Pulses in singularly perturbed reaction-diffusion systems

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Hopf bifurcations for localised pulses

4.1 Introduction

The study of localised patterns in systems of reaction-diffusion equations has been a very active field of research for the last couple of decades. In canonical model systems such as the Gray-Scott [23] or Gierer-Meinhardt [22] model, far-from-equilibrium patterns were constructed and analysed in the presence of an asymptotically small parameter, giving the system under consideration a singularly perturbed nature [11, 26]. This singularly perturbed structure induces a spatial scale separation, which can be used to obtain explicit leading order expressions for the pattern under consideration (e.g. [6]). These techniques were applied in full generality in chapter 3 in the context of single pulse patterns, going beyond the existing analysis in the context of the canonical Gray-Scott and Gierer-Meinhardt models. In chapter 2, this extended theory was applied in the context of an explicit model, exhibiting new, previously unobserved behaviour.

In chapters 2 and 3, the stability analysis of the pulse solutions under consideration led to observation that, under certain general conditions, the most general pulse destabilisation scenario corresponds to a Hopf bifurcation of the pulse eigenvalues, an observation that is also known from the extensive literature on Gray-Scott/Gierer-Meinhardt models [5, 6, 7, 14, 17, 26, 32, 47, 52, 57]. This Hopf bifurcation, and in particular its unfolding, is the main topic of this last chapter. The aim of this chapter

is to develop a mechanism for the weakly nonlinear analysis of the aforementioned Hopf destabilisation scenario, through local analysis of the associated centre manifold.

The chapter is structured as follows. In section 4.2, relevant results from chapter 3 on the existence and stability of pulse solutions are summarised, (re)introducing notation which will be used throughout the text. In section 4.3, the Hopf centre manifold is introduced. Also, the issue of the translational eigenmode is addressed. Since the systems of reaction-diffusion equations studied in the field of pattern formation – and in extension localised pattern solutions thereof – exhibit translational invariance, the translational eigenmode with corresponding (central) eigenvalue $\lambda = 0$ is always present when the stability of the pattern under consideration is assessed. However, the centre manifold can be foliated along the direction spanned by the translational mode (Theorem 4.8), and it follows that the dynamics along the translational direction are trivial.

Section 4.4 is dedicated to the explicit local expansion of the centre manifold established in section 4.3, using direct expansions in the Hopf eigenmodes. The main result is an explicit, albeit elaborate, expression for the first Lyapunov coefficient of the (normal form of) the Hopf bifurcation (Corollary 4.12 and equation (4.58)). Leading order expressions for the pulse and its eigenmodes as developed in chapter 3 (summarised in Theorem 4.2 resp. Theorem 4.3), combined with a specific choice for the inner product, are then used to obtain an explicit leading order expression for this first Lyapunov coefficient, which is used to decide whether the Hopf bifurcation under consideration is subcritical or supercritical. Given the intricate nature of the problem, this result is quite remarkable: the formal centre manifold expansion leads to concrete, explicitly computable results, based on explicit leading order expressions for the Hopf eigenfunctions.

In section 4.5, an alternative method for the calculation of the first Lyapunov coefficient is presented, based on the analysis in [24]. While the algebraic manipulations leading to the explicit expression of the first Lyapunov coefficient are less cumbersome than those in section 4.4, this method introduces a number of new inverse problems to be solved, along with the analysis of the adjoint linear operator associated to the linearisation of the pulse solution. It is argued that, while both methods are in essence equivalent, either one of the two approaches can be preferable in terms of algebraic and analytic tractability, depending on the specific choice of (nonlinear) reaction terms in the system under consideration.

The developed theory is in section 4.6 applied to a model example, the slowly nonlinear Gierer-Meinhardt equation (4.109), which was considered in chapter 2. The explicit leading order expressions available for the pulse and its eigenfunctions allow one to obtain directly computable eigenvalues, and in extension directly computable values of the associated first Lyapunov coefficients. For this example, it is shown that the extension of the canonical Gierer-Meinhardt model with a slowly nonlinear term introduces a Hopf bifurcation which can change its nature from sub- to supercritical, depending on the parameter values (Theorem 4.16). As an aside, a relatively old observation from the literature, based on numerical simulations, is confirmed analytically [11, 57], namely that Hopf bifurcation associated with the pulse in the canonical Gierer-Meinhardt equation is always subcritical (Corollary 4.17).

4.2 Preliminaries

In chapter 3, a general theory for establishing the existence and stability of stationary single pulses was presented in a general setting of a singularly perturbed, two component system of reaction-diffusion equations on the real line, with asymptotically small parameter $0 < \varepsilon \ll 1$. It was shown that the most general context in which these pulse solutions could be constructed led to the following system:

$$\begin{cases} U_t &= U_{xx} &- (\mu U - \nu_1 F_1(U;\varepsilon)) &+ \frac{\nu_2}{\varepsilon} F_2(U,V;\varepsilon) \\ V_t &= \varepsilon^2 V_{xx} &- V &+ G(U,V;\varepsilon) \end{cases} . \tag{4.1}$$

System (4.1) is considered on the unbounded domain such that $U, V : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$; moreover, we restrict ourselves to positive solutions. A stable homogeneous trivial background state is assumed. The range of the model parameters $\mu, \nu_{1,2}$ and mild regularity assumptions on the nonlinear reaction terms $F_{1,2}$ and G are specified in (A1 - A4) of Assumptions 4.1. In the following subsections 4.2.1 and 4.2.2, a very concise overview of the results obtained in chapter 3 are given. The necessary ingredients for establishing localised pulses and their eigenfunctions are given, in order to be able to set up the theory for a Hopf bifurcation of such a localised pulse, which is the main subject of this chapter. By nature, this overview is far from complete and very brief: the reader is encouraged to consider chapter 3 for a complete exposition.

4.2.1 Existence

Introducing the 'fast' (or short scale) coordinate $\xi = \frac{x}{\varepsilon}$, (4.1) can be transformed into

$$\left\{ \begin{array}{rcl} U_t &=& \frac{1}{\varepsilon^2} U_{\xi\xi} - (\mu\, U - \nu_1 F_1(U;\varepsilon)) + \frac{\nu_2}{\varepsilon} F_2(U,V;\varepsilon) \\ V_t &=& V_{\xi\xi} - V + G(U,V;\varepsilon) \end{array} \right. . \tag{4.2}$$

Establishing the existence of a stationary pulse solution in (4.2) (or equivalently (4.1)) which is asymptotic to the (stable) trivial background state of (4.2) is equivalent to constructing a homoclinic orbit in the associated ODE system

$$\left\{ \begin{array}{rcl} u_\xi &=& \sqrt{\varepsilon} p \\ p_\xi &=& \sqrt{\varepsilon}\,(-\nu_2 F_2(u,v;\varepsilon) + \varepsilon\,(\mu u - \nu_1 F_1(u;\varepsilon))) \\ v_\xi &=& q \\ q_\xi &=& v - G(u,v;\varepsilon) \end{array} \right. . \tag{4.3}$$

Since ε is taken to be asymptotically small, (4.3) can be analysed using geometric singular perturbation (or Fenichel) theory [18, 19]. Taking the limit $\varepsilon \to 0$ in (4.3) yields the fast reduced system

$$v_{f,\xi\xi} = v_f - G(u_0, v_f; 0) \;\text{ or }\; \left\{ \begin{array}{rcl} v_{f,\xi} &=& q_f \\ q_{f,\xi} &=& v_f - G(u_0, v_f; 0) \end{array} \right. , \; u_0 > 0 \text{ constant}, \tag{4.4}$$

together with the normally hyperbolic invariant manifold

$$\mathcal{M} = \{(u,p,v,q)\,|\,v = q = 0, u > 0\}\,.$$

Note that, by Assumptions 4.1, (A4), $\mathcal{M}$ is also invariant for the full system (4.3); there, its unstable and stable manifolds are denoted by $\mathcal{W}^{\mathrm{u/s}}(\mathcal{M})$.

On $\mathcal{M}$, the slow dynamics can be represented to leading order by the slow reduced system

$$u_{s,xx} = \mu u_s - \nu_1 F_1(u_s;\varepsilon), \;\text{ or }\; \left\{ \begin{array}{rcl} u_{s,x} &=& p_s \\ p_{s,x} &=& \mu u_s - \nu_1 F_1(u_s;\varepsilon) \end{array} \right. , \; u > 0. \tag{4.5}$$

For this slow reduced system, the unstable and stable manifolds of the origin are denoted by $\mathcal{W}_s^{\mathrm{u/s}}((0,0);\varepsilon)$. These manifolds are by definition spanned by the solutions $(u_s^{\mathrm{u}}(x;\varepsilon), p_s^{\mathrm{u}}(x;\varepsilon))$ resp. $(u_s^{\mathrm{s}}(x;\varepsilon), p_s^{\mathrm{s}}(x;\varepsilon))$ of (4.5). Note that $u_s^{\mathrm{s}}(x) = u_s^{\mathrm{u}}(-x)$ and $p_s^{\mathrm{s}}(x) = -p_s^{\mathrm{u}}(-x)$ by the reversibility symmetry of (4.5).

The existence of a symmetric stationary pulse solution of (4.2) was established in chapter 3 under certain conditions; first, Assumptions 4.1, (A5) below ensures the existence of a homoclinic orbit in the fast reduced system (4.4). Second, introducing

$$D_p(u_0) = \int_{-\infty}^{\infty} F_2(u_0, v_{f,h}(\xi; u_0); 0)\,\mathrm{d}\xi, \tag{4.6}$$

it was seen that the solutions to the equation

$$\mu u^2 - 2\nu_1 \int_0^u F_1(\tilde{u}; 0)\,d\tilde{u} = \tfrac{1}{4}\nu_2^2\, D_p^2(u) = \tfrac{1}{4}\nu_2^2 \left[\int_{-\infty}^{\infty} F_2(u, v_{f,h}(\xi; u); 0)\,\mathrm{d}\xi\right]^2 \tag{4.7}$$

play a central role in the pulse construction process. To remove a number of sign ambiguities, it is necessary to gauge F_2 such that the function D_p obeys Assumptions 4.1, (A6). For a more detailed and extended presentation of the above, see chapter 3, section 3.2. We restate the assumptions from chapter 3; see also Definition 2.3.

Assumptions 4.1. *The following is assumed to hold:*

(A1) μ, $\nu_{1,2}$ *are real and nonsingular in* ε*; furthermore,* $\mu > 0$.

(A2) $F_1(U;\varepsilon) \rightsquigarrow U^{f_1}$ *as* $U \downarrow 0$ *for some* $f_1 > 1$*;*
F_1 *is smooth both on its domain and as a function of* ε.

(A3) *Writing* $F_2(U, V; \varepsilon) = F_{2,1}(U;\varepsilon)\,V + F_{2,2}(U, V; \varepsilon)$,
$F_{2,1}(U;\varepsilon) \rightsquigarrow \tilde{F}_{2,1}(\varepsilon)\,U^{\gamma_1}$ *as* $U \downarrow 0$ *for some* $\gamma_1 \geq 0$ *and* $\tilde{F}_{2,1}(\varepsilon) \in \mathbb{R}$*;*
$F_{2,2}(U, V;\varepsilon) \rightsquigarrow \tilde{F}_{2,2,u}(V;\varepsilon)\,U^{\alpha_1}$ *as* $U \downarrow 0$ *for some* $\alpha_1 \in \mathbb{R}$*;*
$F_{2,2}(U, V;\varepsilon) \rightsquigarrow \tilde{F}_{2,2,v}(U;\varepsilon)\,V^{\beta_1}$ *as* $V \to 0$ *for some* $\beta_1 > 1$*;*
F_2 *is smooth both on its domain and as a function of* ε.

(A4) $G(U, V;\varepsilon) \rightsquigarrow \tilde{G}_u(V;\varepsilon)\,U^{\alpha_2}$ *as* $U \downarrow 0$ *for some* $\alpha_2 \in \mathbb{R}$*;*
$G(U, V;\varepsilon) \rightsquigarrow \tilde{G}_v(U;\varepsilon)\,V^{\beta_2}$ *as* $V \to 0$ *for some* $\beta_2 > 1$*;*
G *is smooth both on its domain and as a function of* ε.

(A5) *For all* $u_0 > 0$ *there exists a positive solution* $v_{f,h}(\xi; u_0)$ *to* (4.4) *which is homoclinic to* $(v_f, q_f) = (0, 0)$.

(A6) $D_p(u) \rightsquigarrow 1 \cdot u^{d_p}$ *as* $u \downarrow 0$ *for some* $d_p \in \mathbb{R}$, *c.f.* (4.6).

The Gray-Scott and Gierer-Meinhardt models are examples of systems obeying these assumptions. However, the full class of systems described in this way is far more encompassing.

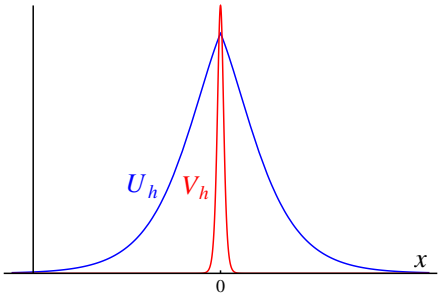

Figure 4.1: A sketch of the stationary, symmetric pulse solution to (4.2), whose existence and structure is established in Theorem 4.2.

The slow-fast structure present in the ODE system (4.3) leads to a scale separation between the U- and V-components of the pulse. Loosely speaking, one can within the unbounded domain define an inner, 'fast' region

$$I_f = \left[-\frac{1}{\varepsilon^{\frac{1}{4}}}, \frac{1}{\varepsilon^{\frac{1}{4}}} \right] \tag{4.8}$$

where the U-component of the pulse is constant to leading order in ε, while outside I_f the V-component of the pulse is exponentially small; see Figure 4.1. The exact leading order pulse structure and existence conditions were stated in chapter 3, Theorem 3.7, reformulated here for reference purposes:

Theorem 4.2. *Assume that conditions 4.1 hold and let $\varepsilon > 0$ be small enough. Let K be the number of non-degenerate solutions $u = u_{*,k} > 0$ of (4.7) such that $(u_{*,k}, p_{*,k}) = (u_{*,k}, \frac{1}{2}\nu_2 D_p(u_{*,k})) \in \mathcal{W}_s^{\mathrm{u}}((0,0);0)$.*

1. *If $K = 0$ then there are no symmetric, positive, one-circuit homoclinic solutions to $(0,0,0,0)$ in (4.3).*

2. *If $K \neq 0$, there are K distinct positive, symmetric, one-circuit homoclinic orbits $\Gamma_{h,k}(\xi) = (u_{h,k}(\xi), p_{h,k}(\xi), v_{h,k}(\xi), q_{h,k}(\xi)) \subset \mathcal{W}^{\mathrm{u}}(\mathcal{M}) \cap \mathcal{W}^{\mathrm{s}}(\mathcal{M})$, $k = 1, 2, ..., K$, with internal reflection point $\xi = 0$, so that $\Gamma_{h,k}(0) = (u_{h,k}(0), 0, v_{h,k}(0), 0)$ in (4.3). In the fast field (4.8), $\Gamma_{h,k}(\xi)$ is to leading order determined by the homoclinic solution $v_{f,h}(\xi; u_{*,k})$ of (4.4):*

$$\Gamma_{h,k}(\xi) = \left(u_{*,k}, 0, v_{f,h}(\xi; u_{*,k}), \tfrac{\mathrm{d}}{\mathrm{d}\xi} v_{f,h}(\xi; u_{*,k}) \right) + O(\sqrt{\varepsilon}) \quad \text{for} \quad \xi \in I_f.$$

In the slow field, $\Gamma_{h,k}(\xi)$ approaches $\mathcal{W}_s^{\mathrm{u}}((0,0);\varepsilon) \subset \mathcal{M}$, resp. $\mathcal{W}_s^{\mathrm{s}}((0,0);\varepsilon) \subset \mathcal{M}$ exponentially fast for $\xi \to \pm\infty$: there are $O(1)$ constants $C_{1,2}$ such that

