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L.E.J. Brouwer's 'Unreliability of the logical principles'. A new translation, with an introduction

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Dedicated to the memory of Georg Kreisel, 1923–2015

Abstract

We present a new English translation of L.E.J. Brouwer's paper 'De onbetrouwbaarheid der logische principes' (The unreliability of the logical principles) of 1908, together with a philosophical and historical introduction. In this paper Brouwer for the first time objected to the idea that the Principle of the Excluded Middle is valid. We discuss the circumstances under which the manuscript was submitted and accepted, Brouwer's ideas on the principle of the excluded middle, its consistency and partial validity, and his argument against the possibility of absolutely undecidable propositions. We note that principled objections to the general excluded middle similar to Brouwer's had been advanced in print by Jules Molk two years before. Finally, we discuss the influence on George Griss' negationless mathematics.

keywords: Luitzen Egbertus Jan Brouwer, George Griss, intuitionism, Jules Molk, principle of the excluded middle

1 Rationale for this translation

In his seminal paper 'The unreliability of the logical principles' (1908b), Brouwer draws for the first time the revisionistic consequences of the general view on logic that he had presented in his dissertation (Brouwer, 1907, pp. 125–132), by rejecting the principle of excluded middle. The paper appeared in Dutch; Brouwer's first published remarks in more widely read languages on the unreliability of the principle

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of excluded middle occur in Brouwer (1913, p. 92n2, p. 96n1) in English, and in Brouwer (1914, p. 80) in German.¹

An English translation, by Heyting and Gibson, appeared in 1975, in volume 1 of Brouwer's *Collected Works* (Brouwer, 1975, pp. 107–111). The project of attempting a novel translation seemed to us a worthwhile one against the following background. Brouwer's original text must have struck already a Dutch reader in 1908 as a difficult and unusual one, whose author nevertheless retains a full mastery of his sentences. In order to preserve for a reader of English at least part of what the original text thus conveys to a reader of Dutch, we believe that one must, to put it in Schleiermacher's memorable terms, move the reader towards the author, instead of moving the author towards the reader. We have therefore aimed to translate as literally as possible; to translate Dutch words by English cognates, and to preserve relations between Dutch cognates among their English translations, wherever appropriate; to preserve the Germanic structure of the original Dutch to the extent that English, likewise a Germanic language, allows for it; and to preserve Brouwer's idiomatic idiosyncracies.

2 Brouwer's submission of his manuscript

From its novel treatment of the principle of excluded middle, it is clear that Brouwer drafted his manuscript after the thesis (Brouwer, 1907), which was defended on February 19, 1907; and towards the end of that year, he submitted it to the *Tijdschrift voor Wijsbegeerte*.² In a letter to one of its editors, the philosophically inclined man of letters Johannes Diderik Bierens de Haan, of December 7, Brouwer had promised to explain matters further in subsequent papers that would be longer and better understandable to non-mathematicians. That letter has not survived, but this element of it is taken up in another letter that did. On January 3, 1908,³ another editor, the physicist P. Kohnstamm, informed Brouwer that the paper had been accepted that day, in spite of most editors confessing to have understood very little of it. Van Dalen (1999, p. 108) suspects that the editorial board had also hesitated to publish Brouwer's paper because he was not a professional philosopher. Moreover, the Dutch professional philosophers had not appreciated the attempt by the student Brouwer, two years earlier, to found a philosophy journal together with Mannoury. To complicate matters further, the *Tijdschrift voor Wijsbegeerte* was the

¹The Brouwer archive contains an as yet unpublished German translation of the paper at hand, most likely prepared by Karl Menger in 1925 or 1926, who in those two years was Brouwer's assistant. In 'My memories of L.E.J. Brouwer', he writes in footnote 11: 'Brouwer's very moderate assignments to me were essentially confined to translations of some older writings of his on intuitionism from Dutch into German.' (Menger, 1979, p. 252).

²From 1933, that journal appeared as *Algemeen Nederlands Tijdschrift voor Wijsbegeerte en Psychologie*, and from 1970 onwards as *Algemeen Nederlands Tijdschrift voor Wijsbegeerte*. It must not be confused with the *Tijdschrift voor Filosofie* that has been published in Leuven since 1939.

³Van Dalen (2011, Online Supplement, pp. 225–226).

very journal that was founded in reaction to their initiative. On the other hand, it seems that any storm there might have been had blown over soon, as the title page of the first volume of the *Tijdschrift voor Wijsbegeerte* lists Brouwer and Mannoury among the people ‘who have promised to contribute’.

In his letter, Kohnstamm added that he had succeeded in making the case for acceptance mainly because of Brouwer’s promise to Bierens de Haan. It is not clear whether Brouwer ever undertook to write the projected sequels. Also, Kohnstamm gave Brouwer the option of adding elucidations to the accepted manuscript; for lack of relevant archive material, we cannot tell whether the published version differs from the manuscript originally submitted.

Kohnstamm had just published a criticism of psychologism in logic in the *Tijdschrift* (Kohnstamm, 1907), in the form of a negative review of Gerard Heymans’ *Die Gesetze und Elemente des wissenschaftlichen Denkens* (Heymans, 1890/1894).⁴ In a letter of January 18, 1907 to his thesis adviser, Diederik Johannes Korteweg, Brouwer had likewise expressed an anti-psychologistic stance:

From your characterization of theoretical logic as part of psychology I gathered that I had expressed myself rather vaguely, because it was actually my intention to show that theoretical logic on no account has a psychological meaning, even though it is a science. (Van Dalen, 2011, p. 37, trl. Van Dalen)

Brouwer did not discuss the matter in the ‘Unreliability’ paper,⁵ which is the more regrettable since the issue of the *Tijdschrift* in which it appears contains also Heymans’ reply to Kohnstamm.⁶

3 Brouwer’s conception of logic

The conception of logic involved in Brouwer’s remarks on the use of the principle of the excluded middle in mathematics is that formulated in his dissertation. Logic, according to Brouwer, is the study of patterns in linguistic records of mathematical acts of construction, and, as such, a form of applied mathematics. Mathematical

⁴Among the references in Kohnstamm (1907) we note on pp. 387–388 Husserl’s *Logische Untersuchungen* and Meinong’s *Gegenstandstheorie*, and, indeed, on p. 403, Brouwer’s dissertation – ‘a stern Dutch book’ (*een ernstig Hollandsch boek*). However, that reference just concerns Brouwer’s adoption (Brouwer, 1907, p. 104) of Poincaré’s remark that the world started rotating only with Copernicus.

⁵Much later, in 1949, Brouwer mentioned in a letter to David van Dantzig his ‘belief that psychological pictures of intuitionistic mathematics, however interesting they may be, never can be adequate’ (Van Dalen, 2011, p. 439). For anti-psychologistic readings of Brouwer, see Placek (1999) and Van Atten (2004, ch. 6).

⁶Heymans (1908), where Brouwer, Husserl, and Meinong are not mentioned.

constructions out of the intuition of time are themselves not of a linguistic nature. Language cannot play a creative role in mathematics; there are no mathematical truths that can be arrived at by linguistic means (such as logic) that could not, at least in principle, have been arrived at in acts of languageless mathematical construction (Brouwer, 1907, p. 133).

A correct inference is one where the construction required by its conclusion can be found from hypothetical actual constructions for its premisses. The hypotheses here are epistemic ones, in that the premisses are *known*. Thus, they differ from assumptions of the usual natural deduction kind, which merely assume that propositions are true. For Brouwer's conception of truth, however, only these epistemic assumptions play a role, since for him to assume that a proposition is true is to assume that one has a demonstration of it, that is, that one knows that it is true.⁷

Our use of logical signs in what follows is meant only as an abbreviatory device. Although Brouwer in his dissertation had remarked of the language accompanying logical reasonings that 'As well as any mathematical language this language can without much trouble be condensed into symbols' (Brouwer, 1975, p. 159),⁸ in his own writings he persisted in preferring sometimes prolix non-symbolic language. We will write $A \rightarrow B$ for 'A (hypothetical) actual construction for A can be continued into a construction for B'.⁹

The logical principles referred to in the title of Brouwer's paper are those of the Aristotelian tradition: the principles of the syllogism (in the paper defined by modus Barbara), of contradiction, and of the excluded third.¹⁰ Of course, from lectures by Gerrit Mannoury Brouwer knew about further developments, in particular those by Frege and Peano;¹¹ but for his principled criticism it suffices to consider the

⁷For a discussion, see Sundholm and van Atten (2008, sections 1 to 4).

⁸As an example, he refers to Whitehead (1898, p. 35ff). 'Zoo goed als alle wiskundige taal is ook deze taal zonder moeite te condenseeren tot symbolen' (Brouwer, 1907, p. 159n).

⁹Heyting's later works on logic do employ symbolism, but here we leave it open whether they are committed to this meaning. See for example Sundholm (1983).

¹⁰Brouwer's choice of terminology here is different from that of his likely logic teacher at Amsterdam, C.F. Bellaar-Spruyt. The latter's posthumously published book on formal logic (Bellaar-Spruyt, 1903) lists the principium identitatis, principium contradictionis, principium exclusi medii (also 'tertium vel medium non datur', Bellaar-Spruyt 1903, p. 18), and the dictum de omni et nullo, Bellaar-Spruyt 1903, p. 14. Brouwer's principle of the syllogism seems to comprise the principium identitatis and the dictum de omni et nullo. The principles of identity and of syllogism are also discussed by Poincaré in 'Sur la nature du raisonnement mathématique' (Poincaré, 1894), which also appears, in abridged form, in the first chapter of *Science et Hypothèse* (Poincaré, 1902); Brouwer knew in any case the latter of the two.

