Strings and AdS/CFT at finite density

Goykhman, M.

Author: Goykhman, Mikhail
Title: Strings and AdS/CFT at finite density
Issue Date: 2014-06-24

Chapter 4

Stringy holography at finite density

4.1 Introduction

In the usual AdS/CFT setting gauge theory on the boundary has a dual description in terms of closed string theory in the bulk. Most often, a limit of small curvature is taken to yield a low energy theory of strings, supergravity. In the $\mathcal{N} = 4$ supersymmetric Yang-Mills case this limit implies strong 't Hooft coupling of field theory. A distinct example of non-gravitational theory with a holographically dual description is the Little String Theory [1, 2]. It can be viewed as the theory of N coincident NS5-branes, taken at vanishing string coupling, $g_s = 0$, where the bulk degrees of freedom decouple. The coupling constant of the low-energy $U(N)$ gauge degrees of freedom, living on the NS5-branes world-volume, stays unaffected by taking this limit, and is equal to $g_5 = \ell_s$, where ℓ_s is string length in type-IIB string theory (see [3] for a review).

The holographic dual of the Little String Theory [2, 4, 5] is the theory of closed strings in the background of NS5-branes, with the geometry $R^{5,1} \times R^{\phi} \times SU(2)_N$, the two-form field and the linear dilaton. The CFT on $SU(2)$ is described by WZW action at level N. The bulk physics (in the double scaling limit) can be reformulated as the string theory on $R^5 \times \frac{SL(2,R)_N}{U(1)} \times SU(2)_N$ space-time. This is due to the fact that the gauged WZW model on $SL(2,R)_N/U(1)$ gives rise to the classical "cigar" geometry of the two-dimensional black hole with the asymptotically linear dilaton [6, 7]. In the large N limit the bulk theory reduces to supergravity[1].

Generally one expects that a lot of nontrivial physics drastically simplifies in the limit of infinitely many degrees of freedom (large N limit), both in the boundary field theory and from the dual bulk perspective. For example, one expects the large N physics of a field theory at finite temperature and density to have "classical" nature, resulting, in particular, in the mean field critical exponents. (Another recent example of this is given by the stringy nature of finite-momentum and zero-frequency singularity of the current-current two-point functions, observed in [8], where the results of [9–15] were extensively used.)

The low energy excitations in Little String Theory at finite temperature have been considered in [16] [2]. The closed string description involves the gauged WZW (gWZW) action with the $SL(2,R)/U(1)$ target space-time and $\mathcal{N}=2$ world-sheet supersymmetry. In [16] the two-point functions of the stress-energy tensor and the $U(1)$ current have been computed holographically; their pole structure indicates the presence of hydrodynamic modes. This has been also verified by solving fluctuation equations in supergravity approximation in the background of a large number of NS5-branes.

In this chapter we study string theory in the background of a direct product of the two-dimensional charged black hole [23] and flat space. The string theory in the two-dimensional charged black hole background is described by the gWZW action with the $\frac{SL(2,R)\times U(1)_x}{U(1)}$ target space-time [24–26]. Here $U(1)_x$ is a compact circle, which is Kaluza-Klein reduced, and $U(1)$ subgroup of $SL(2,R)\times U(1)_x$ is gauged asymetrically. The left-moving sector of the gauged $U(1)$ is a linear combination of the left-moving sector of the $U(1)_x$ and the left-moving sector of the $U(1)$ subgroup of the $SL(2,R)$. The coefficient of this linear combination determines the charge to mass ratio of the resulting black hole. The right-moving sector of the gauged $U(1)_x$ is the right-moving sector of the $U(1)$ subgroup of the $SL(2,R)$.

This bulk system is holographically dual to the boundary quantum field theory at finite temperature and charge density. (One can think of the resulting system as little string theory at finite density, but we do

[1]The radius of the $SU(2)$ sphere is $R_{sph}=\sqrt{N}\ell_s$. Therefore the large N limit is equivalent to the limit of small ℓ_s/R_{sph}.