- $v_{h,k}(\xi) \rightsquigarrow C_1 e^{-|\xi|}$ *and* $q_{h,k}(\xi) \rightsquigarrow \mp C_1 e^{-|\xi|}$ *as* $\xi \to \pm\infty$*;*
- *there are shifts* $x_{*,k} \in \mathbb{R}$ *and solutions* $u^{\mathrm{u}}_{*,k}(x) = u^{\mathrm{u}}_s(x - x_{*,k})$, $p^{\mathrm{u}}_{*,k}(x) = p^{\mathrm{u}}_s(x - x_{*,k})$ *of* (4.5)*, such that* $\Big(u^{\mathrm{u}}_{*,k}(-\varepsilon^{\frac{3}{4}}), p^{\mathrm{u}}_{*,k}(-\varepsilon^{\frac{3}{4}})\Big) = \Big(u_{h,k}(-\varepsilon^{-\frac{1}{4}}), \frac{1}{\sqrt{\varepsilon}} p_{h,k}(-\varepsilon^{-\frac{1}{4}})\Big) = \Big(u_{*,k}, p_{*,k}\Big) + O(\sqrt{\varepsilon})$ *and*

$$\Big(u_{h,k}(\xi), \tfrac{1}{\sqrt{\varepsilon}} p_{h,k}(\xi)\Big) = \Big(u^{\mathrm{u}}_{*,k}(\varepsilon\xi), p^{\mathrm{u}}_{*,k}(\varepsilon\xi)\Big) + O\Big(e^{C_2\xi}\Big) \quad \text{for} \quad \xi < -\varepsilon^{-\frac{1}{4}};$$

- *similarly,* $\Big(u^{\mathrm{s}}_{*,k}(\varepsilon^{\frac{3}{4}}), p^{\mathrm{s}}_{*,k}(\varepsilon^{\frac{3}{4}})\Big) = \Big(u_{h,k}(\varepsilon^{-\frac{1}{4}}), \frac{1}{\sqrt{\varepsilon}} p_{h,k}(\varepsilon^{-\frac{1}{4}})\Big) = \Big(u_{*,k}, -p_{*,k}\Big) + O(\sqrt{\varepsilon})$ *with* $\Big(u^{\mathrm{s}}_{*,k}(x), p^{\mathrm{s}}_{*,k}(x)\Big) = \Big(u^{\mathrm{u}}_{*,k}(-x), -p^{\mathrm{u}}_{*,k}(-x)\Big)$ *and*

$$\Big(u_{h,k}(\xi), \tfrac{1}{\sqrt{\varepsilon}} p_{h,k}(\xi)\Big) = \Big(u^{\mathrm{s}}_{*,k}(\varepsilon\xi), p^{\mathrm{s}}_{*,k}(\varepsilon\xi)\Big) + O\Big(e^{-C_2\xi}\Big) \quad \text{for} \quad \xi > \varepsilon^{-\frac{1}{4}}.$$

The orbits $\Gamma_{h,k}(\xi)$ *correspond to the homoclinic pulse patterns* $(U_{h,k}(\xi), V_{h,k}(\xi))$ *in (4.2) that are symmetric with respect to* $\xi = 0$ *through* $U_{h,k}(\xi) = u_{h,k}(\xi)$, $V_{h,k}(\xi) = v_{h,k}(\xi)$.

The situation described in the theorem is illustrated in chapter 3, Figure 3.2.

4.2.2 Linearisation and eigenfunctions

In order to set up an analysis of Hopf bifurcations of the localised pulses considered in the previous section, it is necessary to highlight some aspects of the stability analysis of the pulse, as carried out in chapter 3, section 3.3. The main purpose of the current section is to obtain a leading order expression for the eigenfunctions of such a localised pulse.

In the following, assume that $K \geq 1$ and fix k (see Theorem 4.2); the associated homoclinic pulse pattern is denoted by $\Gamma_h(\xi) = (U_h(\xi), V_h(\xi))$. The stability analysis of this pulse is closely related to the study of the linear operator

$$\mathcal{L}(\xi;\varepsilon) = \begin{pmatrix} \varepsilon^{-2} & 0 \\ 0 & 1 \end{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \mathcal{A}(\xi;\varepsilon), \tag{4.9}$$

where

$$\mathcal{A}(\xi;\varepsilon) = \begin{pmatrix} \mu - \nu_1 \frac{\mathrm{d}F_1}{\mathrm{d}U} - \frac{\nu_2}{\varepsilon}\frac{\partial F_2}{\partial U} & -\frac{\nu_2}{\varepsilon}\frac{\partial F_2}{\partial V} \\ -\frac{\partial G}{\partial U} & 1 - \frac{\partial G}{\partial V} \end{pmatrix}\Bigg|_{(U,V)=(U_h(\xi),V_h(\xi))}. \tag{4.10}$$

Since $\lim_{\xi\to\pm\infty}\Gamma_h(\xi) = (0,0)$, the matrix $\mathcal{A}(\xi;\varepsilon)$ is asymptotically constant as $\xi \to \pm\infty$: $\lim_{\xi\to\pm\infty}\mathcal{A}(\xi;\varepsilon) = A_\infty(\varepsilon)$. The eigenvalues of this constant matrix determine the essential spectrum σ_e of the operator $\mathcal{L}$, which is given by

$$\sigma_e = \{\lambda \in \mathbb{R} : \lambda \leq \max(-\mu, -1)\} \subset \mathbb{C}. \tag{4.11}$$

The slow-fast structure of the homoclinic pulse Γ_h, made explicit in Theorem 4.2, is inherited by the linear operator $\mathcal{L}$; this operator is the linearisation of (4.2) around Γ_h. One can therefore introduce the 'fast' linear operator

$$\mathcal{L}_f(\xi) = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \left[1 - \frac{\partial G}{\partial V}(u_*, v_{f,h}(\xi; u_*))\right], \quad \xi \in \mathbb{R}, \tag{4.12}$$

with $u_* = u_{*,1}$ and $v_{f,h}$ as in Theorem 4.2, and determine its spectrum. The associated eigenvalue problem $\left(\mathcal{L}_f - \lambda\right)v = 0$ is of Sturm-Liouville type; relevant results from the literature are summarised in chapter 3, Lemma 3.12. Based on those results, let $\lambda_{f,j}$, $j \in \mathbb{Z}_{\geq 0}$ be the eigenvalues of the linear operator $\mathcal{L}_f$ acting on the space of bounded integrable functions on the entire real line.
Similarly, the 'slow' linear operator

$$\mathcal{L}_s(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left[\mu - \nu_1 \frac{\partial F_1}{\partial U}(u_*^{\mathrm{s}}(x), 0)\right], \quad x \geq 0, \tag{4.13}$$

with $u_*^{\mathrm{s}} = u_{*,1}^{\mathrm{s}}$ as in Theorem 4.2, plays a central role in the spectral analysis of $\mathcal{L}$. The eigenvalue problem $(\mathcal{L}_s - \lambda)\,u = 0$ is again of Sturm-Liouville type, albeit on the positive halfline. Let $u_{s,-}(x;\lambda,\varepsilon)$ be the solution to the eigenvalue problem $\mathcal{L}_s u = \lambda\, u$ that is bounded as $x \to \infty$, such that $u_{s,-}(x;\lambda,\varepsilon) \rightsquigarrow 1 \cdot e^{-\sqrt{\mu+\lambda}\,x}$ as $x \to \infty$.

The coupling between the U- and V-components of the pulse (apparent in the off-diagonal entries of $\mathcal{A}$) manifests itself in the spectral analysis of $\mathcal{L}$ through the nonhomogeneous problem

$$\left(\mathcal{L}_f - \lambda\right)v = -\frac{\partial G}{\partial U}(u_*, v_{f,h}(\xi; u_*)), \quad \xi \in \mathbb{R}. \tag{4.14}$$

For $\lambda \neq \lambda_{f,j}$ and $\lambda \notin \sigma_e$, let $v_{\mathrm{in}}(\xi;\lambda,\varepsilon)$ be the unique bounded solution to (4.14). The existence and uniqueness of v_{in} follows from the analysis in chapter 3, section 3.3.3, which is based on the Fredholm alternative. Note that it immediately follows that v_{in} is even as a function of ξ, i.e. $v_{\mathrm{in}}(\xi;\lambda,\varepsilon) = v_{\mathrm{in}}(-\xi;\lambda,\varepsilon)$.

The actual spectral analysis of $\mathcal{L}$ (4.9) does not need to be summarised here: for an overview of this spectral analysis, using an Evans function approach, see chapter 3, sections 3.4 and 3.5. There, a leading order expression for the Evans function was derived (Theorem 3.21), enabling direct calculation of the pulse eigenvalues. It was shown (as a result of Corollary 3.32) that the most general destabilisation scenario for a localised pulse in (4.2) is through a Hopf bifurcation. That observation will be the starting point of the analysis presented in this chapter. However, some comments on the trivial eigenvalue are in order; they can be found in section 4.2.3.

The following theorem, which summarises results from section 3.4.2 in chapter 3, characterises the leading order behaviour of eigenfunctions of the linear operator $\mathcal{L}$ (4.9). This leading order behaviour will be very instrumental in the upcoming analysis.

Theorem 4.3. *Let $\varepsilon > 0$ be small enough. Assume that $\lambda \notin \sigma_e$ and $\lambda \neq \lambda_{f,j}$. If there is a $\lambda \in \mathbb{C}$ for which there is a bounded integrable function $\phi : \mathbb{R} \to \mathbb{C}^2$ such that $\mathcal{L}(\xi;\varepsilon)\,\phi(\xi;\lambda,\varepsilon) = \lambda\,\phi(\xi;\lambda,\varepsilon)$, then $\phi(\xi;\lambda,\varepsilon)$ is determined uniquely. Furthermore, there are $O(1)$ constants $C_{1,2}$ such that the following holds:*

- $\phi(\xi;\lambda,\varepsilon) = \begin{pmatrix} u_{s,-}(\varepsilon\xi;\lambda,\varepsilon) \\ 0 \end{pmatrix} + C_1 e^{-C_2\xi}$ *for* $\xi > \varepsilon^{-\frac{1}{4}}$;
- $\phi(\xi;\lambda,\varepsilon) = \phi(-\xi;\lambda,\varepsilon)$ *for* $\xi < -\varepsilon^{-\frac{1}{4}}$;
- $\phi(\xi;\lambda,\varepsilon) = u_{s,-}(0;\lambda,\varepsilon)\begin{pmatrix} 1 \\ v_{\text{in}}(\xi;\lambda,\varepsilon) \end{pmatrix} + O(\varepsilon^{\frac{3}{4}})$ *for* $\xi \in I_f$.

Proof. The leading order expressions for the eigenfunction of $\mathcal{L}$ are based on the proof of Theorem 3.21 in chapter 3. □

Remark 4.4. From Lemma 3.20 in chapter 3, we know that the 'true' fast eigenvalues of the full linear operator $\mathcal{L}$ (4.9) are only to leading order in ε determined by the fast eigenvalues $\lambda_{f,j}$ of the fast operator $\mathcal{L}_f$ (4.12): these 'true' fast eigenvalues $\lambda_j(\varepsilon)$ of $\mathcal{L}$ obey $\lim_{\varepsilon\to 0}\lambda_j(\varepsilon) = \lambda_{f,j}$. Since the above Theorem 4.3 only determines the leading order expression for the eigenfunction ϕ, it is sufficient to exclude the λ-values for which v_{in}, the unique bounded solution to the inhomogeneous problem (4.14), is not defined. We therefore assume $\lambda \neq \lambda_{f,j}$ instead of $\lambda \neq \lambda_j(\varepsilon)$.

4.2.3 Translational symmetry and the trivial eigenvalue

A general n-component reaction-diffusion equation

$$\tilde{u}_t = D\tilde{u}_{xx} + f(\tilde{u}), \quad \tilde{u} \in \mathbb{R}^n,\ D \in \mathrm{Mat}(n,\mathbb{R}),\ f : \mathbb{R}^n \to \mathbb{R}^n \tag{4.15}$$

is equivariant under the continuous one-parameter group of isometries $(T_\alpha)_{\alpha\in\mathbb{R}}$ which acts as

$$T_\alpha \tilde{u}(x) = \tilde{u}(x+\alpha). \tag{4.16}$$

Every stationary solution $\tilde{u}_0$ to (4.15), for which

$$D\tilde{u}_{0,xx} + f(\tilde{u}_0) = 0, \tag{4.17}$$

can therefore be thought of as representing a continuous family of stationary solutions $(T_\alpha\tilde{u}_0)_{\alpha\in\mathbb{R}}$, obtained under the group action T_α. Since the infinitesimal generator of this underlying translational symmetry group (4.16) is $\tau = \frac{\partial}{\partial x}$, it follows from (4.17) that $\tau\tilde{u}_0$ obeys the linear equation

$$\left[D\frac{\partial^2}{\partial x^2} + \frac{\mathrm{d}f}{\mathrm{d}\tilde{u}}(\tilde{u}_0)\right]\tau\tilde{u}_0 = 0. \tag{4.18}$$

The above considerations apply to the PDE system (4.1)/(4.2); in particular, for the homoclinic pulse solution whose existence was established in Theorem 4.2. In this context, (4.18) takes the form

$$\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h = 0,$$

from which follows that $\lambda = 0$ is always an eigenvalue of the operator $\mathcal{L}$ (4.9), with eigenfunction $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$. Note that Theorem 4.3 does not apply for this eigenvalue, since $\lambda_{f,1} = 0$ (see chapter 3; also, the above argument can be applied to (4.4) with its linearisation (4.12) around the orbit $v_{f,h}$, Assumptions 4.1 (A5)). However, the eigenfunction $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$ does have the same scale separated structure as the eigenfunctions described by Theorem 4.3, see Theorem 4.2. Both considerations will be of importance in the next section.

4.3 The Hopf centre manifold

We will focus on the situation where the pulse Γ_h undergoes a Hopf bifurcation. In other words, let $(\mu,\nu_1,\nu_2) = (\mu_H,\nu_{1,H},\nu_{2,H})$ be such that there is a bounded integrable function $\phi_H : \mathbb{R} \to \mathbb{C}^2$ as in Theorem 4.3 for which

$$\mathcal{L}\phi_H = i\,\omega_H\phi_H, \tag{4.19}$$

with $\omega_H > 0$. Since the operator $\mathcal{L}$ is real, it immediately follows that the Hopf bifurcation (4.19) yields a complex conjugate pair of eigenvalues $\pm i\,\omega_H$ with associated eigenfunction pair $\left\{\phi_H, \overline{\phi_H}\right\}$ – here and henceforth, complex conjugation will be denoted by an overline.

Since the linear operator $\mathcal{L}$ (4.9) is sectorial and its continuous spectrum is completely determined by the essential spectrum σ_e (4.11), we can infer that its central spectrum

$$\sigma_0 = \{\lambda \in \mathbb{C} : \lambda \text{ is in the spectrum of } \mathcal{L},\ \mathrm{Re}(\lambda) = 0\}$$

consists of finitely many eigenvalues. Moreover, $\lambda = 0 \in \sigma_0$ (see subsection 4.2.3); we assume that this trivial eigenvalue is nondegenerate. As mentioned before, it was argued in chapter 3 that this is the most general destabilisation scenario for a given pulse whose existence is ensured by Theorem 4.2. Indeed, this destabilisation through a Hopf bifurcation is typical for pulses in both the Gierer-Meinhardt equation [6] and its slowly nonlinear counterpart, see chapter 2, sections 2.4 and 2.5. The associated Hopf bifurcation has codimension 1.
Henceforth, we assume $\pm i\,\omega_H$ are the only nontrivial central eigenvalues, i.e. that the central spectrum of $\mathcal{L}$ is given by

$$\sigma_{0,H} = \{\pm i\,\omega_H, 0\}\,. \tag{4.20}$$

Moreover, based our insight in the general pulse destabilisation mechanisms from chapter 3, we assume that the Hopf bifurcation under consideration is of codimension 1, such that the Hopf eigenvalues are simple.

To carry out a centre manifold analysis for this central spectrum, one would naively aim for an expansion in the three associated eigenvectors. However, the translational symmetry (4.16) of (4.1)/(4.2), being the source of the trivial central eigenvalue $\lambda = 0$, induces a transversal structure for the centre manifold, enabling one to effectively ignore the translational eigenmode in the local centre manifold expansion (see upcoming Theorem 4.8). To set the stage, we first focus on the ambient function space where the centre manifold will be embedded in.