¹¹These lectures were published in shortened and revised form as *Methodologisches und Philosophisches zur Elementar-Mathematik* (Mannoury, 1909).

Aristotelian case.

4 Unreliability in the natural sciences and in wisdom

Brouwer introduces the main question of his paper, that of the reliability of logic in pure mathematics, by arguing that in two other domains logic is not reliable: the natural sciences and wisdom.

The problem with the use of logic in the natural sciences, as Brouwer describes it, is the familiar problem of induction. There is no guarantee that a mathematical model that explains a given set of observations will correctly predict further observations. But logic leads from statements in the mathematical model to other statements in that model. Hence, it may well lead from premisses that agree with observations to conclusions that do not, and is, in that sense, unreliable.

In wisdom, logic is not reliable for a different type of reason. Logic presupposes the presence of mathematical constructions, but in wisdom such constructions are absent. Mathematics embraces time awareness, whereas wisdom discards it.¹² Since time awareness introduces the subject-object distinction, it keeps consciousness out of what Brouwer later called its ‘deepest home’ (Brouwer, 1949, p. 1235). An attempt to apply logic to wisdom would require one to impose a mathematical structure on it, thereby distorting its content. Logic is unreliable in this domain, for logical conclusions from distorted content cannot be expected to reflect that content accurately.¹³

The question then arises whether in pure mathematics, where, in contrast to natural science, abstraction has been made from all observational content, and, in contrast to wisdom, logic is applied to something that does have mathematical structure, the

¹²Compare this remark in ‘Will, knowledge and speech’:

Mathematical attention is not a necessity but a phenomenon of life subject to the free will, everyone can find this out for himself by internal experience: every human being can at will either dream-away time-awareness and the separation between the Self and the World-of-perception or by his own powers bring about this separation and call into being in the world-of-perception the condensation of separate things. (Van Stigt, 1990, pp. 418–419)

(‘Dat wiskundige beschouwing geen noodzaak, doch een aan den vrijen wil onderworpen levensverschijnsel is, daarvan kan ieder bij zichzelf de inwendige ervaring opdoen: ieder mensch kan naar willekeur hetzij zich zonder tijdsgewaarwording en zonder scheiding tusschen Ik en Aanschouwingswereld verdroomen, hetzij de genoemde scheiding door eigen kracht voltrekken en in de aanschouwingswereld de condensatie van aparte dingen in het leven roepen’, Brouwer 1933, p. 2.)

¹³On Brouwer’s interest in religion, mysticism, and their relations to science, see Van Dalen (1999, sections 1.3 and 1.6); Van Stigt (1996); and Koetsier (2005). A comparison with Gödel on this point is presented in Van Atten and Tragesser (2003).

use of logic is reliable. The main point of this paper is that it is not.

5 Unreliability in mathematics

Brouwer had already made a case for the possibility of unreliable logical principles in his dissertation:

And if one succeeds in the construction of *linguistic* buildings, sequences of sentences proceeding according to the logical laws, thereby departing from linguistic images which could accompany basic mathematical truths in actual mathematical buildings, and if it turns out that those linguistic buildings can never produce the linguistic form of a contradiction, then all the same they belong to mathematics only in their quality of a linguistic building, and have nothing to do with mathematics outside of that building, e.g. with ordinary arithmetic or geometry.

So the idea that by means of such linguistic buildings we can obtain any knowledge of mathematics apart from that which can be constructed directly on the basis of intuition, is mistaken. And more so is the idea that in *this* way we can lay the *foundations* of mathematics, in other words that we can ensure the reliability of the mathematical theorems. (Brouwer, 1975, pp. 132–133, original emphasis)¹⁴

And in entry XX in the list of propositions submitted to the public defence together with it, according to Dutch custom that is still today observed in some universities, he had said:

To secure the reliability of mathematical reasonings one cannot succeed solely by starting from some sharply formulated axioms and further strictly adhering to the laws of theoretical logic. (Brouwer, 1975, p. 101)¹⁵

¹⁴En wanneer het gelukt *taal*gebouwen op te trekken, reeksen van volzinnen, die volgens de wetten der logica op elkaar volgen, uitgaande van taalbeelden, die voor werkelijke wiskundige gebouwen, wiskundige grondwaarheden zouden kunnen begeleiden, en het blijkt dat die taalgebouwen nooit het taalbeeld van een contradictie zullen kunnen vertoonen, dan zijn ze toch alleen wiskunde als taalgebouw en hebben met wiskunde buiten dat gebouw, bijv. met de gewone rekenkunde of meetkunde niets te maken.

Dus in geen geval mag men denken, door middel van die taalgebouwen iets van andere wiskunde, dan die direct intuïtief op te bouwen is, te kunnen te weten komen. En nog veel minder mag men meenen, op *die* manier de *grondslagen* der wiskunde te kunnen leggen, m.a.w. de betrouwbaarheid der wiskundige eigenschappen te kunnen verzekeren. (Brouwer, 1907, pp. 132–133)

¹⁵Het kan niet gelukken, de betrouwbaarheid der wiskundige redeneeringen te verzekeren, enkel door uit te gaan van eenige scherp gestelde axioma's en verder streng vast te houden aan de wetten der theoretische logica. (Brouwer, 1907, Stellingen)

The reliability of logical reasoning depends on the mathematical context in which it is applied: it is the context that determines whether the logical reasonings can be traded in for corresponding mathematical constructions. In the dissertation Brouwer rejected the attempt to come to know, by the use of logic, something mathematical that is nonconstructive; he there considered reliable within constructive mathematics not only the principles of the syllogism (Brouwer, 1907, p. 131) and of contradiction, but also the principle of excluded middle. The reason is that at the time he read $A \vee \neg A$ as $\neg A \rightarrow \neg A$.¹⁶

While in the syllogism a mathematical element could be discerned, the proposition:

A function is differentiable or is not differentiable
says *nothing*; it expresses the same as the following:

If a function is not differentiable, then it is not differentiable.

But the logician, looking at the *words* of the former sentence, and discovering a regularity in the combination of words in this and in similar sentences, here again projects a mathematical system, and he calls such a sentence an *application of the tertium non datur*. (Brouwer, 1975, p. 75, original emphasis)¹⁷

6 The principle of excluded middle is unreliable

In ‘Unreliability’, Brouwer will advance upon the dissertation in two ways: he corrects his reading of the principle of excluded middle, and he shows that this corrected understanding entails the unreliability of a traditional principle within constructive mathematics itself.

This is the (silently) corrected understanding of the principle of excluded middle:

Now the principium *tertii exclusi*: this demands that every supposition¹⁸ is either correct or incorrect, mathematically: that of every

¹⁶As Van Dalen (1999, pp. 106–107) has pointed out, Brouwer most likely arrived at this reading under the influence of the logic lectures by Bellaar-Spruyt.

¹⁷Was in het syllogisme nog een wiskundig element te onderkennen, de stelling:

Een functie is of differentieerbaar of niet differentieerbaar
zegt *niets*; drukt hetzelfde uit, als het volgende:

Als een functie niet differentieerbaar is, is ze niet differentieerbaar.
Maar de *woorden* van eerstgenoemde volzin bekijkend, en een regelmatig gedrag in de opvolging der woorden van deze en van dergelijke volzinnen ontdekkend, projecteert de logicus ook hier een wiskundig systeem, en noemt zulk een volzin een *toepassing van het principe van tertium nondatur*. (Brouwer, 1907, p. 131)

¹⁸Suppositions should here not be taken in the sense of abstract propositions in a Platonic realm of abstract entities, as in Bolzano or in Frege. What seems to be meant is rather: Every mathematical assumption that we can make is either correct or incorrect.

supposed fitting in a certain way of systems in one another, either the termination or the blockage by impossibility, can be constructed. (Brouwer, 1908b, p. 156, trl. ours)¹⁹

The change of mind is acknowledged in ‘Addenda and corrigenda to “On the Foundations of Mathematics” ’ (Brouwer, 1917, p. 1).

Note that $\neg A$ does not merely mean that no proof of A exists, but that from an assumed actual demonstration of A one can ‘construct the blockage by impossibility’ (see also Brouwer 1907, p. 127). In this sense, intuitionistic negation is unlike the classical notion a positive notion, as it involves the existence of a blockage.²⁰

Thus understood, the principle of excluded middle is not reliable, for we do not have a general decision method as required by the constructive reading. Brouwer’s claim is not that we can never have such a method: ‘in infinite systems the principium tertii exclusi is *as yet* not reliable’ (our emphasis). Brouwer states the first so-called ‘Brouwerian counterexamples’ or ‘weak counterexamples’ to the principle of excluded middle, which illustrate its unreliability. These are propositions of which we are in a position to assert the weak negation, but not the truth or the strong negation. Of course any open problem is, as such, a weak counterexample to the principle of excluded middle; the importance of weak counterexamples comes from the fact that they can be used to show that certain highly general principles have not yet been established, such as ‘Every set is finite or infinite’ or ‘The continuum is totally ordered’. Brouwer published weak counterexamples to the principle of excluded middle also in international journals, but only much later (Brouwer, 1921, 1924, 1925, 1929). By then he had found a uniform technique for constructing weak

¹⁹Nu het principium *tertii exclusi*: dit eischt, dat iedere onderstelling of juist of onjuist is, wiskundig: dat van iedere onderstelde inpassing van systemen op bepaalde wijze in elkaar hetzij de beëindiging, hetzij de stuiting op onmogelijkheid kan worden geconstrueerd.