[2]See also e.g. [17–22] for some preceding holographic study of the Little String Theory.

not study the field theoretic interpretation here in detail.) The inverse temperature is equal to $\beta = 2\pi\sqrt{k}\frac{\cos^2(\psi/2)}{\cos\psi}$, where $\psi \in [0, \pi/2]$ is the parameter of asymmetric gauging. The finite charge density in the field theory is described holographically by the background $U(1)$ potential in the bulk, $A_t(u) \simeq -\frac{q}{u}$, where $q = M \sin\psi$ is the charge and M is the mass of the black hole; u is the radial coordinate in the bulk.

The vertex operator of the string ground state in this model was constructed in [25]. In this chapter we construct the vertex operators which describe massless closed string excitations in this model, which constitute the NS-NS sector of type-II supergravity. We also construct the gauge field vertex operators, which are obtained by Kaluza-Klein reduction on $U(1)_x$ from graviton and antisymmetric tensor field vertex operators. The graviton in the bulk is dual to the stress-energy tensor on the boundary; the gauge field in the bulk is dual to the charge current on the boundary. We study the low energy excitations of the system by computing holographically the two-point functions for the charge current and the stress-energy tensor and reading off the dispersion relation from their poles. We find two distinct gapless modes in the shear channel; the dispersion relation of one of them is independent of the charge to mass ratio of the black hole. The two modes merge in the limit of vanishing charge, producing the shear mode which was observed in [16]. We confirm these results by solving fluctuation equations of the type-II supergravity. The situation in the sound channel is similar.

Finally we study fluctuation equations in the low-energy limit in heterotic gravity [23]. We find one gapless mode in the shear channel. Comparing this result with the thermodynamics of the charged black hole [27] we find that the ratio of shear viscosity to entropy density is equal to $\eta/s = 1/(4\pi)$, independently of the charge to mass ratio of the black hole.

The rest of this chapter is organized as follows. In section 2 we review the thermodynamics of the two-dimensional charged black hole and derive the dispersion relation of the shear hydrodynamic mode. In section 3 we apply the BRST quantization method of the coset models, and the covariant quantization of the string to construct the holomorphic and anti-holomorphic physical vertex operators of the massless states on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset. In section 4 we use these vertex operators and write down the vertex operators of graviton, antisymmetric tensor field and gauge fields. In that section we also compute the two-point functions of these vertex operators and discuss the low-energy excitation modes. We also briefly discuss finite-momentum and zero-frequency singularity of the correlation functions. In section 5 we solve fluctuation equations in type-II supergravity to verify the dispersion relations, derived in section 4. We discuss our results in section 6. Appendix A is devoted to a review of some rudimentary conformal field theory and derivation of the gWZW action on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset. In Appendix B we solve fluctuation equations in heterotic gravity. We find one mode in the shear channel. Matching its dispersion relation to the one, written in section 2, we obtain that $\eta/s = 1/(4\pi)$ for any charge to mass ratio.

4.2 Thermodynamics of the charged black hole

The metric of the two-dimensional charged black hole [23] with mass M and charge q in suitable coordinates can be written as [26, 27]

$$\begin{aligned} ds^2 &= -f(u)dt^2 + \frac{\hat{k}}{4}\frac{du^2}{u^2 f(u)}\,, \\ f(u) &= \frac{(u-u_+)(u-u_-)}{u^2}\,, \quad u_\pm = M \pm \sqrt{M^2-q^2} \end{aligned} \tag{4.1}$$

with the background $U(1)$ gauge field and the dilaton field being equal to

$$A_t(u) = q\left(\frac{1}{u_+} - \frac{1}{u}\right)\,, \tag{4.2}$$

$$\Phi = \Phi_0 - \frac{1}{2}\log\left(\frac{u\sqrt{\hat{k}}}{2}\right)\,, \quad \Phi_0 = -\frac{1}{2}\log\left(\frac{Mu_+\sqrt{\hat{k}}}{u_+ + u_-}\right)\,.$$

The gauge potential vanishes at the outer horizon, $A_t(u_+) = 0$. Define parameter ψ by the equation

$$\frac{u_-}{u_+} = \tan^2\frac{\psi}{2}\,. \tag{4.3}$$

For the full description of thermodynamics of two-dimensional charged black hole the reader is referred to [27], we just review their results which are useful for us. It is convenient for further purposes to denote the background dilaton slope (see eq. (4.2)) as $Q = 2/\sqrt{\hat{k}}$. Requiring the metric (4.1) near external horizon $u = u_+$ to be regular, we find the temperature of the charged black hole

$$\beta = \frac{4\pi}{Q} \frac{u_+}{u_+ - u_-} = \frac{4\pi}{Q} \frac{\cos^2 \frac{\psi}{2}}{\cos\psi} . \tag{4.4}$$

