmüller character.

Lemma 65. *Let A be a finite group and let $\lambda, \mu : A \to \mathbb{Z}_p^*$ be distinct group homomorphisms. Assume that p is odd. Then there exists $a \in A$ such that the element $\lambda(a) - \mu(a)$ belongs to $\mathbb{Z}_p^*$.*

Proof. Let $\pi : \mathbb{Z}_p \to \mathbb{F}_p$ denote the canonical projection and let $\omega : \mathbb{F}_p^* \to \mathbb{Z}_p^*$ be the Teichmüller character. The group A being finite, the images of λ and μ live in the torsion of $\mathbb{Z}_p^*$, which is equal to $\omega(\mathbb{F}_p^*)$. Let now $a \in A$ be such that $\lambda(a) \neq \mu(a)$. As a consequence of Lemma 64, each element of $\omega(\mathbb{F}_p^*)$ is uniquely determined by its image modulo p and, the characters being distinct, $\pi(\chi(a) - \psi(a)) \in \mathbb{F}_p^*$. It follows that $\chi(a) - \psi(a)$ is invertible in $\mathbb{Z}_p$. ∎

Lemma 66. *Let A be a finite group and let G be a finite p-group that is also an A-module. Let moreover $\lambda, \mu : A \to \mathbb{Z}_p^*$ be distinct group homomorphisms. Assume that p is odd and that A acts on G through both λ and μ. Then $G = \{0\}$.*

Proof. Let $x \in G$ and let $a \in A$ be as in Lemma 65. Then $\lambda(a)x = ax = \mu(a)x$ and $(\lambda(a) - \mu(a))x = 0$. The element $\lambda(a) - \mu(a)$ being invertible in $\mathbb{Z}_p$, it follows that $x = 0$. As the choice of x was arbitrary, we get $G = \{0\}$. ∎

Definition 67. *Let G be a group and let N be a normal subgroup of G. A subgroup H of G is a* complement *of N in G if $N \cap H = \{1\}$ and $NH = G$.*

Theorem 68. *Assume that p is odd. Let A be a finite abelian group and let*

$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} G/N \longrightarrow 0$$

be a short exact sequence of A-modules. Let moreover $\lambda, \mu : A \to \mathbb{Z}_p^*$ be two distinct group homomorphisms and assume that the following hold.

1. *The group G is a finite p-group.*
2. *The group A acts on N through λ.*
3. *The group A acts on G/N through μ.*

Then $\iota(N)$ has a unique A-stable complement in G.

We will devote the remaining part of Section 2.1 to the proof of Theorem 68. For this purpose, let $R = \mathbb{Z}_p A$ be the group algebra of A over $\mathbb{Z}_p$ and let σ_λ and σ_μ be the homomorphisms of $\mathbb{Z}_p$-algebras $R \to \mathbb{Z}_p$ that are respectively induced, via linear extension, by λ and μ. We define $I_\lambda = \ker \sigma_\lambda$ and $I_\mu = \ker \sigma_\mu$.

Lemma 69. *One has $R = I_\lambda + I_\mu$.*

Proof. We will construct an invertible element in $I_\lambda + I_\mu$. Let $a \in A$ be as in Lemma 65. The element $\lambda(a) - \mu(a) = -(a - \lambda(a)) + (a - \mu(a))$ belongs to $I_\lambda + I_\mu$, because $a - \lambda(a) \in I_\lambda$ and $a - \mu(a) \in I_\mu$, and $\lambda(a) - \mu(a)$ is invertible because of the choice of a. ∎