²⁰Becker (1927, pp. 498–500), with reference to the section ‘Evidence and truth’ (*Evidenz und Wahrheit*) in the sixth of Husserl’s *Logische Untersuchungen* (Husserl, 1984). This is clearly the passage by Becker that Heyting has in mind in Königsberg (Heyting, 1931, p. 113):

Eine logische Funktion ist ein Verfahren, um aus einer gegebenen Aussage eine andere Aussage zu bilden. Die Negation ist eine solche Funktion; ihre Bedeutung hat Becker, im Anschluß an Husserl, sehr deutlich beschrieben. Sie ist nach ihm etwas durchaus Positives, nämlich die Intention auf einen mit der ursprünglichen Intention verbundenen Widerstreit.

(‘A logical function is a method for turning a given statement into another statement. Negation is such a function; Becker, following Husserl, has described its meaning very clearly. It is according to him something wholly positive, namely the intention directed to a conflict bound up with the original intention.’ Trl. ours.) Heyting does not give a reference here, but had already mentioned Becker’s *Mathematische Existenz* on p. 107.

counterexamples that depended only on the fact *that* open problems of a certain simple type still exist, not on the exact content of these problems.

7 Are there absolutely undecidable propositions?

Brouwer adds the following remark to his explanation of the principle of excluded middle:

The question of the validity of the principium tertii exclusi is thus equivalent to the question concerning the *possibility of unsolvable mathematical problems*. For the already proclaimed conviction that unsolvable mathematical problems do not exist, no indication of a demonstration is present.²¹ (Brouwer, 1908b, p. 5)

The claim that every mathematical problem is solvable is of course constructively stronger than the claim that there are no unsolvable problems.²² The former is equivalent to the principle that for any A , $A \vee \neg A$, the latter to the principle that for any A , $\neg\neg(A \vee \neg A)$; and Brouwer had demonstrated the validity of the latter in the same paper. Indeed, in the Brouwer archive there is a note from about the same period 1907–1908 in which the point is made explicitly:

Can one ever demonstrate of a question, that it can never be decided? No, because one would have to do so by *reductio ad absurdum*. So one would have to say: assume that the question has been decided in sense a , and from that deduce a contradiction. But then it would have been demonstrated that *not- a* is true, and the question remains decided. (Van Dalen, 2001, p. 174n. a, trl. ours)²³

Brouwer never published this note. Wavre in 1926 gave the argument for a particular case, while clearly seeing the general point:

²¹Compare entry XXI in the list of theses in the dissertation: ‘Ongegrond is de overtuiging van Hilbert (Gött. Nachr. 1900, pag. 261): “dass ein jedes bestimmte mathematische Problem einer strengen Erledigung notwendig fähig sein müsse, sei es, dass es gelingt, die Beantwortung der gestellten Frage zu geben, sei es dass die Unmöglichkeit der Lösung und damit die Notwendigkeit des Misslingens aller Versuche dargetan wird”.’ (Brouwer, 1907, Stellingen)

²²See also Wittgenstein (1922, 6.5), Schlick (1935), McCarty (2005), and Martin-Löf (1995) (in particular the postscript in the reprint in Van der Schaar 2012).

²³Zal men nu ooit van een vraag kunnen bewijzen, dat ze nooit uitgemaakt kan worden? Neen, want dat zou moeten uit het ongerijmde. Men zou dus moeten zeggen: Gesteld dat het was uitgemaakt in zin a en daaruit affeiden, tot een contradictie kwam. Dan zou echter bewezen zijn, dat niet a waar was, en de vraag bleef uitgemaakt.

It suffices to give an example of a number of which one does not know whether it is algebraic or transcendent in order to give at the same time an example of a number that, until further information comes in, *could* be neither the one nor the other. But, on the other hand, it would be in vain, it seems to me, to want to define a number that *is* neither algebraic nor transcendent, as the only way to show that it is not algebraic consists in showing that it is absurd that it would be, and then the number would be transcendent. (Wavre, 1926, p. 66, trl. ours, original emphasis)²⁴

The general, schematic point was explicitly noted by Heyting in 1934:

Further, the formula $\vdash \neg\neg(a \vee \neg a)$ should be highlighted. It has the same meaning²⁵ as $\vdash \neg(\neg a \wedge \neg\neg a)$ and expresses Brouwer's theorem on the absurdity of the absurdity of the excluded third, and amounts to saying that a demonstrably unsolvable problem cannot exist. (Heyting, 1934, p. 16)²⁶

8 The principle of excluded middle is consistent

Brouwer also observes that, although the principle of excluded middle is not schematically valid, none of its instances is false, since $\neg(A \vee \neg A)$ implies the contradiction $\neg A \wedge \neg\neg A$. This demonstrates the correctness of the principle that, for any A , $\neg\neg(A \vee \neg A)$. Brouwer concludes that it is always consistent to use (this form of) the principle of excluded middle but that it does not always lead to truths. Later, Brouwer gave a refutation of the schema $\forall x(P(x) \vee \neg P(x))$ using specifically intuitionistic principles regarding choice sequences and continuity (Brouwer, 1928).²⁷

²⁴Il suffit donc de fournir l'exemple d'un nombre dont on ne sache s'il est algébrique ou transcendant pour fournir en même temps l'exemple d'un nombre qui, jusqu'à plus ample information, *pourrait* n'être ni l'un ni l'autre. Mais, d'autre part, il serait vain, me semble-t-il, de vouloir définir un nombre qui ne *soit* ni algébrique ni transcendant, car la seule manière de prouver qu'il n'est pas algébrique consistant à prouver qu'il serait absurde qu'il le fût, ce nombre serait transcendant. [original emphasis]

²⁵Heyting writes 'gleichbedeutend'. Note that the two formulas are equi-assertible, but have different assertion conditions.

²⁶Es sei noch die Formel $\vdash \neg\neg(a \vee \neg a)$ hervorgehoben, die mit $\vdash \neg(\neg a \wedge \neg\neg a)$ gleichbedeutend ist und den Brouwerschen Satz von der Absurdität der Absurdität des Satzes vom ausgeschlossenen Dritten zum Ausdruck bringt. Sie besagt, daß es ein nachweisbar unlösbares Problem nicht geben kann.

²⁷In the Bishop tradition, some versions of the principle of excluded middle that Brouwer devised counterexamples to have been given a systematic place: LPO, WLPO, LLPO. See Bridges and Richman (1987, ch. 1, section 1).

Hence Brouwer's proposal to divide the theorems that are usually considered as having been demonstrated into the correct and the non-contradictory ones (Brouwer, 1908b, 7n. 2), that is, those whose reduction to absurdity has been refuted. That is not a suggestion that there are three truth values, true, non-contradictory, false;²⁸ for a non-contradictory proposition might be proved one day and thereby become true. This observation on the consistency of the principle of excluded middle would, in the 1920s, be at the basis of Brouwer's optimism, expressed in print but nevertheless often neglected, concerning the success of the Hilbert Program, a success that Brouwer would consider of no general mathematical value (Brouwer 1924, p. 3; Brouwer 1928, p. 377).

9 Partial validity of the principle of excluded middle

The principle of excluded middle is valid, Brouwer points out, in finite domains, for questions whether a given construction of finite character²⁹ is possible. Only finitely many attempts at that construction can be made, and each will succeed or fail in finitely many steps (see also Brouwer 1955, p. 114). Brouwer came to explain the genesis of the belief in the validity of the principle of excluded middle as follows:

I am convinced that the axiom of solvability and the principle of excluded third are both false,³⁰ and that historically the belief in these dogmas has been caused thusly. First, one has abstracted classical logic from the mathematics of subsets in a certain finite set, then ascribed to this logic an a priori existence independent of mathematics, and finally, on the basis of this alleged apriority, applied it rightlessly to the mathematics of infinite sets. (Brouwer, 1922, n. 4)³¹

²⁸Church makes this point clearly in 1928 in 'On the law of excluded middle' (Church, 1928), criticising Barzin and Errera (1927).

²⁹Not, obviously, in the sense of the classical equivalent of Zermelo's axiom of choice known as the Teichmüller-Tukey lemma.

³⁰Brouwer came to identify those, but after they had been presented separately.

³¹Meiner Ueberzeugung nach sind das Lösbarkeitsaxiom und der Satz vom ausgeschlossenen Dritten beide falsch und ist der Glaube an diese Dogmen historisch dadurch verursacht worden, dass man zunächst aus der Mathematik der Teilmengen einer bestimmten endlichen Menge die klassische Logik abstrahiert, sodann dieser Logik eine von der Mathematik unabhängige Existenz a priori zugeschrieben und sie schliesslich auf Grund dieser vermeintlichen Apriorität unberechtigtweise auf die Mathematik der unendlichen Mengen angewandt hat.

See on this point also Brouwer (1924, p. 2; 1929, pp. 423–424; 1949, p. 492; and 1952B, pp. 510–511).

10 Brouwer’s concern with meaning and truth

We wish to discuss one further aspect of the text itself. In it, Brouwer uses neither the term ‘(wiskundige) waarheid’ ((mathematical) truth), nor ‘betekenis’ (meaning); but a careful consideration of his Dutch and some of his other writings will reveal that these really are the notions under consideration.

We begin with truth. In place of ‘waar’ (true) and ‘onwaar’ (false), Brouwer uses ‘juist’ and ‘onjuist’. The term ‘juist’, however, is most commonly translated by ‘right’ and/or ‘correct’, which raises the question as to how this relates to (propositional) truth. The largest and most authoritative dictionary of the Dutch language, the *Woordenboek der Nederlandsche Taal*, lists among the meanings of ‘juist’: ‘Met de waarheid —, met het wezen van iets in overeenstemming; de waarheid weergevende; aan de waarheid beantwoordende’ (in agreement with the truth, with the essence of something; representing the truth; corresponding to the truth), and gives among its historical examples this sentence from 1897: ‘Met “waarheid” kan men bedoelen de meest juiste voorstelling van de dingen’ (By ‘truth’ one can mean the most correct representation of things) – by a happenstance, written by an author who would become one of Brouwer’s best friends, Frederik van Eeden. Here, the sense of ‘juist’ is given by the translation ‘true’, chosen by Heyting in the *Collected Works* (Brouwer, 1975, p. 110). We prefer the alternative ‘correct’, however, in order to translate the different words ‘juist’ and ‘waar’ differently.