Asymptotical $u \gg 1$ value of the gauge potential (see eq. (4.2)) is equal to the chemical potential:

$$\mu = \frac{q}{u_+} = \sqrt{\frac{u_-}{u_+}} = \tan\frac{\psi}{2} . \tag{4.5}$$

The entropy of the two-dimensional charged black hole is given by [27]

$$S_{bh}(M, q) = \frac{2\pi}{Q} (M + \sqrt{M^2 - q^2}) . \tag{4.6}$$

Using (4.6) and evaluating the grand canonical partition sum $\mathcal{Z}$ one observes [27] that the grand canonical potential $\Omega \sim -\log \mathcal{Z}$ vanishes, and therefore the pressure vanishes.

Consider black brane background space-time $CBH_2 \times R^{d-1}$, which is a direct product of the two-dimensional charged black hole and flat $d-1$-dimensional space. Denote by X the direction of R^{d-1} of propagation of all the excitation, and denote by Y some transverse direction of R^{d-1}. In the shear channel excitation modes appear as poles of the two-point function $\langle T_{XY} T_{XY} \rangle$ of the stress-energy tensor T_{MN}, with the dispersion relation of the low-energy mode given by

$$\omega = -\frac{i\eta}{(M+P)/V} p^2 , \tag{4.7}$$

where p is the momentum and ω is the frequency of the mode; η is the shear viscosity, M/V and P/V are energy and pressure densities.

Because for the two-dimensional charged black hole the pressure vanishes, we obtain

$$\omega = -\frac{i\eta}{M/V} p^2 . \tag{4.8}$$

Using (4.6) one can express,

$$M = \frac{Q S_{bh}}{2\pi(1 + \cos\psi)} , \tag{4.9}$$

and therefore the hydrodynamics predicts a shear pole with the dispersion relation

$$\omega = -4\pi i \frac{\eta}{s} \frac{\sqrt{\hat{k}}\cos^2(\psi/2)}{2} p^2 \,, \tag{4.10}$$

where $s = S/V$ is the entropy density. Bellow we are going to compare this result with the computation in heterotic gravity and derive the value of η/s.

4.3 The physical state conditions and vertex operators

4.3.1 BRST quantization

The gauging of the $U(1)$ subgroup from the $SL(2,R)\times U(1)$ group in the gWZW model on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset is realized by adding the $U(1)$ non-dynamical gauge field to the system, and adding corresponding action terms to the $SL(2,R)\times U(1)$ WZW action. The $U(1)$ subgroup is gauged left-right asymetrically, and anomaly-free condition must be satisfied. The details of the construction are reviewed in Appendix A. The end product is the gWZW action

$$\begin{aligned} S_g = S[g] &+ \frac{1}{2\pi}\int d^2z \partial x \bar{\partial} x \\ &+ \frac{1}{2\pi}\int d^2z \left[A\,\tilde{k} + \bar{A}\,k + A\bar{A}\left(2 + \mathrm{Tr}(g^{-1}\sigma^3 g\sigma^3)\cos\psi\right)\right] . \end{aligned} \tag{4.11}$$

Here we have denoted the currents of the gauged $U(1)$ subgroup as

$$k = \sqrt{\hat{k}}\mathrm{Tr}(\partial g g^{-1}\sigma^3)\cos\psi + 2\sin\psi\partial x = \frac{2}{\sqrt{\hat{k}}} j^3\cos\psi + 2\sin\psi \partial x \,, \tag{4.12}$$

$$\tilde{k} = \sqrt{\hat{k}}\mathrm{Tr}\left(g^{-1}\bar{\partial} g\sigma^3\right) = -\frac{2}{\sqrt{\hat{k}}}\tilde{j}^3 \,. \tag{4.13}$$