4.3.1 Choosing a function space

In order to properly set up the centre manifold theory for the pulse Γ_h at the Hopf bifurcation (4.19), we need choose an appropriate function space to work in. Since both components of the eigenfunction ϕ_H are eventually exponentially decreasing (see

Theorem 4.3), bounded and thus certainly (square) integrable, the space $(L_2(\mathbb{R},\mathbb{C}))^2$ seems a natural choice. The scale separation between the two eigenfunction components forces us to make a choice for the integration variable, be it ξ or $x = \varepsilon\xi$. However, this choice cannot be made in a uniform way, as can be seen if we try to establish the norm of ϕ_H by integrating the sum of squares of its components. If we would choose ξ, the integral over the square of the first component $(\phi_H)_1$ is

$$\int_{\mathbb{R}} \left[(\phi_H)_1(\xi)\right]^2 \mathrm{d}\xi = 2\int_{\varepsilon^{-\frac{1}{4}}}^{\infty} \left[(\phi_H)_1(\varepsilon\xi)\right]^2 \mathrm{d}\xi + \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} \left[(\phi_H)_1(\xi)\right]^2 \mathrm{d}\xi$$

The second term of the above expression will cause the integral to become unbounded as $\varepsilon \to 0$, since in particular, the value of $(\phi_H)_1$ at the boundaries of the fast interval I_f will not vanish.
On the other hand, if we would choose x as our integration variable, the integral over the square of the second component of ϕ_H is

$$\int_{\mathbb{R}} \left[(\phi_H)_1(x)\right]^2 \mathrm{d}x = 2\int_{\varepsilon^{\frac{3}{4}}}^{\infty} \left[(\phi_H)_2(x)\right]^2 \mathrm{d}x + \int_{-\varepsilon^{\frac{3}{4}}}^{\varepsilon^{\frac{3}{4}}} \left[(\phi_H)_2\left(\frac{x}{\varepsilon}\right)\right]^2 \varepsilon \mathrm{d}\frac{x}{\varepsilon},$$

which will vanish in the limit $\varepsilon \to 0$. This would eliminate the contribution of the second component of ϕ_H to the inner product. Therefore, one would be unable to use this inner product to succesfully project onto the finite-dimensional subspace spanned by ϕ_H.

To circumvent this problem, we introduce the function space

$$\mathcal{X} = L_2(\mathbb{R},\mathbb{C}^2;\mu_\varepsilon) \tag{4.21}$$

with the partly scaled Lebesgue measure μ_ε defined such that the associated inner product $\langle\cdot,\cdot\rangle$ can be defined as

$$\langle\phi,\psi\rangle = \int_{\mathbb{R}} \phi^T S\,\overline{\psi}\,\mathrm{d}\xi \quad \text{with} \quad S = \begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix} \tag{4.22}$$

for $\phi,\psi \in \mathcal{X}$. In other words: the product of the first components is integrated over $x = \varepsilon\xi$, the product of the second components over ξ. Note that the norm induced by the inner product (4.22) is for all $\varepsilon > 0$ equivalent to the 'standard' norm on $(L_2(\mathbb{R},\mathbb{C}))^2$, making $\mathcal{X}$ and $(L_2(\mathbb{R},\mathbb{C}))^2$ isometrically isomorphic as metric spaces. A similar norm was introduced in [12].

4.3.2 Foliation of the centre manifold along the translational eigenmode

In this section, we show that the influence of the translational eigenmode can be separated completely from the other eigendirections. We follow the general approach in [24], section 2.3.3 therein. To make full use of the translational symmetry of (4.2), we choose local tubular coordinates to separate the perturbation of the stationary pulse solution Γ_h into a perturbation along resp. perpendicular to the orbits of the translation group $(T_\alpha)_{\alpha\in\mathbb{R}}$, as follows:

$$\begin{pmatrix} U(\xi,t) \\ V(\xi,t) \end{pmatrix} = \Gamma(\xi,t) = T_{\alpha(t)}\left(\Gamma_h(\xi) + \rho(\xi,t)\right), \tag{4.23}$$

where

$$\langle \rho, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle = 0. \tag{4.24}$$

In the tubular coordinates (4.23), the left-hand side of (4.2) yields

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} = \frac{\partial}{\partial t}\Gamma = \frac{\mathrm{d}\alpha}{\mathrm{d}t}\, T_{\alpha(t)}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h(\xi) + \frac{\partial}{\partial\xi}\rho(\xi,t)\right) + T_{\alpha(t)}\frac{\partial}{\partial t}\rho(\xi,t); \tag{4.25}$$

multiplication with $T_{-\alpha(t)}$ gives

$$T_{-\alpha(t)}\begin{pmatrix} U_t \\ V_t \end{pmatrix} = \frac{\mathrm{d}\alpha}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h(\xi) + \frac{\partial}{\partial\xi}\rho(\xi,t)\right) + \frac{\partial}{\partial t}\rho(\xi,t). \tag{4.26}$$

Since the right-hand side of (4.2) is equivariant under the translation $T_{\alpha(t)}$, using the tubular coordinates (4.23) and subsequently multiplying with the inverse translation $T_{-\alpha(t)}$ is equivalent to substitution of $\Gamma_h(\xi) + \rho(\xi,t)$:

$$T_{-\alpha(t)}\begin{pmatrix} U_t \\ V_t \end{pmatrix} = \mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi,t)) \tag{4.27}$$

with

$$\mathrm{RHS}(U,V) = \begin{pmatrix} \frac{1}{\varepsilon^2}U_{\xi\xi} & - & (\mu\, U - \nu_1 F_1(U;\varepsilon)) & + & \frac{\nu_2}{\varepsilon}F_2(U,V;\varepsilon) \\ V_{\xi\xi} & - & V & + & G(U,V;\varepsilon) \end{pmatrix}. \tag{4.28}$$

Based on the orthogonality condition (4.24), we can project both sides of (4.27) onto the subspace spanned by $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$ to obtain separate dynamical equations for $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ and $\frac{\partial}{\partial t}\rho(\xi,t)$. Introducing the projection

$$\Pi_0 = \frac{\langle \cdot, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle}{\langle \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle}\, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h, \tag{4.29}$$

we see that projecting (4.26) onto the subspace spanned by $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$ yields

$$\Pi_0\, T_{-\alpha(t)} \begin{pmatrix} U_t \\ V_t \end{pmatrix} = \frac{\mathrm{d}\alpha}{\mathrm{d}t} \left(1 + \langle \frac{\partial}{\partial\xi}\rho, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h\right). \tag{4.30}$$

Combining this with (4.27), we can express $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ as

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \left(1 + \langle \frac{\partial}{\partial\xi}\rho, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle\right)^{-1} \langle \mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi,t)), \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle. \tag{4.31}$$

Note that (4.31) does not depend explicitly on $\alpha(t)$: this is a direct consequence of the equivariance of (4.2) under the translation group $(T_\alpha)_{\alpha\in\mathbb{R}}$.
The projection onto the orthogonal complement of the subspace spanned by $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$, given by $\mathbb{I} - \Pi_0$, can be used to obtain a dynamical equation for $\rho(\xi,t)$. Applying $\mathbb{I} - \Pi_0$ on (4.27) and using (4.31) yields

$$\begin{aligned} \frac{\partial}{\partial t}\rho(\xi,t) &= (\mathbb{I} - \Pi_0)\,\mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi,t)) - \frac{\mathrm{d}\alpha}{\mathrm{d}t}\,(\mathbb{I} - \Pi_0)\,\frac{\partial}{\partial\xi}\rho \qquad (4.32)\\ &= \mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi,t)) - \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h + \frac{\partial}{\partial\xi}\rho\right) \frac{\langle \mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi,t)), \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle}{1 + \langle \frac{\partial}{\partial\xi}\rho, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle}. \end{aligned}$$

Up to this point, our analysis only used the translational equivariance of (4.2). It will become clear that the results on the (specific form of) the eigenfunctions of $\mathcal{L}$ (4.9) as stated in Theorem 4.3 will enable us to drastically simplify (4.31). The following observation will be important enough in the following to state it as a Lemma:

Lemma 4.5. *Let the conditions of Theorem 4.3 be fulfilled, and let λ be an eigenvalue of $\mathcal{L}$ with eigenfunction $\phi(\xi;\lambda,\varepsilon)$. Then*

$$\langle \phi, \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h \rangle = 0. \tag{4.33}$$

Proof. Since the stationary pulse solution Γ_h is symmetric, it is even as a function of ξ, see Theorem 4.2. Therefore $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$ is odd as a function of ξ. Now, from the definition of $\mathcal{L}$ (4.9) it is clear that $\mathcal{L}(\xi)$ is invariant under reflection: $\mathcal{L}(-\xi) = \mathcal{L}(\xi)$. This means that, if $\phi(\xi;\lambda,\varepsilon)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue λ, then $\phi(-\xi;\lambda,\varepsilon)$ must also be an eigenfunction of $\mathcal{L}$ for that same eigenvalue; furthermore, $\phi(-\xi;\lambda,\varepsilon)$ is also bounded. Since the eigenfunction $\phi(\xi;\lambda,\varepsilon)$ is determined uniquely (see Theorem 4.3), it follows that $\phi(-\xi;\lambda,\varepsilon) = \phi(\xi;\lambda,\varepsilon)$. From the observation that the product of an even function and an odd function is odd, and that the integral of an odd function vanishes identically, the statement (4.33) follows from the definition of the inner product (4.22). □

From Lemma 4.5, we see that every eigenfunction ϕ as in Theorem 4.3 fulfills the orthogonality condition (4.24). This motivates us to use the local Ansatz

$$\rho(\xi, t) = A(t)\,\phi(\xi; \lambda, \varepsilon), \tag{4.34}$$

with $A(t) \in \mathbb{C}$, keeping in mind that we will specify the eigenfunction ϕ to be the Hopf eigenfunction ϕ_H at a later stage.

Corollary 4.6. *For any perturbation of the form* (4.34)*, equation* (4.31) *simplifies to* $\frac{\mathrm{d}\alpha}{\mathrm{d}t} = 0$.

Proof. Since ϕ is even as a function of ξ (see the proof of Lemma 4.5), ρ is even as a function of ξ. The pulse Γ_h is symmetric (Theorem 4.2), so $\Gamma_h(\xi) + \rho(\xi, t)$ is even as a function of ξ. That means that $\mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi, t))$ (4.28) is even as a function of ξ. Subsequently, the inner product $\langle \mathrm{RHS}(\Gamma_h(\xi)+\rho(\xi, t)), \frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h\rangle$ vanishes identically, since the translational eigenmode $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$ is odd in ξ, see the proof of Lemma 4.5. Hence, the right-hand side of (4.31) vanishes. □

Since for the Ansatz (4.34) the expression $\mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi, t))$ (4.28) lies in the orthogonal complement of the span of $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$, equation (4.32) drastically simplifies to

$$\frac{\partial}{\partial t}\rho(\xi, t) = \mathrm{RHS}(\Gamma_h(\xi) + \rho(\xi, t)), \tag{4.35}$$

or equivalently

$$\frac{\mathrm{d}A}{\mathrm{d}t}\,\phi = \mathrm{RHS}(\Gamma_h(\xi) + A\,\phi). \tag{4.36}$$

To state the main result of this subsection, we invoke Theorem 3.19 from chapter 2 in [24], reformulated here:

Theorem 4.7 (**Centre manifolds in presence of continuous symmetry**). *Let* $\mathcal{X} = \Pi_0\mathcal{X} \oplus \mathcal{X}' = \mathrm{span}\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h\right\} \oplus \mathcal{X}'$, $\mathcal{L}' = (\mathbb{I} - \Pi_0)\,\mathcal{L}$ *and let* σ_0' *be the central spectrum of* $\mathcal{L}'$*. Assume that* σ_0' *is finite, and let* $\mathcal{E}_0' \subset \mathcal{X}'$ *be the associated spectral subspace. Let* $\mathcal{U}' \subset \mathcal{X}'$ *be a neighbourhood of the origin in* $\mathcal{X}'$*. Consider the tubular neighbourhood*

$$\mathcal{U} = \{T_\alpha(\Gamma_h + \rho);\, \rho \in \mathcal{U}',\, \alpha \in \mathbb{R}\} \subset \mathcal{X}$$

of the line of equilibria $\{T_\alpha\,\Gamma_h,\, \alpha \in \mathbb{R}\}$.
There exists a map Ψ *which has the same degree of smoothness as the right-hand side of* (4.2)*,* $\Psi : \mathcal{E}_0' \to \mathcal{X}' - \mathcal{E}_0'$*, with* $\Psi(0) = 0$*,* $D\Psi(0) = 0$ *such that the manifold*

$$\mathcal{M}_0 = \left\{T_\alpha\,(\Gamma_h + \rho + \Psi(\rho))\,;\, \rho \in \mathcal{E}_0',\, \alpha \in \mathbb{R}\right\} \subset \mathcal{X}$$

has the following properties:

1. *The manifold* $\mathcal{M}_0$ *is locally invariant under* (4.2)*, in other words, if* $\Gamma(\xi,t) = (U(\xi,t), V(\xi,t))$ *is a solution of* (4.2) *satisfying* $\Gamma(\xi,0) \in \mathcal{M}_0 \cap \mathcal{U}$ *and* $\Gamma(\xi,t) \in \mathcal{U}$ *for all* $t \in [0,T]$*, then* $\Gamma(\xi,t) \in \mathcal{M}_0$ *for all* $t \in [0,T]$*.*

2. $\mathcal{M}_0$ *contains the set of solutions of* (4.2) *staying in* $\mathcal{U}$ *for all* $t > 0$*, in other words, if* Γ *is a solution of* (4.2) *satisfying* $\Gamma(\xi,t) \in \mathcal{U}$ *for all* $t > 0$*, then* $\Gamma(\xi,0) \in \mathcal{M}_0$*.*

The solutions to (4.2) *which stay close to the line of equilibria for all* $t > 0$ *are of the form* (4.23)*, with* $\alpha(t)$ *satisfying* (4.31) *and* $\rho(\xi,t)$ *satisfying* (4.32)*.*

Based on the above results (which in particular hold for our codimension 1 central spectrum (4.20)), we can formulate a theorem on the local structure of the centre manifold associated to the Hopf bifurcation (4.19) and the associated central spectrum (4.20).

Theorem 4.8. *Let the central spectrum of* $\mathcal{L}$ *be given by* (4.20)*. The associated centre manifold* $\mathcal{M}_{0,H}$ *can be foliated along the line of equilibria* $\{T_\alpha \Gamma_h,\ \alpha \in \mathbb{R}\}$*, and has a locally trivial product structure:*

$$\mathcal{M}_{0,H} = \mathbb{R} \times \mathcal{M}'_{0,H}.$$

Moreover, the full dynamics on $\mathcal{M}_{0,H}$ *can be represented by the reduced dynamics on* $\mathcal{M}'_{0,H}$*, given by* (4.35)*.*

Proof. We adopt the notation of Theorem 4.7. For the central spectrum (4.20), the reduced spectral subspace $\mathcal{E}'_0$ is spanned by the eigenvectors of the Hopf eigenvalues $\pm i\,\omega_H$, i.e. $\mathcal{E}'_0 = \operatorname{span}\left\{\phi_H, \overline{\phi_H}\right\}$. Therefore, any $\rho(\xi,t) \in \mathcal{E}'_0$ can be written as $\rho(\xi,t) = A(t)\,\phi_H(\xi) + \overline{A(t)}\,\overline{\phi_H(\xi)}$. By Corollary 4.6, we see that (4.31) reduces to $\frac{\mathrm{d}\alpha}{\mathrm{d}t} = 0$, so $\alpha(t) = \alpha_0$. That means that the full dynamics on $\mathcal{M}_{0,H}$ are represented by (4.32), which in turn can be reduced to (4.35). □

Remark 4.9. Although one would expect, based on the central spectrum (4.20), that the translational eigenmode $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$ in general is excitable, Theorem 4.8 shows that this is not the case. In other words, the pulse Γ_h does not move when perturbed under Hopf bifurcation conditions. Theorem 4.8 therefore enables us to 'neglect' the translational eigenmode in the centre manifold expansion. Moreover, this Theorem analytically confirms the numerical results on the 'pinning' of a periodically oscillating pulse in chapter 2, section 2.5.