Lemma 70. *The subgroup $\iota(N)$ has an A-stable complement.*

Proof. Let $(e, f) \in I_\lambda \times I_\mu$ be such that $e + f = 1$ in R; the pair (e, f) exists thanks to Lemma 69. As a direct consequence of the definition of I_μ, the group G/N is annihilated by f and $f(G) \subseteq \iota(N)$. From the fact that $f \equiv 1 \bmod I_\lambda$, it follows that $f(G) = \iota(N)$. With a similar argument, one shows that $e(G)$ is isomorphic to $e(G/N) = G/N$. We now have that

$$G = (e + f)G = e(G) + f(G) = e(G) + \iota(N)$$

and, the cardinalities of $e(G)$ and $\iota(N)$ being respectively $|G : N|$ and $|N|$, it follows that $G = e(G) \oplus \iota(N)$. The subgroup $e(G)$ is thus a complement of $\iota(N)$ in G. Furthermore, the ring R being commutative, for all $a \in A$, one has that $ae(G) = ea(G)$ is contained in $e(G)$ and therefore $e(G)$ is A-stable. ∎

Lemma 71. *There exists a unique A-stable complement of $\iota(N)$.*

Proof. The subgroup $\iota(N)$ has an A-stable complement in G, by Lemma 70; assume it has two. Then there exist maps $f, f' : G \to N$ respecting the action of A such that $f \circ \iota = f' \circ \iota = \mathrm{id}_N$. We fix such f, f' and write $r = f - f'$; we will show that $r = 0$. Since $f \circ \iota = f' \circ \iota$, the subgroup $\iota(N)$ is contained in the kernel of r. It follows that $r \in \mathrm{Hom}(G/\iota(N), N)$, and so, thanks to Lemma 63, the group A acts on the image of r through μ. On the other hand, the image of r is contained in N and hence the action of A on $r(G)$ is also through λ. It follows from Lemma 66 that $r = 0$, as claimed. In particular, $f = f'$, and so $\iota(N)$ has a unique A-stable complement in G. ∎

In view of Lemma 71, Theorem 68 is proven.

2.2 Involutions

Let G be a finite group of odd order and let $A = \langle \alpha \rangle$ be a multiplicative group of order 2. It follows that the orders of G and A are coprime. Assume that A acts on G and define

$$G^+ = \{g \in G : \alpha(g) = g\} \quad \text{and} \quad G^- = \{g \in G : \alpha(g) = g^{-1}\}.$$

We will keep the notation we just introduced until the end of Section 2.2. We remind the reader about the List of Symbols, at the beginning of this thesis.

Lemma 72. *One has $G^+ \cap G^- = \{1\}$.*

Proof. Let $a \in G^+ \cap G^-$. Then we have $a = \alpha(a) = a^{-1}$, so $a^2 = 1$. The order of G being odd, it follows that $a = 1$. ∎

Proposition 73. *The set G^+ is a group. Moreover, G^+ acts by conjugation on the set G^-.*

Proof. The subset G^+ is a group, because α is a homomorphism. Let now g be in G^+ and a in G^-. Then we have that

$$\alpha(gag^{-1}) = \alpha(g)\alpha(a)\alpha(g)^{-1} = ga^{-1}g^{-1} = (gag^{-1})^{-1},$$

and so gag^{-1} belongs to G^-. ∎

Lemma 74. *The map $G/G^+ \to G^-$ that is defined by $xG^+ \mapsto x\alpha(x)^{-1}$ is a bijection. Moreover, $|G| = |G^+||G^-|$.*

Proof. Denote by ϕ the map $G/G^+ \to G^-$ that is defined by $xG^+ \mapsto x\alpha(x)^{-1}$. To show that ϕ is injective is a straightforward exercise. To prove that it is surjective, we take $b \in G^-$. Since the order of G is odd, there exists a unique a in G such that $a^2 = b$. The element a belongs to $\langle b \rangle$, ando so $\alpha(a) = a^{-1}$. As a consequence, we have that $a\alpha(a)^{-1} = a^2 = b$. We have proven that ϕ is a bijection, from which it follows that $|G|/|G^+| = |G^-|$. ∎

Lemma 75. *The map $G^+ \times G^- \to G$, defined by $(x, y) \mapsto xy$, is a bijection.*