Of course correctness can be relative to something else than truth, for example a convention, a value, or an ideal; but that here truth is meant is clear from the fact that in his dissertation Brouwer indeed is willing to speak of ‘wiskundige grondwaarheden’ (basic mathematical truths),³² in his reply to Mannoury’s review of his dissertation of ‘wiskundige waarheden’ (mathematical truths),³³ and in his draft letter to De Vries, dated February 15, 1907, of ‘de waarheid van de wiskundige stellingen’ (the truth of the mathematical theorems).³⁴ An explicit identification is found in ‘Willen, weten, spreken’ (Will, knowledge and speech) of 1932: ‘juiste (d.w.z. daadwerkelijk wiskundige beschouwingen doeltreffend indicerende) affirmaties’ (Brouwer, 1933, p. 54), translated by Van Stigt³⁵ as ‘correct affirmations (i.e. effectively indi-

³²Brouwer (1907, p. 132).

³³Brouwer (1908a, p. 328).

³⁴Van Dalen (2011, Online Supplement, p. 201).

³⁵The *Collected Works* do not give an English translation of the first two sections of ‘Will, knowledge and speech’ because of their proximity to the corresponding sections of the first Vienna lecture (Brouwer, 1929), which they include in German.

cating actual mathematical viewing)’ (Van Stigt, 1990, p. 424),³⁶ where the viewing takes place in languageless intuition.³⁷

We now turn to meaning. As we have seen, Brouwer begins his discussion of the principle of the excluded middle as follows:

Now the principium *tertii exclusi*: this demands that every supposition is either correct or incorrect. . .

In the original:

Nu het principium *tertii exclusi*: dit eischt, dat iedere onderstelling òf juist òf onjuist is. . .

Brouwer writes ‘demands’ (*eischt*), not, as one might have expected, ‘means’ (*betekent*) or ‘asserts’ (*beweert*). However, the *Woordenboek der Nederlandsche Taal* lists among the meanings of ‘eischen’: ‘Tot voorwaarde voor zijn bestaan, welvaren of welslagen hebben’: ‘to have as a condition for its existence, prospering, or success’. Hence, ‘dit principe eischt dat. . .’ can naturally be understood as ‘for this principle to hold, what is required is that. . .’. But this condition clearly amounts to a meaning specification of the principle.

That Brouwer intends this sense of ‘eischt’ is brought out by a comparison with his discussion of his first principle, to wit syllogism, where at the corresponding place he uses ‘leest in’ (reads . . . as . . .). The latter unambiguously expresses a concern with meaning. A coherent interpretation of Brouwer’s remarks should accord the

³⁶At the corresponding place in the first Vienna lecture, Brouwer had written: ‘zutreffenden (d.h. tatsächliche mathematische Betrachtungen andeutenden) Aussagen’ (Brouwer, 1929, p. 158). We here note that the range of meaning of the German ‘zutreffend’ is included in and much narrower than that of the Dutch ‘juist’.

³⁷The visual metaphor is not often used in Brouwer’s writings. In the dissertation one finds:

Now we have seen that classical logic studies the linguistic counterpart of logical reasonings, i.e. of reasonings on *relations of whole and part* for arbitrary mathematically constructed systems; from the fact that we *see* these mathematical systems we may conclude that *here* the sentences succeeding one another according to classical logic, will never show contradictions, because they correspond to acts of mathematical construction. (Brouwer, 1975, p. 88, original emphasis)

(‘Nu hebben we gezien, dat de klassieke logica bestudeert de taalbegeleiding der logische redeneeringen, d.w.z. der redeneeringen *in relaties van geheel en deel* voor willekeurige wiskundig opgebouwde systemen; en we weten uit het feit, dat we die wiskundige systemen *zien*, dat *daar* de volgens de klassieke logica elkaar opvolgende volzinnen, die immers wiskundige bouwhandelingen begeleiden, nooit contradicties zullen vertoonen’, Brouwer 1907, pp. 159–160, original emphasis.)

same sense to these two verbs;³⁸ the disambiguation of ‘eischt dat’ should pick out the sense in which it has the same meaning as ‘leest in’.³⁹

11 Precursors

Brouwer was not the first who voiced criticism or hesitations about either the usefulness or the validity of the principle of excluded middle in a purely mathematical context. From the 1870s, Kronecker objected to the unlimited use of the principle of excluded middle and of definition by undecided separation of cases. For example, in his treatise on algebraic numbers of 1882, he wrote on the factorization of polynomial functions:

The definition of irreducibility drawn up in section 1 lacks a secure grounding as long as no method has been indicated by which it can be decided whether a definite given function is irreducible according to that definition or not. (Kronecker, 1882, pp. 10–11)⁴⁰

adding in a footnote,

The analogous need, which as a matter of fact has often remained neglected, arises in many other cases, in definitions as in demonstrations, and on another occasion I will come back to this generally and thoroughly. (Kronecker, 1882, p. 11n)⁴¹

His student Jules Molk⁴² gave voice to the doubts of his Doktorvater in the printed version of his Berlin dissertation from 1885:

The definitions should be algebraic and not only logical. It does not suffice to say: ‘A thing exists or it does not exist’. One has to show what being and not being mean, in the particular domain in which we are moving. Only thus do we make a step forward. (Molk, 1885, p. 8, trl. ours)⁴³

³⁸Brouwer’s remark on the second principle does not contain a corresponding verb at all.

³⁹Koss (2013, p. 75) reaches the conclusion that in this paper Brouwer’s concern was neither with meaning nor with truth. As we show above, the linguistic facts do not bear him out.

⁴⁰Die im Artikel 1 aufgestellte Definition der Irreduktibilität entbehrt solange einer sicheren Grundlage, als nicht eine Methode angegeben ist, mittels deren bei einer bestimmten vorgelegten Funktion entschieden werden kann, ob dieselbe der aufgestellten Definition gemäß irreduktibel ist oder nicht.

⁴¹Das analoge Bedürfnis, welches freilich häufig unbeachtet geblieben ist, zeigt sich in vielen anderen Fällen, bei Definitionen wie bei Beweisführungen und ich werde bei einer anderen Gelegenheit in allgemeiner und eingehender Weise darauf zurückkommen.

⁴²December 8, 1857, Strasbourg – May 7, 1914, Nancy.

⁴³Les définitions devront être algébriques et non pas logiques seulement. Il ne suffit pas de dire : ‘Une chose est ou elle n’est pas’. Il faut montrer ce que veut dire être et ne pas être, dans le domaine particulier dans lequel nous nous mouvons. Alors seulement nous faisons un pas en avant.

Molk became professor in Nancy, and was the editor-in-chief and driving force behind the French version of Felix Klein's *Enzyklopädie der mathematischen Wissenschaften und ihren Grenzgebiete*. He translated and augmented, especially concerning foundational matters, Pringsheim's beautiful surveys of topics in elementary analysis. In Book I, volume I.3, section 10, 'Point de vue de L. Kronecker', Molk considerably elaborated upon the above brief remark from his dissertation:

Analysis should, on the other hand, refrain from general considerations of a logical kind alien to its object. In Analysis definitions may introduce nothing but auxiliary notions that facilitate the study of the various natural groups that one forms to study the properties of numbers. These auxiliary notions must have an arithmetical character and not a merely logical one, whence they can only be about *groups of which each element can be effectively obtained by means of a finite number of operations*, and not about groups simply determined by a non-contradictory logical convention.

Similarly, the logical evidentness⁴⁴ of a reasoning does not suffice to legitimize the use of that reasoning in Analysis. In order to give a mathematical demonstration of a proposition, it does not suffice, for example, to establish that the contrary proposition implies a contradiction. One has to give a procedure that, operating on the elements under consideration, by means of a finite number of arithmetical operations in the old sense of the word, permits one to obtain the result formulated by the proposition to be demonstrated. *This procedure constitutes the essence of the demonstration; it is not an addition to it.*

...

The principle of economy in science – economy of time, economy of efforts – in Analysis leads us to the absolute and relative rational numbers: that introduction is legitimate, because its only effect is to shorten the deductions without changing their character. To every proposition about rational numbers, for example expressed by an equation, corresponds a congruence taken according to an easily determined module or system of modules.

...

The character of demonstrations is, on the contrary, completely changed by the introduction of *arbitrary* irrational numbers. One cannot, moreover, give any definition of these numbers except a logical one, determining them, but *not mathematically defining them*. It is that logical (but not mathematical) definition that confers on (infinite) sets of

⁴⁴The French here has 'évidence', in the sense of Cartesian clarity. This meaning, 'evidence of', is the first given by the OED. The more familiar, and in Anglo-American philosophy all-pervasive 'evidence for' exists only in English. For further discussion, see Sundholm (2014).

rational numbers that define so-called arbitrary irrational numbers, the character of an organic sequence. Therefore, those numbers, according to L. Kronecker, cannot rightfully occur in the definitive demonstration of a proposition of Analysis. (Molk, 1904, pp. 159–61, trl. ours, original emphasis)⁴⁵

Any reader of Brouwer’s ‘Unreliability’ will be struck by the coincidence of the views expressed, even down to some of the finer details: the rejection of indirect existence proofs; the prohibition of blind, merely symbolic reasoning; the explicit separation between demonstrated propositions and non-contradictory ones. In view of this, the question arises whether Brouwer was aware of Molk’s treatment. According to the central Dutch library catalogue,⁴⁶ in the Netherlands one copy of the original edition of the article by Pringsheim and Molk was present, namely at the library of the University of Amsterdam. Unfortunately, that library has not been able to answer our question exactly when this fascicule of the Encyclopédie became available.⁴⁷ On the other hand, we wish to note that neither in Brouwer’s extensive notebooks

⁴⁵L’Analyse doit, d’autre part, se garder de considérations générales d’ordre logique étrangères à son objet. Les définitions ne doivent introduire en Analyse que des notions auxiliaires facilitant l’étude des divers groupes naturels que l’on forme pour étudier les propriétés des nombres. Ces notions auxiliaires doivent avoir un caractère arithmétique et non logique seulement, en sorte qu’elles ne sauraient porter que sur des groupes dont chaque élément puisse être effectivement obtenu au moyen d’un nombre fini d’opérations, et non sur des groupes simplement déterminés par une convention logique non-contradictoire.

