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VESTIGES OF REALISM

Göran Sundholm

In the Trinity Term of 1977 I had the privilege of hearing Michael Dummett lecture in the Old Library at All Souls on the justification and criticism of logical laws, and subsequently I also assisted Mark Helme in the production of some unauthorised, not altogether successful, notes on these lectures. I can vividly recall my surprise when Prof. Dummett stated, in connection with the idea with which he had up till then been mainly concerned, namely the idea that meaning is primarily determined in terms of the Gentzen introduction rules, that my old tutor, the Swedish logician Per Martin-Löf, in correspondence, held that "the meanings of the logical constants CAN IN NO WAY be given by the introduction rules, rather they are given by the elimination rules".¹ This was, as far as I knew from tutorials and conversations with Martin-Löf, not a position he would defend at the time and certainly not today. In fact, he has steadfastly held by the introduction-rule orthodoxy in all relevant publications from 1979 onwards.² Taken jointly those publications constitute, as far as I know, the only sustained effort towards a realisation of a constructivist theory of meaning for a sizeable interpreted language serving the needs of pure mathematics on a scale comparable to that of Frege's *Grundgesetze*. From this circumstance alone it seems clear that much of Dummettian interest can be found in the works of Martin-Löf. Indeed, in my opinion, it would have been most valuable if Prof. Dummett had found space for a confrontation with some of Martin-Löf's views in his recent treatise on *The Logical Basis of*

¹See p. 43 of the Helme notes. This is the text of a lecture given at the First International Philosophy Conference, Mussomeli, Sicily, devoted to the philosophy of Michael Dummett. The paper was written for oral delivery with Prof. Dummett in the audience and that format has been retained in this published version.

²See [108], in [26]; [109] (notes from lectures delivered in 1980); [110], in: *Atti degli incontri di logica matematica 2*, Scuola di specializzazione in Logica Matematica, Dipartimento di Matematica, Università di Siena 1985 (Notes from lectures delivered in 1983); and especially: [111] (lecture delivered at the workshop *Theories of meaning* at the Villa di Mondeggi, Florence, 1985) and [112], in [27].

Metaphysics.³ Alas, only a brief mention is made of Martin-Löf's position and, unfortunately, in slightly erroneous terms.⁴ Accordingly, when I received the kind invitation from the organisers of the present conference, my choice of topic was easy: a critical examination of Prof. Dummett's views from the constructivist philosophical perspective outlined in Martin-Löf's works. Many professed realists have felt it incumbent upon themselves to offer criticisms of Prof. Dummett's anti-realism. The present critical examination will be essayed from a less common position: I largely share his constructivist inclinations and my points belong inside the constructivist framework.

A discussion of Prof. Dummett's work will, as likely as not, take some theme from within the Theory of Meaning as its point of departure and the present exposition is no exception. In particular, I want to consider his use of various mathematical theories of constructions as possible blueprints for core theories in a meaning theory. Prof. Dummett refers with approval to the work of Kreisel and Goodman,⁵ which is directed towards a formalised

mathematical theory of constructions which should serve precisely the purpose of providing a semantical theory incorporating the intuitive explanations of the logical constants. ... [T]here is no doubt that these standard intuitive explanations of the logical constants determine their intended intuitionistic meanings, so that anything which can be accepted as the correct semantics for intuitionistic logic must be shown either to incorporate them or, at least, to yield them under suitable supplementary assumptions.⁶

and furthermore he notes that

their efforts have not as yet been fully successful: if they had, then we should undoubtedly have, in the theory of constructions, what all would recognise as being the standard semantics for intuitionistic logic in the same sense as that in which the two-valued semantics is standard for classical logic.⁷

³Published by Duckworth, London, 1991.

⁴See [55], p. 280.

⁵The relevant works by Kreisel are: [91], in [116]; and [92], in [138] (especially pages 119-133); and by Goodman his doctoral dissertation [73] and later papers derived from it.

⁶See [47], p. 389. Hereinafter this work will be cited as *E. of I.*

⁷See [55], p. 27. Given that these efforts are, by now, more than twenty years

The efforts of Kreisel (and consequently also those of Goodman, which derive from Kreisel's) have a chequered pre-history which one might conveniently start in 1837, when Bolzano's *Wissenschaftslehre* appeared in four mighty tomes. Here a radical break with the logical tradition is carried out: the traditional 'Aristotelian' two-term form of judgement

S is P

is rejected in favour of a more simple form

A is true

where A is a propositional content (a *Satz an sich* in the terminology of Bolzano). Frege, probably in complete independence from Bolzano, used what comes down to the same form, and thus there is no advance to be found in the works of Frege regarding the *form* of judgement: his brilliant innovative insight is concerned with the propositional content of the judgements, which he analysed in terms of the mathematical function/argument structure, whereas Bolzano had just transformed (a version of) the Aristotelian form of judgement into the form of content

(object) A has (property) b.

It must be stressed, though, that the notion of truth that is involved in this form of judgement, was left essentially unanalysed by Frege: indeed, in *Der Gedanke* he even claims truth to be indefinable.

There are two traditional principles which play a central role in the interplay between logic and epistemology. The first is, of course, the law of Excluded Middle, or for that matter, as Prof. Dummett, more than anyone else has made us aware, the law of Bivalence. The other principle, which is as time-honoured as that of Bivalence, is that principle to which Prof. Dummett has referred as K, presumably for *knowledge*: if a proposition is true, then it is in principle possible to know that it

old, it seems safe to conclude that we are not going to learn more from either Kreisel or Goodman about the theory of constructions. Accordingly, the above passage of Prof. Dummett's is slightly misleading in creating, as it does, the impression that both authors are still hot in pursuit of the one true theory of constructions. Indeed, since that time, the main efforts have been at the hands of others, but Martin-Löf is the only author who has pursued the theme with vigour and persistence.

is true.⁸ It is only in 1908 that the tension between this principle of Knowability and that of Bivalence is felt, simultaneously in two different places, namely Cambridge and Amsterdam. G.E. Moore is the first to contemplate unknowably true propositions in view of the fact that, as a Realist, he could not allow for the abolition of the law of Bivalence.⁹ L.E.J. Brouwer, on the other hand, took the opposite way out in his reaction to this dilemmatic tension and chose to retain the principal knowability of truth, but had then to refrain from acknowledging the law of Bivalence, and the ensuing law of Excluded Middle, in their full generality.¹⁰

The next (and here I deliberately ignore Russell's works on the theory of judgement)¹¹ step in the development towards the theories of construction was taken by Wittgenstein in the *Tractatus* through a consequent application of a maxim which Prof. Dummett has labelled Principle C (for correspondence), namely the principle that if a proposition is true there must be something in virtue of which it is true.¹² In other words, when a proposition is true, there is a 'truth-maker' for the proposition in question.¹³ The Tractarian analysis of truth is well-known: a proposition presents a certain *Sachverhalt* (state of affairs) and the proposition is true if the relevant state of affairs exists and false otherwise.¹⁴ Thus we may say that the *Tractatus* provides the paradigm for how an extreme logical realist might construe the truth-maker analysis of truth.

Roughly ten years after Wittgenstein's realist analysis of the concept

⁸See [46], in [58], p. 99. I have slightly changed the formulation of K. Prof. Dummett formulates his principle for statements rather than for propositions, presumably so as not to be bothered by the charge of neglecting the phenomena of indexicality. Since I am mainly concerned with the language of mathematics this is a worry to which I can safely assign a low priority. Furthermore, since the term 'statement' is heavily overburdened in contemporary philosophical logic — it has been used (though not by Prof. Dummett) for the act of judgement/assertion, the judgement passed/assertion made, the propositional content, the sentence expressing the propositional content, and the state of affairs that serves as truth-maker for the proposition when it is true — it seems prudent to avoid it altogether and bite the bullet of indexicality in some other way.

⁹See [113], reprinted in [114].

¹⁰See [20].

¹¹Interesting and relevant information on Russell can be found in [95].

¹²M. Dummett, *Ibid.*, p. 89.

¹³See [115].

¹⁴See [166], §§2.202, 2.21 and 4.021.

of truth, Arend Heyting offers an anti-realist analysis, also by means of an application of the Principle C: if a proposition is true, then there exists a proof of the proposition.¹⁵ The further vicissitudes of the realist applications of the Principle C in Tarski's theory of truth and the ensuing model-theoretic notion of 'semantics' has to be left for another occasion; it is Heyting's use of the proofs of propositions to provide systematic meaning-explanations that is of main concern to us now.¹⁶ Heyting construes his propositions with explicit reference to the phenomenological tradition of Edmund Husserl, in the particular shape of the book by Oskar Becker on *Mathematische Existenz*.¹⁷ Accordingly, a proposition is, or expresses, an Intention towards a mathematical object, namely a construction that proves the proposition. The proposition is true if the Intention can be fulfilled, or realised, that is, if there exists a certain construction which satisfies certain conditions, depending on the proposition in question. In particular, it seems clear that one has to take two notions of proof into account to do justice to Heyting's formulations, given their specific, phenomenological background. Thus the proofs of propositions, in terms of which propositions are explained, as when one says that the conjunction $A \wedge B$ of the propositions A and B is a proposition which has got constructions (a, b) as its proofs, where a is a proof of A and b is a proof of B, are *objects* and, as such, the objects of *acts* of construction. The proof, on the other hand, in the sense of that through which one gets to know the truth of a proposition, that is, the proof of a judgement/assertion of the form

A is true,

is not an object, but an act. In this latter sense we must allow that Brouwer is perfectly right in his claim that mathematical proofs (like any other proofs, in fact) are mental acts.¹⁸ Thus, given that, according

¹⁵[83] and the famous Königsberg address [84] are the two important Heyting-papers here. In my [150], I have dealt at some length with Heyting's formulations and how they should best be read.