Proof. Let (x, y) and (z, t) be elements of $G^+ \times G^-$ satisfying $xy = zt$. Then $ty^{-1} = z^{-1}x$. By Lemma 73, the set G^+ is a subgroup of G, so $z^{-1}x \in G^+$. As a consequence, we get that $ty^{-1} = \alpha(ty^{-1}) = t^{-1}y$. It follows that $t^2 = y^2$ so, the order of G being odd, the elements t and y coincide. Consequently, $(x, y) = (z, t)$ and the map is injective. Now by Lemma 74, the cardinalities of $G^+ \times G^-$ and G are the same and the given multiplication is also surjective. ∎

Corollary 76. *Assume G is abelian. Then $G = G^+ \oplus G^-$.*

Proof. The sets G^+ and G^- are both subgroups of G, because G is abelian, and $G^+ \cap G^- = \{1\}$, by Lemma 72. It follows from Lemma 75 that $G = G^+ \oplus G^-$. ∎

Lemma 77. *Let N be a normal A-stable subgroup of G such that the restriction of α to N equals the map $x \mapsto x^{-1}$. Assume moreover that the automorphism of G/N that is induced by α is equal to the inversion map $g \mapsto g^{-1}$. Then $G = G^-$ and G is abelian.*

Proof. From the assumptions it follows that, for each $x \in G$, the element $\alpha(x)x$ belongs to N. Since the order of G is odd, if $x \in G^+$, then x is actually an element of N. It follows that G^+ is contained in N, but N is contained in G^- by assumption. As a consequence of Lemma 72, the subgroup G^+ is trivial so, as a consequence of Lemma 75, the group G is equal to G^-. Let now x and y be elements of G. Then we have that

$$y^{-1}x^{-1} = (xy)^{-1} = \alpha(xy) = \alpha(x)\alpha(y) = x^{-1}y^{-1},$$

and therefore $[x, y] = 1$. The choice of x and y being arbitrary, the group G is abelian. ∎

Lemma 78. *Let N be a normal A-stable subgroup of G. Assume that the action of A on N and the induced action of A on G/N are both trivial. Then $G = G^+$.*

Proof. Let $x \in G$. Then $\alpha(x)x^{-1}$ is an element of N. If x belongs to G^-, then x^{-2} belongs to N so, the order of G being odd, x itself is an element of N. It follows that G^- is a subset of N. The group N is however contained in G^+ and, as a consequence of Lemma 72, one gets $G^- = \{1\}$. It follows from Lemma 75 that $G = G^+$. ∎

Lemma 79. *Let $1 \to N \xrightarrow{f} G \xrightarrow{g} \Gamma \to 1$ is a short exact sequence of A-groups. Denote by f' and g' the restrictions of f and g respectively to N^+ and G^+. Then $1 \to N^+ \xrightarrow{f'} G^+ \xrightarrow{g'} \Gamma^+ \to 1$ is a short exact sequence of A-groups.*

Proof. Since f and g are homomorphisms respecting the action of A, it suffices to prove the surjectivity of g'. Let $\gamma \in \Gamma^+$. Then there exists $x \in G$ such that $g(x) = \gamma$ and, by Lemma 75, there exists $(a, b) \in G^+ \times G^-$ such that $x = ab$. Now, $(g(a), g(b)) \in \Gamma^+ \times \Gamma^-$, because g respects the action of A, and $\gamma = g(x) = g(ab) = g(a)g(b)$. It follows that $g(b) = g(a)^{-1}\gamma$, and so $g(b) \in \Gamma^+ \cap \Gamma^-$. Thanks to Lemma 72, we get that $g(b) = 1$, so γ is equal to $g(a)$. ∎