4.4 Unfolding the Hopf bifurcation

In this section, we use the results of the previous section, to perform a direct centre manifold expansion around the Hopf bifurcation (4.19). Based on Theorem 4.8, we can choose local coordinates such that a perturbation of the pulse Γ_h can be written as

$$\begin{pmatrix} U(\xi,t) \\ V(\xi,t) \end{pmatrix} = \begin{pmatrix} U_h(\xi) \\ V_h(\xi) \end{pmatrix} + A(t)\,\phi_H(\xi) + \overline{A(t)}\,\overline{\phi_H(\xi)} \tag{4.37}$$

with $A(t) \in \mathbb{C}$ acting as a (small) order parameter, and where ϕ_H is the Hopf eigenfunction as in Theorem 4.3, obeying (4.19). It is worthwhile to emphasise that, choosing local coordinates as in (4.37), we indeed not take the 'irrelevant' $\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h$ direction into account in the upcoming weakly nonlinear analysis. Substitution of (4.37) in (4.2) yields

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t}\phi_H + \overline{\frac{\mathrm{d}A}{\mathrm{d}t}}\overline{\phi_H} &= \mathcal{L}\left(A\,\phi_H + \overline{A}\,\overline{\phi_H}\right) + \mathcal{R}\left(\Gamma_h | A, \phi_H\right) \\ &= i\,\omega_H\left(A\,\phi_H - \overline{A}\,\overline{\phi_H}\right) + \mathcal{R}\left(\Gamma_h | A, \phi_H\right) \end{aligned} \tag{4.38}$$

where the remainder terms $\mathcal{R}\left(\Gamma_h | A, \phi_H\right)$ are specified as

$$\mathcal{R}\left(\Gamma_h | A, \phi_H\right) = \begin{pmatrix} \nu_1 \mathcal{N}F_1\left(U_h | A, \phi_H; \varepsilon\right) + \frac{\nu_2}{\varepsilon}\mathcal{N}F_2\left(U_h, V_h | A, \phi_H; \varepsilon\right) \\ \mathcal{N}G\left(U_h, V_h | A, \phi_H; \varepsilon\right) \end{pmatrix}. \tag{4.39}$$

Here, $\mathcal{N}$ selects the nonlinear part of the function it is acting on, as follows:

$$\begin{aligned} \mathcal{N}F_1\left(U_h | A, \phi_H; \varepsilon\right) = \; & F_1\left(U_h + A\left(\phi_H\right)_1 + \overline{A}\left(\overline{\phi_H}\right)_1; \varepsilon\right) - F_1\left(U_h; \varepsilon\right) \\ & - \frac{\mathrm{d}F_1}{\mathrm{d}U}\left(U_h; \varepsilon\right)\left(A\left(\phi_H\right)_1 + \overline{A}\left(\overline{\phi_H}\right)_1\right); \end{aligned} \tag{4.40a}$$

$$\begin{aligned} \mathcal{N}F_2\left(U_h, V_h | A, \phi_H; \varepsilon\right) = \; & F_2\left(U_h + A\left(\phi_H\right)_1 + \overline{A}\left(\overline{\phi_H}\right)_1, V_h + A\left(\phi_H\right)_2 + \overline{A}\left(\overline{\phi_H}\right)_2; \varepsilon\right) \\ & - F_2\left(U_h, V_h; \varepsilon\right) - \frac{\partial F_2}{\partial U}\left(U_h, V_h; \varepsilon\right)\left(A\left(\phi_H\right)_1 + \overline{A}\left(\overline{\phi_H}\right)_1\right) \\ & - \frac{\partial F_2}{\partial V}\left(U_h, V_h; \varepsilon\right)\left(A\left(\phi_H\right)_2 + \overline{A}\left(\overline{\phi_H}\right)_2\right); \end{aligned} \tag{4.40b}$$

$$\begin{aligned} \mathcal{N}G\left(U_h, V_h | A, \phi_H; \varepsilon\right) = \; & G\left(U_h + A\left(\phi_H\right)_1 + \overline{A}\left(\overline{\phi_H}\right)_1, V_h + A\left(\phi_H\right)_2 + \overline{A}\left(\overline{\phi_H}\right)_2; \varepsilon\right) \\ & - G\left(U_h, V_h; \varepsilon\right) - \frac{\partial G}{\partial U}\left(U_h, V_h; \varepsilon\right)\left(A\left(\phi_H\right)_1 + \overline{A}\left(\overline{\phi_H}\right)_1\right) \\ & - \frac{\partial G}{\partial V}\left(U_h, V_h; \varepsilon\right)\left(A\left(\phi_H\right)_2 + \overline{A}\left(\overline{\phi_H}\right)_2\right). \end{aligned} \tag{4.40c}$$

The next step is to introduce a projection onto the linear subspace spanned by the eigenfunction ϕ_H, using the inner product (4.22). This projection, or rather the component along the span of ϕ_H, is given by

$$\Pi_H = \frac{\langle \phi_H, \phi_H \rangle \langle \cdot, \phi_H \rangle - \langle \overline{\phi_H}, \phi_H \rangle \langle \cdot, \overline{\phi_H} \rangle}{\langle \phi_H, \phi_H \rangle^2 - |\langle \phi_H, \overline{\phi_H} \rangle|^2}, \tag{4.41}$$

and is chosen such that $\Pi_H \phi_H = 1$ and $\Pi_H \overline{\phi_H} = 0$. Applying Π_H on (4.38) yields a first order ODE for the order parameter $A(t)$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i\,\omega_H A + \Pi_H \mathcal{R}\,(\Gamma_h, A, \phi_H)\,. \tag{4.42}$$

To study the nonlinear behaviour of $A(t)$ in detail, we need to expand the nonlinear component of (4.42), $\Pi_H \mathcal{R}\,(\Gamma_h | A, \phi_H)$, in the order parameter A. This –quite elaborate– enterprise will be carried out in the following subsection.

4.4.1 Normal form reduction

For ease of notation, we specify the (two component) Hopf eigenfunction ϕ_H as

$$\phi_H = \begin{pmatrix} (\phi_H)_1 \\ (\phi_H)_2 \end{pmatrix} = \begin{pmatrix} u_H \\ v_H \end{pmatrix}. \tag{4.43}$$

By Assumptions 4.1 (A2 - A4), the nonlinear functions $F_{1,2}$ and G are smooth as a function of positive U, V. We can therefore use a regular Taylor expansion in the order parameter A. For the single variable function F_1, this yields

$$\begin{aligned} F_1(U_h + A u_H + \overline{A}\,\overline{u_H}) &= \sum_{j=0}^{N} \frac{\mathrm{d}^j F_1}{\mathrm{d}U^j}(U_h) \left(A u_H + \overline{A}\,\overline{u_H} \right)^j \\ &= \sum_{k+l \geq 0} \frac{1}{k!l!} f_{1,kl} A^k \overline{A}^l \end{aligned} \tag{4.44}$$

with

$$f_{1,kl} = \frac{\mathrm{d}^{k+l} F_1}{\mathrm{d}U^{k+l}}(U_h)\,(u_H)^k (\overline{u_H})^l. \tag{4.45}$$

Using this expansion, the nonlinear component $\mathcal{N}F_1\,(U_h | A, \phi_H; \varepsilon)$ (4.40a) can now readily be expressed as

$$\mathcal{N}F_1\,(U_h | A, \phi_H; \varepsilon) = \sum_{k+l \geq 2} \frac{1}{k!l!} f_{1,kl} A^k \overline{A}^l. \tag{4.46}$$

For the multivariate functions F_2 and G, we would like to obtain a similar expression (here applied to F_2):

$$\begin{aligned} F_2(U_h + Au_H + \overline{A}\overline{u_H}, V_h + Av_H + \overline{A}\overline{v_H}) &= \sum_{j=1}^{N}\sum_{k=0}^{j} \frac{\partial^j F_2}{\partial U^k \partial V^{j-k}}(U_h, V_h)(Au_H + \overline{A}\overline{u_H})^k \\ &\qquad \times (Av_H + \overline{A}\overline{v_H})^{j-k} \\ &= \sum_{k+l\geq 0} \frac{1}{k!l!} f_{2,kl} A^k \overline{A}^l. \end{aligned} \tag{4.47}$$

In order to express the expansion coefficients $f_{2,kl}$ in terms of the components of Γ_h and ϕ_H, we introduce the following tensor definition:

Definition 4.10. *Given a two component function $F(U, V)$, consider the total derivative of F of order k, $D^k F$, as a linear mapping from $\left(\mathbb{C}^2\right)^k$ to $\mathbb{C}$. For every pair of nonnegative integers (m, n) for which $m + n = k$, we can define the (m, n)-tensor $(D^{m+n}F) : \left(\mathbb{C}^2\right)^m \times \left(\left(\mathbb{C}^2\right)^n\right)^* \to \mathbb{C}$ as*

$$(D^{m+n}F)^{j_1\cdots j_m}_{i_1\cdots i_n} = \frac{\partial^{m+n} F}{\partial X_{i_1} \cdots \partial X_{i_m} \partial X_{j_1} \partial X_{j_n}}(U_h, V_h),$$

where $X = (U, V)$ and $i_1 \ldots i_m, j_1 \ldots j_n \in \{1, 2\}$. We have

$$(D^{m+n}F) : (z_1, \ldots, z_m, w^1, \ldots, w^n) \mapsto (D^{m+n}F)^{j_1\cdots j_m}_{i_1\cdots i_n} (z_1)_{i_1} \ldots (z_m)_{i_m} (w^1)_{j_1} \ldots (w^m)_{j_m};$$

where the repeated indices are summed over.

Using Definition 4.10, we can express the coefficients $f_{2,kl}$ in (4.47) as

$$f_{2,kl} = \left(D^{k+l}F_2\right)^{j_1\cdots j_l}_{i_1\cdots i_k} (\phi_H)_{i_1} \cdots (\phi_H)_{i_k} (\overline{\phi_H})_{j_1} \cdots (\overline{\phi_H})_{j_l} \tag{4.48}$$

where $(\phi_H)_1 = u_H$, $(\phi_H)_2 = v_H$, see (4.43). Analogously, we can expand G as

$$\begin{aligned} G(U_h + Au_H + \overline{A}\overline{u_H}, V_h + Av_H + \overline{A}\overline{v_H}) &= \sum_{j=1}^{N}\sum_{k=0}^{j} \frac{\partial^j G}{\partial U^k \partial V^{j-k}}(U_h, V_h)(Au_H + \overline{A}\overline{u_H})^k \\ &\qquad \times (Av_H + \overline{A}\overline{v_H})^{j-k} \\ &= \sum_{k+l\geq 0} \frac{1}{k!l!} g_{kl} A^k \overline{A}^l, \end{aligned} \tag{4.49}$$

where

$$g_{kl} = \left(D^{k+l}G\right)^{j_1\cdots j_l}_{i_1\cdots i_k} (\phi_H)_{i_1} \cdots (\phi_H)_{i_k} (\overline{\phi_H})_{j_1} \cdots (\overline{\phi_H})_{j_l}. \tag{4.50}$$

Now, the remaining (multivariate) nonlinear components $\mathcal{N}F_2(U_h, V_h|A, \phi_H; \varepsilon)$ and $\mathcal{N}G(U_h, V_h|A, \phi_H; \varepsilon)$ ((4.40b) resp. (4.40c)) can be expressed as

$$\mathcal{N}F_2(U_h, V_h|A, \phi_H; \varepsilon) = \sum_{k+l\geq 2} \frac{1}{k!l!} f_{2,kl} A^k \overline{A}^l \tag{4.51}$$

$$\mathcal{N}G(U_h, V_h|A, \phi_H; \varepsilon) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl} A^k \overline{A}^l \tag{4.52}$$

with the coefficients $f_{2,kl}$ and g_{kl} specified in (4.48) resp. (4.50). The remainder terms $\mathcal{R}(\Gamma_h|A, \phi_H)$ (4.39) can now be expanded in A as

$$\mathcal{R}(\Gamma_h|A, \phi_H) = \sum_{k+l\geq 2} \frac{1}{k!l!} R_{kl} A^k \overline{A}^l, \tag{4.53}$$

with

$$R_{kl} = \begin{pmatrix} \nu_1 f_{1,kl} + \frac{\nu_2}{\varepsilon} f_{2,kl} \\ g_{kl} \end{pmatrix}. \tag{4.54}$$

Finally, we can expand (4.42) as

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i\,\omega_H A + \sum_{k+l\geq 2} \frac{1}{k!l!}\, \Pi_H R_{kl}\, A^k \overline{A}^l. \tag{4.55}$$

Equation (4.55) can be brought into normal form via a series of locally invertible complex coordinate changes. We follow [36] and use Lemma 3.6 therein, restated here:

Lemma 4.11 (**Poincaré normal form for the Hopf bifurcation**). *The equation*

$$\dot{z} = i\,\omega\, z + \sum_{2\leq k+l\leq 3} g'_{kl} z^k \overline{z}^l + O\left(|z|^4\right),$$

can be transformed into an equation with only the resonant cubic term:

$$\dot{w} = i\,\omega\, w + c_1 w^2 \overline{w} + O\left(|w|^4\right)$$

where

$$c_1 = \frac{i}{2\omega}\left(g'_{20} g'_{11} - 2\left|g'_{11}\right|^2 - \frac{1}{3}\left|g'_{02}\right|^2\right) + \frac{1}{2} g'_{21}. \tag{4.56}$$

Corollary 4.12. *The nonlinear behaviour of z resp. w is determined by the sign of the first Lyapunov, or Landau, coefficient*

$$\ell_1 = \frac{1}{\omega}\mathrm{Re}\, c_1 = \frac{1}{2\omega^2}\mathrm{Re}\left(i\, g'_{20} g'_{11} + \omega\, g'_{21}\right). \tag{4.57}$$

The Hopf bifurcation is supercritical if $\ell_1 < 0$ and subcritical if $\ell_1 > 0$.

Using Lemma 4.11 and Corollary 4.12, we see that the nonlinear behaviour of $A(t)$ determined by (4.55) can be characterised by the sign of the associated (first) Lyapunov coefficient

$$\ell_1 = \frac{1}{2\omega^2}\mathrm{Re}\left(i\,(\Pi_H R_{20})\,(\Pi_H R_{11}) + \omega\,\Pi_H R_{21}\right) \tag{4.58}$$

with R_{kl} as in (4.54).

4.4.2 Calculating the first Lyapunov coefficient to leading order

The scale separated structure of the eigenfunctions ϕ_H as given in Theorem 4.3, combined with the inner product adapted to this scale separation (4.22), can be used to obtain leading order expressions for $\Pi_H R_{kl}$, yielding an explicit leading order expression for the first Lyapunov coefficient ℓ_1 (4.58). This is the key to making the preceding general approach work in the context of the localised pulse solution of (4.2), for which we have a leading order expression (see Theorem 4.2). More importantly, having a leading order expression for the (Hopf) eigenfunctions (see Theorem 4.3), we are able to come to concrete conclusions about the nature of the Hopf bifurcation. The way the leading order eigenfunction expressions can be used to obtain explicit results on the coefficients of the centre manifold expansion is demonstrated below.