¹⁶In [153], invited contributed paper at the IXth International Conference on Logic, Methodology and Philosophy of Science, Uppsala, August 1991, and intended for a special volume of such invited papers, I offer a further examination of the later versions of the realist application of Principle C.

¹⁷See [8], also as a separate study, Max Niemeyer, Tübingen, 2nd edition 1973.

¹⁸Prof. Dummett's well-known argument against realism can be squared with this Brouwerian opinion, if we recall that Brouwer was writing during the period when

to Heyting, one is only allowed to claim the existence of a proof of the proposition A on the basis of having found one, it holds that the form of the theorem proved by the act of proof, when fully elucidated, will be

c is a proof of A,

where c is the construction found in the act of construction/proof. Heyting is insistent that the assertion of a proposition A is not itself propositional in nature and consequently, on his view, the relation between construction-object and proposition could not be propositional.¹⁹ In conclusion we can clearly state that Heyting's use of the Principle C conforms to the semantic principle that meaning is given via truth-conditions: you explain the meaning of propositions in terms of truth, but the notion of truth in its turn is construed according to the constructivist reading of the Principle C. A proposition A is true, if a proof-object exists as demanded by the proposition. Here the notion of existence has to be taken constructively. Thus one would only claim the existence of such a proof-object on the basis of actually having carried out the act of construction whereby one was found.²⁰

Kreisel and Goodman have cast their theories very much in the style of a standard first order theory with one single Fregean, or Scotist (after Duns Scotus), universe that serves as the range of definition, respectively of significance, for the functions and predicates of their respective theories. In particular, for each given proposition A, the proof-relation

a psychologistic view of the mental prevailed. The stress that Prof. Dummett lays on the outward manifestation of knowledge and other propositional attitudes, can be seen as an application of the Wittgensteinian insight that the external anchoring in a form of life of our mental acts is primary to their internal aspects. Again, the same point can be applied also to accommodate Oskar Becker's use of Husserl's meaning-giving and other acts in the theory of meaning.

¹⁹This account of Heyting is based on a close reading of the *Erkenntnis* article from 1931. The last point concerning the non-propositional status of the proof-relation may be inferred from Heyting, but is not explicitly stated in his text. Obviously, the absence of the inference in the text does not plead for my reading. It does, however, not plead against the reading either, since there was really no reason for him to draw such an inference, given his immediate concerns. In Prof. Dummett's writings the act/object distinction with reference to proofs is generally not dealt with and, accordingly, not applied.

²⁰For further elaboration of these ideas, see the discussion of the judgement
 α exists
 in connection with the Fifth Vestige of Realism.

among constructions is construed as an ordinary propositional function which has to be *decidable* in the sense that the following scheme should hold:²¹

$$\forall c(\prod(c, A) \vee \neg \prod(c, A)).$$

On the mathematical side this was motivated by the primitive recursiveness of the (Gödelized) proof-predicates for standard formal systems, whereas the philosophical motivation was given in terms of the slogan

We recognise a proof when we see one.

Prof. Dummett has been a staunch upholder of the decidability of proof-relations throughout his writings on anti-realist matters, and I here confine myself only to a very recent quote professing adherence to this doctrine:

a proposition is a decidable classification of constructions (into those that are and those that are not proofs of the statement).²²

This reads very well as a meaning-theoretical version of Kreisel's opinion that

[t]he sense of a mathematical assertion denoted by a linguistic object A is intuitionistically determined (or understood) if we have laid down what constructions constitute a *proof* of A , i.e. if we have a construction r_A such that, for any construction c , $r_A(c) = 0$ if c is a proof of A and $r_A(c) = 1$ if c is not a proof of A .²³

²¹ Kreisel uses a certain three-place predicate which specialises to the present use. The present use of the Π -predicate simplifies matters with respect to Kreisel's theory, but not misleadingly so.

²² See [55], p. 29. In my [150], p. 155, a retrospective list of places is given where Prof. Dummett insists on the decidability of the proof-relation. Here, I take it, either the use of 'statement' or of 'proposition' must be just as a slip of the pen.

²³ See [91], p. 201. The gist of the quote is clear: the r_A construction is the characteristic function of the proof relation with respect to the proposition A . The detailed meaning poses fascinating problems of interpretation, though. Do *assertions* have sense? Is it not rather the proposition expressed by the linguistic object A , the utterance of which could be used to make an assertion, which is the sense of the expression (= linguistic object)? Furthermore, does the understanding of a proposition presuppose the idea of the characteristic function of the proof-predicate for the proposition in question? Is not that an extra idea, namely the idea of definitions of functions by cases, over and above the idea of a proof of the proposition and the recognition of what constructions can serve to prove the proposition in question? These questions

Kreisel has always had a fondness for the philosophical point well put in terms of an *obiter dictum*.²⁴ One of his best efforts in this vein was cast in the shape of a question:

Was the (logical) language of current intuitionistic systems obtained by uncritical transfer from languages which were, tacitly, understood classically?²⁵

Concerning Kreisel's own early formulations of his theories of constructions, with their concomitant remarks on the decidability of the proof-relation, it seems to me that the answer to this most pertinent question has to be a resounding YES. This remark would rebound on Kreisel on the (meta)mathematical level concerning his formalism, but Prof. Dummett, whose philosophical, meaning-theoretical remarks are based on the metamathematical formulations of Kreisel, stands as open to the implicit charge in Kreisel's Question as Kreisel does. Accordingly this is where I want to mark the **First vestige of realism** in Prof. Dummett's work: the proof-predicate is viewed as a, more or less standard, propositional function over a unified Scotist universe of constructions, but this is not the only way to read the dictum "We recognise a proof when we see one". Indeed, one of the most attractive features of the Philosophy of Logic offered in Wittgenstein's *Tractatus* (6.113) is its insistence on the mechanical calculability from the symbol itself of various properties. My suggestion is now that this is the right way to view the decidability of the proof-relation: given a symbol, it must be possible by means of mechanical, syntactic, calculation on the sign alone, to decide whether or not the given symbol is a proof of a certain proposition, and in the type theory of Per Martin-Löf this is indeed the case.²⁶ He explicitly

may serve to underline the intricacies that are commonly inherent in the standard writings on the borderline between (meta)-mathematics, foundations of mathematics and the philosophy of language (meaning theory).

²⁴ Prof. Dummett has given wide currency to *Kreisel's dictum*: "The point is not the existence of mathematical objects, but the objectivity of mathematical statements" (See [50], pp. xxviii and 228, and many other places), concerning which questions were set in the B.phil. examinations during my time at Oxford. Prof. Dummett refers to a review of Wittgenstein for the quote in question. What Kreisel states in footnote 1 on p. 138 of his review of Wittgenstein's 'Remarks on the Foundations of Mathematics', in the *British Journal for the Philosophy of Science* Vol. no. 9, 1958, is that "Wittgenstein argues against a mathematical object (presumably: substance), but, at least in places [...] not against the objectivity of mathematics, especially through the recognition of formal facts". *Se non è vero, è ben trovato...*

²⁵ See [93], in [154], p. 268.

²⁶ I here would like to withdraw the tentative remarks I made as to the possible

opts for a Russellian, or Thomistic, typed universe in which every object belongs to a *category* (Aristotle) [*type* (Russell), *sort* (Locke)], or falls under a *universal* (general concept) (Thomas Aquinas), depending on which terminology one might prefer.

Martin-Löf returns to the basic traditional form of judgement

S is P

referred to above, but adds some extra twists, taking the later developments from Bolzano onwards into account. The form of judgement is now

a: α ,

in words, 'a is an object of category α '. As Prof. Dummett has stressed, it is not enough to know these application-conditions in order to know a category of entities; one must also know the identity-conditions (or *criteria*) for objects of the category in question.²⁷ Accordingly, Martin-Löf also operates with a second basic form of judgement, namely

$a = b$: α ,

in words, 'a and b are equal objects of category α '. A particularly important category is the category of sets. The application criterion for set is that in order to know an element of this category, that is in order to know a set, one must know how canonical elements of the set may be formed and, secondly, when two canonical elements of the set are equal elements of the set in question. The identity condition, or criterion, for the category set is that two sets A and B are equal elements of the category set, when canonical elements of one are canonical elements of the other and vice versa, and equal canonical elements of the one are equal canonical elements of the other, and vice versa. In other words,

$A = B$: set

when the rules

semi-decidability of proofs in my [152], in [65], vol. VIII, Ch. III:8, p. 493.

²⁷See [41].

$$\frac{a : A}{a : B}$$

and

$$\frac{a = b : A}{a = b : B}$$

as well as their inverses, are correct for canonical objects. According to the so called propositions-as-types doctrine, which is nothing but a precise version of Heyting's notion of proposition as explained above, one now defines

$$\text{prop} = \text{set: cat},$$

that is, the category of propositions is nothing but the category of sets. In this way every proposition is viewed as a set, namely as the set of its proof-objects and conversely a set is viewed as a proposition, namely as the proposition which is proved by exhibiting an element of the set, that is the proposition which has the elements of the given set as its proof-objects.²⁸ In essence Martin-Löf's work can be seen as a sustained effort to develop a sizeable mathematical language from this fundamental idea, and its, by now quite considerable, philosophical elaboration. Thus he can be characterised, in terms first introduced by Sir Isaiah Berlin, as a hedgehog who knows that he is a hedgehog, that is a thinker whose work is guided by one basic idea that is worked out over a wide field and in great depth.²⁹

In order to make the above abstract explanations slightly more manageable I will consider the well-known example of conjunction. There are four sorts of rules involved here, namely Formation, Introduction, Elimination, and Equality rules. The rule of \wedge -Formation states that $A \wedge B$ is a proposition, when A and B are both propositions, in symbols

$$\frac{A : \text{prop}, B : \text{prop}}{A \wedge B : \text{prop}}$$

The rule of \wedge -Introduction gives meaning to this rule by laying down how canonical elements of the set $A \wedge B$ may be formed:

²⁸ See [86], in [141].