De même l’évidence logique d’un raisonnement ne suffit pas pour légitimer l’emploi de ce raisonnement en Analyse. Pour avoir donné une démonstration mathématique d’une proposition, il ne suffit pas, par exemple d’avoir établie que la proposition contraire implique contradiction. Il faut donner un procédé permettant d’obtenir, au moyen d’un nombre fini d’opérations arithmétiques au sens ancien du mot, effectuées sur les éléments que l’on envisage, le résultat qu’énonce la proposition à démontrer. C’est ce procédé qui constitue l’essence de la démonstration ; il ne vient pas s’y ajouter.

...

Le principe de l’économie dans la science, économie de temps, économie d’efforts, nous amène en Analyse les nombres rationnels absolus et relatifs : cette introduction est légitime, puis-elle n’a pour effet que d’abrèger les déductions sans en changer le caractère. A toute proposition concernant les nombres rationnels, exprimée par une égalité par exemple, correspond une congruence prise suivant un module ou un système de modules toujours faciles à déterminer.

...

Le caractère des démonstrations est, au contraire, complètement changé par l’introduction de nombres irrationnels quelconques. On ne peut d’ailleurs donner de ces nombres qu’une définition logique, les déterminant, mais ne les définissant pas mathématiquement. C’est cette définition logique (mais non mathématique) qui confère aux ensembles (infinis) de nombres rationnels, que l’on dit définir des nombres irrationnels quelconques, le caractère d’une suite organique. Ces nombres ne peuvent donc, suivant L. Kronecker, figurer à aucun titre dans la démonstration définitive d’une proposition d’Analyse.

⁴⁶Nederlandse Centrale Catalogus.

⁴⁷Brouwer occasionally made similar inquiries. Brouwer to his University Library, August 3, 1929:

1904–1907, which show Brouwer to have been an omnivorous reader, nor in his remaining correspondence, nor in the Dissertation have we found a reference to the article by Pringsheim and Molk, nor to any other article in the *Encyclopédie*, nor to any other of Molk’s writings.

Also some younger French mathematicians were sensitive to the issues later raised by Brouwer. For example, Lebesgue (1875–1941) had stated, in a letter to Borel published in 1905:⁴⁸

Although I strongly doubt that one will ever name a set that is neither finite nor infinite, the impossibility of such a set seems to me not to have been demonstrated.⁴⁹

Note that one of the examples in Brouwer’s paper of a principle that has not been demonstrated is ‘every number is finite or infinite’. In the intuitionistic setting, an example of a set that is neither finite nor infinite was given in Brouwer (1924, pp. 3–4).

However, in spite of the early efforts by Kronecker, only with Brouwer do we get a comprehensive development of mathematics excluding any ‘unreliable’ use of the principle of excluded middle.

12 Direct influence

In spite of its historical significance, Brouwer’s paper has apparently had surprisingly little direct influence on others, apart from sporadic references in Brouwer’s

Undersigned would much appreciate it if he could learn the exact date on which Heft 2 of Band 142 of the *Journal für die reine und angewandte Mathematik* (Crelle’s Journal) was received at the University Library. The date probably lies in the first months of 1913.

Many thanks in advance

Sincerely

Your obedient servant

L.E.J. Brouwer. (Van Dalen, 2011, Online Supplement, p. 1743, trl. ours)

(‘Ondergeteekende zou het op hoogen prijs stellen, indien hij den preciezen datum kon vernemen, waarop Heft 2 van Band 142 van het *Journal für die reine und angewandte Mathematik* (Crelle’s Journal) ter Universiteits-bibliotheek is ontvangen. De datum ligt waarschijnlijk in de eerste maanden van 1913. Met beleefden dank bij voorbaat, Hoogachtend, Uw Dienstwillige L.E.J. Brouwer.’) The issue in question contained Brouwer’s paper ‘Über den natürlichen Dimensionsbegriff’ (Brouwer, 1913); the answer to his question here is January 27, 1913 (Van Dalen, 2008, p. 358).

⁴⁸Baire et al. (1904, p. 269).

⁴⁹Bien que je doute fort qu’on nomme jamais un ensemble qui ne soit ni fini, ni infini, l’impossibilité d’un tel ensemble ne me paraît pas démontré.

own work and that of Heyting. The exception is George François Cornelis Griss, as will now be explained.⁵⁰

There is a direct connection between Brouwer's 'Unreliability' and Griss' development, in a series of papers published from 1944 to 1951, of a version of intuitionism without negation.⁵¹ Griss had first explained his rationale to Brouwer directly, in a letter of April 19, 1941:

Showing that something is not true, i.e. showing the incorrectness of a supposition is not an intuitively clear act. For it is impossible to have an intuitively clear concept of an assumption that later turns out to be even wrong. One must maintain the demand that only building things up from the foundations makes sense in intuitionistic mathematics. (Van Dalen, 2011, p. 402, trl. Van Dalen)⁵²

This was repeated almost verbatim in Griss' publication of 1946:

On philosophic grounds I think the use of the negation in intuitionistic mathematics has to be rejected. Proving that something is not right, i.e. proving the incorrectness of a supposition, is no intuitive method. For one cannot have a clear conception of a supposition that eventually proves to be a mistake. Only construction without the use of negation has some sense in intuitionistic mathematics. (Griss, 1946, p. 675)

However, where in the article Griss prefers not to go further into the philosophical issue and goes on to discuss mathematical consequences, in the letter he first offers a justification for his basic idea. It takes the form of a comment on Brouwer's 'Unreliability':

⁵⁰The 'Unreliability' paper may further have had an indirect influence already on Husserl. In 1928 with his 'Intuitionistische Betrachtungen über den Formalismus' (Brouwer, 1928), Brouwer returned, with explicit reference, to the themes of the earlier paper. The relevance of Brouwer's 1928 paper for Husserl's *Formale und transzendente Logik* (Husserl, 1929) is clear and, several years ago, was emphasized to one of us by Byung-Hak Ha. Afterwards Thomas Vongehr at the Husserl Archives in Louvain found an entry in an old card catalogue that showed that Husserl had owned an offprint of that paper. Unfortunately, the offprint itself was no longer to be found. Brouwer and Husserl met in Amsterdam in April 1928 (Van Dalen, 2011, Online Supplement, p. 1515; Husserl, 1994, vol. 5, p. 156), and it is likely that Brouwer gave the offprint to Husserl then, or sent it in the aftermath. However, in spite of its topical closeness to some of the main themes of *Formale und transzendente Logik*, the latter contains no reference to Brouwer. (We express our thanks to Ha and Vongehr.)

⁵¹ Griss (1944, 1946, 1950, 1951a,b,c). For more on Griss and his work, see Heyting (1955) and Franchella (1993).

⁵²Aantonen, dat iets niet waar is, d.w.z. de onjuistheid van een veronderstelling aantonen, is niet een intuïtief-duidelijke handelwijze. Van een veronderstelling, die later zelfs blijkt fout te zijn, kan men namelijk onmogelijk een intuïtief-duidelijke voorstelling hebben. Men moet de eis handhaven, dat alleen het opbouwen vanaf de grondslagen in de intuïtionistische wiskunde betekenis heeft. (Van Dalen, 2011, Online Supplement, p. 2142)

Although my ideas about the foundations of mathematics are not completely identical to yours, the differences are unimportant for what follows, so, for example, I can agree completely with your considerations in the *Tijdschrift voor Wijsbegeerte*, 2nd volume, 1908. Let me just remark that the concept of negation does not explicitly occur in the formulation of the foundations of mathematics, but only in the examination of the validity of the logical principles. You say there:

The principle of contradiction is just as little in dispute; the execution of the fitting of a system a in a particular way into a system b , and finding that this fitting turns out to be impossible are mutually exclusive.⁵³

What does impossibility of a ‘fitting in’ mean here?

In the first place this can mean that one assumes the possibility of fitting, and that this assumption leads to a contradiction. This manner far exceeds the construction of mathematical systems on the basis of the ur-intuition, and as I remarked in the beginning, one cannot clearly obtain a conception of it. If one still accepts it, then one takes in principle a similar step, as when one accepts the principle of the excluded third. An element of arbitrariness enters in our idea about what is and what is not admissible in mathematics, if one does not stick strictly to the requirement that one only builds up mathematical systems from the foundations which are given in the ur-intuition.

Another meaning which can be given to ‘finding that this fitting of a system a into a system b turns out to be impossible’ might be this: that the system a demonstrably differs (in that case this concept has to be defined) from every system that can be fitted into b . One asks for example whether e is an algebraic number and one finds that e is positively transcendent so e demonstrably differs from each algebraic number. If need be, one can even answer the question whether e is algebraic by: e is not algebraic, but then we have assigned a new meaning to the word ‘not’.⁵⁴

⁵³The translation of this passage that we will give below is a little different from this one by Van Dalen, but not substantially so. Note that another passage that Griss could have referred to is Brouwer (1907, p. 127).

