Etude sur quelques sémantiques dialogiques

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Annexes

Annexe A

On Dialogues and Natural Deduction Shahid Rahman, Nicolas Clerbout, Laurent Keiff

Le texte ci-dessous est un article co-écrit avec Shahid Rahman et Laurent Keiff, publié en 2009 dans un volume dédié au Pr. Sundholm.[89] L'article étudie la relation entre dialogique et déduction naturelle en fournissant un algorithme de traduction permettant de passer d'une **P** stratégie de victoire à une preuve en déduction naturelle. Du fait de son antériorité, la présentation des jeux dialogiques dans cet article est assez différente de celle adoptée dans cette dissertation. Parmi d'autres aspects intéressants, cet article énonce l'argument de l'immunité des jeux dialogiques face au connecteur *tonk*, dont il est question dans le Chapitre 1.

Abstract. The dialogical approach to logic has well known relations with other proof methods, like sequent calculi and semantic tableaux. In this paper we explore the connection between dialogues and natural deduction, an area that is much less understood. We define a system of dialogical games that captures the notion of validity of Johansson's minimal logic, and show how to extend the system for intuitionistic logic and classical logic. Then we describe an algorithm that transforms a dialogical proof into a Fitch-style natural deduction proof, and discuss the relation between the two approaches.

For you, dear Herr Urteil

A.1 Dialogue Games and Dialogical Logic

The dialogical tradition, as it stems from Lorenzen et Lorenz (1978) aimed at providing a new approach to the notion of *meaning* in logic that should build a

89. Rahman *et al.* (2009).

conceptual link between languages games, argumentation and validity. The point was to understand logic as a special kind of linguistic interaction. Nowadays, a very dynamic and powerful stream of research explores this notion of interaction in the interface between mathematical game theory and logic. In our paper we place dialogical logic in the framework of what are called in mathematical game theory "extensive games" in order to develop some points that were not systematically clarified in the dialogical tradition. It should be noted that such games are not a proof system proper. They are meant to give an account of how a rational argumentation about a given (logically complex) claim should be conducted, in the sole virtue of its logical form. The system assumes a notion of justification for logically elementary statements, then proceeds in showing how these elementary justifications can be used in the construction of a justification for complex statements. In this section we present the basic game system, [90] then show how the usual notion of validity of a formula A (in minimal, intuitionistic and classical logic) can be expressed in terms on the existence of a certain kind of strategy in the game associated with A.

A.1.1 Speech Acts

The fundamental idea behind the dialogical approach to logic is that a proof is a certain kind of very simple language game. Such games are built out of two fundamental types of speech act, namely *assertion* and *request*. The first can be thought, as for instance in Sellars (1997) and Brandom (2000), as a commitment to provide justifications of a certain kind, and the latter as an imperative to fulfill an assertive commitment. The dialogical tradition takes it that speech acts, and consequently proofs, are best understood when conceived as fundamental forms of *interaction*. This idea is the very core of the growing influence of game-theoretical ideas in logic, witnessed for instance by active research programs such as Hintikka-style Game-Theoretical Semantics, of e.g. Hintikka et Sandu (1997), the computation oriented Game Semantics of, e.g., Abramsky (1997), Blass (1992), Hyland et Ong (2000), Japaridze (2003), Girard's new research program called Ludics, in e.g. Girard (2003), and the dialogical tradition steming from Lorenzen et Lorenz (1978).

Let us first define a language the well-formed formulas (wff) of which are an adequate content for the assertions in our dialogical games.

Définition A.1 (The assertive language)**.** Our language $\mathcal{L}$ is built upon a countable set of elementary formulas $\mathbb{P} = \{p_0, p_1, ...\}$ together with a set of connectives $\{\wedge, \vee, \rightarrow\}$, and a couple of brackets (and). Let $\perp \in \mathbb{P}$ be a distinguished elementary formula. Assume $p \in \mathbb{P}$. The set of wff is as usual freely generated over $\mathbb{P}$ by the grammar :

$$A \quad := \quad p \quad | \quad \perp \quad | \quad (A \vee B) \quad | \quad (A \wedge B) \quad | \quad (A \rightarrow B)$$

90. For a textbook presentation (in French), see Fontaine et Redmond (2008)

We sometimes write $\neg A$ as a shorthand for $A \to \bot$

Moves in a dialogical game are speech acts, and are referred to by expressions specifying an *agent* (i.e. the player making the move), a *force* of the move that can be either an assertion (for which we use the fregean notation $\vdash$) or a request (noted ?), and a *content*. Formally :

Définition A.2 (Dialogical expressions)**.** A *dialogical expression* is an instance of $\langle \mathbf{X}\ f\ e\rangle$ where $\mathbf{X} \in \{\mathbf{O}, \mathbf{P}\}$, $f \in \{\vdash, ?\}$ and $e \in \{L, R, {}_\vee\} \cup \mathcal{L}$.

We will now define the rules for a system of dialogical games that we call *minimal dialogues*, which we will use as a basis for expressing the notions of validity in Johansson's minimal logic, and for the translation algorithm into Fitch-style natural deduction proofs.

A.1.2 Local Semantics

Argumentation forms (or particle rule) give the dialogical semantics of the connectives. Such forms give an abstract description of the way a formula, according to its outmost form, can be criticized, and how to answer the critique. The description is abstract or *local* in the sense that it can be carried out without reference to the context other than the presence of an assertion of a given formula in it. Informally, a *dialogical history* is the history of the dialogue, i.e. a sequence of dialogical expressions, together with an indication of the player who is to play. We give first two general definitions.

Définition A.3 (Dialogical History)**.** A dialogical history $\mathbb{H}$ is a tuple $\langle \Sigma, \mathbf{X}\rangle$ where Σ is a set of dialogical expressions and $\mathbf{X} \in \{\mathbf{O}, \mathbf{P}\}$.

Définition A.4 (argumentation Form)**.** An *argumentation form* is an ordered triple (p, c, d) of dialogical expressions where p is the *precondition*, c the *challenge* and d the *defence*.

Argumentation forms should be understood as follows. In any history $\mathbb{H} = \langle \Sigma, \mathbf{Y}\rangle$ in which player $\mathbf{X}$ asserted the precondition p, player $\mathbf{Y}$[91] may challenge this assertion, yielding a new history $\mathbb{H}' = \langle \Sigma \frown c, \mathbf{X}\rangle$. In a history $\mathbb{H}$, where a $\mathbf{X}$-assertion has been challenged according to some argumentation form (p, c, d), $\mathbf{X}$ may answer to the challenge, yielding a new history $\mathbb{H}'' = \langle \Sigma \frown d, \mathbf{Y}\rangle$.

Precondition	$\langle \mathbf{X} \vdash A \wedge B\rangle$	$\langle \mathbf{X} \vdash A \wedge B\rangle$	$\langle \mathbf{X} \vdash A \to B\rangle$
Challenge	$\langle \mathbf{Y}\ ?\ L\rangle$	$\langle \mathbf{Y}\ ?\ R\rangle$	$\langle \mathbf{Y} \vdash A\rangle$
Defence	$\langle \mathbf{X} \vdash A\rangle$	$\langle \mathbf{X} \vdash B\rangle$	$\langle \mathbf{X} \vdash B\rangle$

91. Through the whole paper we assume $\mathbf{X} \in \{\mathbf{O}, \mathbf{P}\}$, $\mathbf{Y} \in \{\mathbf{O}, \mathbf{P}\}$ and $\mathbf{X} \neq \mathbf{Y}$. We will often refer to dialogical moves as $\mathbf{X}$-moves, or when suitable as $\mathbf{X}$-assertions.

Precondition	$\langle \mathbf{X} \vdash A \vee B \rangle$	$\langle \mathbf{X} \vdash A \vee B \rangle$
Challenge	$\langle \mathbf{Y} \; ?_{\vee} \rangle$	$\langle \mathbf{Y} \; ?_{\vee} \rangle$
Defence	$\langle \mathbf{X} \vdash A \rangle$	$\langle \mathbf{X} \vdash B \rangle$

Local semantics and choice : Let # be a propositional connective. The set of the argumentation forms the precondition of which is an assertion of a formula with # as the main connective is the dialogical local semantics for #. There are two rules for conjunction and for disjunction, and players may choose which one they will use. In a history where a conjunction has been asserted by **X**, **Y** may request any of the conjuncts and in a history where his assertion of a conjunction has been challenged by such a request, **X** may assert the relevant conjunct. In the case of disjunction, both rules admit the same challenge, so **X** will have the choice of the rule he wants to follow in order to defend, which amounts to choose and assert one of the disjuncts. There is only one rule for the conditional, but notice that in a history where **Y** asserted a complex formula (say A) in order to challenge the **X**-assertion of a conditional (say $A \to B$), **X** has a choice between defending the conditional according to the rule, and challenging the antecedent according to the rule that corresponds to A's main connective.

A.1.3 Structural Rules

The dialogical games are meant to capture situations where a protagonist in a language-game commits himself to justify a claim (that we call the *thesis*) in a context where his opponent is committed to the justification of a set of *initial hypotheses*. The following structural rules define the two main aspects of the game :

- what a game is, i.e. the way argumentation forms may be used in order to produce a dialogue ;
- the payoff function of the game, i.e. a winning criterion for game histories : a player who must move and can not has lost.

In order to deal with both aspects, we first need a couple of definitions. As most other games, dialogical games should prevent loops, i.e. the indefinite repetition of the same situation. We call a move initiating a loop a redundant move. Formally, redundancy is defined with respect to types of moves : repetition of a challenge is not redundant if a new type of move has been made between the first and the second occurences.

Définition A.5 (Redundant Moves)**.** We distinguish between redundancy of a challenge and redundancy of a defence :

Challenge Let $A, B \in \mathcal{L}$. Let $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ be a dialogical history such that $\langle \mathbf{Y} \vdash A \rangle \in \Sigma$. Let $\langle \mathbf{X} \; f \; e \rangle \in \Sigma$ be a challenge against $\langle \mathbf{Y} \vdash A \rangle$. Let $\mathbb{H}_0$ be the prefix of $\mathbb{H}$ with $\langle \mathbf{X} \; f \; e \rangle$ as last element. We say that challenge $\langle \mathbf{X} \; f \; e \rangle$

is *redundant in* $\mathbb{H}$ iff there is no assertion $\langle \mathbf{Y} \vdash B \rangle \in \mathbb{H} - \mathbb{H}_0$ such that $\langle \mathbf{Y} \vdash B \rangle \notin \mathbb{H}_0$.[92]

Defence Any repetition of a defense is redundant.

We also need the following terminology :

Définition A.6. Let $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ be a dialogical history. We say that $\mathbb{H}$ is $\mathbf{X}$-*terminal* with game rules $\mathbf{D}$ iff there is no move available to $\mathbf{X}$ according to the rules in $\mathbf{D}$.

[SR-0] (Initial History) Let $\Delta \subset \mathcal{L}$ be a finite set of formulas and $A \in \mathcal{L}$ a formula. The *initial position* of a dialogue for A under hypotheses Δ (notation : $\mathcal{D}(\Delta, A)$) is a history $\mathbb{H}_0 = \langle (\langle \mathbf{O} \vdash \Delta \rangle, \langle \mathbf{P} \vdash A \rangle), \mathbf{O} \rangle$.[93]

[SR-1] (Gameplay) Let $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ be a dialogical history. $\mathbf{X}$ is to play in $\mathbb{H}$. The set of available moves for player $\mathbf{X}$ in history $\mathbb{H}$ is the set of non-redundant challenges specified by argumentation forms applicable to the $\mathbf{Y}$-assertions in $\mathbb{H}$, together with the set of non-redundant defences against $\mathbf{Y}$'s *last* challenge in $\mathbb{H}$, as specified by the argumentation forms. No other move is allowed.

[SR-2] (Winning) Player $\mathbf{Y}$ wins in a terminal history $\mathbb{H}$ iff $\mathbb{H}$ is $\mathbf{X}$-terminal. In a terminal history where $\mathbf{X}$ wins, $\mathbf{Y}$ looses.

Let $\mathbb{AF}$ denote the set of argumentation forms given in the previous section. The game system for minimal dialogues is the set of rules

$$\mathbf{D}_{min} = \mathbb{AF} \cup \{\textbf{SR-0}, \textbf{SR-1}, \textbf{SR-2}\}.$$

Définition A.7. A dialogue $\mathcal{D}(\Delta, A)$ in a rule system $\mathbf{D}$ is the set of all terminal histories with the initial position of $\mathcal{D}(\Delta, A)$ as a prefix and such that any dialogical expression in it is legal in virtue of the rules of $\mathbf{D}$.

Histories of the game correspond to what one understands as dialogues in the usual (non logical) sense of the term. Terminal histories are (usual) dialogues which are complete as far as the logical form of the thesis is concerned. Dialogues as defined here contain all possible complete (usual) dialogues. We discuss our motivations in the following section.

Notice that the argumentation forms as we defined them in the previous section ensure that all moves following the initial position will be played by $\mathbf{O}$ and $\mathbf{P}$ alternately. A terminal history of a dialogue $\mathcal{D}(\Delta, A)$ is thus a pair $\langle \Sigma, \mathbf{X} \rangle$ where

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92. That is, if there is no new assertion by $\mathbf{Y}$ after challenge $\langle \mathbf{X}\ f\ e \rangle$, where an assertion is new only if it did not occur in $\mathbb{H}_0$

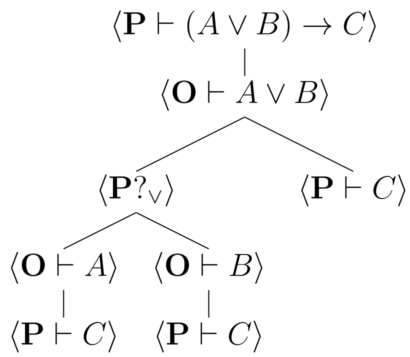
93. For any finite set of formula $\Delta = \{A_0, ..., A_n\}$, we write $\langle \mathbf{X} \vdash \Delta \rangle$ as a shorthand for the sequence $(\langle \mathbf{X} \vdash A_0 \rangle, ..., \langle \mathbf{X} \vdash A \rangle)$.