Recalling the definition of Π_H (4.41), the first step is to obtain leading order expressions for the inner products $\langle \phi_H, \phi_H\rangle$ and $\langle \phi_H, \overline{\phi_H}\rangle$. Using (4.22), we see that

$$\langle \phi, \psi\rangle = \int_{\mathbb{R}} \phi^T S\,\overline{\psi}\,\mathrm{d}\xi = \int_{\mathbb{R}} (\phi)_1(\overline{\psi})_1\,\mathrm{d}(\varepsilon\xi) + \int_{\mathbb{R}} (\phi)_2(\overline{\psi})_2\,\mathrm{d}\xi \tag{4.59}$$

so that, see (4.43),

$$\langle \phi_H, \phi_H\rangle = \int_{\mathbb{R}} |u_H|^2\,\mathrm{d}(\varepsilon\xi) + \int_{\mathbb{R}} |v_H|^2\,\mathrm{d}\xi. \tag{4.60}$$

Using the leading order expression for ϕ_H from Theorem 4.3, we can specify the

above as

$$\begin{aligned}\int_{\mathbb{R}} |u_H|^2 \,\mathrm{d}(\varepsilon\xi) &= \int_{-\infty}^{-\varepsilon^{-\frac{1}{4}}} |u_{s,-}(-\varepsilon\xi; i\,\omega_H, \varepsilon)|^2 \mathrm{d}(\varepsilon\xi) + \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} |u_{s,-}(0; i\,\omega_H, \varepsilon)|^2 + O(\varepsilon^{\frac{3}{4}})\,\mathrm{d}(\varepsilon\xi) \\ &\quad + \int_{\varepsilon^{-\frac{1}{4}}}^{\infty} |u_{s,-}(\varepsilon\xi; i\,\omega_H, \varepsilon)|^2 \mathrm{d}(\varepsilon\xi) \\ &= \int_{-\infty}^{-\varepsilon^{\frac{3}{4}}} |u_{s,-}(-x; i\,\omega_H, \varepsilon)|^2 \mathrm{d}x + \int_{-\varepsilon^{\frac{3}{4}}}^{\varepsilon^{\frac{3}{4}}} |u_{s,-}(0; i\,\omega_H, \varepsilon)|^2 + O(\varepsilon^{\frac{3}{4}})\,\mathrm{d}x \\ &\quad + \int_{\varepsilon^{\frac{3}{4}}}^{\infty} |u_{s,-}(x; i\,\omega_H, \varepsilon)|^2 \mathrm{d}x \\ &= 2\int_0^{\infty} |u_{s,-}(x; i\,\omega_H, \varepsilon)|^2 \mathrm{d}x + O(\varepsilon^{\frac{3}{4}}), \end{aligned} \tag{4.61}$$

while

$$\begin{aligned}\int_{\mathbb{R}} |v_H|^2 \,\mathrm{d}\xi &= \int_{-\infty}^{-\varepsilon^{-\frac{1}{4}}} \left|C_1 e^{C_1\xi}\right|^2 \mathrm{d}\xi + \int_{-\varepsilon^{-\frac{1}{4}}}^{\varepsilon^{-\frac{1}{4}}} |u_{s,-}(0; i\,\omega_H, \varepsilon) v_{\mathrm{in}}(\xi; i\,\omega_H, \varepsilon)|^2 + O(\varepsilon^{\frac{3}{4}})\,\mathrm{d}\xi \\ &\quad + \int_{\varepsilon^{-\frac{1}{4}}}^{\infty} \left|C_1 e^{-C_1\xi}\right|^2 \mathrm{d}\xi \\ &= |u_{s,-}(0; i\,\omega_H, \varepsilon)|^2 \int_{-\infty}^{\infty} |v_{\mathrm{in}}(\xi; i\,\omega_H, \varepsilon)|^2 \,\mathrm{d}\xi + O(\varepsilon^{\frac{1}{2}}), \end{aligned} \tag{4.62}$$

so that

$$\langle \phi_H, \phi_H \rangle = 2\int_0^{\infty} |u_{s,-}(x; i\,\omega_H, \varepsilon)|^2 \mathrm{d}x + |u_{s,-}(0; i\,\omega_H, \varepsilon)|^2 \int_{-\infty}^{\infty} |v_{\mathrm{in}}(\xi; i\,\omega_H, \varepsilon)|^2 \,\mathrm{d}\xi + O(\varepsilon^{\frac{1}{2}}). \tag{4.63}$$

Similarly,

$$\langle \phi_H, \overline{\phi_H} \rangle = 2\int_0^{\infty} u_{s,-}(x; i\,\omega_H, \varepsilon)^2 \mathrm{d}x + u_{s,-}(0; i\,\omega_H, \varepsilon)^2 \int_{-\infty}^{\infty} v_{\mathrm{in}}(\xi; i\,\omega_H, \varepsilon)^2 \,\mathrm{d}\xi + O(\varepsilon^{\frac{1}{2}}). \tag{4.64}$$

The inner product of R_{kl} with ϕ_H resp. $\overline{\phi_H}$ can be made explicit as well. Using (4.54), we see that

$$\langle R_{kl}, \phi_H \rangle = \int_{\mathbb{R}} \nu_1 f_{1,kl}\, \overline{u_H}\, \mathrm{d}(\varepsilon\xi) + \int_{\mathbb{R}} \frac{\nu_2}{\varepsilon} f_{2,kl}\, \overline{u_H}\, \mathrm{d}(\varepsilon\xi) + \int_{\mathbb{R}} g_{kl}\, \overline{v_H}\, \mathrm{d}\xi, \quad (4.65)$$

$$\langle R_{kl}, \overline{\phi_H} \rangle = \int_{\mathbb{R}} \nu_1 f_{1,kl}\, u_H\, \mathrm{d}(\varepsilon\xi) + \int_{\mathbb{R}} \frac{\nu_2}{\varepsilon} f_{2,kl}\, u_H\, \mathrm{d}(\varepsilon\xi) + \int_{\mathbb{R}} g_{kl}\, v_H\, \mathrm{d}\xi. \quad (4.66)$$

Now, using (4.45), Theorem 4.2 and Theorem 4.3,

$$\begin{aligned} \int_{\mathbb{R}} f_{1,kl}\, \overline{u_H}\, \mathrm{d}(\varepsilon\xi) &= \int_{\mathbb{R}} \frac{\mathrm{d}^{k+l} F_1}{\mathrm{d}U^{k+l}}(U_h)\, (u_H)^k (\overline{u_H})^{l+1}\, \mathrm{d}(\varepsilon\xi) \\ &= 2 \int_0^\infty \frac{\mathrm{d}^{k+l} F_1}{\mathrm{d}U^{k+l}}(u_s^*(x)) \left(u_{s,-}(x; i\,\omega_H, \varepsilon)\right)^k \left(\overline{u_{s,-}(x; i\,\omega_H, \varepsilon)}\right)^{l+1} \mathrm{d}x \\ &\quad + O(\varepsilon^{\frac{1}{2}}). \end{aligned} \quad (4.67)$$

Similarly,

$$\int_{\mathbb{R}} f_{1,kl}\, u_H\, \mathrm{d}(\varepsilon\xi) = 2 \int_0^\infty \frac{\mathrm{d}^{k+l} F_1}{\mathrm{d}U^{k+l}}(u_s^*(x)) \left(u_{s,-}(x; i\,\omega_H, \varepsilon)\right)^{k+1} \left(\overline{u_{s,-}(x; i\,\omega_H, \varepsilon)}\right)^{l} \mathrm{d}x \quad (4.68)$$

up to $O(\varepsilon^{\frac{1}{2}})$.

The expressions for $f_{2,kl}$ (4.48) and g_{kl} (4.50) are substantially more involved than that for $f_{1,kl}$ (4.45). However, based on their initial definition as expansion coefficients (4.47) resp. (4.49), we can determine their behaviour inside and outside the fast region I_f (4.8).

Lemma 4.13. *For $\xi \notin I_f$, both $f_{2,kl}$ and g_{kl} are exponentially small in ξ, i.e. for all $k, l \geq 0$ there are $C_{1,2} > 0$ such that*

$$\max\left\{\left|f_{2,kl}\right|, |g_{kl}|\right\} \leq C_1\, e^{-C_2 \xi} \quad \textit{for all } \xi \notin I_f. \quad (4.69)$$

Proof. Based on Theorem 4.2 and Theorem 4.3, we see that outside I_f, $V_h + A\, v_H + \overline{A}\, \overline{v_H}$ is exponentially small in ξ. By Assumptions 4.1, (A3) resp. (A4), we can infer that both $F_2(U_h + A u_H + \overline{A}\,\overline{u_H}, V_h + A v_H + \overline{A}\,\overline{v_H})$ and $G(U_h + A u_H + \overline{A}\,\overline{u_H}, V_h + A v_H + \overline{A}\,\overline{v_H})$ must be exponentially small as well for $\xi \notin I_f$. By (4.47) and (4.49), it follows that both $f_{2,kl}$ and g_{kl} are exponentially small in ξ. □

From Lemma 4.13, it follows that an integral of the form $\int_{\mathbb{R}} \frac{1}{\varepsilon} f_{2,kl}\, \overline{u_H}\, \mathrm{d}(\varepsilon\xi) = \int_{\mathbb{R}} f_{2,kl}\, \overline{u_H}\, \mathrm{d}\xi$ converges, even though both U_h and u_H decay asymptotically slowly outside I_f. Therefore, we can conclude that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\mathbb{R}} f_{2,kl}\, \overline{u_H}\, \mathrm{d}(\varepsilon\xi) &= \int_{\mathbb{R}} f_{2,kl}\, \overline{u_H}\, \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \left(D^{k+l}F_{2,f}\right)^{j_1\cdots j_l}_{i_1\cdots i_k} (\phi_{H,f})_{i_1}\cdots(\phi_{H,f})_{i_k}(\overline{\phi_{H,f}})_{j_1}\cdots(\overline{\phi_{H,f}})_{j_l}\cdot(\overline{\phi_{H,f}})_1\, \mathrm{d}\xi\end{aligned} \tag{4.70}$$

up to $O(\varepsilon^{\frac{1}{2}})$, where $\phi_{H,f} = \phi_H|_{\xi\in I_f}$, i.e. (see Theorem 4.3)

$$\phi_{H,f} = \begin{pmatrix} u_{H,f} \\ v_{H,f}\end{pmatrix} = u_{s,-}(0; i\,\omega_H, \varepsilon)\begin{pmatrix} 1 \\ v_{\mathrm{in}}(\xi; i\,\omega_H, \varepsilon)\end{pmatrix} \tag{4.71}$$

and $D^{k+l}F_{2,f}$ is the equal to the tensor defined in 4.10 applied to F_2, evaluated in $(U_h, V_h) = \left(u_*, v_{f,h}(\xi, u_*)\right)$, see Theorem 4.2. For the same reason,

$$\int_{\mathbb{R}} g_{kl}\, \overline{v_H}\, \mathrm{d}\xi = \int_{-\infty}^{\infty} \left(D^{k+l}G_f\right)^{j_1\cdots j_l}_{i_1\cdots i_k} (\phi_{H,f})_{i_1}\cdots(\phi_{H,f})_{i_k}(\overline{\phi_{H,f}})_{j_1}\cdots(\overline{\phi_{H,f}})_{j_l}\cdot(\overline{\phi_{H,f}})_2\, \mathrm{d}\xi, \tag{4.72}$$

where $\phi_{H,f}$ is as in (4.71) and again $D^{k+l}G_f$ is the equal to the tensor defined in 4.10 applied to G, evaluated in $(U_h, V_h) = \left(u_*, v_{f,h}(\xi, u_*)\right)$. For completeness, we state

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\mathbb{R}} f_{2,kl}\, u_H\, \mathrm{d}(\varepsilon\xi) &= \int_{-\infty}^{\infty} \left(D^{k+l}F_{2,f}\right)^{j_1\cdots j_l}_{i_1\cdots i_k} (\phi_{H,f})_{i_1}\cdots(\phi_{H,f})_{i_k} \\ &\qquad \times(\overline{\phi_{H,f}})_{j_1}\cdots(\overline{\phi_{H,f}})_{j_l}\cdot(\phi_{H,f})_1\, \mathrm{d}\xi + O(\varepsilon^{\frac{1}{2}}),\end{aligned} \tag{4.73}$$

$$\begin{aligned}\int_{\mathbb{R}} g_{kl}\, v_H\, \mathrm{d}\xi &= \int_{-\infty}^{\infty} \left(D^{k+l}G_f\right)^{j_1\cdots j_l}_{i_1\cdots i_k} (\phi_{H,f})_{i_1}\cdots(\phi_{H,f})_{i_k} \\ &\qquad \times(\overline{\phi_{H,f}})_{j_1}\cdots(\overline{\phi_{H,f}})_{j_l}\cdot(\phi_{H,f})_2\, \mathrm{d}\xi + O(\varepsilon^{\frac{1}{2}}).\end{aligned} \tag{4.74}$$

The expressions derived above can be used to calculate the first Lyapunov coefficient ℓ_1 (4.58) explicitly. A systematic approach to obtain all the necessary terms yields

expressions such as

$$\begin{aligned}\int_{\mathbb{R}} g_{20}\,\overline{v_H}\,\mathrm{d}\xi &= \int_{-\infty}^{\infty}\left(D^{2+0}G_f\right)_{i_1\,i_2}\,(\phi_{H,f})_{i_1}(\phi_{H,f})_{i_2}\cdot(\overline{\phi_{H,f}})_2\,\mathrm{d}\xi\\ &= \int_{-\infty}^{\infty}\Bigg[\frac{\partial^2 G}{\partial U^2}(U,V)\,u_{H,f}^2 + 2\frac{\partial^2 G}{\partial U\partial V}(U,V)\,u_{H,f}\,v_{H,f}\\ &\qquad +\frac{\partial^2 G}{\partial V^2}(U,V)v_{H,f}^2\Bigg]_{(U,V)=\left(u_*,v_{f,h}(\xi,u_*)\right)}\times\ \overline{v_{H,f}}\,\mathrm{d}\xi \qquad (4.75)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} f_{2,11}\,\overline{u_H}\,\mathrm{d}\xi &= \int_{-\infty}^{\infty}\left(D^{1+1}F_{2,f}\right)_i^j\,(\phi_{H,f})_i(\overline{\phi_{H,f}})_j\cdot(\overline{\phi_{H,f}})_1\,\mathrm{d}\xi\\ &= \int_{-\infty}^{\infty}\Bigg[\frac{\partial^2 F_2}{\partial U^2}(U,V)\,|u_{H,f}|^2 + \frac{\partial^2 F_2}{\partial U\partial V}(U,V)\,\left(u_{H,f}\,\overline{v_{H,f}}+\overline{u_{H,f}}\,v_{H,f}\right)\\ &\qquad +\frac{\partial^2 F_2}{\partial V^2}(U,V)|v_{H,f}|^2\Bigg]_{(U,V)=\left(u_*,v_{f,h}(\xi,u_*)\right)}\times\ \overline{u_{H,f}}\,\mathrm{d}\xi. \qquad (4.76)\end{aligned}$$

The above concretisations of the formal expressions from the first part of section 4.4 are summarised in the following key Lemma.

Lemma 4.14. *The leading order expression for the first Lyapunov coefficient ℓ_1 associated to the normal form of Hopf bifurcation* (4.19)*, as given in* (4.58)*, is obtained by combining* (4.67)*,* (4.70) *and* (4.72) *with* (4.65)*, resp.* (4.68)*,* (4.73) *and* (4.74) *with* (4.66)*, and subsequently using the resulting expression combined with* (4.63) *and* (4.64) *in the projection $\Pi_H R_{kl}$* (4.41) *for the integer pairs $(k,l)=(2,0)$, $(k,l)=(1,1)$ and $(k,l)=(2,1)$.*

Although the expressions thus obtained may not be quite insightful in their full generality, for specific choices of the model functions $F_{1,2}$ and G, the above approach yields explicit quantities which can be evaluated directly for a specific Hopf bifurcation. This will the subject of section 4.6, where the explicit eigenfunction expressions derived in chapter 2, section 2.3 for the slowly nonlinear Gierer-Meinhardt equation will enable us to obtain (parameter dependent) values for the first Lyapunov coefficient.

4.5 An alternative approach

In the 'direct' approach followed in section 4.4, which was based on the normal form transformation referred to in Lemma 4.11, we used the fact that the local coordinates (4.37) could be used to describe the associated centre manifold. The actual 'shape' of the manifold however was never made explicit: the information contained in Ψ (Theorem 4.7) was hidden in the transformation $z \mapsto w(z)$ indicated in Lemma 4.11. In this section, we again derive an expression for c_1 (for its definition, see Lemma 4.11). We follow a less explicit approach compared to that of the previous section: this upcoming, alternative approach is based on [24], section 3.4.2 therein. The relative elegance of the obtained expressions (see upcoming Lemma 4.15) comes with a cost, however: we will need extra information on the adjoint of $\mathcal{L}$. Depending on the specific choice of the model functions $F_{1,2}$ and G (4.1), this alternative approach might be less or more cumbersome to carry out, compared with the previous, direct approach presented in section 4.4.