To determine physical spectrum of the quantum model on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset, we are going to use BRST quantization method [28] (see [7] where this method was applied to build the $SL(2,R)/U(1)$ model). The path integral for the theory is

$$
\begin{aligned}
Z &= \int [dG][dA][d\bar{A}] \exp\left(-S_g[G, A, \bar{A}]\right) \\
&= \int [dG][du][dv] \det \partial \det \bar{\partial} \; \exp\left(-S[G] + S[w]\right) . \qquad (4.14)
\end{aligned}
$$

Represent functional determinants in terms of the gauge ghost fields

$$
\det \partial \det \bar{\partial} = \int [db][dc][d\tilde{b}][d\tilde{c}] \exp\left(-\frac{1}{2\pi}\int d^2z (b\bar{\partial}c + \tilde{b}\partial\tilde{c})\right) . \qquad (4.15)
$$

Ghosts satisfy OPEs

$$
c(z)b(w) \sim \frac{1}{z-w} + \cdots , \qquad \tilde{c}(\bar{z})\tilde{b}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}} + \cdots . \qquad (4.16)
$$

Fix the gauge symmetry, for concreteness fix $v = 1$, therefore $u = w$. Consequently the path integral is given by

$$
Z = \int [dG][dw][db][dc][d\tilde{b}][d\tilde{c}] \exp\left(-S[G] + S[w] - \frac{1}{2\pi}\int d^2z (b\bar{\partial}c + \tilde{b}\partial\tilde{c})\right) \qquad (4.17)
$$

and the total action is given by

$$
S_q = S[G] - S[w] + \frac{1}{2\pi}\int d^2z (b\bar{\partial}c + \tilde{b}\partial\tilde{c}) \qquad (4.18)
$$

Notice from this action that the correlation function for w has the wrong sign:

$$
\langle \partial w(z_1)\partial w(z_2)\rangle = \frac{1}{2(z_1 - z_2)^2} . \qquad (4.19)
$$

Perform variations $(\delta G, \delta w, \delta b)$ in the action (4.18),

$$
\delta S_q = -\frac{1}{2\pi}\int d^2z \left[\hat{k}\mathrm{Tr}(\partial G G^{-1}\bar{\partial}(G^{-1}\delta G)) + 2\partial w\bar{\partial}\delta w - \delta b\bar{\partial}c\right] . \qquad (4.20)
$$

For the transformations with a local Grassmann parameter η:

$$
\delta G = \eta c G T_L , \quad \delta w = \eta c , \quad \delta b = \eta(k + 2\partial w) \qquad (4.21)
$$

we therefore obtain

$$
\delta S_q = \frac{1}{2\pi}\int d^2z (\bar{\partial}\eta)\, c(k + 2\partial w) . \qquad (4.22)
$$

If η is a global parameter, then S_q is invariant. Corresponding transformation is BRST symmetry transformations. Then from (4.22) for anti-holomorphic $\eta(\bar{z})$ we can read off holomorphic component of the corresponding conserved Noether current:

$$j_{BRST} = c(k + 2\partial w)\,. \tag{4.23}$$

Notice that

$$\langle (k(z_1) + 2\partial w(z_1))(k(z_2) + 2\partial w(z_2))\rangle = 0\,. \tag{4.24}$$

Therefore corresponding BRST charge

$$Q_{BRST} = \frac{1}{2\pi i}\oint dz j_{BRST} \tag{4.25}$$

is nilpotent. Similarly one finds the anti-holomorphic component of the BRST current

$$\tilde{j}_{BRST} = \tilde{c}(\tilde{k} + 2\bar{\partial}\tilde{w})\,. \tag{4.26}$$

Physical states of the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset model are the BRST-closed states of the $SL(2,R)\times U(1)$ model:

$$Q_{BRST}|\mathrm{phys}\rangle = 0\,, \quad \tilde{Q}_{BRST}|\mathrm{phys}\rangle = 0\,, \tag{4.27}$$

and are defined up to BRST-exact states.

Denote null bosonic currents as

$$J = k + 2\partial w\,, \qquad \tilde{J} = \tilde{k} + 2\bar{\partial}\tilde{w}\,. \tag{4.28}$$

BRST physical state conditions (4.27) therefore become

$$J_n|\mathrm{phys}\rangle = 0\,,\ \ n\geq 0\,, \quad \tilde{J}_n|\mathrm{phys}\rangle = 0\,,\ \ n\geq 0\,. \tag{4.29}$$

The BRST-exact massless state is obtained by acting with J_{-1} and $\tilde{J}_{-1}$ on the BRST-closed ground state.