Σ has $(\langle \mathbf{O} \vdash \Delta \rangle, \langle \mathbf{P} \vdash A \rangle)$ as a prefix, followed by a (possibly empty) sequence of alternating **O**- and **P**-moves. The following theorem shows that the payoff function is correctly defined, i.e. that any history in $\mathcal{D}(\Delta, A)$ will reach a terminal position :

THÉORÈME A.1.1. *Let $\Delta \subset \mathcal{L}$ be a finite set of formulas and $A \in \mathcal{L}$ a formula. Any terminal history in the dialogue $\mathcal{D}(\Delta, A)$ is of finite length.* [94]

Démonstration. The proof relies on the following considerations. First of all, we remark that any terminal history has $\mathbb{H}_0$ as a prefix, which is of finite length. Now the local rules feature the subformula property in this sense : for any argumentation form (p, c, d) where the degree [95] of the precondition p is n, the degree of the defence d is at most $n-1$. So any sequence of moves where a player systematically challenges the defence against his previous challenge ends up with an elementary defence after finitely many moves. Now there is no local rules for elementary assertions, so the sequence of challenges must end there. Finally the rules taking redundancies in charge ensure that no infinite loop may occur in $\mathcal{D}(\Delta, A)$. ■

Example We give here the dialogue $\mathcal{D}(\varnothing, (A \vee B) \to C)$:



A notational remark Readers familiar with the dialogical literature (as e.g. Rahman et Keiff (2005)) may be surprised not to find the usual notation for dialogues. Such a notation displays a dialogue in the form of a 2-column table, one for each player. Each column is divided in three sub-columns, the outer bearing a number for the move, the central bearing the formula or the attack marker, and the inner bearing the number of the challenged move when the move is a challenge. A challenge and the corresponding defence are written on the same line. Here's an example :

94. The *length* of a history $\mathbb{H} = (\Sigma, \mathbf{X})$ is the number of elements of Σ.
95. By *degree* we mean the number of connectives in it.

	O			**P**	
				$(A \wedge B) \to A$	0
1	$A \wedge B$	0		A	4
3	A		1	$?_L$	2

Actually such a notation seems to be the most natural way to represent a dialogue, as it shows how the argumentation forms structure the dialogue. The most important point is that it makes it clear when a given assertion is a defence, and when it is a challenge (against a conditional). Our notation does not show the difference in the syntax.[96] However, the table notation is not very convenient when the aim is to describe strategies, where splittings occur. Since dialogical validity is defined in terms of strategies and since we would like to show how to extract strategies from the extensive form of games, we have chosen to use the tree-like notation.

A.1.4 Dialogues and Validity

The basic game-theoretical tool to study the properties of games is their representation in extensive form. A simple version of the usual definition, retaining only what is useful for our purpose would have the following definition :

Définition A.8 (extensive form)**.** The *extensive form* $\mathcal{E}$ of a dialogical game $\mathcal{D}(\Delta, A)$ is the smallest rooted tree[97] such that :

i. The root is labelled with the initial history of $\mathcal{D}(\Delta, A)$, with $\langle \mathbf{P} \vdash A \rangle$ as its *active* expression.
ii. Let h be a branch in $\mathcal{E}$, n the last node of h, $\langle \mathbf{X}\ f\ e \rangle$ the active expression of n, and Σ the sequence of dialogical expressions labelling the nodes of h. For any dialogical expression e' such that e' denotes a legal move in a history $(\Sigma, \mathbf{Y})$ according to the rules of $\mathcal{D}(\Delta, A)$, there is a node in the tree which is a successor of n and labelled with e.
iii. Any leaf n of $\mathcal{E}$ bears as an extra label (1,-1) if n belongs to a branch that is $\mathbf{P}$-terminal ($\mathbf{O}$ wins) and (-1,1) otherwise.

It is easy to see that a dialogue $\mathcal{D}(\Delta, A)$ as we defined it in the previous section *is* an extensive form. Each terminal history in $\mathcal{D}(\Delta, A)$ is a *play* of the game, that is a possible course of the actual argumentation. Let us now give a formal definition of a strategy :

96. See the example at the end of section A.1.5.

97. A *tree* is a set of nodes together with an irreflexive relation of successor, such that (i) there is a single node that is the successor of no other node ; (ii) for any two distinct nodes in the tree, it is not the case that they share a successor. A *branch* of the tree is any sequence of nodes begining with the root and linearly ordered by the successor relation. A *leaf* of the tree is a node that has no successor.

Définition A.9 (Strategy)**.** A *strategy* for $\mathbf{X}$ (or $\mathbf{X}$-strategy) $\mathcal{S}_{\mathbf{X}}$ in $\mathcal{D}(\Delta, A)$ is a subtree of $\mathcal{D}(\Delta, A)$ such that :

i. $\mathcal{S}_{\mathbf{X}}$ contains only maximal branches of $\mathcal{D}(\Delta, A)$.

ii. Any node labelled with a $\mathbf{X}$-move which has at least one successor in $\mathcal{D}(\Delta, A)$ has exactly one successor in $\mathcal{S}_{\mathbf{X}}$.

iii. For any node n labelled with a $\mathbf{Y}$-move, if m is a successor of n in $\mathcal{D}(\Delta, A)$ then m belongs to $\mathcal{S}_{\mathbf{X}}$.

A $\mathbf{X}$-strategy is *winning* iff it contains no $\mathbf{X}$-terminal history. In other terms, anytime in the course of the game where $\mathbf{X}$ is to choose a move, the strategy indicates a move such that if $\mathbf{X}$ plays it, he remains in a play of the game where he is sure to win. We can now give a precise formulation of the triviality of $\mathbf{D}_{min}$:

THÉORÈME A.1.2. *In any dialogue $\mathcal{D}(\Delta, A)$ with the rules $\boldsymbol{D}_{min}$, any $\boldsymbol{P}$-strategy is winning.*

The reason for this is simple. In a dialogue $\mathcal{D}(\Delta, A)$, $\mathbf{O}$ moves first, and the only moves available to him (if any) are challenges. The argumentation forms always allow for a defense against a challenge, and as long as $\mathbf{O}$ will have non-redundant challenges, $\mathbf{P}$ will have non-redundant defences. Since the argumentation forms ensure that the defence of an assertion is always logically simpler than the precondition, and since there is no challenge against elementary assertions, $\mathbf{O}$ will necessarily run out of challenges, and $\mathbf{P}$ wins asserting the last non-redundant defence. The meaning of a winning strategy $\mathcal{S}_{\mathbf{P}}$ in $\mathcal{D}(\Delta, A)$ is that statement A is justifiable in the context of hypotheses Δ *provided* $\mathbf{P}$ *knows how to justify the set of elementary assertions he made in the course of the game.*[98] Clearly, winning strategies talk about (conditional) justifiability and not (actual) justification.

The notion of winning strategy is thus not enough to define the class of statements that one may consider as logically valid. In the conceptual framework of dialogical logic, validity is demonstrated by a property of the game associated with a formula. The idea is that some statements trigger an argumentation process that is, *for inherently interactive reasons*, sufficient to consider them *actually* justified. This property of the argumentation comes from the dynamics of the committment to elementary justifications.

Assume that I enter a debate against someone with respect to some claim A. My strategy in the debate tells me what are the elementary claims I should be able to justify in order to justify my claim. Assume that one of them, say p, is such that in the history $\mathbb{H}$ where I assert p, my opponent has already asserted p. If we grant that argumentation games for elementary proposition are of perfect information and that if a player is showed how to make a move he can always reproduce it, then in $\mathbb{H}$ I have what we may call after Abramsky a *copycat* strategy. Such a

98. More precisely, $\mathbf{P}$ must know how to justify every elementary assertion he must make in any maximal history of $\mathcal{S}_P$.

strategy consists simply in replicating against his assertion of p any challenge from my opponent against my own, and answering his challenges by replicating his answers. In such game situations, either my opponent will fail to justify his claim, or he will succeed, but doing so he will show me how to justify my own claim.

From these considerations stems the dialogical appraoch to *formality*. We say that an elementary assertion $\langle \mathbf{X} \vdash p\rangle$ in a dialogical history $\mathbb{H} = \langle \Sigma, \mathbf{X}\rangle$ is *contentious* iff $\langle \mathbf{Y} \vdash p\rangle \notin \Sigma$. So an elementary assertion of $\mathbf{X}$ is not contentious iff $\mathbf{Y}$ is already committed to the justification of p by having asserted it. Thus a winning $\mathbf{P}$-strategy will give an actual justification to the thesis provided it does not contain any contentious claim. Such a strategy would show that any critical party trying to refute the thesis will in the course of the argumentation commit itself to accept the justifications for the thesis. We say that a strategy is *formal* just when it contains no contentious assertion.

The reader familiar with the dialogical tradition will notice that the approach of this paper diverges here with the standard view on dialogical logic, in that we consider formality as a property of *strategies* while it is usually seen as a property of *games*. However this does not mean that we take it that the signification of formal winning strategies *reduces* to validity. As we will discuss in the last section of this paper, the usual Introduction/Elimination rules of natural deduction do not reflect all that one finds in a $\text{FWS}_{\mathbf{P}}$.

Although the term "copycat" denotes a certain way to build a strategy, it is also a property of the games. It can be stated thus :

Définition A.10 (Copycat games)**.** Let $\Delta \subset \mathcal{L}$ be a finite set of formulas and $A \in \mathcal{L}$. If $A \in \Delta$, then there is a $\text{FWS}_{\mathbf{P}}$ in $\mathcal{D}(\Delta, A)$.

What the copycat really says is that in any game situation where $\mathbf{X}$ should defend an assertion of a formula A that the other player has also asserted, he can provide a complete justification [99] of A without any contentious move. The reason is simple : any $\mathbf{Y}$-challenge in the complete justification of $\langle \mathbf{X} \vdash A\rangle$ can be replicated by $\mathbf{X}$ against $\langle \mathbf{Y} \vdash A\rangle$, and any $\mathbf{Y}$-defense can also be replicated, with the difference that $\mathbf{Y}$ always moves first, so none of $\mathbf{X}$'s replication of a $\mathbf{Y}$-move is contentious. In that respect, one may well extend the definition of redundant move with any challenge against a non-contentious assertion.

Our interest in Johansson's *minimal logic* (**ML**) is determined by the fact that it gives a basis from which one can build intuitionistic and classical logics by extension of the set of inference rules or, equivalently, of the set of axioms. Let us now state the connection between strategies and validity in the case of **ML**. We will come in the next section to its Fitch-style natural deduction system. As an axiomatic system, **ML** is defined as the set of axioms :

99. A complete justification of A is the set of all the defences of A and all the defences of these defences and so on.

1. $p \to (q \to p)$
2. $((p \to (p \to q))) \to (p \to q)$
3. $(p \to q) \to ((q \to r) \to (p \to r))$
4. $(p \wedge q) \to p$
5. $(p \wedge q) \to q$
6. $p \to (q \to (p \wedge q))$
7. $p \to (p \vee q)$
8. $q \to (p \vee q)$
9. $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$
10. $(p \to q) \to ((q \to \bot) \to (p \to \bot))$
11. $(p \wedge (p \to \bot)) \to \bot$

together with the inference rules Uniform Substitution and Modus Ponens.

We claim that the notion of winning formal **P**-strategies (FWS$_{\mathbf{P}}$) in $\mathbf{D}_{min}$ is correct for minimal logic in the following sense :

THÉORÈME A.1.3. *Let $\Delta \subset \mathcal{L}$ be a finite set of formulas and $A \in \mathcal{L}$. There is a proof of $\Delta \Rightarrow A$* [100] *in minimal logic iff there is a FWS$_{\mathbf{P}}$ in $\mathcal{D}(\Delta, A)$.*

Démonstration. The left to right part is fairly easy. It suffices to show that (i) there is a FWS$_{\mathbf{P}}$ in the dialogue $\mathcal{D}(\varnothing, A)$ for any axiom A of minimal logic and (ii) inference rules of minimal logic preserve the existence of a FWS$_{\mathbf{P}}$.

The right to left part is an immediate corollary of the correction of our translation algorithm. Assume there is a FWS$_{\mathbf{P}}$ in $\mathcal{D}(\Delta, A)$. Then we translate it into a Fitch-style proof of A from hypotheses Δ. If the Fitch-style natural deduction system for minimal logic is correct, then there is an axiomatic proof of $\Delta \Rightarrow A$. ■

A.1.5 Intuitionistic logic and Classical Dialogues

When presented in Natural Deduction, the relation between Johansson minimal logic and intuitionistic logic is very simple : the latter is the result of the addition of *Ex Falso Sequitur Quodlibet* to the inference rules. [101] This rules stipulates that any formula can be infered from $\bot$. There is a way to extend our dialogical games that is equally simple.

100. As usual, a proof of $\Delta \Rightarrow A$ in an axiomatic system $\mathbf{S}$ is a sequence of formulas such that each of them is either an axiom, or the result of applying an inference rule to previous formulas in the sequence.

101. See section A.2 for details.

[SR-3] (***Ex Falso Quodlibet***) Let $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ be a dialogical history such that $\langle \mathbf{Y} \vdash \bot \rangle \in \Sigma$. Player $\mathbf{X}$ may challenge $\langle \mathbf{Y} \vdash \bot \rangle$ with a move $\langle \mathbf{X} \ ? \ A \rangle$ for any $A \in \mathcal{L}$. In any dialogical context $\mathbb{H}' = \langle \Sigma', \mathbf{Y} \rangle$ such that $\langle \mathbf{X} \ ? \ A \rangle \in \Sigma'$, $\mathbf{Y}$ may play $\langle \mathbf{Y} \vdash A \rangle$.