²⁹ See [12]. According to Berlin, Tolstoy was a hedgehog who thought he was a fox, that is a thinker who explores many ideas along various avenues of thought.

$$\frac{a : A, b : B}{(a, b) : A \wedge B}$$

and

$$\frac{a = a' : A, b = b' : B}{(a = b) = (a', b') : A \wedge B}$$

that is, given proof-objects for A , respectively B , their ordered pair is a canonical proof-object for the proposition $A \wedge B$. Once the meaning of conjunction has been laid down via this meaning-giving rule other aspects of the use of conjunction may be justified. Thus, for instance, the \wedge -Elimination rules

$$\frac{c : A \wedge B}{p(c) : A}$$

and

$$\frac{a : A \wedge B}{q(c) : B}$$

are correct, where p and q are the left, respectively right, *projections*, obeying the following \wedge -Equality rules that serve to give substance to the Elimination rules in the same way that the Introduction rule gave substance to the Formation rule:³⁰

³⁰In fact, in order to ensure the required syntactic decidability of the proof-relation, some additional type information needs to be included in the proof-objects. Thus the Introduction rule should properly read

$$\frac{a : A, b : B}{\wedge I(A, B, a, b) : A \wedge B}$$

and the Equality rule

$$\frac{c : A \wedge B}{\wedge E(A, B, \wedge I(A, B, a, b)) = a : A}$$

Note that the additional type-information is not of the 'second clauses' kind, well-known from the writings of Kreisel and Goodman and their followers. The second clauses introduce constructions that have to prove that the functions involved in the explanations of implication and the universal quantifier really do what they are supposed to do. Thus the second construction will have to be a proof-object for a proposition of the form

$$c : A,$$

but this is impossible, since the proof-relation is not a propositional function, an option I share with Heyting, Scott and Martin-Löf. The most important reason is that one gets caught in an infinite regress of ever descending meaning-explanations, somewhat along the same line as the regress-argument offered by Frege against the

$$\frac{a : A, b : B}{p((a, b)) = a : A}$$

and

$$\frac{a : A, b : B}{p((a, b)) = b : B}$$

We should note here that this way of presenting propositions in terms of their canonical proof-objects allows us to illuminate some of the remarks that Prof. Dummett makes concerning the meaning of implication:

A proof of $A \rightarrow B$ is a construction of which we can recognise that, applied to a proof of A , it yields a proof of B . Such an operation is therefore an operation carrying proofs into proofs. Note that it would be incorrect to characterise ... a proof of $A \rightarrow B$ as 'a construction which transforms every proof of A into a proof of B ' since we should then have no right to suppose that we could effectively recognise a proof whenever we were presented with one. We therefore have to require explicitly that a construction is to count as a proof of ... $A \rightarrow B$ only if we can recognise it as effecting the required transformation of proofs of A into proofs of B .³¹

This worry that the implication (and Universal quantifier) should somehow demand more of the constructions serving as their proof-objects than, for instance, conjunction is in my opinion entirely misplaced. From the first line of this quote it is clear that proofs of implications can be applied to other proofs; thus, here we are dealing with proof-objects rather than proof-acts. For *any* proposition, be it an implication, a conjunction or what have you, it must be possible to recognise its proof-objects as such (and this *am Symbol allein* as argued above). This property has to be ensured through the way in which the relevant proof-object is given (the act of proof/construction).

Furthermore, the alleged impredicativity of implication is also taken care of when we explain propositions via their canonical proof-objects.³²

correspondence theory of truth in *Der Gedanke*. The proof that the construction does what it is supposed to do is not itself a construction object, but rather the act of proof/construction whereby the construction object is given. See my paper, *Ibid.*.

³¹See [47], p. 13.

³²See [47], p. 394 ff..

For instance, in a canonical proof-object (a, b) for a conjunction $A \wedge B$ the proof-objects a and b need not themselves be canonical. It must, however, be possible to evaluate them to canonical form by means of syntactical calculation alone. Herein lies a way to block the vicious circularity of meaning-explanations that threatens in the case of implication (negation, universal quantification). Prof. Dummett imposes a stratification of the universe of constructions in terms of the complexity of the propositions they prove, and the function which transforms proofs of A into proofs of B only has to be defined for proofs of complexity less than or equal to that of A .³³ This, however, does not do justice to the fact that one would like the function f which serves to prove the proposition $A \rightarrow B$ to be defined on any proof of A , irrespective of complexity. This we can effect by using the standard clause

$$\begin{array}{l} f \text{ is a proof of the proposition } A \rightarrow B \\ \text{iff} \\ f(a) \text{ is a proof of } B, \text{ provided } a \text{ is a proof of } A. \end{array}$$

This clause takes any proof of A into account, using whatever methods of proof one might think of, including the elimination rule for implication, provided only that they are valid, that is, provided that the proofs obtained by means of these principles are evaluable to canonical form. This provides one with a way to have one's cake and eat it too: when defining the function f , I only need to consider arguments in canonical form, and with this the impredicativity is also taken care of. Note, however, that the parallel between canonical proofs and normal derivations is lost. A canonical proof does not necessarily have canonical parts: it is just a proof ending with an application of an introduction rule. (canonical proofs are proofs on I-form.) A normal derivation, on the other hand, will have only normal parts and so it will have a sort of sub-formula property and thus impose a stratification on derivations.³⁴

Prof. Dummett remarks that

because of the peculiarities of the intuitionistic interpretation

³³M. Dummett, *Ibid.*.

³⁴Presumably this is the origin of Prof. Dummett's stratification of proofs, canonical and non-canonical. Another possible source is Goodman's use of a stratified universe of construction in his thesis (*Ibid.*). For the record, the impredicativity also shows that one should not construe the proof-clauses quantificationally, that is the reference to *all* constructions should not be effected through the universal quantifier.

of \rightarrow , provability is not a stable property: we cannot think of an addition to our stock of methods of proof as merely allowing us to prove more than we could before, while all the proofs we had already given remain intact, since such an addition may lead to a rejection of certain earlier proofs.³⁵

I cannot endorse this view: one only needs to insist that the additional methods of proof are correct, in the sense that it must be possible to evaluate proofs obtained by means of them to canonical form, in order to ensure that the old proofs remain proofs and that provability is a stable notion.³⁶

Furthermore, we can now easily deal with the problem of defining \perp . Prof. Dummett is of the opinion that we can always choose \perp to be equal to $\text{Id}(N, 0, 1)$.³⁷ This, however, only works well for the case of arithmetic where the intuitionistic \perp -rule is derivable for $\text{Id}(N, 0, 1)$, under essential use of the principle of induction. If the language should comprise other types of propositions than merely those of arithmetic, the above method for deriving the intuitionistic \perp -rule will not work and it is not at all obvious that the proposition $\text{Id}(N, 0, 1)$ will serve in the role of \perp . A better approach is to *define* the absurd proposition which has *no* canonical proof-objects. This is perfectly legitimate on the propositions-as-sets (of proof-objects) conception. One will then have to show in the context of arithmetic that any proof-object for the proposition $\text{Id}(N, 0, 1)$ can be transformed into one for \perp , that is, one will have to derive the fourth Peano axiom.

We should note that the identity criterion for sets offered above, essentially in terms of co-extensionality, induces one for propositions via the formal identification of the two categories set and prop. This criterial, intensional, identity for propositions is completely determined hereby: roughly speaking, two propositions are equal when they have the same proof-objects. Hence a solution has been given to a problem Quine claimed was unsolvable, namely the problem of providing a precise definition of identity between propositions, or, equivalently, of sameness of meaning or synonymy. The price one has to pay for this extensional treatment of an intensional identity relation is that the notion of ob-

³⁵See [47], p. 402.

³⁶See also below on Martin-Löf's discussion of truth from a constructivist standpoint.

³⁷See [47], p. 13. For an explanation of the $\text{Id}(N, x, y)$ conception of identity, see the next paragraph but one.

ject is an intensional one, owing to the use of the distinction between canonical and non-canonical ways of giving the objects.

This topic is also of relevance for the constructivist treatment of identity. As already stressed above Martin-Löf's universe is a typed, or categorised, one. When using such a universe it is certainly reasonable to accept Russell's doctrine of types in the form that each propositional function has a certain category as its range of significance. This must hold also for the propositional function of equality $\text{Id}(A, x, y)$ with respect to the set A . The constant Id has the formation rule

$$\frac{a : A, b : A}{\text{Id}(A, a, b) : \text{prop}}$$

with proof-objects formed according to the introduction rule

$$\frac{a : A}{r(a) : \text{Id}(A, a, a)}$$

The corresponding elimination rule must, of course, be a version of Leibniz's law. It is important to realise here that the identity-criteria for the set A cannot be given in terms of the propositional function $\text{Id}(A, x, y)$. This can be seen as follows: the identity criteria are given in terms of criterial identity, and thus this notion is conceptually prior to the set A itself, since A is given in terms of it. The set A , on the other hand, is conceptually prior to the propositional function of identity regarding its elements. The same type of reasoning can also be used concerning the proof relation between proposition and proof, which, for basically the same reasons, namely, the respective conceptual priorities, will not be a propositional function.