⁵⁴Hoewel mijn ideeën over de grondslagen van wiskunde niet volkomen gelijk zijn aan de Uwe, zijn de verschillen voor het volgende niet van belang, zodat ik bijv. geheel kan aansluiten bij Uw beschouwingen in het *Tijdschrift voor Wijsbegeerte*, 2de jaargang, 1908. Alleen merk ik op, dat het begrip negatie bij het formuleren van de grondslagen der wiskunde niet expliciet optreedt, maar pas bij het onderzoek naar de geldigheid der logische principes. U zegt daar:

Brouwer's paper 'Essentially negative properties' (Brouwer, 1948) was written in response to Griss. In his letter of 1941, Griss had remarked that 'no real number a is known about which it has been proved that it cannot possibly be equal to 0 ($a \neq 0$), while at the same time it has not been proven that the number differs positively from 0 ($a \neq 0$)'. Brouwer in his paper constructed a real number a with just that property; but he did not provide an accompanying philosophical account as an alternative to Griss' view.

An occasion for Brouwer, Griss and others to debate these matters in public would have been a meeting planned by S.I. Dockx.⁵⁵ A letter of Beth to Dockx of July 8, 1949, suggests that also Freudenthal, Heyting, and Van Dantzig were invited, but at the same time makes it clear that Brouwer declined because he did not want to participate in an event with Freudenthal (Van Dalen, 2011, Online Supplement, p. 2446).⁵⁶ To the best of our knowledge, the meeting never took place.

Heyting published a reaction in 1955, 'G.F.C. Griss and his negationless intuitionistic mathematics'. While Heyting noted that 'unrealized suppositions' are implicit in all general statements, so that banishing such suppositions would reduce mathematics to an 'utterly unimportant and uninteresting subject' (Heyting, 1955, p. 95), he did not provide a detailed confrontation with the arguments of Griss. It can be argued that Brouwer's dissertation in effect contains an answer to Griss' objection: according to Van Atten (2009), the view expressed on the hypothetical judgement at the beginning of chapter 3 of Brouwer's thesis (Brouwer, 1907, pp. 125–127) is that logical reasoning does not operate on constructions, let alone hypothetical ones,

Evenmin is aanvechtbaar het principe van contradictie: het volvoeren van de inpassing van een systeem a op een bepaalde wijze in een systeem b , en het stuiten op de onmogelijkheid van die inpassing sluiten elkaar uit.

Wat betekent hierin de onmogelijkheid van een inpassing?

Ten eerste kan dit betekenen, dat men van de mogelijkheid van een inpassing uitgaat en die veronderstelling ad absurdum voert. Deze wijze van doen gaat echter ver uit boven het opbouwen van wiskundige systemen op grond van de oerintuïtie, en men kan er zich, zoals ik reeds in het begin opmerkte, geen duidelijke voorstelling van maken. Aanvaardt men ze toch, dan doet men in principe een dergelijke stap, als wanneer men het beginsel van het uitgesloten derde zou aanvaarden. Er komt een element van willekeur in onze opvatting over wat al of niet toelaatbaar is in de wiskunde, als men niet streng vasthoudt aan de eis alleen wiskundige systemen op te bouwen vanaf de grondslagen, die in de oerintuïtie gegeven zijn.

Een andere betekenis, die aan de uitdrukking 'stuiten op de onmogelijkheid van een inpassing van een systeem a in een systeem b ' gehecht kan worden, zou deze kunnen zijn, dat het systeem a aanwijsbaar verschilt (dit begrip moet dan gedefinieerd zijn) van ieder systeem, dat in het systeem b kan worden ingepast. Men vraagt bijvoorbeeld, of e een algebraïsch getal is en vindt, dat e positief transcendent is, zodat e aanwijsbaar van ieder algebraïsch getal verschilt. Desnoods kan men zelfs op de vraag, of e algebraïsch is, antwoorden: e is niet algebraïsch, maar dan hebben we aan het woord 'niet' een nieuwe betekenis toegekend.

⁵⁵Stanislas Isnard Dockx, Antwerp 1901–Brussels 1985.

⁵⁶For an account of Brouwer's by then long-standing conflict with Freudenthal, see Van Dalen (2005, pp. 721–728, 753–757, and 794–799).

but on conditions on constructions. The difference is that these conditions, whether fulfillable or not, can themselves be represented as actual objects.

De onbetrouwbaarheid der logische principes

door
L.E.J. Brouwer

[Noot Brouwer voorafgaand aan de herdruk van 1919] *Dit opstel zou ook thans nog in denzelfden vorm geschreven kunnen zijn. Medestanders hebben de er verdedigde opvattingen nog weinig gevonden.*

1. De *wetenschap* beschouwt herhaling in den tijd van als onderling gelijk stelbare volgreeksen van kwalitatieve verscheidenheid in den tijd. Dit vereenzamen der idee tot waarneembaarheid, en als zoodanig tot herhaalbaarheid, verschijnt na religielooze scheiding¹ tusschen subject en tot *iets anders* geworden onbereikte bereikbaarheid. De drang tot bereiking dezer bereikbaarheden wordt in het intellect volgens een wiskundig systeem van gestelde stelbaarheden, geboren uit abstractie van herhaling en herhaalbaarheden, gestuurd langs onmiddellijke bereiktheden.

Alles wat verschijnen kan als onbereikte bereikbaarheid, laat zich in systemen van gesteldheden intelligeeren, zoo ook religie; maar dan is de religieuze *wetenschap* religieloos: gewetensussend, of ijdel spel, of slechts van doelnajagende beteekenis.²

En, als alle religieloosheid, heeft wetenschap nòch religieuze betrouwbaarheid, nòch betrouwbaarheid in zich. | Allermint kan een wiskundig systeem van gesteldheden, los van de waarnemingen, die het intelligeerde, onbepaald vervolgd, betrouwbaar blijven in het richten langs die waarnemingen.

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Zoodat onafhankelijk van de waarneming volvoerde logische redeneeringen, die immers beteekenen wiskundige transformaties in het intelligeerende wiskundig systeem, uit wetenschappelijk aanvaarde praemissen onaannemelijke conclusies kunnen afleiden.³

De klassieke opvatting, die in de ervaringsgeometrie uit aanvaarde praemissen door volgens de logische principes gevoerde redeneeringen slechts onaanvechtbare conclu-

¹een vermogen, voortgekomen uit de oerzonde van vrees of begeerte, maar wederkeerend, ook zonder levende vrees of begeerte. vgl. L.E.J. Brouwer. *Leven, Kunst en Mystiek*. pag. 13–23.

²t.a.p. pag. 27.

³t.a.p. pag. 20, 21.

The unreliability of the logical principles

by

L.E.J. Brouwer

[Brouwer's note preceeding the 1919 reprint] *This essay could also today still be written in the same form. The opinions defended in it have, as yet, found few supporters.*

1. *Science* considers repetition in time of interidentifiable succession-sequences of qualitative differentiation through time. This isolating of the idea into an observable, and as such a repeatable, emerges after areligious^a separation¹ between the subject and an unreached reachable that has become *something separate*. In the intellect, the urge to reach these reachables is conducted along things immediately reached, according to a mathematical system of posited positables, born out of abstraction of repetition and repeatables.

Everything that can emerge as unreached reachable lets itself be intelligized in systems of posits,^b thus also religion; but then religious *science* is areligious: conscience-numbing, or idle play, or of merely goal-chasing significance.²

And science, as everything areligious, possesses *neither* religious reliability, *nor* reliability in itself. Least of all can a mathematical system of posits, separated from the observations it made intelligible, when continued indefinitely, remain reliable when directing along those observations.

Consequently logical argumentations, which, after all, consist in mathematical transformations in the mathematical system that makes [the observations] intelligible, may derive unlikely conclusions from scientifically accepted premises, when carried out independently of observation.³

The classical conception, which in experiential geometry witnessed reasonings — from accepted premises, carried out according to logical principles — derive only

¹A capacity, rooted in the original sin of fear or desire, but reappearing also without living fear or desire. Cf. Brouwer (1905, pp. 13–23).

²Brouwer (1905, p. 27).

³Brouwer (1905, pp. 20, 21).

^aNB not ‘antireligious’.

^b‘Posit’ not in Quine’s sense, but rather like Kant’s ‘Setzung’.

sies zag afleiden, induceerde de logische redeneeringen als methode van opbouw der wetenschap en de logische principes als menselijke vermogens tot opbouw van wetenschap.

Maar de geometrische redeneeringen gelden slechts voor een onafhankelijk van eenige ervaring in het intellect opbouwbaar wiskundig systeem, en dat een zoo populaire groep van waarnemingen als de geometrie het bedoelde wiskundig systeem zoo blijvend verdraagt, verdient, als alle proefhoudende natuurwetenschap, met wantrouwen te worden aangezien.

Het inzicht van de wetenschappelijke onbetrouwbaarheid der logische redeneeringen maakt, dat de conclusiën van Aristoteles omtrent de constitutie der natuur zonder praktische verifieering niet overtuigen; dat de waarheid, die bij Spinoza opengaat, geheel onafhankelijk wordt gevoeld van zijn logische systematiek; dat men niet gehinderd wordt door de antinomieën van Kant, en evenmin door het ontbreken van in al haar consequenties door te voeren physische hypothesen.

| Bovendien zijn bij de betoogen betreffende op wiskundige systemen gespannen 154
ervaringswerkelijkheden de logische principes niet het richtende, maar in de begeleidende taal achteraf opgemerkte regelmatigheid, en zoo men los van wiskundige systemen spreekt volgens die regelmatigheid, is er altijd gevaar voor paradoxen als die van Epimenides.