Notice that SR-3 amount litterally to consider $\bot$ as a nullary connective with the following argumentation form :

Precondition	$\langle \mathbf{X} \vdash \bot \rangle$
Challenge	$\langle \mathbf{Y} \ ? \ A \rangle$
Defence	$\langle \mathbf{X} \vdash A \rangle$

The obvious drawback of rule SR-3 is that it introduces potentially redundant challengees against $\bot$ that should be ruled out. We extend definition A.5 with the following clause :

Définition A.11 (Redundant challenge against $\bot$)**.** Let $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ be a dialogical history such that $\langle \mathbf{Y} \vdash \bot \rangle \in \Sigma$. $\langle \mathbf{X} \ ? \ B \rangle$ is redundant in $\mathbb{H}$ if there is a challenge $\langle \mathbf{X} \ ? \ A \rangle \in \Sigma$ against $\langle \mathbf{Y} \vdash \bot \rangle$ and $\langle \mathbf{X} \vdash A \rangle \notin \Sigma$.

That is, in order to challenge $\bot$ again, player $\mathbf{X}$ should find a way to use the first formula he requested. Given definitions A.5 and A.11, we define our system for intuitionistic dialogues as :

$$\mathbf{D}_{int} = \mathbf{D}_{min} \cup \{\textbf{SR-3}\}$$

Clearly, in any dialogical context where $\mathbf{X}$ has asserted contentiously $\bot$ and $\mathbf{Y}$ has made no contentious assertion, $\mathbf{Y}$ has a formal strategy. So if in a dialogical game $\mathbf{P}$ has a strategy to force $\mathbf{O}$ to concede $\bot$ contentiously, he has a FWS. This explains in particular why $\bot \to A$ is valid for any A.

To transform the natural deduction presentation of intuitionistic logic into classical logic, it suffices to add the rule of excluded middle.[102] The usual interpretation of the difference between intuitionistic logic and classical logic is very similar for dialogue systems and sequent calculi : the latter vary in the number of formulae to the right, while the former vary in the number of challenges a player can recall.[103] But our goal here is to see to what extent dialogues and natural deduction can be understood in the same perspective. So we will follow the natural deduction route to classical logic :

[SR-4] (Excluded Middle) Let $p \in \mathbb{P}$ be an elementary formula, $A \in \mathcal{L}$, and $\Delta \subset \mathcal{L}$ a finite set of formulas. Let $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ be a dialogical history in $\mathcal{D}(\Delta, A)$.

102. See section A.2 for details.
103. Recall that SR-1 says that one can only defend the last challenge. This makes $\mathbf{D}_{min}$ fundamentally intuitionistic.

Then $\mathbf{X}$ may play $\langle \mathbf{X} \ ?\ p \vee (p \to \bot) \rangle$. In any dialogical history $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ such that $\langle \mathbf{Y} \ ?\ p \vee (p \to \bot) \rangle \in \Sigma$, $\mathbf{X}$ may play either $\langle \mathbf{X} \vdash p \rangle$ or $\langle \mathbf{X} \vdash (p \to \bot) \rangle$.

SR-4 is clearly equivalent to a presupposition of determinacy for the atoms of the language, in the sense that for any atom p, either p or $p \to \bot$ should be justifiable. Of course, since a rule such as SR-4 will introduce infinitely many possible redundant moves in the game, we shall update our definition of redundancy :

Définition A.12 (Redundant Excluded Middle moves)**.** A challenge $\langle \mathbf{X} \ ?\ p \vee (p \to \bot) \rangle$ is redundant in a dialogical context $\mathbb{H} = \langle \Sigma, \mathbf{X} \rangle$ iff one of the following holds :

i. (*relevance*) p is not a subformula of the formulas in the initial history ;
ii. (*repetition*) $\langle \mathbf{X} \ ?\ p \vee (p \to \bot) \rangle \in \Sigma$.

Given definitions A.5, A.11 and A.12, we define our system for classical dialogues as :

$$\mathbf{D}_{cl} = \mathbf{D}_{int} \cup \{\textbf{SR-4}\}$$

The core of strategies In spite of the fact that strictly redundant moves are forbidden by the rules of a dialogue, a $\text{FWS}_{\mathbf{P}}$ contains a lot of moves that are redundant *from the point of view of validity*, i.e. when what is at stake is to show that a given strategy is *formal*. While it is desirable to formulate dialogues rules with respect only to considerations of argumentation, when one aims at translating a dialogical proof (that is to say a $\text{FWS}_{\mathbf{P}}$) into some other kind of proof, it is much better to eliminate all those undesirable moves. We call the result of this elimination the *core* of a strategy.

The core of a $\text{FWS}_{\mathbf{P}}$ should only retain such moves that are important to determine wether the inference from premisses to the thesis is valid or not, i.e. wether the **P**-strategy is formal or not. Clearly, any (non redundant) repetition of a **O**-challenge would make no difference, for if the first defence was formal, so will be the second. Any **O**-move that is legal in virtue of SR-3 or SR-4 will also be without incidence. For SR-3, consider a dialogical history containing $\langle \mathbf{P} \vdash \bot \rangle$. Then either this move was contentious, and the strategy is not formal whatever happens next, or it is not contentious, hence the history contains also $\langle \mathbf{O} \vdash \bot \rangle$. Then for any challenge against his own assertion of $\bot$, **P** can challenge **O** in the same way and win formally by copycat. For SR-4, the reasoning is the same : any **O** challenge would be followed by the same **P** move, and **P** will have a copycat strategy to defend formally. We introduce a last simplification on $\text{FWS}_{\mathbf{P}}$ to define its core : everytime **O** has a choice between two moves, he will (when the rules allow) try the second move in the same history where he tried the first before the game reaches a terminal position. In order to show that a winning **P**-strategy is formal, it is sufficient to consider after each **O**-split only the moves relevant to one of **O**'s options. Indeed, the concatenation of a two sequences of moves where **P** plays formally is also a sequence of **P**-formal moves. Figure 1 below is an example of the core of a $\mathrm{FWS}_{\mathbf{P}}$ in $\mathbf{D}_{cl}$ for Peirce's Law. The extensive form of the game is much too rich to be convieniently displayed as an example, since it contains every possible strategy for both players. Here we are only concerned with a small fragment of it, namely one specific **P**-strategy, from which the only moves we retain are the one that are relevant with respect to validity.

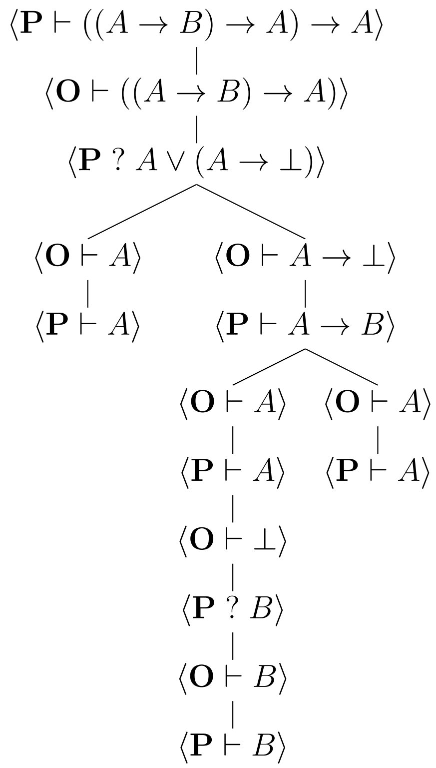


Figure A.1

An interesting feature of this strategy is the split after $\langle \mathbf{P} \vdash A \to B \rangle$. In both branches **O** moves with $\langle \mathbf{O} \vdash A \rangle$, but in the left branch this is a challenge against $\langle \mathbf{P} \vdash A \to B \rangle$, while in the right branch it is a defence of $\langle \mathbf{O} \vdash (A \to B) \to A \rangle$.[104] In the right branch, the last **O**-challenge is the second move and **P** can defend the thesis, while on the left branch the last challenge from **O** is against $\langle \mathbf{P} \vdash A \to B \rangle$, leaving no other *formal* option than the use of **SR-3** to defend the conditional.

It is a well known fact that formal strategies are the dialogical equivalent for the usual notion of validity, in the sense that there exists a $\mathrm{FWS}_{\mathbf{P}}$ in $\mathcal{D}(\Delta, A)$

104. This difference does not show up here. This is the price we have to pay for the tree-like notation. The problem could easily be fixed, adding to the definition of a dialogue a function associating to each move a unique number, to each challenge the number of the precondition allowing it, and to each defence the number of the challenge it answers to. This would involve us in some kind of hybrid or labelled system. A move that, for the sake of simplicity, we would like to avoid.

in $\mathbf{D}_{int}$ (resp. $\mathbf{D}_{cl}$) iff $\Delta \Rightarrow A$ is valid in intuitionistic (resp. classical) logic. See Felscher (1985a) and Rahman (1993) for the proofs.

A.1.6 Dialogues and Tableaux

Our purpose in the remaining of the section is to show the way one can extract from a FWS$_\mathbf{P}$ a tree-like structure of assertions that actually is a tableau proof.

All through our presentation of dialogical rules, we insisted in giving strictly *symmetrical* rules, in the sense that – except for the definition of the initial history – players in a dialogue have exactly the same rights. But the study of validity, i.e. of winning formal **P**-strategies, introduces an obvious asymmetry : the only splits in the strategy are **O**'s choices. But such choices can be understood in two different ways. On the one hand, as we did up to now, one can define the objects of the choices as moves. On the other hand, as we suggested when we explained the notion of choice with respect to the conditional, one can think that when choosing a move, the player is actually choosing a *rule*.

This change of perspective may be motivated by the following consideration. If one accepts that the business of logic is to provide a theory of *inference*, and that inference is a relation between assertions, then our argumentation forms may be considered as building blocks for such a theory. The relation of justification is defined in a dialogue as holding between an assertion and the set of its defences against all possible challenges. As a consequence, from this point of view, challenges are transitory indications about the course of the game that, while crucially important when one considers (dialogical) proof as a *process*, play no role in the *result* of the process. So one may want to represent a complete dialogue from the perspective of inference understood as a set of justificatory relations between assertions and their defences.

Actually, for disjunction and conjunction there is no difference between both perspectives. But there is an important one for the conditional. The extensive form of a local game for a conditional is :

$$
\begin{array}{c}
\langle \mathbf{X} \vdash A \to B \rangle \\
| \\
\langle \mathbf{Y} \vdash A \rangle \\
\swarrow \quad \searrow \\
\langle \mathbf{X}\ f\ e \rangle \quad \langle \mathbf{X} \vdash B \rangle
\end{array}
$$

(where $\langle \mathbf{X}\ f\ e \rangle$ stands for the relevant challenge against $\langle \mathbf{Y} \vdash A \rangle$, according to A's outmost form.) But from the perspective of argumentation forms, $\langle \mathbf{Y} \vdash A \rangle$ is ambiguous as it is both a precondition in the left branch and a challenge in the right branch. Thus the correct topology of a tree representing the relation of inference as defined above (that is with respect to preconditions and defences only) should be :

$$
\begin{array}{ccc}
 & \langle \mathbf{X} \vdash A \to B \rangle & \\
\langle \mathbf{Y} \vdash A \rangle & & \langle \mathbf{X} \vdash B \rangle \\
\ldots \quad \langle \mathbf{Y} \ f \ e \rangle & &
\end{array}
$$

(where $\langle \mathbf{Y} \ f \ e \rangle$ stands for one of the possible *defences* of A.)

Now when one is interested in proofs, the extensive forms are no longer considered *in abstracto* but are, borrowing the term from Girard's Ludics,[105] *incarnated*. Indeed, from the point of view of $\text{FWS}_{\mathbf{P}}$, the splits reflect **O**'s choices. **P** may well play both possible rules in the same course of the game, and actually, *it is in his strategical interest to do so.*

From the preceding considerations, it is easy to give rules for building a tree that represent the inference relation for a dialogue $\mathcal{D}(\Sigma, A)$. The only complication comes from the SR-1 rule stipulating that only the last challenge may be answered. From the point of view of argumentation forms, this amounts to consider that one can apply a **X**-rule only if its precondition is the last **X**-assertion, and we consider all applicable **X**-rules as applied at once. Here are the rules :

O-rules	**P**rules
$\langle \mathbf{O} \vdash A \vee B \rangle$ $\langle \mathbf{O} \vdash A \rangle \quad \langle \mathbf{O} \vdash B \rangle$	$\langle \mathbf{P} \vdash A \vee B \rangle$ $\mid$ $\langle \mathbf{P} \vdash A \rangle$ $\mid$ $\langle \mathbf{P} \vdash B \rangle$
$\langle \mathbf{O} \vdash A \wedge B \rangle$ $\mid$ $\langle \mathbf{O} \vdash A \rangle$ $\mid$ $\langle \mathbf{O} \vdash B \rangle$	$\langle \mathbf{P} \vdash A \wedge B \rangle$ $\langle \mathbf{P} \vdash A \rangle \quad \langle \mathbf{P} \vdash A \rangle$
$\langle \mathbf{O} \vdash A \to B \rangle$ $\langle \mathbf{P} \vdash A \rangle \quad \langle \mathbf{O} \vdash B \rangle$	$\langle \mathbf{P} \vdash A \to B \rangle$ $\mid$ $\langle \mathbf{O} \vdash A \rangle$ $\mid$ $\langle \mathbf{P} \vdash B \rangle$

One will obviously recognise here the tableaux rules for signed formulas as in Smullyan (1968) when one interprets **O** as T and **P** as F. See Rahman (1993) for a thorough presentation of the connection between dialogues and tableaux in the propositional and first-order cases of intuitionistic and classical logic.