Prof. Dummett discusses the notion of criterial identity in various guises, e.g. 'strict' or 'intensional' identity.³⁸ He stresses that the notion must be decidable and expresses this decidability by means of the

³⁸See [55], p. 125, resp. [47], pp. 25-26.

formula³⁹

$$(*) \quad \forall x \forall y (x \equiv y \vee \neg x \equiv y).$$

This, however, treats of intensional identity as a propositional function, which it is not. The decidability involved here is not that of a propositional function which, constructively or non-constructively, either is, or is not, applicable to each pair of arguments, but rather that of recognizability *am Symbol allein*, that served above also to ensure the decidability of the proof-relation: that is, given two objects *s* and *t* in category α , we can, on the level of syntactic reflection, check mechanically whether or not they are intensionally equal by means of syntactic, mechanical, calculation on the expressions 's' and 't' alone. Indeed, when we know that

$$s = t : \alpha,$$

that is, when *s* and *t* are criterially, or intensionally, equal, then clearly, on the level of syntactic reflection, the expressions 's' and 't' are definitionally equal, that is, they express intensionally equal objects of category α , and *vice versa*. As Prof. Dummett himself observes, the objects of intuitionistic mathematics are intensional objects, and therefore their identity is sensitive to the way in which they are presented. In the quantified proposition used by Prof. Dummett to express the decidability of strict identity, free variables occur in the quantified matrix that is meant to express strict identity. The notion of strict identity, however, does not really make sense when applied between variables, since variables present no objects and therefore, on the level of syntactic reflection, one cannot carry out the necessary calculations on the symbols alone.⁴⁰ Accord-

³⁹See [47], p. 28. In [41], p. 632, Prof. Dummett treats of this decidability in the guise of the decidability of the equality of sense and attributes this principle to Frege:

Sense is so conceived by Frege that synonymy must be an effectively decidable relation: if the senses of two expressions are the same, and someone knows the sense of each expression, then he must know that they have the same sense.

⁴⁰In the *Begriffsschrift* Frege treats 'Inhaltsgleichheit', definitional identity between expressions, much in the same way that Prof. Dummett treated strict identity in (*). It is well-known, however, that several of the formulae in the *Begriffsschrift* that involve identity do not make sense since they presuppose that definitional identity is treated as a propositional function (which it is not) and that its argument places are open to free variables and the application of quantification. Prof. Dummett treats of the matter in [41], p. 544.

ingly the treatment of intensional identity in the formula (*) constitutes a **Second vestige of Realism** in Prof. Dummett's work.

Furthermore, a **Third vestige of Realism** can be found in his use of the realist notion of a state of affairs: this notion, classically, for instance in Wittgenstein's *Tractatus* and in Husserl, serves the role of propositional truth-maker, and as such it is also treated by Prof. Dummett. In particular, Prof. Dummett holds that it must be decidable, or 'determinate, for any recognisable states of affairs, whether or not that state of affairs shows [a certain] assertion to have been correct'.⁴¹ It is, however, hard to see how the notion of state of affairs makes sense constructively, since it is the realist key notion par excellence. Indeed, the decidability, or determinateness, in question is readily transformed into the constructive decidability of the notion of equality between propositions, that is, when

A: prop

and

B: prop

both are given, it is decidable *am Symbol allein*, whether or not

$A = B$: prop.

The constructive validity of this principle allows one to put the decidability argument against classical logic in a particularly attractive form. Consider the type of explication offered for the universal quantifier by Frege in the *Begriffsschrift*. There we find an explanation which runs as follows when the domain of quantification has been made explicit:

$$\begin{aligned} (\forall x : A)B(x) &= T : \text{prop, if } B(a) = T : \text{prop, for all } a:A, \\ (\forall x : A)B(x) &= F : \text{prop, otherwise,} \end{aligned}$$

and, as is well-known, we shall in general not be able to calculate mechanically which of two alternatives will hold when the set A is infinite. Thus definitional (intensional) equality between sentences (propositions) will turn out to be non-decidable on this classical view that puts the set

⁴¹ See [46], in [58], pp. 120-123, especially p. 121.

of propositions equal to the two Boolean truth values $\{T, F\}$.⁴²

The **Fourth** relevant **vestige of Realism** in Prof. Dummett's work concerns the same principle, but now for names.⁴³ Thus, for an object and a name it must be decidable whether the term refers to the given object or not. Also this realist principle can be accounted for constructively in terms of the mechanical decidability, *am Symbol allein*, of the notion of definitional identity between objects of a category, that is when

$$a : \alpha$$

and

$$b : \alpha$$

it is decidable whether or not⁴⁴

$$a = b : \alpha.$$

The final vestige of realism in Prof. Dummett's work, that I shall deal with, is concerned with the intuitionist notion of truth. Prior to the discussion of this notion, a terminological clarification and recommendation might not be out of place. In my opinion, 'constructivism' deserves by far the preference above 'intuitionism' as an appellation for the sort of mathematics to which Prof. Dummett is attracted. Indeed, Heyting gives no meaning-explanations for typically intuitionist notions such as choice sequences or propositions involving the creative subject and, as Troelstra observes, it is uncertain, whether the Brouwerian notion of choice sequence permits a molecular meaning theory of the sort envisaged by Prof. Dummett.⁴⁵ Furthermore, the *intuitionist* doctrine of

⁴²I owe this way of viewing the matter to Dr. Aarne Ranta of the Academy of Finland. One could also refine Frege's position to let the prop's consist not of the truth-values themselves but let them be expressions which can be *evaluated* to such truth-values. In this case the evaluation-function will be incalculable.

⁴³Strictly speaking, this is not a vestige of realism in Prof. Dummett's work, but, rather, a feature of realism discussed in his works. I have chosen to include it on my list simply because of its being the exact analogue of the corresponding principle for states of affairs and propositions that was discussed above. Cf. also the following footnote.

⁴⁴I hesitate to attribute the present principle to Prof. Dummett. It is discussed in [41], Ch. 14, but then only as an important principle, and perhaps not as something Prof. Dummett would uphold himself. Similarly, there is a relevant passage in [46], in [58], pp. 135-136.

⁴⁵See [158] in [27]. See also my paper, *Ibid.*, pp. 163-164. The remarks that Prof. Dummett makes on the elimination of choice sequences and the meaning of the quantifiers, [47], pp. 450-451, are clearly relevant here.

the languagelessness of mathematics seems very hard to square philosophically with Prof. Dummett's meaning-theoretical concerns.⁴⁶

Turning now to the constructivist, or (in order to stay with Prof. Dummett's terminology) intuitionistic notion of truth, the following scheme might serve as a starting point for the discussion:

the proposition A is true
 =
 there exists a proof for A .

First we note that, as was already remarked above in connection with Heyting's work, this is also an application of the general truth-maker analysis of truth, in accordance with Prof. Dummett's Principle C. As was also remarked, 'proof' can here be taken in two senses. The first of these would take proof in the sense of the mental act which confers evidence on the judgement

A is true,

and the existence would here have to be actual existence in the sense that someone had carried out such a mental act. The ensuing notion of truth, however, would not be a pleasant one, since such natural clauses as

the proposition $A \vee B$ is true
 iff
 A is true or B is true

would no longer be correct. Indeed, notoriously, one can have carried out a constructive proof of $A \vee B$ without yet having carried out such a proof for either of the two disjuncts.⁴⁷ Thus it seems a more promising line

⁴⁶See my paper [151].

⁴⁷Prof. Dummett's formulation, in [47], p. 12, of the meanings of conjunction and disjunction is slightly out of focus, it seems to me: "A proof of $A \& B$ is anything that is a proof of A and B ", "A proof of $A \vee B$ is anything that is a proof either of A or of B ". This means that a proof of a conjunction will have to be a proof of three propositions, namely the conjunction and the two conjuncts, and similarly a proof of a disjunction will have to be a proof of at least two propositions, namely the disjunction itself and one of its disjuncts. This would be most counter-intuitive, since the proof would not determine its conclusion: from the proof one could not read off *what* it is a proof of and that is hardly acceptable.

to read the above elucidation of truth in terms of proof-objects rather than in terms of proof-acts. The first thing which must be stressed then, though, is that the notion of existence involved cannot be that of the existential quantifier, since that quantifier, being itself essentially propositional in nature, will itself be explained in terms of the truth-conditions for existential-quantifier propositions, and these truth-conditions are all formulated in terms of the relevant notion of existence, which thus, on pain of a vicious conceptual circularity, will have to be different from that of the propositional notion of existence as expressed by the quantifier. The notion of existence involved here is that of the existence of a general concept: when α is a general concept (category), then

α exists

is a judgement. In order to explain a novel form of judgement, one has to explain what knowledge is expressed by a judgement of the form in question, that is, one must explain what one has to know in order to have the right to make the judgement.⁴⁸ In the particular case of the judgement

α exists,

what one must know, in order to have the right to make it, is that an object falls under the concept α , that is one must have made a judgement

a: α .

Truth is then readily explained in terms of existence of a proof-object: when A is a proposition, that is, when it is known what a proof-object of A has to be, one simply puts

A is true

⁴⁸Cf. [110], p. 227:

What is a judgement before it has become evident, or known? That is, of the two, judgement and evident judgement, how is the first to be defined? The characteristic of a judgement in this sense is merely that it has been laid down what knowledge is expressed by it, that is, what you must know in order to have the right to make, or utter, it.

=

the concept
proof-object for A
exists.