4.3.2 Ground state vertex operator

The ground state vertex operator V_t of the $\frac{SL(2,R)\times U(1)}{U(1)}$ model was constructed in [25] as a ground state vertex operator of the $SL(2,R)\times U(1)$ model invariant under gauge $U(1)$ transformations. This vertex operator describes a tachyon, which due to GSO projection is projected out of the NS-NS sector. The lowest NS states are massless, and they are described by the vertex operator $V^M = j^M_{-1} V_t$ (and similarly for anti-holomorphic vertex operator). We derive these massless vertex operators bellow in this section. In this subsection we find it useful to reproduce the result of citeGiveon:2003ge4 using BRST quantization, developed in the previous subsection.

Suppose $\Phi(z,\bar{z})$ is a vertex operator on $SL(2,R)\times U(1)$. Then

$$V_t(z,\bar{z}) = \Phi(z,\bar{z}) \exp\left(im_L w_L + im_R w_R\right) \tag{4.30}$$

where

$$w(z,\bar{z}) = w_L(z) + w_R(\bar{z}) \tag{4.31}$$

is a vertex operator on the coset $\frac{SL(2,R)\times U(1)}{U(1)}$ if the physical state condition (4.27) are satisfied. We obtain

$$k_0 \cdot V_t(z,\bar{z}) = -im_L V_t(z,\bar{z})\,, \tag{4.32}$$

$$\tilde{k}_0 \cdot V_t(z,\bar{z}) = -im_R V_t(z,\bar{z})\,. \tag{4.33}$$

The non-compact w-circle contains only momentum modes and does not contain any winding modes, therefore

$$m_L = \frac{M}{R} - WR\,, \quad m_R = \frac{M}{R} + WR \quad \Rightarrow \quad m_L = m_R\,. \tag{4.34}$$

Let us denote $m_L = m_R = N$. The ground state on $SL(2,R)\times U(1)$ is described by the vertex operator [3]

$$\Phi(z,\bar{z}) = V_{jm\bar{m}} e^{2in_L x_L + 2in_R x_R}\,, \tag{4.35}$$

therefore

$$k_0 \cdot \Phi(z,\bar{z}) = 2\left(\frac{m\cos\psi}{\sqrt{\hat{k}}} - in_L \sin\psi\right)\Phi(z,\bar{z})\,, \tag{4.36}$$

$$\tilde{k}_0 \cdot \Phi(z,\bar{z}) = -\frac{2\bar{m}}{\sqrt{\hat{k}}}\Phi(z,\bar{z})\,. \tag{4.37}$$

[3] Define $x \sim x+\pi$, so that $n_{L,R}$ are integers. The $V_{jm\bar{m}}$ is the $SL(2,R)$ ground state primary field, see details in Appendix A.

The BRST physical state conditions (4.32), (4.33) due to (4.34) therefore imply

$$2\left(\frac{m\cos\psi}{\sqrt{\hat{k}}} - in_L\sin\psi\right) = -iN \tag{4.38}$$

$$\frac{2\bar{m}}{\sqrt{\hat{k}}} = iN \tag{4.39}$$

and consequently

$$m\cos\psi + \bar{m} - i\sqrt{\hat{k}}n_L\sin\psi = 0\,, \tag{4.40}$$

as was derived in [25].

4.3.3 Vertex operators of massless states in type-II superstring theory

The stress-energy tensor, which follows from the action (4.18), has the following holomorphic (left-moving) component [4]

$$T(z) = \frac{1}{\hat{k}}\eta_{AB}j^A j^B - \partial x\partial x + \partial w\partial w \tag{4.41}$$

and similarly for anti-holomorphic component. Therefore $\mathcal{O}((z-w)^{-2})$ terms of the OPEs of the stress-energy tensor and the ground state primary V_t are given by

$$T(z)V_t(w,\bar{w}) = \frac{1}{(z-w)^2}\left(-\frac{j(j+1)}{\hat{k}} + \frac{n_L^2 - m_L^2}{2}\right)V_t(w,\bar{w}) + \cdots \tag{4.42}$$

$$\tilde{T}(\bar{z})V_t(w,\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^2}\left(-\frac{j(j+1)}{\hat{k}} + \frac{n_R^2 - m_R^2}{2}\right)V_t(w,\bar{w}) + \cdots \tag{4.43}$$

In what follows we are going to perform Kaluza-Klein reduction of the $U(1)_x$ circle, therefore $n_L = n_R = 0$.