4.5.1 Expanding the centre manifold

Based on Theorem 4.7 and Theorem 4.8, we restrict ourselves without loss of generality to $\mathcal{X}' = \mathcal{X} - \operatorname{span}\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h\right\}$, i.e. we focus on a single leaf of the foliation of the entire centre manifold. Dropping the tildes, we see that solutions on the centre manifold $\mathcal{M}_{0,H}$ can be expressed as

$$\begin{pmatrix} U(\xi,t) \\ V(\xi,t) \end{pmatrix} = \Gamma(\xi,t) = \Gamma_h(\xi) + A(t)\,\phi_H(\xi) + \overline{A(t)}\,\overline{\phi_H(\xi)} + \Psi(A,\overline{A},\phi_H,\overline{\phi_H}). \tag{4.77}$$

Also, since we analyse a Hopf bifurcation, we know that the dynamics of $A(t)$ should obey the normal form equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i\,\omega_H\,A + c_1\,A\,|A|^2 + O(|A|^4). \tag{4.78}$$

Adapting the notation from section 4.4, we apply (4.2) on a perturbation of the pulse Γ_h. Substitution of $(U,V) = \Gamma_h + \rho$ in (4.2) yields

$$\frac{\partial\rho}{\partial t} = \mathcal{L}\rho + \mathcal{R}(\Gamma_h;\rho) \tag{4.79}$$

with $\mathcal{L}$ as in (4.9), and

$$\begin{aligned}\mathcal{R}(\Gamma_h;\rho) &= \begin{pmatrix}\nu_1 F_1(U_h+(\rho)_1;\varepsilon)+\frac{\nu_2}{\varepsilon}F_2(U_h+(\rho)_1,V_h+(\rho)_2;\varepsilon)\\ G(U_h+(\rho)_1,V_h+(\rho)_2;\varepsilon)\end{pmatrix}\\ &\quad -\begin{pmatrix}\nu_1 F_1(U_h;\varepsilon)+\frac{\nu_2}{\varepsilon}F_2(U_h,V_h;\varepsilon)\\ G(U_h,V_h;\varepsilon)\end{pmatrix}+\mathcal{A}\rho \end{aligned} \tag{4.80}$$

with $\mathcal{A}$ as in (4.10). Substituting

$$\rho = A\,\phi_H + \overline{A}\,\overline{\phi_H} + \Psi(A,\overline{A},\phi_H,\overline{\phi_H}) \tag{4.81}$$

in (4.79), we obtain for its left-hand side

$$\frac{\mathrm{d}A}{\mathrm{d}t}\left(\phi_H+\frac{\partial}{\partial A}\Psi(A,\overline{A},\phi_H,\overline{\phi_H})\right)+\frac{\mathrm{d}\overline{A}}{\mathrm{d}t}\left(\overline{\phi_H}+\frac{\partial}{\partial \overline{A}}\Psi(A,\overline{A},\phi_H,\overline{\phi_H})\right) \tag{4.82}$$

and for its right-hand side

$$\begin{aligned} &i\,\omega_H\,A\,\phi_H - i\,\omega_H\,\overline{A}\,\overline{\phi_H} + \mathcal{L}\Psi(A,\overline{A},\phi_H,\overline{\phi_H})\\ &\qquad + \mathcal{R}\left(\Gamma_h; A\,\phi_H + \overline{A}\,\overline{\phi_H} + \Psi(A,\overline{A},\phi_H,\overline{\phi_H})\right). \end{aligned} \tag{4.83}$$

Since (U,V) (4.77) are real, $\Psi(0)=0$ and $D\Psi(0)=0$, see Theorem 4.7, we can expand $\Psi(A,\overline{A},\phi_H,\overline{\phi_H})$ in powers of A and $\overline{A}$ as

$$\begin{aligned}\Psi(A,\overline{A},\phi_H,\overline{\phi_H}) &= h_{20}A^2 + h_{11}A\,\overline{A} + \overline{h_{20}}\overline{A}^2\\ &\quad + h_{30}A^3 + h_{21}A^2\overline{A} + \overline{h_{21}}A\,\overline{A}^2 + \overline{h_{30}}\overline{A}^3 + O(|A|^4). \end{aligned} \tag{4.84}$$

Note that h_{11} is real since $\Psi(A,\overline{A},\phi_H,\overline{\phi_H})$ is invariant under complex conjugation, see (4.77). Moreover, from (4.80) we see that $\mathcal{R}(\Gamma_h;0)=0$ and $D_\rho\mathcal{R}(\Gamma_h;0)=0$, so we can expand $\mathcal{R}(\Gamma_h;\rho)$ in powers of ρ in a similar way as Ψ was expanded in powers of A and $\overline{A}$, yielding

$$\mathcal{R}(\Gamma_h;\rho) = \hat{\mathcal{R}}_{(2)}(\rho,\rho) + \hat{\mathcal{R}}_{(3)}(\rho,\rho,\rho) + O(|\rho|^4), \tag{4.85}$$

where $\hat{\mathcal{R}}_{(2)}(\cdot,\cdot)$ and $\hat{\mathcal{R}}_{(3)}(\cdot,\cdot,\cdot)$ are fully symmetric 2- and 3-tensors.

Using (4.81) and (4.84), we can expand $\mathcal{R}$ in powers of A and $\overline{A}$ as

$$\begin{aligned}
&\mathcal{R}\left(\Gamma_h; A\,\phi_H + \overline{A}\,\overline{\phi_H} + \Psi(A,\overline{A},\phi_H,\overline{\phi_H})\right) = \\
&\qquad \hat{\mathcal{R}}_{(2)}(\phi_H,\phi_H)\,A^2 + 2\,\hat{\mathcal{R}}_{(2)}(\phi_H,\overline{\phi_H})\,A\,\overline{A} + \hat{\mathcal{R}}_{(2)}(\overline{\phi_H},\overline{\phi_H})\,\overline{A}^2 \\
&\qquad\quad + \left[2\,\hat{\mathcal{R}}_{(2)}(\phi_H,h_{20}) + \hat{\mathcal{R}}_{(3)}(\phi_H,\phi_H,\phi_H)\right]A^3 \\
&\qquad + \left[2\,\hat{\mathcal{R}}_{(2)}(\phi_H,h_{11}) + 2\,\hat{\mathcal{R}}_{(2)}(\overline{\phi_H},h_{20}) + 3\,\hat{\mathcal{R}}_{(3)}(\phi_H,\phi_H,\overline{\phi_H})\right]A^2\overline{A} \\
&\qquad + \left[2\,\hat{\mathcal{R}}_{(2)}(\overline{\phi_H},h_{11}) + 2\,\hat{\mathcal{R}}_{(2)}(\phi_H,\overline{h_{20}}) + 3\,\hat{\mathcal{R}}_{(3)}(\phi_H,\overline{\phi_H},\overline{\phi_H})\right]A\,\overline{A}^2 \\
&\qquad\quad + \left[2\,\hat{\mathcal{R}}_{(2)}(\overline{\phi_H},\overline{h_{20}}) + \hat{\mathcal{R}}_{(3)}(\overline{\phi_H},\overline{\phi_H},\overline{\phi_H})\right]\overline{A}^3 + O(|A|^4). \qquad (4.86)
\end{aligned}$$

Using (4.84) and (4.78) in (4.82) yields

$$\begin{aligned}
&\left(i\,\omega_H\,A + c_1\,A^2\overline{A}\right)\left(\phi_H + 2h_{20}A + h_{11}\overline{A} + 3h_{30}A^2 + 2h_{21}A\,\overline{A} + \overline{h_{21}}\overline{A}^2\right) + \text{c.c.} + O(|A|^4) \\
&= i\,\omega_H\,A\,\phi_H + 2i\,\omega_H\,h_{20}A^2 + i\,\omega_H\,h_{11}A\,\overline{A} + 3i\,\omega_H\,h_{30}A^3 + \left[2i\,\omega_H\,h_{21} + c_1\phi_H\right]A^2\overline{A} \\
&\qquad + i\,\omega_H\,\overline{h_{21}}A\,\overline{A}^2 + \text{c.c.} + O(|A|^4) \\
&= i\,\omega_H\,A\,\phi_H + 2i\,\omega_H\,h_{20}A^2 + 3i\,\omega_H\,h_{30}A^3 + \left[i\,\omega_H\,h_{21} + c_1\phi_H\right]A^2\overline{A} + \text{c.c.} + O(|A|^4). \qquad (4.87)
\end{aligned}$$

Similarly, using (4.84) and (4.86) in (4.83) yields

$$\begin{aligned}
&i\,\omega_H\,A\,\phi_H + \mathcal{L}\left(h_{20}A^2 + \frac{1}{2}h_{11}A\,\overline{A} + h_{30}A^3 + h_{21}A^2\overline{A}\right) + \hat{\mathcal{R}}_{(2)}(\phi_H,\phi_H)\,A^2 \\
&\qquad + \hat{\mathcal{R}}_{(2)}(\phi_H,\overline{\phi_H})\,A\,\overline{A} + \left[2\,\hat{\mathcal{R}}_{(2)}(\phi_H,h_{20}) + \hat{\mathcal{R}}_{(3)}(\phi_H,\phi_H,\phi_H)\right]A^3 \\
&\qquad + \left[2\,\hat{\mathcal{R}}_{(2)}(\phi_H,h_{11}) + 2\,\hat{\mathcal{R}}_{(2)}(\overline{\phi_H},h_{20}) + 3\,\hat{\mathcal{R}}_{(3)}(\phi_H,\phi_H,\overline{\phi_H})\right]A^2\overline{A} \\
&\qquad\quad + \text{c.c} + O(|A|^4) \\
&= i\,\omega_H\,A\,\phi_H + \left[\mathcal{L}h_{20} + \hat{\mathcal{R}}_{(2)}(\phi_H,\phi_H)\right]A^2 + \left[\frac{1}{2}\mathcal{L}h_{11} + \hat{\mathcal{R}}_{(2)}(\phi_H,\overline{\phi_H})\right]A\,\overline{A} \\
&\qquad + \left[\mathcal{L}h_{30} + 2\,\hat{\mathcal{R}}_{(2)}(\phi_H,h_{20}) + \hat{\mathcal{R}}_{(3)}(\phi_H,\phi_H,\phi_H)\right]A^3 \\
&\qquad + \left[\mathcal{L}h_{21} + 2\,\hat{\mathcal{R}}_{(2)}(\phi_H,h_{11}) + 2\,\hat{\mathcal{R}}_{(2)}(\overline{\phi_H},h_{20}) + 3\,\hat{\mathcal{R}}_{(3)}(\phi_H,\phi_H,\overline{\phi_H})\right]A^2\overline{A} \\
&\qquad\quad + \text{c.c} + O(|A|^4). \qquad (4.88)
\end{aligned}$$

Since (4.87) and (4.88) are the expansions of the left-hand side resp. the right-hand side of (4.79) in powers of A and $\overline{A}$, we can compare their expansion coefficients. The

first order terms (for A and $\overline{A}$) coincide; for the second order terms (in A^2, $A\overline{A}$ and $\overline{A}^2$), equating the coefficients yields

$$(\mathcal{L} - 2\,i\,\omega_H)\,h_{20} = -\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H), \tag{4.89}$$

$$\mathcal{L}\,h_{11} = -2\hat{\mathcal{R}}_{(2)}(\phi_H, \overline{\phi_H}). \tag{4.90}$$

The central spectrum of $\mathcal{L}$ on the foliation leaf under consideration is $\{\pm\, i\,\omega_H\}$, so $2\,i\,\omega_H$ nor 0 is an eigenvalue of $\mathcal{L}$. Therefore, the operators $\mathcal{L}$ and $\mathcal{L} - 2\,i\,\omega_H$ are invertible, yielding unique solutions for h_{20} and h_{11}.

To obtain an equation for c_1, the only third order expansion coefficient we need to consider is that of $A^2\overline{A}$. Equating the respective coefficients in (4.87) resp. (4.88) yields

$$(\mathcal{L} - i\,\omega_H)\,h_{21} = c_1\phi_H - 2\,\hat{\mathcal{R}}_{(2)}(\phi_H, h_{11}) - 2\,\hat{\mathcal{R}}_{(2)}(\overline{\phi_H}, h_{20}) - 3\,\hat{\mathcal{R}}_{(3)}(\phi_H, \phi_H, \overline{\phi_H}). \tag{4.91}$$

This allows us to formulate the following Lemma:

Lemma 4.15. *Let ϕ_H^* be the unique bounded solution for which $\mathcal{L}^*\phi_H^* = i\,\omega_H\,\phi_H^*$. Then c_1 (4.78) is given by*

$$c_1 = \frac{1}{\langle \phi_H, \overline{\phi_H^*} \rangle} \langle 2\,\hat{\mathcal{R}}_{(2)}(\phi_H, h_{11}) + 2\,\hat{\mathcal{R}}_{(2)}(\overline{\phi_H}, h_{20}) + 3\,\hat{\mathcal{R}}_{(3)}(\phi_H, \phi_H, \overline{\phi_H}), \overline{\phi_H^*} \rangle. \tag{4.92}$$

Proof. By the Fredholm alternative, (4.91) has a unique solution for h_{21} if and only if the right-hand side is orthogonal to the kernel of the adjoint operator $(\mathcal{L} - i\,\omega_H)^*$. The spectra of $\mathcal{L}$ and $\mathcal{L}^*$ are each others complex conjugates. Since $\mathcal{L}$ is real, its spectrum is invariant under complex conjugation. Therefore, if λ is an eigenvalue of $\mathcal{L}$, then λ is also an eigenvalue of $\mathcal{L}^*$. That means in particular that the central spectrum of $\mathcal{L}^*$ coincides with the central spectrum of $\mathcal{L}$, which is given by the pair $\pm\, i\,\omega_H$. Let the associated adjoint eigenfunctions be denoted as ϕ_H^*, $\overline{\phi_H^*}$. Since the (one-dimensional) kernel of $(\mathcal{L} - i\,\omega_H)^* = (\mathcal{L}^* + i\,\omega_H)$ is spanned by $\overline{\phi_H^*}$, the solvability condition obtained from the Fredholm alternative yields (4.92). □

While this expression for c_1 is a lot less involved than the expression derived in Lemma 4.11, it cannot be calculated directly using the eigenfunction expressions stated in Theorem 4.3. The expressions for h_{20} (4.89) and h_{11} (4.90) are still implicit; moreover, we have not yet analysed the adjoint operator $\mathcal{L}^*$. Both are the subject of the next section.