These elaborations may serve as background to Prof. Dummett's valuable discussion of these matters:

To say this is, in effect, to equate 'A is true' with 'We can prove A' rather than with 'A has been proved', and 'A is false' with 'We cannot prove A'. Such an interpretation of 'true' and 'false' remains faithful to the basic principles of intuitionism only if 'We can prove A' (A is provable) is not interpreted to mean, either at one extreme, that, independently of our knowledge, there exists something which, if we became aware of it, we should recognise as a proof of A, nor, at the other, that as a matter of fact we either have proved A or shall at some time prove it. In the former case, we should be appealing to a platonistically conceived objective realm of proofs; in the latter we should be entitled to deny that A was provable on non-mathematical grounds (e.g. if the obliteration of the human race were imminent). 'We can prove A' must be understood as being rendered true only by our actually proving A, but as being rendered false only by our finding a purely mathematical obstacle to proving it. From any standpoint, therefore, there can again, be no guarantee that every mathematical statement is either true or false.⁴⁹

For a constructivist it is somewhat of a disappointment, in the light of this insightful passage, to see that Prof. Dummett does not always remain faithful to these ideas. In his 'Reply to Prawitz' we find the following:

How should we explain the 'can' in 'can be verified' (?) ... An untensed use of 'is true' is no doubt admissible: but then it should genuinely be in tense of timelessness, and not in that

⁴⁹See [47], p. 19. Indeed, the whole section 1.2 entitled 'The Meaning of the Logical Constants' is, with the exception given in footnote 47, a most impressive piece. It is noteworthy how Prof. Dummett (rightly) rejects the application of what Lovejoy dubbed 'The principle of Plenitude' with respect to provability, that is, possibility of finding a proof: it is not so that every possibility must have been actualised in the future.

of eternity; 'is' ought not to be read as 'always was and always will be'. There is a well-known difficulty about thinking of mathematical proofs — and equally, of verifications of empirical statements — as existing independently of our hitting on them, which insisting that they are proofs we are capable of grasping or of giving fails to resolve. **Namely, it is hard to see how the equation of falsity of a statement (the truth of its negation) with the non-existence of a proof or verification can be resisted: but, then, it is equally hard to see how, on this conception of the existence of proofs, we can resist supposing that a proof of a statement determinately either exists or fails to exist.** We shall then have driven ourselves into a realist position, with a justification of bivalence.⁵⁰

The boldface passage just given constitutes the final and **Fifth vestige of Realism** that I want to raise in Prof. Dummett's work. Only someone who is already convinced of realist doctrine, must take the charge made here seriously. The final line of the first quote contains as adequate an anti-realist answer as can be given to the worry in the second (which is ten years later than the first. Had Prof. Dummett forgotten his earlier, excellent treatment?). Indeed, if the anti-realist, in this case the mathematical constructivist, holds firm to his own principles, there is little the realist can do to find fault *within* his position: its coherence seems perfectly safe. Personally, I think that also the realist position is coherent, and that the well-known reasoning advanced by Prof. Dummett, to the effect that it is not, does not succeed: in spite of a fair number of try-outs, I have yet to meet the realist who becomes convinced by the Dummett argument and, in consequence, who wants to give up his position. Worry and the feeling that there must be something wrong with the argument is the standard first reaction, rather than an abolition of a basic realist stance.⁵¹ Reflected reaction

⁵⁰ See [155], p. 285 (my boldface).

⁵¹ This, however, might also be the intended effect of Prof. Dummett's argument. See the preface to [50], p. xxxix:

I personally have no unshakeable commitment to anti-realism in any of these cases, even the mathematical one. [...] I have urged the claims of the anti-realist position only because it seemed to me that, in most cases, philosophers unthinkingly adopted a realist view without noticing that it required substantiation: to say it is natural to take such a view or that it is

will consist in further refinement of positions, the sharpening of statements, modifications with respect to other areas of discourse, etc., all the while remaining faithful to the basic realist attitude originally adopted. This state of affairs was diagnosed in one of the first substantial discussions of the realist/anti-realist debate, namely that of Fichte's *Erste Einleitung in die Wissenschaftslehre*, where Realism appears under the guise of Dogmatism (Fichte was, of course, an idealist), and Anti-realism is known as Idealism.⁵² Fichte's view, that neither participant in the debate will ever succeed to refute the other, or to convince him that he had better embrace the opposite position, is correct, I think. It is agreeable to see that I am joined in this opinion by a very eminent protagonist of the realist, ontologico-semantical school in the Philosophy of Logic, namely the late Heinrich Scholz.⁵³ Scholz quotes the defence offered by Bertrand Russell:

My argument for the law of excluded middle and against the definition of 'truth' in terms of 'verifiability' is not that it is impossible to construct a system on this basis, but rather that it is possible to construct a system on the opposite basis, and that this wider system, which embraces unverifiable truths, is necessary for the interpretation of beliefs which none of us, if we were sincere, are prepared to abandon.⁵⁴

Scholz then winds up his discussion with the following passage, which I wholly endorse:

Gleichwohl muß eine solche Verteidigung wie die ganze klassische Logik heute gefaßt sein auf Angriffe, die in einer mehr oder weniger herausfordernden Sprache vorgetragen werden. Demgegenüber kann nicht nachdrücklich genug daran erinnert werden, daß auch die Krisenfestigkeit einer noch so konstruktiven Logik, auf die hier Bezug genommen wird, in keinem Fall den Charakter der Absolutheit für sich in Anspruch nehmen kann. Absolute Krisenfestigkeit ist ein Idol, das nur bestehen

'part of our theory of the world' is merely the equivalent of Dr. Johnson's kicking the stone.

⁵²See [61].

⁵³See [140], pp. 11-12, Springer, Berlin, 1961. The book was prepared for publication by Hasenjäger and Scholz's *Vorwort* and *Einleitung* are both particularly interesting from the point of view of the realist/anti-realist debate.

⁵⁴See [139], third edition, p. 682.

kann vor einem Mangel an Selbstkritik. Hieraus folgt, daß es eine objektive, allgemeinverbindliche Rangordnung der Auffassungen der Logik ein für allemal nicht gibt. Über einen Wettbewerb der Möglichkeiten, die überhaupt auf eine sinnvolle Art zur Diskussion gestellt werden können, ist grundsätzlich nicht hinauszukommen. Aus diesem Grunde ist es mit Bezug auf den angedeuteten Spielraum angemessen, von der Logik zu sagen, was FICHTE in einem denkwürdigen Falle von der Philosophie überhaupt gesagt hat: "Was für eine Logik man wähle, hängt davon ab, was für ein Mensch man ist".

This should not be taken as an expression of relativism: we are not free to pick or choose. Only, there are no absolute grounds available to us on which to found such a choice. The notion of correctness one would need here, to decide which of the two positions is the right one, and, indeed, even to make sense of the idea that a certain view is the right view, comes close to the *rectitudo* dealt with by St. Anselm in his *De Veritate* and which, on some views, would itself be the Absolute or God.

Returning now to the intuitionistic notion of truth and its timelessness, one point we must answer is the well-known query as to the status of mathematical truths prior to their being proved. For instance, was Waring's Conjecture true also prior to Hilbert's proof in the first decade of this century? Martin-Löf has refined the above position on truth by means of an application of a threefold Kantian distinction, whereby objects get divided into logically possible, really possible and actual.⁵⁵ As soon as a proposition A has been explained, that is, as soon as it has been laid down what a proof-object for the proposition has to be, a proof-object for the proposition A is a *logically possible* object and the judgement

A is true,

that is, the judgement

the concept *proof of A* exists,

is logically possible. Thus, for instance, the judgement

⁵⁵See [112].

\perp is true

is logically possible, since it has been laid down what counts as a proof object of the absurdity \perp . Obviously, it is not a *really possible* judgement, since, in virtue of the meaning explanation in question, there exists *no* proof-object for the absurdity \perp , that is, proof-objects for the absurdity are logically possible, but not really possible.⁵⁶ Finally, Hilbert's proof-object for Waring's Conjecture is *actual* and the judgement

Waring's Conjecture is true

is actual. Thus, for instance, for each natural number k , the judgement

The proposition
 k stones were used to erect the castle at Mussomeli
 is true

is logically possible, and, for a certain $m \leq 10^{100}$, the judgement *simpliciter*

The proposition
 m stones were used to erect the castle at Mussomeli
 is true

will be really possible, but, as far as I know, not actual, since nobody will have counted the stones actually used.

These distinctions will be inherited also for truth, where we can speak of truth *simpliciter*, potential truth, respectively, actual truth, according to whether the existence of the proof-object in question is, logically possible, really possible or actual. Thus, to a logically possible proof-object corresponds the notion of truth *simpliciter*.

In order to get the terminology in terms of *simpliciter*, potential and actual into perspective, let me carry out the discussion once more on the level of judgements. When A is a proposition, we may consider the judgement

⁵⁶In the current, possible-worlds inspired, view of modality, the truth of absurdity would not be held to be a logically possible option. The present notion is clearly a different one from that of the possible worlds approach.

A is true,

that is,

the concept of *proof of A* exists.

By the explanation offered previously, as soon as the category α has been explained,

α exists

is a judgement *simpliciter*, irrespective of whether it can, or ever will, become evident, or known. If it can become evident, that is, if an object falling under this concept is not merely logically possible, but really possible, then the judgement is potentially evident and finally when an object falling under the concept not only can be found, but actually has been found, the judgement is actually evident. Via the explication of truth as existence of proof-object, the notion of truth *simpliciter* of a proposition A corresponds to the logical possibility of proof-objects. When the judgement *simpliciter*

A is true

is potentially evident (when it can become known), that is, when a proof-object is really possible, the proposition A is potentially true. Finally, when the judgement is actually evident (known), the proposition is actually true and a proof-object will have actual existence in virtue of being known. This leads to the following correspondences with Prof. Dummett's work.⁵⁷

	Martin-Löf		Dummett
proof-object	notion of truth	type of judg.	
logically possible	<i>simpliciter</i>	<i>simpliciter</i>	truth
really possible	potential	knowable	justifiability
actual	actual	known	justifiedness

These distinctions can be used to account for a number of intricate points in the philosophy of constructivism, among them the question

⁵⁷ See [54], in [18]. The match is not perfect; Dummett's truth is a realist notion.

concerning the temporal status of the truth of Waring's Conjecture. To repeat: was Waring's Conjecture true also prior to Hilbert's proof? Concerning potential truth the correct answer is: the question is *senseless*, since the notion of existence involved is *timeless*. For actual truth, prior to Hilbert's proof, the answer, obviously, must be no.