2. In religieuze waarheid, in *wijsheid*, die de splitsing opheft in subject en iets anders, is geen wiskundig intelligen, daar de verschijning van den tijd niet langer wordt aanvaard, nog minder dus betrouwbaarheid van logica. Integendeel, de taal der inkeerende wijsheid verschijnt ordeloos, onlogisch, omdat ze nooit kan voeren langs in het leven gedrukte systemen van gesteldheden, slechts hun breking kan begeleiden, en zoo misschien de wijsheid, die die breking doet, kan laten opengaan.⁴

3. Blijft de vraag, of dan althans de logische principes vaststaan voor van levensinhoud vrije *wiskundige systemen*, voor systemen opgetrokken uit de gestelde abstractie

⁴t.a.p. pag. 47 vlgg., 65 vlgg.

incontrovertible conclusions, induced the logical reasonings as method for building science, and the logical principles as human capacities for building science.

But the geometrical reasonings are valid only for a mathematical system that can be built in the intellect independently of any experience, and that such a popular group of observations as geometry corroborates the mathematical system in question so enduringly, deserves, like all experimental natural science, to be regarded with distrust.

The insight of the scientific unreliability of the logical reasonings has as consequence that Aristotle's conclusions on the constitution of nature are unconvincing without verification in practice; that the truth unveiled in Spinoza is experienced wholly independently of his logical architectonic; that one is not hindered by the antinomies of Kant, nor by the lack of physical hypotheses that can be carried through in all their consequences.

Moreover, regarding discourse concerning experiential realities that have been cast on mathematical systems, the logical principles are not directive, but regularities that have afterward been noticed in the accompanying language, and if one speaks according to these regularities with no link to mathematical systems, there is always the danger of paradoxes such as that of Epimenides.

2. In religious truth, in *wisdom*, which suspends the splitting into subject and something separate, there is no mathematical intellection, as the appearance of time is no longer accepted, even less thus the reliability of logic. On the contrary, the language of inward-turning wisdom appears without order, illogical, because it can never carry along systems of posits pressed upon life, but can only accompany their breakdown, and thus perhaps unveil the wisdom that effects the break.⁴

3. The question remains whether then the logical principles hold at least for *mathematical systems* that are free of living content,^c for systems erected from posited

⁴Brouwer (1905, p. 47ff,65ff).

^cFor the expression 'living content', compare: '... the *existence* of that mathematical reasoning system does not entail that it *lives*, in other words that it accompanies a chain of thoughts ...' Brouwer (1907, p. 138n, emphasis Brouwer, trl. ours) (... volgt uit het *bestaan* van dat wiskundig redeneersysteem nog niet, dat dat taalsysteem *leeft*, m.a.w. een aaneenschakeling van gedachten begeleidt...).

van herhaling en herhaalbaarheid, uit de gestelde inhoudslooze tijdsintuïtie, uit de oer-intuïtie der wiskunde.⁵ Door alle tijden is in wiskunde met vertrouwen logisch geredeneerd; nooit aarzelde men, door logica uit postulaten getrokken conclusies te aanvaarden, waar de postulaten gelden. In dezen tijd zijn echter paradoxen geconstrueerd, die wiskundige paradoxen schijnen⁶, en wantrouwen wekken tegen het | 155 vrije gebruik van logica in wiskunde, zoodat enkele wiskundigen hun vooronderstelling van logica in wiskunde loslaten, en logica en wiskunde tezamen trachten op te bouwen⁷, in aansluiting aan de door Peano gegrondveste school der *logistiek*. Aangeetoond kan echter worden⁸, dat deze paradoxen voortkomen uit dezelfde dwaling als die van Epimenides, dat ze namelijk ontstaan, waar regelmatigheid in de taal, die wiskunde begeleidt, wordt uitgebreid over een taal van wiskundige woorden, die geen wiskunde begeleidt; dat verder de logistiek eveneens zich bezighoudt met de wiskundige taal in plaats van met de wiskunde zelf, dus de wiskunde zelf niet verheldert; dat ten slotte alle paradoxen verdwijnen, als men zich beperkt, slechts te spreken over expliciet uit de oer-intuïtie opbouwbaar systemen, m.a.w. in plaats van logica door wiskunde, wiskunde door logica laat vooronderstellen.

Zoo blijft nu alleen nog de meer gespecialiseerde vraag: “Kan men bij zuiver wiskundige constructies en transformaties de voorstelling van het opgetrokken wiskundig systeem tijdelijk verwaarloozen, en zich bewegen in het begeleidend taalgebouw, geleid door de principes van *syllogisme*, van *contradictie* en van *tertium exclusum*, in vertrouwen dat door tijdelijke oproeping van de voorstelling der bereedeneerde wiskundige constructies telkens elk deel van het betoog zou kunnen worden gewettigd?”

Hier zal blijken, dat dit vertrouwen voor de beide eerste principes wèl, voor het laatste niet gegrond is.

Het *syllogisme* vooreerst leest in de inpassing van een | systeem *b* in een systeem *c* 156 en de daarmee samengaande inpassing van een systeem *a* in het systeem *b* een directe inpassing van het systeem *a* in het systeem *c*, wat niet anders is dan een tautologie.

⁵vgl. L.E.J. Brouwer. Over de Grondslagen der Wiskunde. pag. 8, 81, 98, 179.

⁶Burali-Forti. (Rendiconti del circolo Matematico di Palermo. 1897. p. 164).

Zermelo. (Mathematische Annalen 59). Koenig. (ibid. 61).

Richard. (Revue générale des Sciences. 1905).

Russell. (The Principles of Mathematics. Part I. Chap. X).

Voor pogingen tot oplossing dezer paradoxen vgl., behalve de opstellers zelf: Poincaré. (Revue de Métaphysique et de Morale. 1905 no. 6, 1906 no. 1, 3). Mollerup. (Mathematische Annalen 64).

Schoenflies. (Bericht über die Mengenlehre. II. Kap. I. §7).

⁷in het bijzonder Hilbert in Verhandlungen des internationalen Mathematiker-Congresses in Heidelberg 1904. p. 174.

⁸Grondslagen der Wiskunde. III.

abstraction of repetition and repeatability, from the posited contentless intuition of time, from the [Ur-]intuition of mathematics.⁵ Through all ages, in mathematics one has reasoned logically with confidence; never did one hesitate to accept conclusions drawn from postulates by logic, where the postulates hold. In this time, however, paradoxes have been constructed that seem to be mathematical paradoxes,⁶ and that arouse distrust against the free use of logic in mathematics, so that some mathematicians let go of their presupposition of logic in mathematics, and try to build up logic together with mathematics,⁷ following the school of *logistics* founded by Peano. It can be shown,⁸ however, that these paradoxes result from the same error as that of Epimenides, namely, that they arise where regularities in the language that accompanies mathematics are extended over a language of mathematical words that does not accompany mathematics; that, further, logistics too is concerned with the mathematical language instead of with mathematics itself, thus does not clarify mathematics itself; that, finally, all paradoxes disappear, when one restricts oneself to speaking only of systems that explicitly can be built out of the Ur-intuition, in other words, when instead of letting mathematics presuppose logic, one lets logic presuppose mathematics.

Thus, now only the more specific question still remains: ‘Can one, in the case of purely mathematical constructions and transformations, temporarily neglect the presentation of the mathematical system that has been erected, and move in the accompanying linguistic building, guided by the principles of *the syllogism*, of *contradiction*, and of *tertium exclusum*, always confident that, by momentary evocation of the presentation of the mathematical constructions suggested by this reasoning, each part of the discourse could be justified?’

Here it will turn out that this confidence is well-founded for each of the first two principles, but not for the last.

To begin with, the *syllogism* reads the fitting of a system *b* into a system *c* and the concomitant fitting of a system *a* into the system *b* as a direct fitting of the system *a* into the system *c*, which is nothing but a tautology.^d

Nor can the principle of *contradiction* be assailed: completing the fitting of a

⁵Cf. Brouwer (1907, pp. 8, 81, 98, 179.).

⁶Burali-Forti (1897), Zermelo (1904), Koenig (1905), Richard (1905), Russell (1903, Part I. Chap. X). For attempts at solving these paradoxes see, besides the proposers themselves: Poincaré (1905–1906), Møllerup (1907), Schoenflies (1908, Kap. I. § 7). [Note that the preface to that work is dated ‘im Oktober 1907’.]

⁷In particular Hilbert in Hilbert (1905).

⁸Brouwer (1907, ch. III).

^dA tautology in the sense of, for example, Kant, not Wittgenstein.

Evenmin is aanvechtbaar het principe van *contradictie*: het volvoeren van de inpassing van een systeem *a* op bepaalde wijze in een systeem *b*, en het stuiten op de onmogelijkheid van die inpassing sluiten elkander uit.

Nu het principium *tertii exclusi*: dit eischt, dat iedere onderstelling of juist of onjuist is, wiskundig: dat van iedere onderstelde inpassing van systemen op bepaalde wijze in elkaar hetzij de beëindiging, hetzij de stuiting op onmogelijkheid kan worden geconstrueerd. De vraag naar de geldigheid van het principium tertii exclusi is dus equivalent met de vraag naar de *mogelijkheid van onoplosbare wiskundige problemen*. Voor de wel eens uitgesproken⁹ overtuiging, dat onoplosbare wiskundige problemen niet bestaan, is geen aanwijzing van een bewijs aanwezig.