4.5.2 The coefficient c_1 (4.92)

First, we study the adjoint operator $\mathcal{L}^*$. Combining the definition of $\mathcal{L}$ (4.9) with that of the scaling matrix S used in the definition of the inner product (4.22), we see that $\mathcal{L}$ can be written as

$$\mathcal{L} = S^{-2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \mathcal{A}. \tag{4.93}$$

Now,

$$\begin{aligned}
\langle \phi, \mathcal{L}^*\psi\rangle = \langle \mathcal{L}\phi, \psi\rangle &= \int_{\mathbb{R}} (\mathcal{L}\phi)^T S\overline{\psi}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \left(S^{-2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi - \mathcal{A}\phi\right)^T S\overline{\psi}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \left(S^{-2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi\right)^T S\overline{\psi} - (\mathcal{A}\phi)^T S\overline{\psi}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi\right)^T S^{-2}S\overline{\psi} - \phi^T\mathcal{A}^T S\overline{\psi}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \phi^T S^{-2}S\,\overline{\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\psi} - \phi^T S\, S^{-1}\mathcal{A}^T S\overline{\psi}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \phi^T S\left[S^{-2}\overline{\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\psi} - S^{-1}\mathcal{A}^T S\overline{\psi}\right]\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \phi^T S\overline{\left[S^{-2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - S^{-1}\mathcal{A}^T S\right]\psi}\,\mathrm{d}\xi
\end{aligned}$$

so

$$\mathcal{L}^* = S^{-2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - S^{-1}\mathcal{A}^T S. \tag{4.94}$$

Using (4.10),

$$S^{-1}\mathcal{A}^T S = \left.\begin{pmatrix} \mu - \nu_1\frac{\mathrm{d}F_1}{\mathrm{d}U} - \frac{\nu_2}{\varepsilon}\frac{\partial F_2}{\partial U} & -\frac{1}{\varepsilon}\frac{\partial G}{\partial U} \\ -\nu_2\frac{\partial F_2}{\partial V} & 1 - \frac{\partial G}{\partial V}\end{pmatrix}\right|_{(U,V)=(U_h(\xi),V_h(\xi))}. \tag{4.95}$$

We see that $\mathcal{L}^*$ has the same scale separated structure as $\mathcal{L}$, with only the roles of $\nu_2\frac{\partial F_2}{\partial V}$ and $\frac{\partial G}{\partial U}$ reversed. Therefore, we can treat $\mathcal{L}^*$ in a similar way as $\mathcal{L}$, see subsection 4.2.2. Since the diagonal entries of $\mathcal{A}$ are the same as those of $S^{-1}\mathcal{A}^T S$, the leading order slow and fast linear operators associated to $\mathcal{L}^*$ coincide with those of

$\mathcal{L}$, i.e. $(\mathcal{L}^*)_s(x) = \mathcal{L}_s(x)$ and $(\mathcal{L}^*)_f(\xi) = \mathcal{L}_f(\xi)$, as given in (4.13) resp. (4.12). The leading order fast nonhomogeneous Sturm-Liouville problem for $\mathcal{L}^*$ is

$$\left(\mathcal{L}_f^* - \lambda\right) v = -\nu_2 \frac{\partial F_2}{\partial V}(u_*, v_{f,h}(\xi; u_*)), \quad \xi \in \mathbb{R}. \tag{4.96}$$

compare (4.14). Since F_2 and G obey equivalent conditions, see Assumptions 4.1, (A3) and (A4), the theory developed in chapter 3, section 3.3 can be directly applied to $\mathcal{L}^*$, yielding Hopf eigenfunctions ϕ_H^*, $\overline{\phi_H^*}$ with the scale separated structure as described in Theorem 4.3. Note that outside I_f, the leading order behaviour of ϕ_H^* coincides with that of ϕ_H, since $(\mathcal{L}^*)_s(x) = \mathcal{L}_s(x)$.

Solving (4.89) and (4.90) for h_{20} resp. h_{11} is a lot more cumbersome. In general, one would approach the problem as follows. First, consider (4.89). Since the operator $\mathcal{L} - 2\,i\,\omega_H$ is invertible, we know there is no nontrivial bounded solution to the homogeneous problem

$$(\mathcal{L} - 2\,i\,\omega_H)\,\phi = 0. \tag{4.97}$$

The two-component linear second order differential equation (4.97) can be rewritten as a four-dimensional first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{\phi} = \mathcal{B}\,\hat{\phi}, \tag{4.98}$$

the solution space of which is spanned by four linearly independent solutions $\left\{\hat{\phi}_i\right\}$, $i = 1, \ldots, 4$. To solve (4.89), which can be written in first order form as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{h}_{20} - \mathcal{B}\,\hat{h}_{20} = \begin{pmatrix} 0 \\ -\varepsilon^2 \left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_1 \\ 0 \\ -\left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_2 \end{pmatrix} \tag{4.99}$$

with

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \varepsilon^2\mathcal{A}_{11} + 2\,i\,\omega_H & 0 & \varepsilon^2\mathcal{A}_{12} & 0 \\ 0 & 0 & 0 & 1 \\ \mathcal{A}_{21} & 0 & \mathcal{A}_{22} + 2\,i\,\omega_H & 0 \end{pmatrix}, \tag{4.100}$$

we use the method of variation of constants and write

$$h_{20} = \sum_{i=1}^{4} c_i(\xi)\,\hat{\phi}_i, \tag{4.101}$$

such that (4.99) is transformed into

$$\sum_{i=1}^{4} \frac{\mathrm{d}c_i}{\mathrm{d}\xi}\,\hat{\phi}_i = \begin{pmatrix} 0 \\ -\varepsilon^2 \left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_1 \\ 0 \\ -\left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_2 \end{pmatrix} \tag{4.102}$$

or equivalently

$$\hat{\Phi}\,\frac{\mathrm{d}c}{\mathrm{d}\xi} = \begin{pmatrix} 0 \\ -\varepsilon^2 \left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_1 \\ 0 \\ -\left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_2 \end{pmatrix}, \tag{4.103}$$

where $c = (c_1, c_2, c_3, c_4)^T$ and $\hat{\Phi}$ is the matrix whose columns are given by the eigenvectors $\hat{\phi}_i$, i.e.

$$\hat{\Phi} = \left\{\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3, \hat{\phi}_4\right\}. \tag{4.104}$$

Since the trace of $\mathcal{B}$ (4.100) vanishes, the determinant of $\hat{\Phi}$ is constant; moreover, since $\mathcal{L} - 2\,i\,\omega_H$ is invertible, we know that $\det \hat{\Phi} \neq 0$. Therefore, (4.103) can be solved by inverting $\hat{\Phi}$, yielding

$$\frac{\mathrm{d}c}{\mathrm{d}\xi} = \frac{1}{\det \hat{\Phi}}\,\mathrm{adj}\,\hat{\Phi} \begin{pmatrix} 0 \\ -\varepsilon^2 \left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_1 \\ 0 \\ -\left(\hat{\mathcal{R}}_{(2)}(\phi_H, \phi_H)\right)_2 \end{pmatrix}, \tag{4.105}$$

where $\mathrm{adj}\,\hat{\Phi}$ is the adjugate matrix of $\hat{\Phi}$. Since $h_{20} = \hat{\Phi}\,c$ (4.101), we can integrate (4.105) to obtain for h_{20}:

$$h_{20}(\xi) = \hat{\Phi}(\xi) \int^{\xi} \frac{1}{\det \hat{\Phi}}\,\mathrm{adj}\,\hat{\Phi}(\xi') \begin{pmatrix} 0 \\ -\varepsilon^2 \left(\hat{\mathcal{R}}_{(2)}(\phi_H(\xi'), \phi_H(\xi'))\right)_1 \\ 0 \\ -\left(\hat{\mathcal{R}}_{(2)}(\phi_H(\xi'), \phi_H(\xi'))\right)_2 \end{pmatrix} \mathrm{d}\xi'. \tag{4.106}$$

Solving (4.90) for h_{11} can be done analogously. Solving the four-dimensional first order differential equation associated to $\mathcal{L}\,\hat{h}_{11} = -2\hat{\mathcal{R}}_{(2)}(\phi_H, \overline{\phi_H})$, the same variation of constants approach ultimately yields

$$\hat{h}_{11}(\xi) = \hat{\Phi}_0(\xi) \int^{\xi} \frac{1}{\det \hat{\Phi}_0}\,\mathrm{adj}\,\hat{\Phi}_0(\xi') \begin{pmatrix} 0 \\ -2\varepsilon^2 \left(\hat{\mathcal{R}}_{(2)}(\phi_H(\xi'), \overline{\phi_H(\xi')})\right)_1 \\ 0 \\ -2\left(\hat{\mathcal{R}}_{(2)}(\phi_H(\xi'), \overline{\phi_H(\xi')})\right)_2 \end{pmatrix} \mathrm{d}\xi', \tag{4.107}$$

where $\hat{\Phi}_0$ is the matrix whose columns are given by the four independent solutions to $\mathcal{L}\phi_0 = 0$, when written in its four-dimensional first order form – this would be equivalent to (4.98), with associated $\mathcal{B}_0$ as in (4.99), with $\omega_H = 0$. To obtain a result for the operator $\mathcal{L}$ on the reduced space $\mathcal{X}' = \mathcal{X} - \text{span}\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h\right\}$, we project $\hat{h}_{11}$ onto the orthogonal complement of span $\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}\Gamma_h\right\}$ (c.f. (4.29)), yielding

$$h_{11} = (\mathbb{I} - \Pi_0)\,\hat{h}_{11}. \tag{4.108}$$

The expressions for h_{20} (4.106) and h_{11} (4.108), together with an encompassing analysis of the adjoint operator $\mathcal{L}$ (4.94), can now be combined with the result of Lemma 4.15 to obtain an explicit expression for c_1 (4.92).

Comparing the approach advocated in this section with the more direct approach of section 4.4, we see that the (relatively) short expressions culminating in Lemma 4.15 come with a cost of having to analyse three additional equations: one eigenvalue problem for the adjoint operator $\mathcal{L}^*\phi_H^* = i\,\omega_H\phi_H^*$ and two inverse problems (4.89) and (4.90). Depending on model under consideration, i.e. for a specific choice of $F_{1,2}$ and G, either approach might be preferable over the other. In the next section, we apply the theory developed above to such a specific model, the slowly nonlinear Gierer-Meinhardt equation.

4.6 Application: the slowly nonlinear Gierer-Meinhardt equation

In chapter 2, the existence and stability of pulse solutions as considered in section 4.2 was established for the slowly nonlinear Gierer-Meinhardt equation (2.7), restated here:

$$\begin{cases} U_t &= & U_{xx} &- & \left(\mu\, U - \nu_1 U^d\right) &+ & \frac{\nu_2}{\varepsilon}\, V^2 \\ V_t &= & \varepsilon^2 V_{xx} &- & V &+ & \frac{V^2}{U} \end{cases}. \tag{4.109}$$

The original Gierer-Meinhardt equation, a canonical model for morphogenesis which is studied extensively in the context of pattern formation [6, 12, 22, 26, 48, 51], can be recovered from (4.109) by setting $\nu_1 = 0$. The system (4.109) is of the form (4.1) with

$$F_1^{\mathrm{nGM}}(U;\varepsilon) = U^d,\ d > 1, \quad F_2^{\mathrm{GM}}(U,V;\varepsilon) = V^2, \quad G^{\mathrm{GM}}(U,V;\varepsilon) = \frac{V^2}{U}. \tag{4.110}$$

The nonlinearities F_2 and G are chosen according to the 'classical' Gierer-Meinhardt model, and are therefore denoted as F_2^{GM} and G^{GM}. The 'slow' nonlinearty F_1 is

absent in the Gierer-Meinhardt model, but was introduced in chapter 2 as U^d, to study the influence of such a slow nonlinearity on the pulse construction and stability. The slowly nonlinear term F_1 is therefore denoted as F_1^{nGM}.

4.6.1 Analytical preliminaries

This section summarises the analysis on the slowly nonlinear Gierer-Meinhardt model (4.109), as derived in chapter 2, sections 2.2 and 2.3.

It can easily be verified that the above choice for $F_{1,2}$ and G (4.110) satisfies Assumptions 4.1 (A1 - A4). The reduced fast system (4.4) is realised as

$$v_{f,\xi\xi} = v - \frac{1}{u_0} v^2, \tag{4.111}$$

which has a homoclinic solution

$$v_{f,h}^{\mathrm{GM}}(\xi; u_0) = \frac{3u_0}{2} \operatorname{sech}^2 \frac{1}{2}\xi, \tag{4.112}$$

satisfying Assumptions 4.1, (A5). Using this homoclinic solution, $D_p(u_0)$ (4.6) can be calculated as

$$D_p^{\mathrm{GM}}(u_0) = \int_{\infty}^{\infty} \left(v_{f,h}^{\mathrm{nGM}}(\xi; u_0)\right)^2 \mathrm{d}\xi = 6u_0^2, \tag{4.113}$$

which means that Assumptions 4.1, (A6) is satisfied once the factor 6 is scaled out by rescaling $\nu_2 \to \hat{\nu}_2 = 6\nu_2$. The choice of F_1 realises the reduced slow system (4.5) as

$$u_{xx} = \mu\, u - \nu_1 u^d, \tag{4.114}$$

which also has an orbit homoclinic to the origin. Therefore, the slow unstable and stable manifolds of the origin $\mathcal{W}_s^{\mathrm{u/s}}((0,0;\varepsilon)$ coincide and are both described by the same function

$$u_s^{\mathrm{nGM}}(x) = \left(\frac{\mu(d+1)}{2\nu_1} \operatorname{sech}^2 \frac{1}{2}(d-1)\sqrt{\mu}\, x\right)^{\frac{1}{d-1}}. \tag{4.115}$$

The pulse existence condition (4.7) becomes

$$\frac{2\nu_1}{d+1} u^{d-1} = \mu - \frac{3}{2}\hat{\nu}_2 u^2, \tag{4.116}$$

which always has precisely one positive solution for $u = u_*$, so Theorem 4.2 is valid for $K = 1$. From Theorem 4.2, it follows that there exists a pulse solution $\Gamma_h^{\mathrm{nGM}}(\xi)$,

which is to leading order given by

$$\Gamma_h^{\mathrm{nGM}}(\xi) = \begin{cases} \begin{pmatrix} u_s^{\mathrm{nGM}}(\varepsilon\xi - x_*) \\ 0 \end{pmatrix} & \text{for} \quad \xi < \varepsilon^{-\frac{1}{4}}, \\ \begin{pmatrix} u_* \\ v_{f,h}^{\mathrm{GM}}(\xi; u_*) \end{pmatrix} & \text{for} \quad \xi \in I_f, \\ \begin{pmatrix} u_s^{\mathrm{nGM}}(\varepsilon\xi + x_*) \\ 0 \end{pmatrix} & \text{for} \quad \xi > \varepsilon^{-\frac{1}{4}}. \end{cases} \tag{4.117}$$

The stability analysis of the pulse Γ_h^{nGM} yields the fast linear operator (cf. (4.12))

$$\mathcal{L}_f^{\mathrm{GM}}(\xi) = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \left[1 - 3\,\mathrm{sech}^2\,\frac{1}{2}\xi\right], \quad \xi \in \mathbb{R}, \tag{4.118}$$

which has eigenvalues $\lambda_{f,0}^{\mathrm{GM}} = \frac{5}{4}$, $\lambda_{f,1}^{\mathrm{GM}} = 0$ and $\lambda_{f,2}^{\mathrm{GM}} = -\frac{3}{4}$. The slow linear operator (4.13) is realised as

$$\mathcal{L}_s^{\mathrm{nGM}}(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mu\left[1 - \frac{d(d+1)}{2}\,\mathrm{sech}^2\,\frac{1}{2}(d-1)\,\sqrt{\mu}\,x\right], \quad x \geq 0. \tag{4.119}$$

The solutions of its eigenvalue problem $(\mathcal{L}_s^{\mathrm{nGM}} - \lambda)u = 0$ can be determined explicitly using associated Legendre functions, see chapter 2, section 2.3. For the nonhomogeneous fast problem (4.14), realised as

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}v - \left[1 + \lambda - 3\,\mathrm{sech}^2\,\frac{1}{2}\xi\right]v = \frac{9}{4}\mathrm{sech}^4\,\frac{1}{2}\xi, \tag{4.120}$$

the unique bounded solution v_{in} can also be explicitly represented using associated Legendre functions. These explicit expressions manifest themselves in the leading order eigenfunction behaviour described in Theorem 4.3.