Similarly — a point raised above with respect to the truth-condition for disjunction — the T-sentence clause

$$\begin{array}{c} A \vee B \text{ is true} \\ \text{iff} \\ A \text{ is true or } B \text{ is true} \end{array}$$

holds for potential, but not actual, truth.

As a final example consider yet another of Prof. Dummett's discussions of implication:⁵⁸

In some very vague intuitive sense one might say that the intuitionistic connective \rightarrow was stronger than the classical \rightarrow . This does not mean that the intuitionistic statement $A \rightarrow B$ is stronger than the classical $A \rightarrow B$, for intuitively the antecedent of the intuitionistic conditional is also stronger. The classical antecedent is such that A is true, irrespective of whether we can recognise it as such or not. Intuitionistically, this is unintelligible: the intuitionistic antecedent is that A is intuitionistically provable, and this is a stronger assumption. *We have to show that we could prove B on the supposition, not merely that A happens to be the case (an intuitionistically meaningless supposition), but that we have been given a proof of A.*

The italicized phrase will have to be understood as referring to the possibility of transforming proof-objects. That is, one must give a uniform way of obtaining a proof-object for B in terms of a proof-object for A. How this works can best be seen in working out a simple example. Proofs of conjunctions are pairs and their respective components can be extracted using projection functions. In a proof of $A \wedge \neg A$, the first component will be a proof of A and the second component will be a proof of $\neg A$, that is, a function transforming proofs of A into proofs of \perp . Therefore, application of the second component to the first will yield a proof

⁵⁸See [47], pp. 16–17, (my italics).

of \perp . Hence, in this fashion we have provided a uniform method for how to find a proof for the absurdity, given a proof of $A \wedge \neg A$, and so we have proved the implication of the second from the first. A more formal representation of the above reasoning would be: $p(z) : A$ and $q(z) : \neg A$, on the supposition $z : A \wedge \neg A$. Hence, $ap(q(z), p(z)) : \perp$ (where $ap(x, y)$ is the application function), still on the same supposition. Therefore,

$$\lambda z.ap(q(z), p(z)) : (A \wedge \neg A) \rightarrow \perp,$$

now on no supposition, and the truth of the proposition has accordingly been established, through the construction of this proof-object. Note however that the antecedent is an empty set; no proof-object of the proposition $A \wedge \neg A$ can ever be found. This is a false proposition. The italicised part of Prof. Dummett's passage can also be read in a different way, namely as involving the supposition that I have obtained a proof-object for the antecedent. This would be represented as the supposition that I had proved the judgement $a : A \wedge \neg A$, for a suitable a , that is, that I had carried out a proof-act having this judgement as its object, and, in the case just discussed, this is an impossible situation. I shall *never* find myself in the situation of having found a proof-object for the proposition $A \wedge \neg A$.⁵⁹ Therefore, to sum up: the notion of truth involved in the clause

$$\begin{aligned} A \rightarrow B \text{ is true} \\ \text{iff} \\ B \text{ is true, provided } A \text{ is true} \end{aligned}$$

is that of truth *simpliciter*, that is, the notion of *existence of a proof-object*, applied to the propositions in question, independently of whether a proof-object can ever be found. For potential truth, nonsense will sometimes result, for instance, in such cases where the antecedent is false. (Thus, also in the case discussed above of the truth-clause for disjunction, it seems best to let the notion of truth involved be that of truth *simpliciter*, rather than that of potential truth, though the latter

⁵⁹Note that what is involved here is the difference between the supposition that a proposition is true, which is expressed by means of a free variable for a proof-object, and the supposition that the truth of the proposition has been established by means of a proof-object, which is represented through the use of a closed term for the proof-object, depending on no variable assumptions, and where what is assumed is that an act has been carried out rather than that a proposition is true.

does not make nonsense out of the clause in question.)

With this example my tour of Michael's work has come to an end. My examination has been largely critical, so I very much want to conclude by borrowing a leaf from the Preface to his *Truth and Other Enigmas*, where, concerning his critical study of Goodman's *The Structure of Appearance*, he states that he hopes it is obvious that he would not have spent so much time and effort on something which he did not hold in the highest esteem. The same holds for myself regarding his own work. Above I applied the Berlin hedgehog/fox terminology to Martin-Löf, who is a hedgehog, who knows that he is a hedgehog. It is equally clear that Michael is a fox, who knows that he is a fox, and what a pleasure it is to accompany him on his wanderings through the thorny paths of anti-realism.

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REPLY TO SUNDHOLM

I should be surprised if there were not many vestiges of realism in my writings. A hedgehog, who knows one big thing and sticks to it, can keep himself uncontaminated by alien thoughts; but a fox, who goes snuffling around among the many things about each of which he knows a little, is bound to pick up variegated ideas not consistent with one another. We are all of us brought up to view the world in a realist manner, and it is difficult for us foxes to shake off all the effects of that upbringing. More exactly, I believe that there are several features of our language, and therefore of the way we learn to think, that push us to take the first steps towards realism, and was attempting to explore one of these in one of the papers Sundholm quotes (*The Source of the Concept of Truth*). These features are, in my view, to be respected, not eliminated as defects; it is a test of any version of anti-realism that it can accommodate them without degenerating into full-blown realism. I therefore view it as misleading that Sundholm should remark (footnote 57) that the notion of truth employed in that paper is a realist one. It was not intended to be a specific or full-grown notion at all: only a newborn infant in which we can discern the future lineaments of a realist conception, but which, given a proper upbringing, still might develop into a viable constructivist one.

My intention is not to defend myself from the charge of having expressed thoughts heretical by the canons of constructivist orthodoxy, but to explore what strikes me as a deep disagreement between Sundholm and me: over the distinction between a proposition and a judgement or assertion. Certainly the two are very different. A judgement or an assertion is, as Sundholm says, an *act*: a proposition is the *content* of such an act — the content of an actual or possible judgement — and, as such, may be regarded as an object, if one of a very special kind. This distinction was first clearly drawn, and in almost these terms, by Frege in his *Begriffsschrift*. How, then, does the distinction bear on the concept of truth? In a brief unpublished fragment of 1915, Frege wrote:

One can only say: the word 'true' has a sense, but one that

contributes nothing to the sense of the whole sentence in which it occurs as a predicate.

But it is just in virtue of this that this word appears to be suited to indicate the essence of logic. Every other property-word would be less suited to do this, because of its specific sense. The word 'true' thus seems to make the impossible possible: namely to make what corresponds to the assertoric force appear as a contribution to the thought. And, although it miscarries, or, rather, precisely through its miscarrying, this attempt points to the special character of logic . . . (*Nachgelassene Schriften*, p. 272, *Posthumous Writings*, p. 252.)

Why is using the concept of truth attempting the impossible? Well, what is the difference between a proposition and a judgement? When we express a proposition, we do not thereby adopt any stance towards its truth or falsity: but, when we make or express a judgement, we are assigning truth to the proposition we express. We can, and often must, express a proposition without ever feigning to judge it true, as when we ask whether it is true, or when we assert a complex proposition of which it is a constituent part. So it appears that judging is ascribing the property of truth to a proposition. But, if we attempt to express the property so ascribed by means of the predicate 'is true', in the way that every other property can be expressed by a suitable predicate, the predicate we have framed fails to do what was required of it. What was required was to convert the mere expression of a proposition into a judgement whose content it was. But when we transform the expression of a proposition, say ' π is transcendental', into a statement of its truth — 'it is true that π is transcendental' — we have not succeeded in expressing a judgement: we merely have, once more, the expression of a proposition — of the very same proposition as before, according to Frege — as is shown by the fact that our new sentence can be put into interrogative form or used as the antecedent of a conditional.

If we wish to symbolise what is distinctive of a judgement — or of an assertion, the communication of a judgement — to mark it off from the mere expression of a proposition, we cannot therefore employ a predicate: we must use a sign that does not contribute, or purport to contribute, to the content of the proposition. That is precisely the function of Frege's assertion sign: it symbolises, not any constituent of the proposition, after whose sense it would be meaningful to enquire,

but the assertoric force attached to the proposition in making an assertion whose content that proposition is. Sundholm expresses what he terms a 'judgement/assertion' in the form '**A** is true' (p. 141); we must accordingly view him as using 'is true', not as a predicate properly so called, but as the equivalent of Frege's assertion sign. So far, then, the distinction between a proposition **A** and the judgement expressed by '**A** is true' is clear and incontrovertible.

Propositions are objects, judgements are acts. What is the difference between an object and an act? Well, one can talk *about* objects, that is, frame propositions concerning them; but it does not make sense to speak of performing them. An act, on the other hand, is, pre-eminently, something that can be performed. One can also convey that one is performing it or has performed it. Presumably, one can also talk about an act; but neither performing it nor conveying that one has performed it is a case of saying something about it, but is an activity of quite different kind. A proposition is, however, the *content* of a judgement or of an assertion, and so one can do something with a proposition other than talk about it: namely, one can assert it or judge it true (where here 'true' is as much an integral part of the verb as is 'down' in the phrase 'track him down').