Lemma 80. *Let p be an odd prime number and assume that G is a finite p-group. Let $(G_i)_{i\geq 1}$ denote the lower central series of G and assume that the automorphism of G/G_2 that is induced by α equals the inversion map $x \mapsto x^{-1}$. Let H be an A-stable subgroup of G and let $\mathcal{O}$ and $\mathcal{E}$ be the collections of respectively all odd and all even positive integers. Then the following hold.*

1. *Let $j \in \mathbb{Z}_{\geq 1}$. Then $H^+ \cap G_j \neq H^+ \cap G_{j+1}$ if and only if $H \cap G_j \neq H \cap G_{j+1}$ and $j \in \mathcal{E}$.*
2. *One has $|H^+| = \prod_{j\in\mathcal{E}} |H \cap G_j : H \cap G_{j+1}|$.*
3. *One has $|H^-| = \prod_{j\in\mathcal{O}} |H \cap G_j : H \cap G_{j+1}|$.*

Proof. For all $j \in \mathbb{Z}_{\geq 1}$, we define $V_j = (H \cap G_j)/(H \cap G_{j+1})$ and we consider the short exact sequence

$$1 \to H \cap G_{j+1} \to H \cap G_j \to V_j \to 1$$

of A-groups. We first prove (1) and (2) together. If $j \in \mathbb{Z}_{\geq 1}$, we note that $(H \cap G_j)^+ = H^+ \cap G_j$, so Lemma 79 yields

$$|H^+ \cap G_j : H^+ \cap G_{j+1}| = |V_j^+|.$$

From Lemma 62 it follows that A acts on G_j/G_{j+1} by scalar multiplication by $(-1)^j$, and so, whenever j is an odd positive integer, Lemma 72 yields that the cardinality of $(G_j/G_{j+1})^+$ is equal to 1. Since, for all $j \in \mathbb{Z}_{\geq 1}$, the groups V_j and $(H \cap G_j)G_{j+1}/G_{j+1}$ are isomorphic, it follows that, for each odd j, the cardinality of V_j^+ is equal to 1. Moreover, if j is even, then V_j is equal to V_j^+. The cardinality of H^+ being equal to $\prod_{j\geq 1} |H^+ \cap G_j : H^+ \cap G_{j+1}|$, we get that

$$|H^+| = \prod_{j\geq 1} |V_j^+| = \prod_{j\in\mathcal{E}} |V_j^+| = \prod_{j\in\mathcal{E}} |V_j|.$$

This proves (1) and (2). We now prove (3). By Lemma 74, the cardinality of $|H^-|$ is equal to $|H|/|H^+|$. Since $|H| = \prod_{j\geq 1} |V_j|$, it follows from (2) that

$$|H^-| = \frac{\prod_{j\geq 1} |V_j|}{\prod_{j\in\mathcal{E}} |V_j|} = \prod_{j\in\mathcal{O}} |V_j|.$$

∎

Lemma 81. *Let H be an A-stable subgroup of G and let g be an element of G. Then the following are equivalent.*

1. *The subgroup gHg^{-1} is A-stable.*
2. *The element g belongs to $G^+ \mathrm{N}_G(H)$.*

Proof. Let $I = \mathrm{N}_G(H)$; then I is A-stable, because H is. We first prove that (1) implies (2). Assume that the subgroup gHg^{-1} is A-stable, so $\alpha(gHg^{-1}) = gHg^{-1}$. In particular, the element $g^{-1}\alpha(g)$ belongs to I, and thus $\alpha(gI) = \alpha(g)I = gI$. Since the cardinality of I is odd and A has order 2, there is an element x in I such that $\alpha(gx) = gx$. For such an element x, we then have that $gx \in G^+$, so $g = gx \cdot x^{-1} \in G^+I$. Assume now (2) is satisfied; we prove (1). Since g belongs to $G^+ \mathrm{N}_G(H)$, there exists $(\gamma, n) \in G^+ \times \mathrm{N}_G(H)$ such that $g = \gamma n$. For such pair (γ, n), we have that $gHg^{-1} = \gamma H \gamma^{-1}$, and therefore $\alpha(gHg^{-1}) = gHg^{-1}$. This proves (1). ∎