4.6.2 Hopf bifurcations

As in the general case chapter 3, the eigenvalues for the pulse (4.117) can be determined using Evans function techniques. In chapter 2, an explicit leading order expression for the Evans function was found in terms of the leading order eigenfunction expressions from Theorem 4.3, see Theorem 2.12. This leading order Evans function can be directly numerically evaluated for different parameter values. In Figure

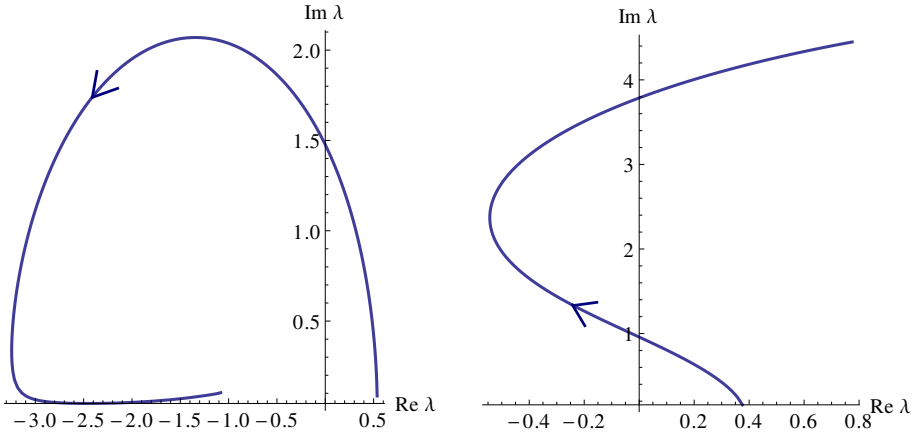

Figure 4.2: The pulse eigenvalues to leading order in ε as a function of increasing μ, indicated by the arrow. Here, $\nu_1 = 2$ and $\nu_2 = \frac{1}{2}$ ($\hat{\nu}_2 = 3$) are fixed. For $d = 2$ (left figure), the pulse undergoes one stabilising Hopf bifurcation for $\mu = \mu_H = 1.47986\ldots$ at $\lambda = i\,\omega_H = 1.47638\ldots i$. For $d = 5$ (right figure), a second, destabilising Hopf bifurcation takes place for $\mu = \mu_{H,2} = 5.134\ldots$ at $\lambda = i\,\omega_{H,1} = 3.78646\ldots i$, while the first Hopf bifurcation is at $\mu = \mu_{H,1} = 0.4173\ldots$ with $\lambda = i\,\omega_{H,1} = 0.958684\ldots i$ for these parameter values.

4.2, the pulse eigenvalues are plotted in the complex plane for fixed $\nu_{1,2}$ and d, while varying μ. Here, the influence of the slow nonlinear term $F_1^{\mathrm{nGM}}(U;\varepsilon) = U^d$ (4.110) can be clearly seen. For $d = 2$, the eigenvalue orbit crosses the imaginary axis for $\mu = \mu_H = 1.47986\ldots$ at $\lambda = i\,\omega_H = 1.47638\ldots i$, and it becomes clear that the pulse is stable for all $\mu > \mu_H$ (see chapter 2, Theorem 2.18). However, for $d = 5$, the eigenvalue orbit exhibits a different behaviour. After a first stabilising Hopf bifurcation for $\mu = \mu_{H,1} = 0.4173\ldots$ at $\lambda = i\,\omega_{H,1} = 0.958684\ldots i$, the eigenvalue orbit turns around and undergoes a second, destabilising Hopf bifurcation for $\mu = \mu_{H,2} = 5.134\ldots$ at $\lambda = i\,\omega_{H,1} = 3.78646\ldots i$.

This turning behaviour is general for $d > 3$, see chapter 2, Theorem 2.19. That means that for all $d > 1$, there is an neighbourhood of $(\nu_1, \nu_2) = (2, \frac{1}{2})$ in parameter space such that there is a (possibly bounded) interval in μ for which the pulse Γ_h^{nGM} is stable. At the boundary of this interval, the pulse destabilises through a Hopf bifurcation. Since our parameter space $\{(\mu, \nu_1, \nu_2, d)\}$ is four-dimensional, we can determine the (boundaries of the) stability region by intersecting it with two-dimensional hy-

perplanes, i.e. by fixing two parameters. In Figure 4.3, the boundary of this stability region is determined for different values of d and ν_1, with μ and ν_2 as free parameters. The two Hopf bifurcation values for $d = 5$, $\nu_1 = 2$, $\nu_2 = \frac{1}{2}$ are indicated on the blue curve in both figures. The Hopf bifurcations merge into a singular Hopf bifurcation where the bifurcation curves fold.

In Figure 4.4, left, the Hopf frequencies for $d = 5$ and $\nu_1 = 2$ are plotted as a function of the parameter ν_2. The merging of Hopf bifurcations can again be observed. For these Hopf eigenvalues, the associated first Lyapunov coefficients ℓ_1 (4.58) were calculated according to Corollary 4.12. It can be seen that the Hopf bifurcations of the lower branch have a positive –even large– first Lyapunov coefficient, and are therefore subcritical (Corollary 4.12). However, for the upper branch of Hopf bifurcations, it is seen that the sign of the first Lyapunov coefficients can change. Note that this upper branch corresponds with the destabilising Hopf bifurcation $\lambda = i\,\omega_{H,2}$ which is present for all $d > 3$, see Figure 4.2 (right). A collection of such curves of first Lyapunov coefficients is shown in Figure 4.5, based on the associated Hopf curves from Figure 4.3. It is clear that this crossing from sub- to supercriticality is a general phenomenon, and is therefore not restricted to the specific choice of parameters used to produce these Figures.

The direct numerical evaluation of the first Lyapunov coefficient, made possible by the results from Lemma 4.14 and Lemma 4.15, enables us to draw conclusions about the sub- or supercriticality of the Hopf bifurcations of pulses in the slowly nonlinear Gierer-Meinhard model (4.109). Based on the curves shown in Figure 4.5, we can take a well-chosen point in parameter space, e.g. $(\nu_1, \nu_2, d) = (2, \frac{5}{2}, 5)$, such that one of the two Hopf bifurcations for this parameter triplet is subcritical, and the other supercritical. By continuous dependence on parameters, we can then state the following Theorem:

Theorem 4.16. *Let $\varepsilon > 0$ be sufficiently small. There exists an open nonempty neighbourhood $\mathcal{V}$ in (μ, ν_1, ν_2)-parameter space such that the following holds. For any $(\nu_1, \nu_2, d) \in \mathcal{V}$, there are two Hopf bifurcation values $\mu_{H,1}(\nu_1, \nu_2, d)$ and $\mu_{H,2}(\nu_1, \nu_2, d)$ with $\mu_{H,1} < \mu_{H,2}$ for which the associated pulse eigenvalues are given by $\lambda_{H,1} = i\,\omega_{H,1}$ resp. $\lambda_{H,2} = i\,\omega_{H,2}$ with $\omega_{H,1}, \omega_{H,2} \in \mathbb{R}$. The Hopf bifurcation $\lambda_{H,1} = i\,\omega_{H,1}$ is subcritical; the Hopf bifurcation $\lambda_{H,1} = i\,\omega_{H,1}$ is supercritical.*

With these results in mind, it is quite straightforward to obtain a result which was suggested, but not confirmed, in previous literature on the 'canonical' Gierer-Meinhardt system, i.e. (4.109) with $\nu_1 = 0$. In [6], the existence and stability of

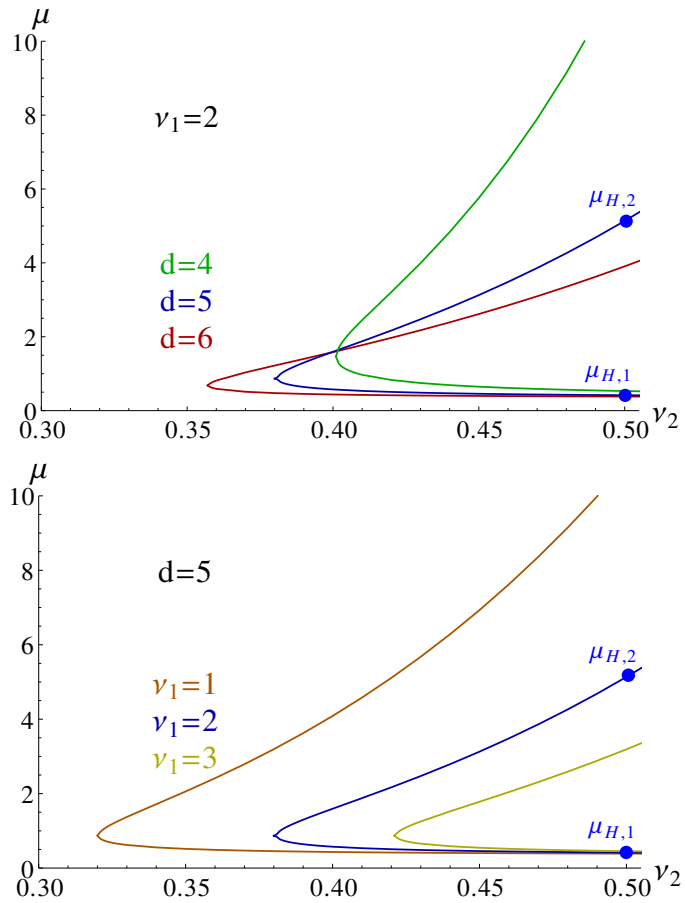

Figure 4.3: Curves of Hopf bifurcation parameter values in the (ν_2, μ)-plane. Left, $\nu_1 = 2$; bifurcation curves are plotted for $d = 4, 5, 6$. Right, $d = 5$; bifurcation curves are plotted for $\nu_1 = 1, 2, 3$. In both figures, the bifurcation values $\mu_{H,1} = 0.4173\ldots$ and $\mu_{H,2} = 5.134\ldots$ for $d = 5, \nu_2 = 2$ are indicated. These curves form the boundary of the region in parameter space for which the pulse Γ_h^{nGM} is stable.

pulse solutions in Gierer-Meinhardt type systems was established using ideas similar to those used in chapters 2 and 3. There, it was shown that for $\mu = \mu_H = 0.36\ldots$, the pulse undergoes a Hopf bifurcation. Numerical simulations [11, 57] suggested that this Hopf bifurcation is subcritical. This observation is confirmed by direct nu-

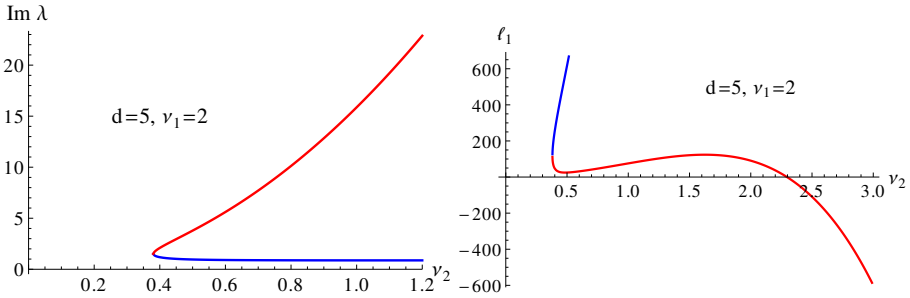

Figure 4.4: In the left figure, the Hopf frequencies are plotted as a function of ν_2, for $d = 5$ and $\nu_1 = 2$. The lower branch, representing $\omega_{H,1}$, is indicated in blue; the upper branch, representing $\omega_{H,2}$, is indicated in red. In the right figure, the associated first Lyapunov coefficients ℓ_1 are plotted, using corresponding colors. It can be seen that the first Lyapunov coefficient corresponding to $\omega_{H,2}$ changes sign at $\nu_2 = \nu_2^* = 2.2955\ldots$; there, the nature of this upper branch Hopf bifurcation changes from subcritical to supercritical.

merical evaluation of the associated first Lyapunov coefficient, which has the value $\ell_1 = 2900.91 > 0$. As a consequence, the following Corollary is a direct result from numerical evaluations equivalent to those underlying Theorem 4.16:

Corollary 4.17. *Let $\varepsilon > 0$ be sufficiently small. The Hopf bifurcation associated to the classical Gierer-Meinhardt pulse is subcritical.*

4.7 Discussion

The research presented in this chapter was inspired by the observation of stable oscillating pulses in the slowly nonlinear Gierer-Meinhardt model, see chapter 2, section 2.5. There, it was shown that numerical simulations of the full PDE system suggested the existence of breathing pulses (possibly with a dynamically modulated amplitude) near parameter values for which the stationary pulse undergoes a Hopf bifurcation. The hypothesis that such a Hopf bifurcation could be the 'birthplace' of these breathing pulses is confirmed in the current chapter. A consequence of the supercriticality of the Hopf bifurcation, established in Theorem 4.16, is that stable periodically modulated pulse amplitudes (i.e. breathing pulses) can and do indeed exist.

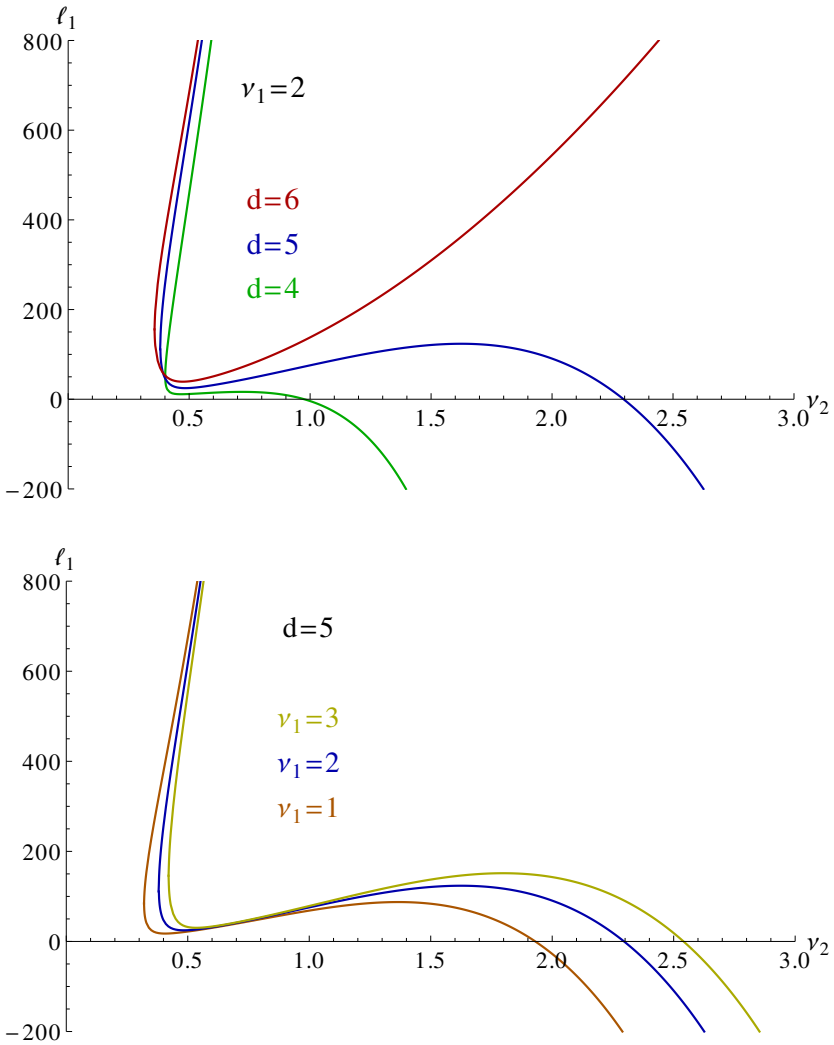

Figure 4.5: An overview of several Lyapunov curves, corresponding to the Hopf bifurcation curves shown in Figure 4.3. The color coding coincides. It can be seen that the transition from sub- to supercriticality is a general phenomenon. However, for $d = 6$, this seems not to occur, at least for values up to $\nu_2 = 6$ (full range not shown).

However, this is not the end of the story. The centre manifold associated to the Hopf bifurcation has only been expanded up to third order. A fifth order expansion, near the generalised Bautin point where the Hopf bifurcation transgresses from sub- to supercriticality (i.e. where the first Lyapunov coefficient ℓ_1 vanishes), can in principle be done. This would entail performing an analysis analogous to that presented in section 4.4 or 4.5, to the extended fifth order normal form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i\,\omega_H\,A + c_1\,A\,|A|^2 + c_2\,A\,|A|^4 + \mathcal{O}(|A|^6), \tag{4.121}$$

compare (4.78) / Lemma 4.11. This way, the first steps towards a more encompassing description of the dynamically modulated pulse amplitude near Hopf bifurcations can be taken. Numerical results from chapter 2, section 2.5 suggest that this amplitude can be quasiperiodically or even chaotically modulated.

It is worthwhile to note that the procedure to obtain explicit expressions for the Hopf normal form, as presented in this chapter, is not restricted to the stationary pulse solution, which was analysed in chapters 2 and 3. The procedure is in principle valid for (multi)pulses and fronts in singularly perturbed reaction-diffusion systems: as long as one is able to obtain an explicit expression for the stationary pattern (and, more importantly, for its eigenfunctions), the techniques presented in this chapter can be used to obtain an explicit expression for the normal form expansion coefficients, which can be directly numerically evaluated, allowing one to gain more insight in the dynamical properties of the pattern under consideration.