Sundholm immediately takes a further step. He distinguishes between constructions which constitute proofs of propositions, and to which we refer when explaining the meanings of propositions, and proofs of judgements, which are 'that through which one gets to know the truth of a proposition'. The former, he tells us, are *objects*, but the latter are not objects, but *acts*. Hence, he says, "the form of the theorem proved by the act of proof, when fully elucidated, will be '**c** is a proof of **A**', where **c** is the construction found in the act of construction/proof". Thus '**c** is a proof of **A**' no more symbolises a proposition than does '**A** is true'; it, too, communicates the performance of an act. Accordingly, 'is a proof of' is no more a relational expression than 'is true' is a predicate: the assertion '**c** is a proof of **A**' cannot allow of being contradicted by '**c** is *not* a proof of **A**', or being placed in the antecedent of a conditional, any more than the negation sign can be placed in front of the assertion sign, or the latter taken to govern only the antecedent of a conditional.

What are we to make of this? Can we accommodate it within the general framework of the distinction between propositions and judgements, as it has here been drawn? Well, we not only make assertions, but indicate the *kind* of basis on which we make them: we say, 'I see

that **A**', 'I remember that **A**', 'I conclude that **A**', and so on. When we say such things, we are not telling our hearers how it is with us, but, in each case, asserting that **A** while indicating on what basis we are doing so. Likewise, when I see something, remember something or infer something, I do not judge that my visual experience, memory-experience or reasoning process warrants my judging that **A**: I simply judge that **A**, on the basis of what I see, remember or infer. Nor is a judgement that such-and-such is good evidence that **A**, or a sound ground for accepting **A** as true, a necessary preliminary to judging that **A**. On the contrary, we make no such preliminary judgement, in the normal case at least: we simply *recognise* the evidence or ground as warranting the judgement, and make the judgement accordingly. We may thus regard Sundholm's 'c is a proof of **A**' as expressing an assertion of **A** accompanied by an explicit indication of the basis for the assertion; 'c is a proof of' functions as an assertion sign with index c to indicate on the basis of what construction the proposition is being asserted.

It appears, nevertheless, that this use of 'is a proof of' as a qualified sign of assertoric force cannot be the only use of the expression Sundholm must allow. He quotes the standard explanation of the conditional proposition $\mathbf{A} \rightarrow \mathbf{B}$ as being that **f** is a proof of it just in case **f(a)** is a proof of **B**, provided **a** is a proof of **A** (p. 149). Even if we take **A** and **B** as specific propositions, the letter **a** must be a variable over constructions. It seems, therefore, that in this situation the expression 'is a proof of' must be a genuine relational expression, standing for a relation between constructions and propositions, and not a sign indicating force, for two reasons: (i) it figures in the antecedent of a conditional; and (ii) it has a variable term in one of its argument-places. The reason (i) appears conclusive, since we have already noted that a force-indicator cannot stand within the scope of a sentential operator. The reason (ii) seems equally compelling. If '**a** : **A**' conveys a judgement, then **A** must be a specific proposition and **a** a specific ground for the assertion: there is no such thing as asserting an indeterminate proposition or as asserting a determinate proposition on an indeterminate ground. If this conclusion is correct, it can only be rated a bad idea to have one and the same expression playing two such different roles, however closely related they may be; that is too great an encouragement to confusion.

Both of these grounds may be countered, however. Ground (i) may be rebutted by the plea that the stipulation is not really itself a conditional proposition, but a lax formulation of a rule. Clearly 'is a proof of' is

the informal equivalent of the colon in Martin-Löf's notation, ' $c : A$ ' being the expression of a judgement that A on the basis of the construction c . The rule in question is then the elimination rule stating that, given the judgements ' $f : A \rightarrow B$ ' and ' $a : A$ ', one is entitled to make the judgement ' $f(a) : B$ '. (This would be an example of the meaning of a logical constant's being given by the elimination rule governing it; but I shall not enter into this controversy here.) It remains that in all such stipulations, the symbols for constructions — here f and a — are *variables*. That is the necessary characteristic of the statement of any general rule: variables are required in order to express the generality. In Frege's formulations of his rules of inference, for example, he uses the assertion sign for premisses and conclusion because an inferential step must lead from asserted premisses to an asserted conclusion: but it is followed, not by symbols for any actual propositions ('names of truth-values'), but by Greek capital letters indicating arbitrary propositions. One cannot object that it is impossible to assert an indeterminate proposition. In stating the rules of inference, Frege is not *making* assertions, but talking *about* assertions that may be made: it is noteworthy that he is careful to enclose the whole, assertion sign and a capital Greek letter, in quotation marks. We may not only perform an act of judgement, and convey to others that we have performed it: we can also frame propositions and lay down stipulations concerning the making of acts of judgement. Acts are thus objects as well, in the sense of that which we can speak and think about. In so doing, we may use the expressions intended to convey acts of judgement, but in tacit or explicit quotation marks, not as conveying them, but as indicating what we are talking about; and we may legitimately quantify into the expressions so used. We should not then confuse the colon or the phrase 'is a proof of' with a genuine predicate, when it merely contributes to indicating a judgement of which we are predicating something.

A mathematical proposition partitions mathematical constructions into those that are and those that are not proofs of it; it therefore determines a unique set of constructions, those, namely, that prove it. Can it be identified with this set? Not if set-membership is a relation, as we normally conceive it as being: a construction is a member of the set, but a ground for asserting the proposition. The equation of propositions with sets is thus not explanatory of the notion of a proposition, since it requires a transformation of our ordinary notion of a set, namely into what we should ordinarily think of as the proposition that the set has

a member. A construction that proves this proposition is an object that can be recognised as a member of the set; the execution of this construction — the proof-act whereby we arrive at the corresponding judgement — will be the production of such an object. Where α is the set, the judgement whose content is the proposition with which it is thus identified can therefore be written as ' α exists' (meaning ' α is inhabited'), and its 'fully elucidated' form as ' $\mathbf{a} : \alpha$ ', where \mathbf{a} is a member of the set α , just as Sundholm says (p. 156). On this understanding of the notion of a set, set-membership is indeed not a relation, but a ground of judgement; to arrive at such an understanding, we must start with the constructive notion of a proposition.

So far, then, Sundholm's use of the Martin-Löfian terminology and apparatus seems entirely coherent. We may not be happy with the use of an apparent predicate like 'is true' and an apparent relational expression like 'is a proof of' as force-indicators, but, as long as we bear in mind what they are meant to be, that is no more than an awkwardness. As it seems to me, it is when Sundholm starts discussing the notion of truth that things start to go awry. The first point to note is that, even if the foregoing arguments failed, the distinction between proof-objects and proof-acts requires that there is, after all, a sense of 'is a proof of' under which it *is* a genuine expression for a relation. For, if a proposition is an object, and a construction is an object, they are both things of the right sort for a relation to obtain between them; and, if so, how can it be denied that one such relation that may obtain consists in the construction's being a proof of the proposition? A construction is an object: the corresponding act is the carrying out of that construction. How are we to understand this? Is it like saying, 'A house is an object, but building a house is an act'? It does not seem so. For there are no unbuilt houses, whereas it seems that there are constructions that have not been carried out. Consider what Sundholm first says about the constructivist notion of truth (p. 155). He begins by stating that a proposition is true just in case there exists a proof of it, and then asks whether 'proof' here should be taken to mean a proof-act or a proof-object. To answer this, he considers the principle that the proposition $\mathbf{A} \vee \mathbf{B}$ is true if and only if either \mathbf{A} is true or \mathbf{B} is true. If, in the characterisation of 'is true', we take 'proof' to mean 'proof-act', Sundholm argues, that intuitively compelling principle will fail, since it is well known that a valid constructive proof of $\mathbf{A} \vee \mathbf{B}$ may be carried out, even though no such proof has been carried out either of \mathbf{A} or of \mathbf{B} . He concludes that we must interpret

'proof' to mean 'proof-object': the clear implication is that, under this interpretation, the principle will hold. If it does, then, in the case considered, that a constructive proof has been given of $A \vee B$, without any such proof's having been given either of A or of B , a proof-object either for A or for B must exist, that is, a construction that proves either one or the other, even though, by hypothesis, no such construction has been carried out. There must, therefore, exist constructions that have not been carried out, whereas there do not exist houses that have never been built.

What is the sense of 'exists' in accordance with which a construction or proof-object may exist even though it has not been carried out? A constructively acceptable proof of a disjunctive proposition $A \vee B$ will provide an effective means of constructing a proof either of A or of B , although, if it was not a canonical proof, we shall not know in advance of carrying out the construction which of the two propositions it will prove. So perhaps a construction of a given kind may be said to exist if we have an effective means of carrying it out, whether or not we have carried it out. This is, however, still a temporal notion, since we acquire such an effective means only at a certain point in time, not having possessed it previously, whereas we subsequently learn from Sundholm (p. 163) that the relevant notion of existence is to be a timeless one; so this explanation will not serve.

Sundholm indeed hastens to comment on the sense in which he is using 'exists', explaining that "the notion of existence involved cannot be that of the existential quantifier" (p. 156). But what is he doing at all, in attempting, from a constructivist standpoint, to characterise the notion of truth? On the face of it, he is attempting to lay down the condition for a proposition to be true. But, if 'is true' is construed, as we previously concluded that it must be, not as a predicate of propositions, but as a misleading form of the assertion sign, it is nonsensical — quite literally ungrammatical — to frame sentences of the form 'The proposition A is true if and only if ...'. You cannot simultaneously assert a proposition and state its equivalence with another proposition; you cannot put the assertion sign within the scope of the sentential connective 'if and only if'. In any case, A is a variable here, and you cannot assert an indeterminate or variable proposition. Well, the fact that A is a variable shows, as we have seen, that we are not here *making* or *communicating* any judgements, but *talking about* them. And though to make a judgement is not to predicate anything of it, we can predicate of

a judgement that it is or is not warranted, and lay down the condition for it to be warranted. That is so, indeed; but whether a judgement is warranted depends upon the epistemic condition of the subject who makes the judgement; when the judgement is a mathematical one, the condition is that the subject knows that the appropriate proof-act has occurred — that such an act exists in a perfectly ordinary sense of 'exists' — that is, that the appropriate construction, in Sundholm's sense of a constructivistically acceptable proof, has actually been carried out. There is then no reason to suppose that, whenever a judgement of the form ' $A \vee B$ is true' will be warranted, either the judgement ' A is true' or the judgement ' B is true' will be warranted; so, if this is what we are talking about, no objection can be raised on this ground to speaking of a proof-act rather than a proof-object.