105. See Girard (1999) for instance.

A.2 Fitch-style Natural Deduction

Définition A.13 (Fitch-style deduction)**.** A Fitch-style deduction is a sequence Σ of tuples, each of the form (l, A, m), where l is taken from some ordered set of labels $\mathbb{L}$, A is a derived formula, m is the justification of the derivation, and such that :

– a justification m is of one of the following foms :

form of m	justification rule
$-$	No justification. The formula is an assumption
$\wedge$E, l	$\wedge$ elimination on the formula of line l
$\wedge$I, l_1, l_2	$\wedge$ introduction on the formulae of lines l_1 and l_2
$\rightarrow$E, l_1, l_2	$\rightarrow$ elimination on the formulae of lines l_1 and l_2
$\rightarrow$I, l_i-l_j	$\rightarrow$ introduction from the block of lines l_i-l_j
$\vee$E, l_1, l_i-l_j, l_k-l_n	$\vee$ elimination on the formula of line l_1, the block of lines l_i-l_j, and the block of lines l_k-l_n
$\vee$I, l	$\vee$ introduction on the formula of line l
R, l	reiteration rule on the formula of the line l

– Σ obeys the following rules :[106]

$$\begin{array}{l|l|l} l & & \underline{A} \\ \vdots & & \vdots \end{array} \qquad \begin{array}{l|ll} l & A & \\ \vdots & \vdots & \\ l' & A & \text{R, } l \end{array}$$

$$\begin{array}{l|ll} l_1 & A & \\ \vdots & \vdots & \\ l_2 & B & \\ \vdots & \vdots & \\ l & A \wedge B & \wedge\text{I, } l_1,\ l_2 \end{array} \qquad \begin{array}{l|ll} l & A \wedge B & \\ \vdots & \vdots & \\ l' & A & \wedge\text{E}_1,\ l \end{array} \quad \begin{array}{l|ll} l & A \wedge B & \\ \vdots & \vdots & \\ l' & B & \wedge\text{E}_2,\ l \end{array}$$

106. We present these rules in graphical notation, for the sake of clarity. For a more formal, non graphical, account of derivation rules in Fitch-style Natural Deduction, see Geuvers et Nederpelt (2004)

$$
\begin{array}{l|l|ll}
l_1 & & A & \\ \cline{3-3}
\vdots & & \vdots & \\
l_2 & & B & \\
l & \multicolumn{2}{l}{A \to B} & \to\mathrm{I},\ l_1,\ l_2
\end{array}
\qquad
\begin{array}{l|ll}
l_1 & A & \\
\vdots & \vdots & \\
l_2 & A \to B & \\
l & B & \to\mathrm{E},\ l_1,\ l_2
\end{array}
$$

$$
\begin{array}{l|ll}
l & A & \\
\vdots & \vdots & \\
l' & A \vee B & \vee\mathrm{I},\ l
\end{array}
\qquad
\begin{array}{l|l|ll}
i & \multicolumn{2}{l}{A \vee B} & \\
\vdots & \multicolumn{2}{l}{\vdots} & \\
j_1 & & A & \\ \cline{3-3}
\vdots & & \vdots & \\
j_2 & & C & \\
k_1 & & B & \\ \cline{3-3}
\vdots & & \vdots & \\
k_2 & & C & \\
l & \multicolumn{2}{l}{C} & \vee\mathrm{E},\ i,\ j_1\text{–}j_2,\ k_1\text{–}k_2
\end{array}
$$

There are different notational conventions for Fitch-style Natural Deduction. One can be found in Gamut (1992). We use one which is used in Garson (2006) or Barwise et Etchemendy (1993) for example. This notation is caracterized by two kinds of graphic features : vertical lines and horizontal lines, the latter being called "‘Fitch bars"’. A Fitch bar is placed right beneath each assumption, with one exception : the premisses of the derivation we want to show are all put together above a sole Fitch bar. A vertical line starts with an assumption and is used to indicate how long the assumption is available.

The rules →I and ∨E make use of one of the most appealing features of Fitch-style Natural Deduction : subderivations. Each time an assumption not belonging to the set of premisses is done, a new subderivation is created. A vertical line is drawn until the assumption is discharged, and the rules →I and ∨E show how such a discharge can occur. We call a subderivation *finished* when the assumption starting it is discharged. No individual step of a finished subderivation can be used to apply a rule outside of the subderivation, but an individual step of an unfinished subderivation can be used in a subderivation of "lower level". Rules →I and ∨E involve the use of finished subderivations as a whole.

We define the (Fitch-style) Natural Deduction system for minimal logic as the set of rules

$$\mathbf{F}_{min} = \{\wedge E, \wedge I, \to E, \to I, \vee E, \vee I, R\}.$$

The Natural Deduction system for intuitionistic logic is defined as the set

$$\mathbf{F}_{int} = \mathbf{F}_{min} \cup \{EFSQ\},$$

where EFSQ is the following rule :

$$\begin{array}{l|ll}
l & \bot & \\
\vdots & \vdots & \\
l' & A & \quad \text{EFSQ, } l
\end{array}$$

Finally, $\mathbf{F}_{cl}$, the Natural Deduction system for classical logic, is defined as

$$\mathbf{F}_{cl} = \mathbf{F}_{int} \cup \{EM\},$$

where EM is the following rule :

$$\begin{array}{l|l|l}
i & & A \\ \cline{3-3}
\vdots & & \vdots \\
j & & C \\
k & & A \to \bot \\ \cline{3-3}
\vdots & & \vdots \\
l & & C \\
m & C & \qquad \text{Excluded Middle, } i\text{–}j,\ k\text{–}l
\end{array}$$

Définition A.14. *(conclusion, premisses, Fitch-style proof)*
Let Σ be a Fitch-style deduction in $\mathbf{F}_{min}$, $\mathbf{F}_{int}$, or $\mathbf{F}_{cl}$

1. The last line of Σ is called the conclusion of Σ.
2. The premisses of Σ are any formulae A such that A is an assumption of the subderivation which ends with the conclusion of Σ.
3. Let Δ be a set of premisses and A a formula. A Fitch-style deduction with A as a conclusion and the members of Δ as premisses is called a (Fitch-style) proof of A from Δ.

A.3 From Dialogues to Fitch-style Proofs

In this section, we give a procedure to translate the core of a $\text{FWS}_\mathbf{P}$ in $\mathcal{D}(\Delta, A)$ into a Fitch-style Proof of A (from Δ).

A.3.1 The procedure

The algorithm takes the core of a formal **P**-winning strategy $\text{FWS}_{\mathbf{P}}$ in one of the game systems we defined ($\mathbf{D}_{min}$, $\mathbf{D}_{int}$, $\mathbf{D}_{cl}$) and translate it into a Fitch-style proof in the corresponding Natural Deduction system ($\mathbf{F}_{min}$, $\mathbf{F}_{int}$, $\mathbf{F}_{cl}$, respectively). The mechanism is rather simple and consists in arranging the content of the assertive moves of $\text{FWS}_{\mathbf{P}}$'s core in a linear order such that the sequence of formulae complies with the natural deduction rules. Each member of the sequence is then labelled with a suitable number and a justification. We describe a procedure that translates the core of a $\text{FWS}_{\mathbf{P}}$ in $\mathbf{D}_{cl}$ into a Fitch-style proof in $\mathbf{F}_{cl}$. The algorithm for intuitionnistic logic is obtained by removing the clauses relative to the rule for Excluded Middle. The algorithm for minimal logic is obtained by removing also the clauses relative to Ex Falso Quodlibet.

Let us begin with some terminology. We say that, in a $\text{FWS}_{\mathbf{P}}$'s core, an assertion $\langle \mathbf{X} \vdash C \rangle$ *depends on* a move $\langle \mathbf{Y} \vdash A \vee B \rangle$ if **X** can formally defend C only after **Y** defended $A \vee B$. In a similar way, we say that $\langle \mathbf{X} \vdash B \rangle$ depends on application of rule SR-4 if **X** can formally defend it only after he played $\langle \mathbf{X}\ ?\ A \vee (A \to \bot) \rangle$ and **Y** answered this move.

Initial stage

First we place the members $\phi_1, \ldots, \phi_n$ of Δ as the premisses of the deduction and the thesis A as the conclusion. If A depends on a **O**-disjunction, then the justification of A is the application of a $\vee$E Rule. If it depends on application of SR-4, it is the result of rule EM. Otherwise it is the introduction rule corresponding to its main connective.

$$\begin{array}{c|l}
p_1 & \phi_1 \\
\vdots & \vdots \\
p_n & \phi_n \\
\hline
\vdots & \vdots \\
n & A
\end{array}$$

Incomplete Justification forms

Until the end of the procedure, no complete justification can be formed. Only the name of the rule applied can be given (see below). The lines of the steps to which the rule is applied will complete the justifications once every move is used by the procedure.

- Any move $\langle \mathbf{O} \vdash A \rangle$ challenging a move $\langle \mathbf{P} \vdash A \to B \rangle$ is introduced as a new assumption.

- Any pair of moves ($\langle \mathbf{O} \vdash A \rangle$, $\langle \mathbf{O} \vdash B \rangle$) played in virtue of the local rule for disjunction is introduced as a pair of assumptions opening parallel subderivations.
- Any pair of moves ($\langle \mathbf{O} \vdash A \rangle$, $\langle \mathbf{O} \vdash A \to \bot \rangle$) played in virtue of rule SR-4 is also introdued as a pair of assumptions opening parallel subderivations.
- Any move $\langle \mathbf{O} \vdash A \rangle$ used in virtue of rule SR-3 is introduced as the result of EFSQ Rule.
- Any other move $\langle \mathbf{O} \vdash A \rangle$ is introduced as the result of an Elimination Rule. The main connective of the formula defended by **O** with this move determines which Elimination Rule is used.
- Any move $\langle \mathbf{P} \vdash A \rangle$ occurring after **O** already performed the move $\langle \mathbf{O} \vdash A \rangle$ is the result of a Reiteration Rule.
- Any move $\langle \mathbf{P} \vdash A \rangle$ which depends on a move $\langle \mathbf{O} \vdash B \vee C \rangle$ is the result of a $\vee$E Rule.
- Any move $\langle \mathbf{P} \vdash A \rangle$ which depends on application of rule SR-4 is introduced as the result of EM Rule.
- Any other move $\langle \mathbf{P} \vdash A \rangle$ is the result of an Introduction Rule. The main connective of A determines which Introduction Rule is used.

Generalities

Once the initial stage is done, the procedure follows the order of the moves but ignores non assertoric expressions, which are specific to the dialogical approach. In general, moves of the form $\langle \mathbf{O} \vdash A \rangle$ will be placed upwards, that is such that its label immediately follows the one of the last assertion placed upwards ; moves of the form $\langle \mathbf{P} \vdash A \rangle$ will be placed downwards, that is such that its label (not necessarily immediately) precedes the last assertion placed downwards in the current subderivation.
A **O**-challenge against a conditional assertion is placed upwards as a new assumption which is available up to the assertion lastly placed downwards, leaving it outside the new subderivation. When there is a split in the dialogical proof, these general conventions about placement must be adapted.

Split : case 0

Whenever a move $\langle \mathbf{P} \vdash A \rangle$ depends on the application of rule SR-4, it is placed as the result of EM Rule. Morever, the moves $\langle \mathbf{O} \vdash B \rangle$ and $\langle \mathbf{O} \vdash B \to \bot \rangle$ must be dealt with immediately after $\langle \mathbf{P} \vdash A \rangle$, even if other moves occur in FWS$_{\mathbf{P}}$'s core between $\langle \mathbf{P} \vdash A \rangle$ and the split. These other moves must be placed in both subdeductions opened with B and $B \to \bot$.

Suppose C is the assertion lastly placed upwards and D the one lastly placed downwards (notice that D can possibly be A). The last formula of both subderivations is an occurrence of A (which incomplete justification form can consist in a reiteration or Introduction rule). The occurrence of A associated with the label l cannot be used again this way. We obtain :

$$
\begin{array}{l|lll}
h & C & & \\
i & & \begin{array}{|l} B \\ \hline \vdots \\ A \end{array} & \\
i_2 & & & \\
j_1 & & \begin{array}{|l} B \to \bot \\ \hline \vdots \\ A \end{array} & \\
j_2 & & & \\
k & D & & \\
\vdots & \vdots & & \\
l & A & & \text{Excluded Middle}
\end{array}
$$

Split : case 1

Suppose we have to place the move $\langle \mathbf{P} \vdash A \rangle$ that counts as a challenge against a move $\langle \mathbf{O} \vdash A \to B \rangle$. This **P** move is followed by a **O** split : the left branch starts with the proper challenging move $\langle \mathbf{O}\text{-}f\text{-}e \rangle$ against $\langle \mathbf{P} \vdash A \rangle$, the right branch starts with the defensive move $\langle \mathbf{O} \vdash B \rangle$. The procedure places A and B in the current subderivation as follows :

$$
\begin{array}{l|ll}
i & C & \ldots \\
\vdots & \vdots & \\
j & A & \ldots \\
k & B & \to\mathrm{E} \\
\vdots & \vdots & \\
l & D & \ldots
\end{array}
$$

where C is the last assertion placed upwards and D is the last assertion placed downwards. The procedure will place the remaining assertions of the left branch between C and A, and the remaining assertions of the right branch between B and D, following the appropriate clauses.

Split : case 2

Consider the case where $\mathrm{FWS}_{\mathbf{P}}$'s core comes to a split in which the left branch starts with the move $\langle \mathbf{O}?_L \rangle$ and the right branch $\langle \mathbf{O}?_R \rangle$ challenging a preceding move $\langle \mathbf{P} \vdash A \wedge B \rangle$. The respective defensive moves are $\langle \mathbf{P} \vdash A \rangle$, $\langle \mathbf{P} \vdash B \rangle$ and are placed in the following way :

$$\begin{array}{l|ll} i & C & \dots \\ \vdots & \vdots & \\ j & A & \dots \\ \vdots & \vdots & \\ k & B & \dots \\ k+1 & D & \dots \end{array}$$

where C is the last assertion placed upwards and D the last assertion placed downwards by the procedure. The remaining assertions of the left branch will be placed between C and A, while the remaining assertions of the right branch will be placed between A and B.

Split : case 3

Suppose FWS$_{\mathbf{P}}$'s core comes to a split where the left branch starts with the move $\langle \mathbf{O} \vdash A \rangle$ and the right branch starts with $\langle \mathbf{O} \vdash B \rangle$, such that these moves are the two possible defences of a preceding move $\langle \mathbf{O} \vdash A \vee B \rangle$. Those two assertions are placed as follows :

$$\begin{array}{l|lll} i & C & & \dots \\ j_1 & & \begin{array}{|l} A \\ \hline \vdots \end{array} & \\ \vdots & & & \\ k_1 & & \begin{array}{|l} B \\ \hline \vdots \end{array} & \\ \vdots & & & \\ l & D & & \dots \end{array}$$

where C is the last assertion placed upwards and D the last assertion placed downwards. The remaining assertions of the left branch are placed within the subderivation opened by A, those of the right branch within the subderivation opened by B.