2.3 Jumps and width

Let p be a prime number and let G be a finite p-group. Denote by $(G_i)_{i\geq 1}$ the lower central series of G (see Section 1.2). If x is a non-trivial element of G, then there exists a positive integer d such that $x \in G_d \setminus G_{d+1}$. The number d is called the *depth* of x (in G) and it is denoted by $\mathrm{dpt}_G(x)$. Let now H be a subgroup of G and let j be a positive integer. The *j-th width of H in G* is

$$\mathrm{wt}_H^G(j) = \log_p |H \cap G_j : H \cap G_{j+1}|.$$

We observe that, if $\pi_j : G_j \to G_j/G_{j+1}$ denotes the canonical projection, then $\pi_j(H \cap G_i)$ has cardinality $p^{\mathrm{wt}_H^G(j)}$. An index j is a *jump of H in G* if $\mathrm{wt}_H^G(j) \neq 0$ and, whenever it will be clear that j is a jump of H in G, we will refer to $\mathrm{wt}_H^G(j)$ as the *width of j in G*. If $G = H$, we denote the *j-th width of G* by $\mathrm{wt}_G(j)$ instead of $\mathrm{wt}_G^G(j)$ and, in several results, we will lighten the notation even further by writing $w_j = \mathrm{wt}_G(j)$. The *width of G* is defined as $\mathrm{wt}(G) = \max_{i\geq 1} \mathrm{wt}_G(i)$; for a generalization to general pro-p-groups, see [KLGP97].

Lemma 82. *Let p be a prime number and let j be a positive integer. Let moreover G be a finite p-group and let H be a subgroup of G. Then j is a jump of H in G if and only if H contains an element of depth j in G.*

Proof. Straightforward. ∎

Lemma 83. *Let p be a prime number. Let moreover G be a finite p-group and let H be a subgroup of G. Then, for all $\alpha \in \mathrm{Aut}(G)$, the groups H and $\alpha(H)$ have the same jumps in G.*

Proof. Let α be an automorpshism of G and let j be a positive integer. By Lemma 82, the integer j is a jump of H if and only if there exists $x \in H$ such that $\mathrm{dpt}_G(x) = j$. As the elements of the lower central series are characteristic in G, we have that $\mathrm{dpt}_G(\alpha(x)) = \mathrm{dpt}_G(x)$, and thus we are done. ∎

Lemma 84. *Let p be a prime number and let G be a finite p-group. Let moreover H be a subgroup of G and call $\mathcal{J}$ the collection of all jumps of H in G. Then $|H| = \prod_{j \in \mathcal{J}} p^{\mathrm{wt}_H^G(j)}$.*

Proof. It follows directly from the definitions of jumps. ∎

In the next proposition, we use the notation introduced in Section 2.2.

Lemma 85. *Let p be an odd prime number and let G be a finite p-group. Let $A = \langle \alpha \rangle$ be a multiplicative group of order 2 acting on G. Let $\chi : A \to \{\pm 1\}$ be an isomorphism. Let $(G_i)_{i \geq 1}$ denote the lower central series of G and assume that the automorphism of G/G_2 that is induced by α equals the inversion map $x \mapsto x^{-1}$. Let H be an A-stable subgroup of G and let $\mathcal{O}$ and $\mathcal{E}$ be the collections of respectively all odd and all even jumps of H in G. Assume that the induced action of A on G/G_2 is through χ. Then the following hold.*

1. *One has $|H^+| = \prod_{j \in \mathcal{E}} p^{\mathrm{wt}_H^G(j)}$ and $\mathcal{E}$ is the set of jumps of H^+ in G.*
2. *One has $|H^-| = \prod_{j \in \mathcal{O}} p^{\mathrm{wt}_H^G(j)}$.*

Proof. Apply the new dictionary to Lemma 80. ∎