In this respect, however, the notion of being warranted differs from that of being true, which, as commonly understood, is an objective condition independent of the state of the subject, who may, through luck, make a true assertion even though he was not warranted in making it. Can we not, therefore, construe 'the proposition A is true', in the context 'The proposition A is true if and only if there exists a proof of A ', as meaning 'the judgement " A is true" is correct', where the correctness of an assertion is an objective condition independent of the state of the subject? In ' A is true', 'is true' functions as the assertion sign, even though standing within quotation marks; but the notion of the correctness of an assertion can be equated with that of the truth of the proposition asserted, where now truth is a genuine property of propositions. Now may we not need such a notion as that of objective correctness or objective truth? The realist certainly needs it: it is of the very essence of his conception. Possibly, a constructivist *may* need a corresponding notion also. But that requires to be made out, and cannot be simply assumed: on the face of it, the constructivist appears to be able to get on very happily without it; indeed its introduction is going to cause him some awkwardness. Even to raise the question, 'How are we to characterise the constructivist notion of truth?', gives the impression of supposing that the constructivist needs to have *some* notion of the objective truth of a proposition, distinct from that of the existence of a warrant for asserting it. With what right can that be presupposed?

But is Sundholm *really* presupposing this? May he not be expressing himself in a misleading way, so that he is not really enquiring under what conditions a proposition is true, but, accepting that ' A is true' is

the general form of a judgement, or, rather, assertion, asking in what equivalent way a judgement may, in general, be expressed? That, indeed, is how it turns out. Sundholm explains (p. 156) that the notion of existence appealed to in 'there exists a proof(-object) for **A**' is 'that of the existence of a general concept'. When α is a general concept or category, he says, then ' α exists' is (not a proposition but) a judgement. We have here, then, an instance of the equation of sets, or, rather, in this instance, of general concepts, with propositions. The concept *proof-object for A* is to be equated with the proposition **A**: the judgement 'The concept *proof-object for A* exists' is then warranted by the actual construction of a proof of **A**, just as is the judgement '**A** is true'. We have been given no *explanation* of 'is true' as used in '**A** is true'; rather, our prior understanding of the latter judgement is appealed to in explanation of the judgement-form 'The concept *proof-object for A* exists'.

Even so, we may remain worried by the case of disjunctive judgements. The judgement '**A** \vee **B** is true' is warranted by carrying out a constructive proof of the proposition **A** \vee **B**; Sundholm is explicit that this need not be a canonical proof. On the score that it is the existence of *proof-objects*, not of *proof-acts*, that is in question, we are supposed to be entitled to equate the judgement with '**A** is true or **B** is true'. But with what are we equating it here? If 'is true' is and remains a sign of assertion, '**A** is true or **B** is true' is simply ill-formed: you cannot disjoin two *judgements*, but only two *propositions*. Perhaps what Sundholm means to say is that, if the judgement '**A** \vee **B** is true' is warranted, then either the judgement '**A** is true' or the judgement '**B** is true' is also warranted. But, if the warrant for a judgement is the carrying out of a construction of a proof-object for the proposition, then this is simply not so: that, indeed, was the whole problem in the first place. We are still left baffled by the notion of existence for proof-objects.

It is when Sundholm comes to apply the trichotomy of logically possible, really possible and actual (pp. 160–165) that my unease reaches the peak. He starts by presenting this as a trichotomy of *objects*, but immediately applies it to *acts*, in particular to judgements. He explains that, 'as soon as it has been laid down what a proof-object for [a] proposition is to be', such a proof-object becomes a logically possible object, and a judgement of its existence a logically possible judgement. For myself, I should not be happy to call a proof that $2 + 2 = 5$ a logically possible object, let alone a proof of absurdity, of which it has been laid down that nothing is to count as a proof. For any proposition of which we do not

know a disproof, a proof of it may be called an (at present) *epistemically* possible object, but it is an abuse of words to call it *logically* possible. Sundholm further tells us that the trichotomy can be extended to truth, allowing us to regard a proposition as true *simpliciter*, potentially true or actually true according as its proof-object is logically possible, really possible or actual (pp. 161–162). Is he now distinguishing grades of truth, considered as attaching to propositions, or is he still regarding ‘is true’ only as an assertion sign, but continuing to use a misleading terminology? In his argument about the meaning of implication (pp. 163–164), he says that the notion of truth involved in the specification that $A \rightarrow B$ is true if and only if B is true, provided A is true, must be truth *simpliciter*, since a proof-object for A may not be really possible. If ‘is true’ is merely an assertion sign, this specification is misbegotten, for the usual reasons: it needs reformulating as ‘We shall have a warrant for the judgement “ $A \rightarrow B$ is true” just in case we should have a warrant for the judgement “ B is true” provided that we had a warrant for the judgement “ A is true”’. Even this formulation does not appear completely happy; it would be better stated as ‘We shall have a warrant for the judgement “ $A \rightarrow B$ is true” just in case we are in a position to recognise that we should have a warrant for the judgement “ B is true” provided that we had a warrant for the judgement “ A is true”’. When it is reformulated in this way, it is evident that Sundholm’s truth *simpliciter* does not come into the matter at all. We shall *not* have a warrant for the judgement ‘ $A \rightarrow B$ is true’ merely by recognising that the judgement ‘ B is true’ will be logically possible, provided that the judgement ‘ A is true’ is logically possible. Given that it has been laid down what is required of a construction for it to be a proof of the proposition A , the judgement ‘ A is true’ *will* be logically possible, on Sundholm’s understanding of ‘logically possible’; that is not what is required. We can, however, recognise that, if we had actually constructed a proof-object for A , we should be able to construct one for B , in advance of knowing whether there is any real possibility of constructing one for A . It is not any special sense of ‘is true’ that is in question; actual truth is all we need to appeal to, but as hypothesis, not as categorically attributed.

The notion of truth *simpliciter* is clear, but is not a notion of truth at all. The notion of actual truth is also clear, and, as Sundholm says (p. 163), is a temporal notion; as more is proved, so more becomes actually true. It is the notion of potential truth, or of the really possible, that is obscure. When is a construction really possible? Sundholm gives

us very little help. He tells us (p. 161) that an (empirical) proof of a proposition of the form 'k stones were used to erect the castle at Mussomeli' is really possible, but probably not actual. He tells us (p. 163) that the existence of a really possible proof-object is timeless. He cannot be meaning merely to say that the assertion sign has no tense, since that would not help us to distinguish potential from actual truth, or real possibility from actuality. He tells us also that the disjunction principle discussed above is valid for potential, though not for actual, truth. From these hints, I am unable to gather what real possibility is. It is the problem of the existence of proof-objects over again. Plainly, it is in this notion that we find a vestige of realism in Sundholm's thinking; and it has neither been clearly explained nor made out that any such notion is needed.

I should prefer to say that a constructive mathematician has no need of any notion of truth distinct both from the assertion sign and from the notion of a warrant for assertion. That is not to say that I do not respect Sundholm's struggles with the concept of truth. On the contrary, I do not believe that a constructive approach to propositions about non-mathematical reality *can* dispense with the concept of truth; that is why it must involve a vestige of realism.

I have some other cavils with what Sundholm has written. In particular, I cannot believe the opposition between realism and idealism to be irresolvable, although it may never be resolved to everyone's actual satisfaction. I also have no faith in the general possibility of recognising proofs of mathematical propositions *am Symbol allein*, which has so much taken Sundholm; I think that the ideal of dispensing with understanding in this regard is a fantasy. But the important matter was the concept of truth, which is why, though I apologise for having discussed it at such length, I cannot regret doing so.

REPLY TO WRIGHT

Crispin Wright's discussion of my paper on Gödel's incompleteness theorem has the great merit of linking the principal topic of that with the more celebrated thesis of Lucas, recently endorsed by Penrose. In my paper I did not refer to Lucas's work, unsurprisingly since his original paper on the subject was published in the same year as mine; but I came close to endorsing his view, saying:

it may be the case that no formal system can ever succeed in embodying all the principles of proof that we should intuitively accept; and this is precisely what is shown to be the case in regard to number theory by Gödel's theorem.¹

Obviously, a formal system that would mimic all the actual proofs of number-theoretic propositions we have ever accepted or ever shall accept is a fantasy. The difficulty is that on which Wright remarks (p. 172), that of making clear what we are talking about when we speak of 'the powers of the human mind'; the idea was that, at the end of human history, it would be possible in theory to go through all number-theoretic proofs we had accepted and isolate the basic assumptions and principles of reasoning appealed to in them. Principles of proof may be divided into (1) those we currently accept; (2) those we shall come to accept; (3) those we should be disposed to accept if anyone had hit on them (or if an angel from heaven had expounded them); and (4) those we should never accept. The quoted observation is hasty. Plainly, if a formal system embodied all the principles in categories (1) and (2), and perhaps in (3) as well, but also embodied some in category (4), we should not recognise it as sound, and hence presumably should not acknowledge it as consistent; we should therefore have no reason to accept as true the Gödel undecidable sentence for such a system. The most that could be claimed is that there could be no formal system embodying all the principles in categories (1) and (2), without embodying any in category (4).

¹See [50], p. 200.