The last formula of both subderivations is an occurrence of the first formula occurring in the sequel of the proof which was given the justification form $\vee$E and which has not been used this way yet. Let E be such a formula, we obtain something of the form :

$$
\begin{array}{l|lll}
j_1 & & \multicolumn{1}{|l}{A} & \\ \cline{3-3}
\vdots & & \multicolumn{1}{|l}{\vdots} & \\
j_2 & & \multicolumn{1}{|l}{E} & \dots \\
k_1 & & \multicolumn{1}{|l}{B} & \\ \cline{3-3}
\vdots & & \multicolumn{1}{|l}{\vdots} & \\
k_2 & & \multicolumn{1}{|l}{E} & \dots \\
\vdots & \vdots & & \\
l & E & & \lor\mathrm{E}
\end{array}
$$

The occurrence of E associated with the label l must not be used again this way.

Completing the justifications

When every assertion in $\mathrm{FWS}_{\mathbf{P}}$'s core has been dealt with by the procedure, the useless dots are removed, and the steps are numbered up-down, starting from 1. Then the justifications of each step can be completed. In what follows, the formula of which we complete the justification is called A. Recall that assumptions get no justification.

- *Ex falso* : The full justification of A is $EFSQ, l$, where l is the label given to the move $\langle \mathbf{O} \vdash \bot \rangle$ which is challenged in virtue of SR-3 to play the move $\langle \mathbf{P} \vdash A \rangle$.
- *Reiteration* : The full justification of A is R, l where l is the label given to the previous move $\langle \mathbf{O} \vdash A \rangle$.
- *Conditional Introduction* : The full justification is $\to\mathrm{I}, l_1\text{-}l_2$ where l_1 is the label given to the move $\langle \mathbf{O} \vdash C \rangle$ challenging A, and l_2 is the label allocated to the move $\langle \mathbf{P} \vdash D \rangle$ which defends it.
- *Other Introduction Rules* : The full justification is obtained by adding the label(s) of the move(s) $\langle \mathbf{P} \vdash D \rangle$ defending A in $\mathrm{FWS}_{\mathbf{P}}$'s core.
- *Disjunction Elimination* : The full justification of A is $\lor\mathrm{E}, i, j\text{-}k, l\text{-}m$ where i is the label given to the disjunctive move $\langle \mathbf{O} \vdash C \lor D \rangle$ A depends on, j-k denotes the labels of the subderivation initiated by the first disjunct and l-m denotes the labels of the subderivation initiated by the second disjunct.
- *Other Elimination Rules* : The full justification is obtained by adding the label given to the move $\langle \mathbf{O} \vdash B \rangle$ defended by the move $\langle \mathbf{O} \vdash A \rangle$.

A.3.2 Correction of the algorithm

Théorème A.3.1. *Let $\mathcal{S}$ be a formal winning strategy for $\boldsymbol{P}$ in $\mathcal{D}(\Delta, A)$. $\mathcal{F}$, the result of applying our algorithm to the core of $\mathcal{S}$, is a correct Fitch-style proof of*

A from Δ.

Démonstration. For the sake of brievety, we only sketch a proof that is fairly easy but fastidiously long. Here are the main ideas :

(i) The result of the procedure is a sequence of tuples, each of the form (l, A, m) with l a label, A a formula and m a justification, linearly ordered in a sequence by their labels.

(ii) Now consider this sequence as obtained from the empty sequence by successively adding the tuples, following the order on the labels. It can be shown by an inductive reasonning that this construction observes at each step the rules of Fitch-style Natural deduction as given in section A.2. [107] Once this is done, we have shown that $\mathcal{F}$ is a Fitch-style deduction.

(iii) After this, it is easy to show that $\mathcal{F}$ also observes the clauses of definition A.14, which allows us to conclude that $\mathcal{F}$ is a correct Fitch-style proof of A from Δ. ■

A.3.3 A simple example

Let us consider the formula $(a \vee b) \rightarrow (a \vee b)$ which is valid in minimal logic. Here is the core of a $\text{FWS}_{\mathbf{P}}$ for it :

$$
\begin{array}{c}
\langle \mathbf{P} \vdash (a \vee b) \rightarrow (a \vee b) \rangle \\
| \\
\langle \mathbf{O} \vdash a \vee b \rangle \\
| \\
\langle \mathbf{P} \vdash a \vee b \rangle \\
| \\
\langle \mathbf{O} ?_{\vee} \rangle \\
| \\
\langle \mathbf{P} ?_{\vee} \rangle \\
\diagup \quad \diagdown \\
\begin{array}{cc}
\langle \mathbf{O} \vdash a \rangle & \langle \mathbf{O} \vdash b \rangle \\
| & | \\
\langle \mathbf{P} \vdash a \rangle & \langle \mathbf{P} \vdash b \rangle
\end{array}
\end{array}
$$

The initial stage is :

$$
\begin{array}{c|ll}
\vdots & \vdots & \\
l & (a \vee b) \rightarrow (a \vee b) & \rightarrow\mathrm{I}
\end{array}
$$

The next assertion is a **O**-assertion that counts as a challenge against the thesis. We obtain :

$$
\begin{array}{c|ll}
l_1 & \quad \begin{array}{|l} a \vee b \\ \hline \vdots \end{array} & \\
\vdots & & \\
l & (a \vee b) \rightarrow (a \vee b) & \rightarrow\mathrm{I}
\end{array}
$$

107. In fact, this is very clear – but very long – when one works with formal definitions similar to the one used in Geuvers et Nederpelt (2004).

The next assertion is the corresponding **P**-defence which depends itself on a **O**-disjunction. So we obtain

$$
\begin{array}{l|ll}
l_1 & \quad \mid \underline{a \lor b} & \\
\vdots & \quad \mid \vdots & \\
l_2 & \quad \mid a \lor b & \lor\mathrm{E} \\
l & (a \lor b) \to (a \lor b) & \to\mathrm{I}
\end{array}
$$

Next we have a **O** split between two possible defences of a disjunction.

$$
\begin{array}{l|ll}
l_1 & \quad \mid \underline{a \lor b} & \\
l_3 & \quad \mid \quad \mid \underline{a} & \\
\vdots & \quad \mid \quad \mid \vdots & \\
l_4 & \quad \mid \quad \mid a \lor b & \lor\mathrm{I} \\
l_5 & \quad \mid \quad \mid \underline{b} & \\
\vdots & \quad \mid \quad \mid \vdots & \\
l_6 & \quad \mid \quad \mid a \lor b & \lor\mathrm{I} \\
l_2 & \quad \mid a \lor b & \lor\mathrm{E} \\
l & (a \lor b) \to (a \lor b) & \to\mathrm{I}
\end{array}
$$

In both branches, there remains a **P** assertion which is a repetition of a previous **O**-assertion and which counts as a defence of the move which depends on the **O**-disjunction. Thus :

$$
\begin{array}{l|ll}
l_1 & \quad \mid \underline{a \lor b} & \\
l_3 & \quad \mid \quad \mid \underline{a} & \\
l_7 & \quad \mid \quad \mid a & \mathrm{R} \\
l_4 & \quad \mid \quad \mid a \lor b & \lor\mathrm{I} \\
l_5 & \quad \mid \quad \mid \underline{b} & \\
l_8 & \quad \mid \quad \mid b & \mathrm{R} \\
l_6 & \quad \mid \quad \mid a \lor b & \lor\mathrm{I} \\
l_2 & \quad \mid a \lor b & \lor\mathrm{E} \\
l & (a \lor b) \to (a \lor b) & \to\mathrm{I}
\end{array}
$$

Finally, we number the steps of this Fitch-style deduction and complete the justification forms :

1	$a \vee b$	
2	a	
3	a	R, 2
4	$a \vee b$	$\vee$I, 3
5	b	
6	b	R, 5
7	$a \vee b$	$\vee$I, 6
8	$a \vee b$	$\vee$E, 1, 2–4, 5–7
9	$(a \vee b) \rightarrow (a \vee b)$	$\rightarrow$I, 1–8

A.3.4 Comments and discussion

Let us resume our take on the definition of meaning as it stems from the previous sections. The dialogical perspective offers means to distinguish between three aspects of meaning :

i. The argumentation forms associated with a connective $\#$ determine what may be called its *local meaning*, by giving rules for an interaction in a sellarsian game of assertions and requests. Such rules are abstract triples (p, c, d) which give constraints on games without defining what a game is. They just describe what one is committing oneself to when performing a $\#$-assertion. One may say that the local meaning of $\#$ is the meaning of the connective *stricto sensu*. Take disjunction as an example. Its local meaning is given by the forms :

Precondition	$\langle \mathbf{X} \vdash A \vee B \rangle$	$\langle \mathbf{X} \vdash A \vee B \rangle$
Challenge	$\langle \mathbf{Y} \ ?\ _{\vee} \rangle$	$\langle \mathbf{Y} \ ?\ _{\vee} \rangle$
Defence	$\langle \mathbf{X} \vdash A \rangle$	$\langle \mathbf{X} \vdash B \rangle$

Local meaning spells out a set of ways to deal with committment in the assertoric language game, independently of any specific form such a game may take (i.e. retaining as the only relevant feature of the game the connexion between commitment and request instanciated by the structure Precondition–Challenge–Defense).

It is extremely important to build the theory of meaning at this abstract level because, as advocated by Wittgenstein, [108] many games can be played without a complete definition of the rules. Hence not only the same argumentation forms are a common basis for many different language games (as

108. See Wittgenstein's *Lectures* 110.

exemplified in this paper), but you can even imagine a game where such forms are all you know about the rules.

ii. The local meaning of the connective # may then be contrasted with its *strategical meaning*, i.e. the way argumentation forms combine into extensive forms of a game associated with a #-assertion. This aspect of meaning is determined by the combination of argumentation forms *and* structural rules, and is given by the set of terminal histories in the dialogue associated with the assertion. Structural rules bear this name for they structure the use of the argumentation forms, in the sense that they give constraints for a set of move to be an actual play of the game. The strategical meaning of a disjunctive statement is thus given by the following schematic extensive form :

$$\begin{tikzcd}[column sep=small]
 & \langle \mathbf{X} \vdash A \vee B \rangle \arrow[d, dash] & \\
 & \langle \mathbf{Y} \; ?_{\vee} \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
\langle \mathbf{X} \vdash A \rangle \arrow[d, dash] & & \langle \mathbf{X} \vdash B \rangle \arrow[d, dash] \\
\dots & & \dots
\end{tikzcd}$$

Strategical meaning recursively combines into an extensive form for an arbitrary statement, which spells out the *conditions on the justifiability* of the statement in terms of a disjunction of justifications for sets of elementary assertions, one such set for each terminal history.

iii. The notions of copycat strategy and of non-contentious assertion allow to single out a specific case of conditions on the justifiability of a thesis, namely when those conditions are null, which corresponds precisely to the cases when the thesis is a valid consequence of the initial concessions. With respect to disjunction and formal strategies, one might observe the following. Let C be a complex formula and $\langle \mathbf{Y} \; ? \; C \rangle$ denote the appropriate challenge against $\langle \mathbf{X} \vdash C \rangle$. Assume a game situation where $\mathbf{P}$ has no formal justification of his assertion of $\langle \mathbf{P} \vdash C \rangle$ before $\mathbf{O}$ defended $A \vee B$. Here's a fragment of a $\mathbf{P}$ strategy :

$$
\begin{array}{ccc}
 & \vdots & \\
 & \mid & \\
 & \langle \mathbf{P} \vdash C \rangle & \\
 & \mid & \\
 & \langle \mathbf{O} \ ?\ C \rangle & \\
 & \mid & \\
 & \vdots & \\
 & \mid & \\
 & \langle \mathbf{O} \vdash A \vee B \rangle & \\
 & \mid & \\
 & \langle \mathbf{P} \ ?\ _{\vee} \rangle & \\
 \swarrow & & \searrow \\
\langle \mathbf{O} \vdash A \rangle & & \langle \mathbf{O} \vdash B \rangle \\
\mid & & \mid \\
\vdots & & \vdots
\end{array}
$$

One may observe that, for $\mathbf{P}$, to have a formal justification of $\langle \mathbf{P} \vdash C \rangle$ exactly amounts to have one given the **O**-assertion of A on the one hand, and to have one given the **O**-assertion of B on the other hand. And this is precisely what the Elimination rule for disjunction says.

Thus in general one may introduce a notion of *inferential behaviour* for the connectives :

Définition A.15 (Inferential behaviour)**.** Let # be a connective and A a #-formula. The *inferential behaviour* of # is compound of :

(a) the relation between the assertion $\langle \mathbf{P} \vdash A \rangle$ and the set of assertions of formulas B such that $\mathbf{P}$ needs a formal justification of $\langle \mathbf{P} \vdash B \rangle$ in order to formally justify the assertion of A ;

(b) the relation between the assertion $\langle \mathbf{O} \vdash A \rangle$ and the set of assertions of formulas B such that $\mathbf{P}$ can formally justify $\langle \mathbf{P} \vdash B \rangle$ given the concession $\langle \mathbf{O} \vdash A \rangle$.

In other terms, the inferential behaviour of # is given by the role a #-formula will play, as either a **P**- or a **O**-assertion, *in a game where **P** wants to stay under nullary conditions on the justification he provides.*

Let us conclude with some remarks on the relation between dialogues and natural deductions. The local semantics in dialogical logic and the Eliminiation-Introduction rules in Natural Deduction may be considered as rival approaches to define the meaning of logical constants, when one understands, in the wittgensteinian tradition, meaning as use.