Zoolang alleen bepaalde eindige discrete systemen gesteld worden, is het onderzoek naar de mogelijkheid of onmogelijkheid eener inpassing steeds beëindigbaar en voerend tot antwoord, is dus het principium tertii exclusi een betrouwbaar redeneer-principe.¹⁰

Dat ook oneindige systemen ten opzichte van zoovele eigenschappen eindig worden beheerscht, geschiedt door overzien van de aftelbaar oneindige reeks der geheele getallen met *volledige inductie*¹¹, namelijk door opmerken van eigenschappen, d.w.z. inpassingen, die voor een *willekeurig geheel getal* gelden, in het bijzonder ook van contradicties, dat zijn onmogelijke inpassingen, die voor een *willekeurig geheel getal* gelden. Dat echter uit de in een vraag gestelde systemen een is af te leiden, dat door een invariant over een aftelbaar oneindige reeks de vraag volledig induceerend leest, en zoo oplost, blijkt eerst a posteriori, als toevallig de constructie van zulk een systeem gelukt is. Want het geheel der uit de vraagstelling te ontwikkelen systemen is *aftelbaar onaf*¹², dus niet a priori methodisch te onderzoeken ten opzichte van de aanwezigheid of afwezigheid van een de vraag beslissend systeem. En het is niet uitgesloten, dat een even gelukkige greep, als zoo dikwijls de beslissing bracht, eens het aftelbaar onaffe systeem der mogelijke ontwikkelingen tot een onoplosbaarheid zou overzien.

Zoodat in oneindige systemen het principium tertii exclusi vooralsnog niet betrouwbaar is. Toch zal men bij ongerechtvaardigde toepassing nooit kunnen stuiten

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⁹vgl. Hilbert. Mathematische Probleme. Göttinger Nachrichten. 1900. Ook Schoenflies (l.c.) wil onvoorwaardelijk de methode van het indirecte bewijs handhaven, die hij ten onrechte uitsluitend van het principium contradictionis afhankelijk acht.

¹⁰Dit onderzoek kan zelfs steeds door een machine worden uitgevoerd, of door een gedresseerd dier, vereischt niet de oer-intuïtie der wiskunde, levend in een menschelijk intellect. Maar tegenover vragen betreffende oneindige verzamelingen wordt die oer-intuïtie telkens weer onmisbaar; door dit voorbij te zien, zijn Peano en Russell, Cantor en Bernstein slechts tot dwalingen gekomen.

¹¹Poincaré is misschien de eenige, die in de volledige inductie 'le raisonnement mathématique par excellence' heeft herkend. Vgl. La Science et l'Hypothese. Chap. I.

¹²vgl. Grondslagen der Wiskunde. p. 148.

system a in a certain way into a system b , and being blocked by the impossibility of that fitting, exclude one another.

Now the principium *tertii exclusi*: this demands that every supposition is either correct or incorrect, mathematically: that of every supposed fitting in a certain way of systems in one another, either the termination or the blockage by impossibility, can be constructed. The question of the validity of the principium *tertii exclusi* is thus equivalent to the question concerning the *possibility of unsolvable mathematical problems*. For the already proclaimed conviction that unsolvable mathematical problems do not exist, no indication of a demonstration is present.⁹

As long as only certain finite discrete systems are posited, the investigation into the possibility or impossibility of a fitting can always be terminated and leads to an answer, whence the principium *tertii exclusi* is a reliable principle of reasoning.¹⁰

That also infinite systems, with respect to so many properties, are controlled by finite means, is achieved by surveying the denumerably infinite sequence of the whole numbers by *complete induction*,¹¹ namely by observing properties, that is, fittings, that hold for an *arbitrary whole number*, and in particular also contradictions, that is, impossible fittings, that hold for an *arbitrary whole number*. However, that from the systems posited in a question, one can be derived that reads the question by means of a complete induction, on the basis of an invariant in a denumerably infinite sequence, and thereby solves it, is found only a posteriori, when accidentally the construction of such a system has succeeded. For the whole of the systems that can be developed from the question posed is *denumerably unfinished*,¹² whence cannot be a priori investigated methodically regarding the presence or absence of a system that decides the question. And it is not excluded, that by a draw as lucky as the ones that have so often led to a decision, we will one day see from the denumerably infinite system of possible developments that it is unsolvable.

So that in infinite systems the principium *tertii exclusi* is as yet not reliable. Still, one can never, in unjustified application, be blocked by a contradiction and thereby discover the groundlessness of one's reasonings. After all, to that end it would have

⁹Cf. Hilbert (1900). Also Schoenflies (1908) wants to uphold the method of indirect proof unconditionally, which he mistakenly considers to depend only on the principium contradictionis.

¹⁰This investigation itself can always be done by a machine or by a trained animal, not requiring the intuition of mathematics living in a human intellect. But in face of questions involving infinite sets, that intuition becomes, again and again, indispensable; by overlooking this, Peano and Russell, Cantor and Bernstein have only arrived at errors. [Brouwer gives an exposition of these errors in the chapter 3 of his dissertation Brouwer 1907, 'Wiskunde en Logica'.]

¹¹Poincaré is perhaps the only one who has recognized mathematical induction as 'le raisonnement mathématique par excellence'. See Poincaré (1902, Chap. I).

¹²Cf. Brouwer (1907, p. 148).

op een contradictie en zoo de ongegrondheid van zijn redeneeringen ontdekken. Immers daartoe zouden de volvoering en de contradictoriteit van een inpassing beide tegelijk contradictoor moeten kunnen zijn, wat het principium contradictionis niet toelaat.

Een sprekend voorbeeld levert de volgende onbewezen stelling, die op grond van het principium tertii exclusi in de gangbare theorie der transfinite getallen algemeen vertrouwd en gebruikt wordt, dat n.l. elk getal is òf eindig òf oneindig, m.a.w. dat voor elk getal γ kan worden geconstrueerd:

| hetzij een afbeelding van γ geheel op de rij der geheele getallen zóó, dat daarbij 158
een getal α uit die rij *het laatste* is (de getallen $\alpha + 1, \alpha + 2, \alpha + 3, \dots$ vrij blijven),
hetzij een afbeelding van γ geheel of gedeeltelijk op de rij der geheele getallen in haar geheel.¹³

Zoolang deze stelling onbewezen is, moet men voor onzeker houden, of vragen als:

“*Is bij de decimale ontwikkeling van π een cijfer, dat duurzaam veelvuldiger op-treedt, dan alle andere?*”

“*Komen bij de decimale ontwikkeling van π oneindig veel paren van gelijke opeen-volgende cijfers voor?*”

een oplossing bezitten.

En evenzoo onzeker blijft, of de algemeenere wiskundige vraag:

“*Is in de wiskunde het principium tertii exclusi onbepaald geldig?*”

een oplossing bezit.¹⁴

Samenvattende:

In wijsheid is geen logica.

In wetenschap is logica vaak, maar niet duurzaam doeltreffend.

In wiskunde is niet zeker, of alle logica geoorloofd is, en is niet zeker of is uit te maken, of alle logica geoorloofd is.

¹³De eventuele onjuistheid dezer stelling zal weer nooit in een contradictie kunnen blijken; immers de contradictoriteit van de constructie der vrij blijvende rij $\alpha + 1, \alpha + 2, \alpha + 3 \dots$, en die van haar contradictoriteit kunnen nooit tezamen optreden.

¹⁴Men behoort dus in wiskunde de gewoonlijk als *bewezen* geldende stellingen te onderscheiden in *juiste* en *niet-contradictore*. Tot de eerste behooren de algebraïsche en analytische gelijkheden, en de geometrische snijpuntsstellingen; ook, dat een puntverzameling geen andere machtigheid bezitten kan, dan de (Grondslagen, pag. 149) genoemde. Tot de laatste, dat een puntverzameling zeker een dier machtigheden bezit; ook, dat een afgesloten puntverzameling zich laat splitsen in een perfecte en een aftelbare.

to be possible for the execution and contradictoriness of a fitting to be simultaneously contradictory, which the principium contradictionis does not allow.

A striking example is provided by the following undemonstrated proposition, which on the ground of the principium tertii exclusi is generally trusted and used in the current theory of transfinite numbers, namely that every number is either finite or infinite, in other words, that for every number γ one can construct:

either a mapping of all of γ to the sequence of the whole numbers, such that a number α in that sequence is *the last one* (the numbers $\alpha + 1, \alpha + 2, \alpha + 3, \dots$ remain free),

or a mapping of all or part of γ to the sequence of the natural numbers in its entirety.¹³

As long as this proposition is undemonstrated, it must be held uncertain whether questions such as:

‘Is there in the decimal expansion of π a digit that occurs enduringly more often than all others?’

‘Do there occur in the decimal expansion of π infinitely many pairs of equal consecutive digits?’

have a solution.

And likewise, it remains uncertain whether the more general mathematical question:

‘Is in mathematics the principium tertii exclusi unconditionally valid?’

has a solution.¹⁴

Summarizing:

In wisdom is no logic.

In science, logic is often, but not enduringly efficacious.

In mathematics it is not certain whether all logic is permissible, and it is not certain whether it can be decided whether all logic is permissible.

¹³A latent incorrectness of this proposition also shall never become clear from a contradiction: after all, the contradictoriness of the construction of the sequence $\alpha + 1, \alpha + 2, \alpha + 3, \dots$ which remains free and that of its contradictoriness can never occur together.

¹⁴One should therefore in mathematics distinguish the propositions that are usually taken to have been *demonstrated* into *correct* and *non-contradictory* ones. To the former belong the algebraic and analytic equalities, and the geometrical incidence theorems; also, that a point set can have no other cardinality than those mentioned in Brouwer (1907, p. 149). To the latter, that a point set does indeed have one of those cardinalities; also, that a closed point set can be split into a perfect and a denumerable one.

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