One may then observe that there is no dialogical equivalent to the introduction rules, and thus that the local semantics is incomplete in so far as it does not reflect completely the behaviour of a connective when it constitutes the outmost form of either a conclusion or a premiss. But this argument is somehow misleading, for dialogical *logic* and natural deduction are in agreement indeed on both aspects

– as our algorithm and the previous remarks show clearly. There is a difference, though, and an important one. The argumentation forms are intended to give an account of the meaning of connectives in the context of sellarsian games of asking for and giving reasons (or justifications), which we take as the fundamental form of language game, or at least one of them. Such forms may then be studied from a strategical point of view, and a requisit of formality may be added in order to reach the level of their inferential behaviour. [109]

We take it as one of the main contribution of the dialogical approach to show how this behaviour is *explained* by argumentation forms (i.e. meaning proper) plus strategical considerations, whereas Introduction/Elimination rules rather *describe* it. Actually, while one can seek in the general patterns of interaction as described by argumentation forms on the one hand and structural rules on the other hand the reasons why inference rules are like they are, such a question is meaningless when one take them as a definition of the meaning of connectives.

In the core of a $\mathrm{FWS}_{\mathbf{P}}$, we consider *incarnated* local rules, in the sense that the tree structure generated by application of argumentation forms to a precondition may be defined differently when the precondition is a **P**- or a **O**-assertion. This is the whole point of our remarks about dialogues and tableaux. In such a strategy, a **O**-argumentative form [110] decomposes a complex **O**-assertion in simpler one(s), which is exactly what does an elimination rule in a Natural Deduction proof. But when we restrict our attention to $\mathrm{FWS}_{\mathbf{P}}$'s cores, there is no more symmetry in the signification of game phases than there is in the rules. **O**-forms eliminate complex assertions in order to reach elementary ones because, in the context of a formal strategy, such **O**-elementary assertions (and the associated justifications) should be understood as *ressources* for **P**. Now **P**-forms have a different signification : they are the proper justifications, since through the whole game the burden of the justification of the thesis lies on **P**'s side. Therefore a **P**-argumentative form gives the justification of the assertion of a complex formula by means of (the assertion of one of) its immediate subformulas, which just amounts to the same as applications of Introduction rules. In other terms, to see how they relate to natural deduction proofs, **P**-forms should be read bottom-up (i.e. from elementary justification to the justification of the thesis) while **O**-forms should be read top-down (i.e. from the concession of complex ressources to the concession of elementary ones). Most of the clauses of the algorithm give an account of these relations between dialogical local rules and Elimination-Introduction rules in Natural Deduction.

There is an exception however to these considerations, of which we have already said something : the elimination rule for disjunction, which is associated with a **P** case in the procedure. Contrary to the other elimination rules, the formula derived may not be a subformula of the disjunction. But such a rule is

109. Notice that such a division between the meaning and the inferential behaviour of a connective relies crucially on the decision to introduce formality as a property of strategies and not of games.

110. That is, a form the precondition of which is a **O**-assertion.

clearly of a meta-logical nature. Since it consists in drawing a formula if it is derived in two parallel subderivations (each starting with one of the disjuncts), it clearly relies on considerations about the combination of proofs and not only on the meaning of disjunction, as witnessed by the three premisses in the (non Fitch) elimination rule.

Clearly, in an *argumentation* (i.e. a terminal history in the dialogue, as represented in iii. above) only one of the disjuncts will be asserted, which is as it should be : whenever I assert a disjunction, I just claim I can provide a justification to one of the disjuncts. It is only when it comes to define conditions on *formal strategies* in the game that the *behaviour* of disjunction coincides with its elimination rule.

We see the difference between explaining and (only) describing an inferential behaviour as a strong argument in favor of the claim that Natural Deduction rules are not, as they stand, meaning-constituting. In fact, other arguments for this claim were already given in discussions about the *tonk* connective introduced in the famous Prior (1961) : the usual introduction of harmony constraints [111] indicates that Introduction/Elimination rules are not sufficient to constitue the meaning of the connectives. The dialogical perspective offers another way to understand what goes wrong with *tonk*, namely an endogenous way.

For the sake a brevity, we restrict ourselves to a summary. [112] First of all, let us recall the Natural Deduction rules for *tonk* :

$$
\begin{array}{c|ll}
l & A & \\
\vdots & \vdots & \\
l' & A \text{ tonk } B & \quad \text{tonk I, } l
\end{array}
$$

$$
\begin{array}{c|ll}
l & A \text{ tonk } B & \\
\vdots & \vdots & \\
l' & A & \quad \text{tonk E}_1\text{, } l
\end{array}
\qquad\qquad
\begin{array}{c|ll}
l & A \text{ tonk } B & \\
\vdots & \vdots & \\
l' & B & \quad \text{tonk E}_2\text{, } l
\end{array}
$$

It is impossible to give local rules which will match the Introduction/Elimination ones through our algorithm. To see this, assume that there were such rules. Then they should specify whose player (challenger or defender) may choose between a continuation of the game with A or with B. So let us suppose that the choice is the challenger's. The local rules for *tonk* would then be of the form :

111. As pointed out by Belnap (1962) and others.
112. A thorough discussion on this subject is to be presented in a forthcoming paper

Precondition	$\langle \mathbf{X} \vdash A$ tonk $B\rangle$	$\langle \mathbf{X} \vdash A$ tonk $B\rangle$
Challenge	$\langle \mathbf{Y}$? $L\rangle$	$\langle \mathbf{Y}$?$R\rangle$
Defence	$\langle \mathbf{X} \vdash A\rangle$	$\langle \mathbf{X} \vdash B\rangle$

But consider what happens when this form is incarnated as a **P**-case in a FWS$_{\mathbf{P}}$ and translated through our algorithm : we obtain the following rule :

$$
\begin{array}{l|ll}
l_1 & A & \\
\vdots & \vdots & \\
l_2 & B & \\
\vdots & \vdots & \\
l' & A \text{ tonk } B & \quad \text{tonk I}, l_1, l_2
\end{array}
$$

However, this is not the Introduction rule for *tonk*, which is of disjunctive type while the one we obtain is of conjunctive type. Well, then how about letting the choice be the defender's ? Unfortunately, a similar problem appears.

Precondition	$\langle \mathbf{X} \vdash A$ tonk $B\rangle$	$\langle \mathbf{X} \vdash A$ tonk $B\rangle$
Challenge	$\langle \mathbf{Y}$? tonk$\rangle$	$\langle \mathbf{Y}$?tonk$\rangle$
Defence	$\langle \mathbf{X} \vdash A\rangle$	$\langle \mathbf{X} \vdash B\rangle$

If we take the incarnated form of a **O**-case in a FWS$_{\mathbf{P}}$ and translate it by means of our procedure, we obtain a rule which is not the Elimination rule for *tonk* : the rule obtained will be much like the Elimination rule for disjunction while we need an Elimination rule of a conjunctive type.

To summarize, from the dialogical point of view, the reason why the inferential behaviour expressed by Natural Deduction rules for *tonk* makes no sense is that these rules do not let us decide who has the choice. Indeed this decision must be taken at the local level, but such a decision will not yield the intended Natural Deduction rules. In other words there are no player-independant rules for *tonk* : this behaviour does not reflect any form of interaction between players in a language game. Player-independance is a necessary feature of local rules since there is no mean to specify who the players are at this level. [113] Since local rules cannot be given for *tonk*, this connective has no meaning *stricto sensu*, and this is straightforward in the dialogical approach : the very notion of local meaning has a form of in-built harmony.

Acknowledgments

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113. Players are defined by the structural rules.

Annexe B

Eléments d'une reconstruction dialogique d'une tradition de l'Inde classique

Le contenu de cette annexe est tiré de l'article Clerbout *et al.* (2011) consacré à un aspect de la philosophie jaine. Notre but est ici d'illustrer encore une fois la flexibilité de l'approche dialogique afin de souligner son potentiel comme théorie de la signification fondée sur les pratiques argumentatives. Dans cet article était présentée une lecture contemporaine de la théorie jaine du *naya-vada* — théorie des points de vue — accompagnée par une reconstruction dialogique. Nous ne faisons ici que de brefs rappels généraux concernant cette théorie : pour une présentation plus détaillée, le lecteur peut consulter Clerbout *et al.* (2011) ainsi que certaines sources qui y sont citées en bibliographie.

Brièvement, le *naya-vada* peut être décrit comme une théorie selon laquelle chaque affirmation est toujours effectuée dans le contexte d'une perspective épistémique donnée — un « point de vue » — dont elle dépend. Dans le *naya-vada*, il y a sept tels points de vue. [114] L'un des buts de l'approche jaine est alors d'accompagner cette épistémologie d'une théorie dans laquelle ces points de vue sont pris en compte lors d'un débat. Pour le logicien moderne, un premier réflexe est sans doute de proposer une lecture modale de cette théorie, dans laquelle l'idée de départ consisterait à considérer que les points de vue sont le pendant des mondes possibles ou contextes que l'on introduit dans les sémantiques modales. Mais il a été noté dans Clerbout *et al.* (2011) que plusieurs éléments justifient de chercher à approfondir la question. Notamment, un élément important est que chaque point de vue représente un type d'accès épistémique aux objets du domaine de discours. Ce type définit à son tour un cadre précis de la manière dont les descriptions de ces objets peuvent ou non être utilisées dans des assertions, et de la manière dont ces assertions peuvent ou doivent être justifiées. Une sémantique modale habituelle, même dialogique, apparaît trop abstraite pour permettre de

114. Il semble en fait que ce nombre dépende de l'époque et du texte considérés : d'autres présentations fournissent des classifications différentes de celle que nous suivons. Des détails sont évoqués dans Clerbout *et al.* (2011).

rendre compte avec finesse des différents accès épistémiques que l'on trouve dans la théorie du naya-vada.

Comme on l'a dit, chaque point de vue dans la théorie représente un type d'accès aux objets du domaine de discours. L'accès dépend de considérations ontologiques et épistémiques : par exemple, l'accès différera selon que certaines hypothèses ayant trait à l'existence des objets sont admises ou non. Au final, les différences entre les sept points de vue se trouvent dans leurs différentes hypothèses ontologiques et épistémiques sous-jacentes. Une reconstruction moderne de la théorie jaine doit donc offrir un moyen d'exprimer et de prendre en compte ces différences. Mais il est crucial que la reconstruction permette également de remplir le but des philosophes jains de proposer aussi une théorie du débat dans laquelle ces différences peuvent être prises en compte. Pour ce double but, le cadre dialogique apparaît comme un candidat plus que prometteur pour cette reconstruction moderne parce que le fondement même de la théorie dialogique de la signification est de placer le débat argumentatif au coeur de l'approche.

Une présentation détaillée de chaque point de vue tel que présenté par les jains dépasse largement les objectifs de cette annexe et peuvent être trouvées non seulement dans Clerbout *et al.* (2011) mais aussi dans de nombreux travaux de spécialistes. Encore plus que leur rapport avec la théorie jaine du *naya-vada*, c'est la présentation des sept sémantiques en elle-même qui nous intéresse ici. Il s'agit là d'une occasion de présenter différentes sémantiques mettant en jeu différentes pratiques argumentatives dont les régulations permettent de rendre compte de diverses hypothèses qui peuvent sous-tendre un débat. Les hypothèses en question concernent, comme nous le verrons ci-dessous, le statut ontologiques de différents types d'objets ou encore le processus de désignation des objets. Les pratiques argumentatives, quant à elle, comprennent entre autres l'usage de la quantification (de premier ou de deuxième ordre), le mécanisme de substitution, la justification de l'utilisation des termes individuels et des prédicats. Au minimum, cette présentation représente quelques pas dans des directions peu ou pas explorées en dialogique : quantification de deuxième ordre, usage des descriptions définies, etc.

Langage et règles locales

Le langage $\mathcal{L}$ est construit sur :

- Un ensemble dénombrable de symboles de variables de premier ordre : x_1, x_2, ...
- Un ensemble dénombrable de symboles de variables monadiques de deuxième ordre : X_1, X_2, ...
- Un ensemble dénombrable de symboles de prédicats unaires : P_1, P_2, ...
- Le symbole d'égalité $=$.
- Les connecteurs propositionnels : $\wedge$, $\vee$, $\rightarrow$, $\neg$.
- Les quantificateurs : $\exists$, $\forall$.

– L'opérateur *iota* : ι.
– Un ensemble dénombrable de symboles de constantes d'individus : k_1, k_2, ...

Un *terme* peut être (i) une variable de premier ordre, (ii) une constante d'individu ou (iii) une expression de la forme $(\iota x)(Px)$, où x est une variable de premier ordre et P un prédicat unaire.

Définition B.1 (Formules)**.** Soient t et u des termes, et P un prédicat unaire. Alors $t = u$ et Pt sont des formules atomiques. Les formules sont données par :

(i) Si φ est une formule atomique alors φ est une formule.
(ii) Si φ et ψ sont des formules alors $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ et $\neg\varphi$ le sont aussi.
(iii) Si φ est une formule et x une variable de premier ordre, alors $\forall x\varphi$ et $\exists x\varphi$ sont des formules.
(iv) Si φ est une formule et X une variable monadique de deuxième ordre, alors $\forall X\varphi$ et $\exists X\varphi$ sont des formules.
(v) Rien d'autre n'est une formule.

Les règles locales sont simplement celles de la dialogique du premier ordre (Chapitre 2) auxquelles on ajoute les règles suivantes pour la quantification de deuxième ordre :[115]

Affirmation	$\mathbf{X}\,!\,\exists X\varphi$	$\mathbf{X}\,!\,\forall X\varphi$
Attaque	$\mathbf{Y}\,?[!\varphi(P_i)]^\infty$	$\mathbf{Y}\,?[!\varphi(P)]$
Défense	$\mathbf{X}\,!\,\varphi(P_i)$	$\mathbf{X}\,!\,\varphi(P)$

Où P est un prédicat unaire et $\varphi(P)$ dénote la formule obtenue en remplaçant chaque occurrence de X par P dans φ.

Le langage et la signification locale des constantes logiques étant fixés, on peut maintenant introduire sept versions différentes des règles structurelles, de manière à obtenir sept différentes sémantiques dialogiques. Celles-ci diffèrent par les pratiques argumentatives qui sont tolérées ou non, elles-mêmes spécifiées par différentes hypothèses d'ordre ontologique ou épistémique. On obtient alors une reconstruction dialogique pour chacun des points de vue de la théorie jaine.

Les sept sémantiques que nous présentons brièvement ci-dessous ont toutes en commun les règles structurelles habituelles de la dialogique de premier ordre.[116]

115. À cause de leur ressemblance avec celles pour la quantification de premier ordre, le lecteur peut se reporter au Chapitre 2 pour quelques éclaircissements.

116. Ici nous nous en tenons aux règles pour la logique classique, mais il est bien sûr possible de considérer la variante intuitionniste. Peut-être qu'une connaissance détaillée de la théorie jaine permet d'effectuer un choix entre les deux versions pour une reconstruction fidèle, mais ceci dépasse nos compétences et notre domaine d'intérêt.

B.1 Premier point de vue

Selon le premier point de vue, la connaissance peut porter sur deux types d'objets : les universaux et les particuliers. La connaissance implique en effet la capacité à reconnaître l'existence de caractéristiques communes aux choses et de les classer selon ses caractéristiques. Il faut donc conclure que les universaux existent, comme les particuliers, et sont donc objets de connaissance. Ainsi, un énoncé de la forme « k est P » concerne deux objets : un particulier, ici noté k, et une propriété, ici notée P. Il faut également pouvoir rendre compte de la relation d'inhérence entre certains universaux et certains particuliers — celle-là même sur laquelle se fonde la classification. Cela dit, le premier point de vue exclut par ailleurs qu'il puisse y avoir une relation d'inhérence entre universaux.

Le langage doit donc pouvoir représenter deux types d'objets : c'est la raison pour laquelle il permet la quantification de premier et de deuxième ordre. Mais puisqu'il n'y a pas de relation d'inhérence possible entre universaux, il ne peut y avoir de prédication sur les prédicats et donc pas de quantification au-delà de celle de deuxième ordre. Dans la mesure où les autres points de vue sont encore plus restrictifs sur ces sujets, le langage $\mathcal{L}$ défini ci-dessus convient donc parfaitement.

Un aspect important dans le contexte de théories indiennes de la connaissance est que pour chaque type de connaissance on considère certains prédicats caractéristiques de ce type, applicables aux objets de connaissance. Il s'agit là d'un point dont la complexité et l'intérêt dépassent l'exposition dans cette annexe. Formuler des règles précises et concordantes avec les théories indiennes est un travail qui demande notamment une expertise du champ que nous ne possédons pas. Mais pour souligner la capacité de l'approche dialogique à tenir compte de tels impératifs nous exposons ici une méthode de contrôle de l'usage des prédicats, bien que trop abstraite pour une reconstruction totalement détaillée des théories en question. Cette méthode de contrôle passe par la règle structurelle suivante, ajoutée aux règles pour la dialogique de premier ordre :

> **SR-P1** : Le Proposant ne peut utiliser un prédicat P que si l'Opposant l'a utilisé auparavant pour attaquer une affirmation $\forall X\varphi$ ou défendre une affirmation $\exists X\varphi$.

Il s'agit là d'un méthode de contrôle que l'on rencontre souvent dans le cadre de l'approche dialogique. En particulier, cette règle est la version au deuxième ordre d'une règle apparaissant dans l'approche dialogique de la logique libre. [117]

B.2 Deuxième point de vue

Dans le cas du deuxième point de vue, les universaux sont également objets de connaissance. Mais à la différence du premier point de vue, il s'agit des seuls

117. Voir Rahman (2001), Fontaine et Ramah (2010).

objets de connaissance. L'argument est que la persistance nécessaire pour qu'il puisse y avoir connaissance ne se trouve qu'au niveau des universaux : le monde perceptible, auquel les particuliers se rapportent, n'est quant à lui que changement. Ces derniers ne sont rien d'autre que des manifestations d'universaux.

Les règles structurelles pour cette deuxième sémantique dialogique sont les mêmes que celles pour le premier point de vue — y compris la règle contrôlant l'usage des prédicats — auxquelles on ajoute deux règles. La première prend en charge le fait que les particuliers ne sont pas objets de connaissance : une manière de le prendre en compte est de considérer qu'il ne peut y avoir de quantification portant sur les particuliers.

> **SR-P2.1** : Aucune quantification de premier ordre n'est autorisée.

Cette règle a pour effet d'exclure toute quantification de premier ordre des expressions qui peuvent être jouées comme affirmation. Cependant, elle n'exclut pas l'usage des termes individuels qui peuvent encore apparaître au sein des affirmations des joueurs. Mais la particularité est ignorée dans le deuxième point de vue : les termes ne sont donc utilisés dans cette sémantique que comme manifestations des propriétés, au point qu'ils ne peuvent pas être distingués quand ils sont manifestations des mêmes propriétés. La prochaine règle s'attaque à cet aspect du deuxième point de vue.

> **SR-P2.2** : Si le joueur **X** a affirmé Pk_i et Pk_j au cours d'une même partie, alors **Y** peut requérir de **X** qu'il affirme $k_i \overset{P}{\approx} k_j$ dans cette partie.

On utilise $k_i \overset{P}{\approx} k_j$ pour le fait que k_i et k_j sont indiscernables par rapport à P. Cette indiscernabilité ouvre la voie à un affaiblissement de la règle formelle, autorisant par exemple **P** à affirmer Pk_i si **O** a affirmé Pk_j et $k_i \overset{P}{\approx} k_j$ auparavant. Il s'agit là d'une forme affaiblie de substitution.

B.3 Troisième point de vue

Le troisième point de vue est pour ainsi dire le pendant du deuxième : cette fois, ce sont les universaux qui ne peuvent être objets de connaissance. Une raison avancée pour cela est que connaître un universel reviendrait à connaître toutes ses instanciations. Mais selon le troisième point de vue il n'y a pas de connaissance de ce genre qui soit possible : la connaissance concerne l'usage pour ainsi dire quotidien.[118] Par conséquent, l'indiscernabilité des particuliers doit évidemment être abandonnée. De plus, la quantification portant sur les particuliers redevient pertinente sous le troisième point de vue.

118. Encore une fois, le lecteur est invité à consulter Clerbout *et al.* (2011) pour une présentation plus approfondie des arguments.

Les règles structurelles pour cette sémantique sont donc celles du premier ordre auxquelles on ajoute une règle pour exclure la quantification de deuxième ordre :

SR-P3 : Aucune quantification de deuxième ordre n'est autorisée.

Il est à noter que cette règle rend obsolète la règle SR-P1. De sorte que si l'on veut maintenir un contrôle de l'usage des prédicats, il faut utiliser un autre moyen. Plusieurs possibilités sont envisageables : par exemple, on pourrait stipuler qu'au début de chaque partie, l'Opposant dresse une liste de prédicats qui peuvent être utilisés.

Du point de vue de la reconstruction de la théorie jaine du *naya-vada*, on pourrait proposer une autre sémantique qui serait un cas spécial de celle que nous allons présenter pour le quatrième point de vue, mais il s'agit d'une complication peu pertinente dans le cas du présent travail. Quelques précisions à ce sujet sont données dans Clerbout *et al.* (2011).

B.4 Quatrième point de vue

On peut résumer le quatrième point de vue comme l'ajout du déni total de persistance au troisième point de vue, même si cette simplification n'est pas tout à fait fidèle. Plus précisément, l'idée est que les particuliers tels qu'ils sont considérés dans les trois premiers points de vue ne peuvent pas être connus parce qu'ils n'existent pas. Leur persistance n'est qu'une illusion. Les objets susceptibles d'être connus sont les unités ultimes dans le processus de décomposition de ce qui existent. Ces unités, qu'on peut appeler « atomes infinitésimaux » ou « particuliers ultimes », n'existent qu'ici et maintenant. Leur existence est la raison de l'illusion de persistance des particuliers.

La sémantique dialogique proposée dans Clerbout *et al.* (2011) pour ce point de vue est directement inspirée de la notion de « world-line » proposée par Hintikka.[119] Dans cette théorie, un terme individuel est considéré comme une fonction partielle qui peut sélectionner un objet dans un scénario, ou contexte, donné t et un autre objet dans un autre scénario t'. La sémantique pour ce point de vue présente donc un aspect modal que les sémantiques pour les autres points de vue n'ont pas. Concrètement, le langage est ici étendu avec un nouveau type de termes, notés f_s, représentant le nom d'une fonction (partielle) f au scénario s. Ce sont ces termes qui sont utilisés pour les particuliers ultimes. À partir de ce système de fonctions, on peut tirer des world-line qui indiquent quels particuliers ultimes se rapportent à quels particuliers.

De la sorte, on peut penser aux particuliers ultimes comme aux manifestations à différents scénarios des particuliers standard. Cependant il convient d'insister sur le fait que, dans le cadre du quatrième point de vue, ce sont ces manifestations

119. Voir Hintikka (1969).

qui sont objets de connaissance, et pas les particuliers. Malgré la possibilité en principe de tirer des world-line, les particuliers ultimes sont considérés comme incomparables. La sémantique dialogique proposée doit donc mettre en jeu des conditions strictes sur l'utilisation des termes, au moyen de la règle suivante :

> **SR-P4** : (i) Supposons que **X** a affirmé $f_{s_i} = f_{s_j}$. À chaque fois que **X** affirme à s_i une formule φ dans laquelle f_{s_i} apparaît, **Y** peut requérir de **X** qu'il remplace des occurrences de f_{s_i} par des occurrences de f_{s_j} à s_i.
> (ii) **P** peut choisir f_{s_i} à un contexte quelconque uniquement si f_{s_i} est nouveau, ou si **O** a déjà choisi f_{s_i} à ce scénario.

C'est surtout la deuxième partie de la règle qui restreint l'utilisation des termes, puisque la première décrit une opération assez standard de substitution. La deuxième partie permet de rendre compte du fait que les particuliers ultimes n'ont aucune persistance : les termes utilisés en cours de partie à un contexte donné ne peuvent en général pas être utilisés par le Proposant aux contextes apparaissant par la suite dans la partie.

B.5 Cinquième point de vue

Dans le cas des trois derniers points de vue, on considère que langage et objet ne sont pas séparables : le langage est le seul accès aux objets et joue donc un rôle épistémique crucial. L'important n'est donc pas tant de savoir si ce sont les particuliers ou les universaux qui existent et sont objets de connaissance, que de considérer la manière dont la connaissance des objets passe par le langage. Cela dit, rien ne semble indiquer le besoin de conserver deux niveaux de quantification pour l'approche de ces points de vue : nous en restons donc à des sémantiques sans quantification de deuxième ordre, où la règle SR-P3 a cours.

La particularité du cinquième point de vue est que l'accès épistémique à l'identité entre objets se réduit à la synonymie.[120] Ainsi, des descriptions différentes peuvent nommer un seul et même objet. Une autre conséquence est qu'une erreur à propos de la synonymie entre descriptions revient à une erreur à propos de l'identité entre objets : c'est là un signe fort du rôle crucial du langage vis-à-vis de la connaissance.

Notre sémantique dialogique doit donc offrir une manière d'exprimer et d'utiliser la relation de synonymie entre termes. L'idée est la suivante : l'Opposant commence une partie en introduisant une liste de synonymies entre prédicats via l'utilisation du symbole $\sim$. De plus, la règle suivante est ajoutée à celles pour le troisième point de vue :

> **SR-P5** : Supposons que $P \sim Q$. À chaque fois que **X** utilise P dans une affirmation au cours d'une partie, **Y** peut requérir de **X** qu'il

120. Il s'agit bien entendu de synonymie en sanskrit.

effectue l'affirmation obtenue en remplaçant chaque occurrence de P par Q.

Ceci s'applique notamment aux prédicats apparaissant dans les termes de la forme $(\iota x)(Px)$.

Quand deux prédicats P et Q sont synonymes, ils sont dialogiquement indiscernables au sens suivant : remplacer P par Q dans un débat ne peut pas en changer l'issue.

B.6 Sixième point de vue

Le sixième point de vue est à l'opposé du cinquième. Aucune synonymie n'y est reconnue, le contenu descriptif d'une expression linguistique caractérise un unique objet. Dans le sixième point de vue, on considère donc qu'il y a une relation une-à-une entre descriptions et objets. Une sémantique dialogique est facilement obtenue par opposition à celle discutée précédemment : aucune synonymie n'est concédée par l'Opposant en début de partie, et aucune substitution de prédicats n'est autorisée. Ce dernier point est pris en charge par la règle explicite suivante au lieu de SR-P5 :

SR-P6 : Les joueurs ne peuvent jamais remplacer un prédicat par un autre prédicat en cours de partie.

B.7 Septième point de vue

Tout comme pour le sixième point de vue, aucune synonymie n'est reconnue dans ce dernier point de vue. Par ailleurs, les conditions d'utilisation de descriptions pour nommer des objets sont particulièrement strictes. Notamment, ces conditions possèdent une dimension temporelle : selon le septième point de vue, un nom peut être utilisé *en ce moment* pour désigner un objet si et seulement si ce nom signifie qu'il y a un unique x tel que Px, et que cet objet a cette propriété *en ce moment*. C'est-à-dire qu'un objet ne peut être décrit d'une certaine manière qu'à l'instant où il est conforme à la description. Il est difficile de proposer des exemples complètement fidèles à cette position, parce que la théorie jaine est directement liée à la langue sanskrite, mais l'idée est la suivante : un dresseur de fauves, par exemple, ne peut être ainsi décrit correctement que lorsqu'il dresse des fauves ; on ne peut pas l'appeler ainsi quand il fait autre chose. Une conséquence de cette théorie de la désignation est qu'un individu ne peut être désigné au moyen de ses différentes propriétés que s'il remplit toutes ses propriétés au moment où on cherche à le nommer.

En l'absence de connaissances du sanskrit et de détails sur ce point de vue, les conditions de désignation que nous venons d'évoquer semblent pour le moins peu claires. Mais notre but est plutôt de montrer comment des conditions de ce genre peuvent être implémentées au moyen de contraintes sur les processus de justification de l'usage des termes individuels au cours d'un débat argumentatif. L'approche dialogique que nous proposons est donc caractérisée par l'usage de contextes temporels, et par la règle suivante :

> **SR-P7** : (i) Toute partie commence avec **O** concédant une liste d'égalités de la forme $k = (\iota x)(Px)$ sans indexes temporels.
>
> (ii) À partir de la thèse, les contextes temporels sont utilisés par les joueurs, et toutes les règles habituelles doivent en tenir compte.
>
> (iii) Quand **X** affirme φ et que k apparaît dans φ, le joueur **Y** a la possibilité d'attaquer en requérant que l'usage de k soit justifié. Si $k = (\iota x)(Px)$ a été concédé, la défense adéquate est d'affirmer Pk.

C'est seulement au niveau de la justification de l'usage des termes que la dimension temporelle est importante, et c'est pourquoi les égalités concédées par **O** en début de partie ne sont pas contextualisées. La première et la troisième partie de la règle fournissent les conditions d'usage que nous avons décrites : il faut que k signifie « Il y a un unique x tel que Px », et k ne peut être utilisé de manière justifiée à un contexte que si Pk est justifié à ce contexte.

Bibliographie

ABRAMSKY, S., 1997 : « Semantics of Interaction : an introduction to Game Semantics ». In DYBJER, P. et PITTS, A. (éds.) : *Proceedings of the 1996 CLiCS Summer School, Isaac Newton Institute*, pages 1–31. Cambridge University Press, Cambridge.

ANGELELLI, I., 1970 : « The techniques of Disputation in the history of logic ». *The Journal of Philosophy*, 67:800–815.

BARWISE, J. et ETCHEMENDY, J., 1993 : *The language of First-Order Logic.* CSLI lecture notes. Center for the Study of Language and Information.

BELNAP, N., 1962 : « Tonk, Plonk and Plink ». *Analysis*, 22:130–134.

BLACKBURN, P., DE RIJKE, M. et VENEMA, Y., 2001 : *Modal Logic*, volume 53 de *Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, Cambridge.

BLASS, A., 1992 : « A game semantics for linear logic ». *Annals of Pure and Applied Logic*, 56:183–220.

BOLANDER, Tomas et BLACKBURN, Patrick, 2007 : "Termination for Hybrid Tableaus". *Journal of Logic and Computation*, 17(3):517–554.

BOOLOS, George S., BURGESS, John P. et JEFFREY, Richard C., 2007 : *Computability and Logic.* Cambridge University Press, 5 édition.

BRANDOM, R. B., 2000 : *Articulating Reasons.* Harvard University Press, Harvard.

CLERBOUT, N., 2012 : « The Dialogic of Actually ». In MAGNIER *et al.* (éd.) : *Logic of Knowledge. Theory and Applications.*

CLERBOUT, N., 2013 : « First-Order Dialogical Games and Tableaux ». A paraître.

CLERBOUT, N., GORISSE, M.-H. et RAHMAN, S., 2011 : « Context-Sensitivity in Jain Philosophy : A Dialogical Study of Siddharsigani's *Commentary on the Handbook of Logic* ». *Journal of Philosophical Logic*, 41(5):633–662.

CROSSLEY, J. N. et HUMBERSTONE, L., 1977 : « The Logic of Actually ». *Reports on Mathematical Logic*, 8:11–29.

D'AGOSTINO, M., GABBAY, D. M., HÄHNLE, R. et POSEGGA, J. (éds.), 1998. *Handbook of Tableau Methods*. Springer.

DAMIEN, L., GORISSE, M.-H. et RAHMAN, S., 2004 : « La dialogique temporelle ou Patrick Blackburn par lui-même ». *Philosophia Scientiae*, 8(2):39–59.

DAVIES, M. et HUMBERSTONE, L., 1980 : « Two Notions of Necessity ». *Philosophical Studies*, 38:1–30.

DUTILH NOVAES, C., 2007 : *Formalizing Medieval Logical Theories*, volume 7 de *Logic, Epistemology and the Unity of Science*. Springer.

FELSCHER, W., 1985a : « Dialogues, Strategies, and Intuitionnistic Provability ». *Annals of Pure and Applied Logic*, 28:217–254.

FELSCHER, W., 1985b : « Dialogues as a Foundation for Intuitionnistic Logic ». In GABBAY, Dov M. et GUENTHNER, Franz (éds.) : *Handbook of Philosophical Logic. Volume 3 : Alternatives in Classical Logic*, volume 166 de *Studies in Epistemology, Logic, Methodology, and Philosophy of Science*, pages 341–372. Kluwer, Kordrecht & Hingham.

FITTING, M., 1990 : *First-Order Logic and Automated Theorem Proving*. Springer-Verlag, New York.

FITTING, M. et MENDELSOHN, R. L., 1998 : *First-Order Modal Logic*, volume 277 de *Synthese Library*. Kluwer Academic Publishers.

FONTAINE, M. et RAMAH, S., 2010 : « Fiction, Creation and Fictionality : an Overview ». *Methodos*, 10. URL `http://methodos.revues.org/2343`. Accès Juillet 2012.

FONTAINE, M. et REDMOND, J., 2008 : *Logique Dialogique : Une Introduction. Volume I. Méthode de Dialogique : Règles et Exercices*, volume 5 de *Cahiers de Logique et d'Epistémologie*. College Publications, Londres.

GALE, D. et STEWART, F. M., 1953 : « Infinite games with perfect information ». In KUHN, Harold W. et TUCKER, Albert W. (éds.) : *Contributions to the Theory of Games Volume II*, volume 28 de *Annals of Mathematics Studies*, pages 245–266. Princeton University Press, Princeton.

GAMUT, L.T.F., 1992 : *Logic, Language, and Meaning. Volume 1 Introduction to Logic*. University of Chicago Press, Chicago.

GARSON, J. W, 2006 : *Modal Logic for Philosophers*. Cambridge University Press, Cambridge.

GEUVERS, H. et NEDERPELT, R., 2004 : « Rewriting for Fitch Style Natural Deductions ». In VAN OOSTROM, Vincent (éd.) : *Rewriting Techniques and Applications*, volume 3091 de *Lecture Notes in Computer Science*, pages 134–154. Springer-Verlag.

GIRARD, J.-Y., 1999 : « On the meaning of logical rules I : syntax vs. semantics ». In BERGER, U. et SCHWICHTENBERG, H. (éds.) : *Computational Logic*, pages 215–272. Springer-Verlag, Heidelberg.

GIRARD, J.-Y., 2003 : « From foundations to ludics ». *Bulletin of Symbolic Logic*, 9(2):131–168.

HAZEN, A. P., 1978 : « The Eliminability of the Actuality Operator in Propositional Modal Logic ». *Notre Dame Journal of Formal Logic*, 19(4):617–622.

HINTIKKA, J., 1969 : « Semantics for propositional attitudes ». In DAVIS, J.W. et ALIA (éds.) : *Philosophical Logic*, pages 21–45. Dordrecht : Reidel.

HINTIKKA, J. et SANDU, G., 1997 : « Game-theoretical semantics ». In VAN BENTHEM, J. et TER MEULEN, A. (éds.) : *Handbook of Logic and Language*, pages 361–410. Elsevier, Amsterdam.

HODGES, W., 2001 : Logic and Games. *Stanford Encyclopedia of Philosophy*. URL `http://plato.stanford.edu/entries/logic-games/`. Accès Juin 2012.

HUGHES, G. E. et CRESSWELL, M. J., 1996 : *A New Introduction to Modal Logic*. Routledge, Londres & New York.

HUMBERSTONE, L., 2008 : « Modal Formulas True at Some Point in Every Model ». *The Australasian Journal of Logic*, 6:70–82.

HYLAND, J. M. E. et ONG, C.-H. L., 2000 : « On Full Abstraction for PCF ». *Information and Computation*, 163:285–408.

JAPARIDZE, G., 2003 : « Introduction to computability logic ». *Annals of Pure and Applied Logic*, 123:1–99.

KEIFF, L., 2007 : *Le Pluralisme Dialogique.* Thèse de doctorat, Université Charles de Gaulle Lille 3.

KEIFF, L., 2009 : « Dialogical Logic ». *Stanford Encyclopedia of Philosophy.* URL `http://plato.stanford.edu/entries/logic-Dialogical`. Accès Septembre 2011.

KLEENE, S. C., 1952 : *Introduction to Metamathematics.* Bibliotheca Mathematica. North-Holland, Amsterdam.

KLEENE, S. C., 1967 : *Mathematical Logic.* John Wiley.

KRABBE, E. C., 1982 : *Studies in Dialogical Logic.* Thèse de doctorat, Rijksuniversiteit Groningen.

KRABBE, E. C., 1985 : « Formal Systems of Dialogue Rules ». *Synthese*, 63:295–328.

KRABBE, E. C., 2006 : « Dialogue Logic ». In GABBAY, Dov M. et WOODS, J. (éds.) : *Handbook of the History of Logic Volume 7*, pages 665–704. Elsevier, Amsterdam.

LETZ, R., 1998 : « First-Order Tableau Methods ». In D'AGOSTINO *et al.* (éd.) : *Handbook of Tableau Methods*, pages 125–196.

LORENZ, K., 1968 : « Dialogspiele als semantische Grundlage von Logikkalkülen ». *Archiv für mathematische Logik und Grundlagenforschung*, 11:32–55 et 73–100. Réimprimé dans Lorenzen et Lorenz (1978).

LORENZ, K., 2001 : « Basic Objectives of Dialogue Logic in Historical Perspective ». *Synthese*, 127:255–263.

LORENZEN, P. et LORENZ, K., 1978 : *Dialogische Logik.* Wissenschaftliche Buchgesellschaft, Darmstadt.

MAGNIER, S., SALGUERO, F. J. et BARÉS, C. (éds.), 2012. *Logic of Knowledge. Theory and Applications*, volume 3 de *Dialogues and the Games of Logic.* College Publications, Londres.

NORTMANN, U., 2001 : « How to Extend the Dialogical Approach to Provability Logic ». *Synthese*, 127:95—103.

POPEK, A., 2012 : « Logical dialogues from Middle Ages ». In MAGNIER *et al.* (éd.) : *Logic of Knowledge. Theory and Applications*, pages 223–244.

PRIEST, G., 2008 : « Jaina Logic : A Contemporary Perspective ». *History and Philosophy of Logic*, 29(3):263–278.

PRIOR, A., 1961 : « The Runabout Inference Ticket ». *Analysis*, 21(2):38–39.

RAHMAN, S., 1993 : *Ueber Dialogue, Protologische Kategorien und andere Seltenheiten*. Campus, Francfort & New York.

RAHMAN, S., 2001 : « On Frege's Nightmare : A Combination of Intuitionnistic, Free and Paraconsistent Logics ». In WANSING, Heinrich (éd.) : *Essays on Non-Classical Logic*, pages 61–89. World Scientific, Singapour.

RAHMAN, S., 2009 : « A non normal dialogic for a wonderful world and more ». In VAN BENTHEM, JOHAN *et al.* (éd.) : *The Age of Alternative Logics*, pages 311–334. Kluwer-Springer, Dordrecht.

RAHMAN, S., 2010 : « Negation in the Logic of First Degree Entailment and Tonk : A Dialogical Study ». In PRIMIERO, GIUSEPPE, MARION, MATTHIEU *et al.* (éd.) : *(Anti)Realism. The Realism-Antirealism Debate in the Age of Alternative Logics*, pages 175–201. Springer, Dordrecht.

RAHMAN, S. et CARNIELLI, W., 2000 : « The Dialogical approach to paraconsistency. *Synthese*, 125(1–2):201–232.

RAHMAN, S., CLERBOUT, N. et KEIFF, L., 2009 : « On Dialogues and Natural Deduction ». In PRIMIERO, Giuseppe et RAHMAN, Shahid (éds.) : *Acts of Knowledge - History, Philosophy and Logic*, volume 7 de *Tributes*, pages 301–336. College Publications, Londres.

RAHMAN, S. et DÉGREMONT, C., 2006 : « The beetle in the box : exploring IF-dialogues ». In AHTO, T. et PIETARINEN, A. V. (éds.) : *Truth and Games : Essays in Honour of Gabriel Sandu*, pages 91–122. Acta Philosophica Fennica, Helsinki.

RAHMAN, S. et KEIFF, L., 2005 : « On How to Be a Dialogician ». In VANDERVEKEN, Daniel (éd.) : *Logic, Thought and Action*, pages 359–408. Springer, New York.

RAHMAN, S. et RÜCKERT, H., 1999 : « Dialogische Modallogik (für T, B, S4 und S5) ». *Logique et Analyse*, 42(167–168):243–282.

RAHMAN, S. et RÜCKERT, H., 2001 : « Dialogical connexive logic ». *Synthese*, 127(1–2):105–139.

RAHMAN, S. et TULENHEIMO, T., 2009 : « From Games to Dialogues and Back. Towards a general frame for validity ». In MAJER, Ondrej, PIETARINEN, Ahti-Veikko et TULENHEIMO, Tero (éds.) : *Games : Unifying Logic, Language, and Philosophy*, pages 153–208. Springer, Dordrecht.

SCHROEDER-HEISTER, P., 2008 : « Lorenzen's operative justification of intuitionistic logic ». In VAN ATTEN, M., BOLDINI, P., BOURDEAU, M. et HEINZMANN, G. (éds.) : *One Hundred Years of Intuitionism (1907-2007) : The Cerisy Conference*, pages 214–240. Basel : Birkhäuser.

SELLARS, W., 1997 : *Empiricism and the Philosophy of Mind*. Harvard University Press, Cambridge.

SMULLYAN, R., 1968 : *First-Order Logic*. Springer Verlag, New York.

TULENHEIMO, T., 2009 : « Review of Humberstone (2008) ». *Mathematical Reviews*. URL `http://www.ams.org/mathscinet`. MR2430969.

TURING, A., 1936 : « On computable numbers, with an application to the *Entscheidungsproblem* ». In *Proceedings of the London Mathematical Society, Series 2*, volume 42, pages 230–265.

WITTGENSTEIN, L., 1953 : *Recherches Philosophiques*. Bibliothèque de Philosophie. NRF Gallimard, Paris. trad. E. Rigal et alia, 2005.

YRJÖNSUURI, M., 2001 : *Consequences, Obligations and Insolubles*, volume 49 de *New Synthese Historical Library*. Kluwer, Dordrecht.