In this chapter we are interested in the NS-NS vertex operators of the massless closed string excitations in type-II superstring theory. These

[4] The term with w corresponds to the coset Kazama-Suzuki construction [29, 30], where $T_{G/H} = T_G - T_H$, in the following way. From BRST condition due to (4.23) one obtains, schematically, $\partial w = -\frac{1}{2}k$. Therefore contribution of w to the stress-energy tensor $T(z)$ is $T_w(z) = \partial w\partial w = \frac{1}{4}kk$. Then we expect $T_H = -T_w$, which is indeed the case: $T_H(z)k(0) = k(z)/z^2$.

operators in the $(-1,-1)$ picture are constructed as (anti)symmetrized direct products of massless holomorphic $\mathbf{V}^\mu(z) = e^{-\varphi}\psi^\mu_{-1/2}\cdot V_t$ and anti-holomorphic $\tilde{\mathbf{V}}^\mu(\bar{z}) = e^{-\tilde{\varphi}}\tilde{\psi}^\mu_{-1/2}\cdot V_t$ vertex operators. Here $\psi^\mu(z)$ and $\tilde{\psi}^\mu(\bar{z})$ are world-sheet fermions, and $\varphi\,,\tilde{\varphi}$ are bosonized superconformal ghosts. The only non-trivial super-Virasoro physical state condition, which one needs to impose on the massless states, is $G_{1/2}\cdot\mathbf{V}^\mu(z) = 0$, and similarly for anti-holomorphic vertex operator, where $G(z) = \sum_r G_r/z^{r+3/2}$ is the supercurrent.

The other option, which is what we are going to use in this chapter, is to consider vertex operators V^μ and $\tilde{V}^\mu$ in zero-ghost picture. They are obtained from $(-1,-1)$ picture vertex operators $\mathbf{V}^\mu$ and $\tilde{\mathbf{V}}^\mu$ by acting with the picture changing operators $e^\varphi G$ and $e^{\tilde{\varphi}}\tilde{G}$. As a result one obtains vertex operator in zero-ghost picture

$$V^\mu = G_{-1/2}\cdot\psi_{-1/2}\cdot V_t = (j^\mu_{-1} + p\cdot\psi_{-1/2}\psi^\mu_{-1/2})\cdot V_t\,, \tag{4.44}$$

where j^μ is the current, supersymmetric to the fermion ψ^μ, and p^μ is the momentum of the state. Similar expression is true for anti-holomorphic vertex operator. The only non-trivial super-Virasoro constraint which one should impose in zero-ghost picture is $L_1\cdot V^\mu = 0$. Moreover, the L_1 here is actually the amplitude of the stress-energy tensor $L_1^{(b)}$ for only bosonic modes: in the r.h.s. of (4.44) contribution of fermions is automatically annihilated by the fermionic stress-energy tensor amplitude $L_1^{(f)} = \psi^\nu_{1/2}\psi_{\nu 1/2}$. Therefore instead of studying massless NS-NS states in type-II superstring theory we can study gravity multiplet in bosonic string theory.

4.3.4 (Anti)holomorphic vertex operators of massless modes in the $R\times\frac{SL(2,R)}{U(1)}$ coset model

In this subsection we review the construction of (anti)holomorphic vertex operators [16], describing massless (right-)left-moving excitations in the gWZW model on $R\times\frac{SL(2,R)}{U(1)}$ [6, 7]. The classical background is the two-dimensional black hole with the linear dilaton in a direct product with a real line. The real line is parametrized by the flat coordinate X, which we choose as a direction of propagation of all the excitations. The momentum is equal to p.

The authors of [16] considered graviton vertex operator in $(-1,-1)$ picture on the world-sheet with $\mathcal{N} = 2$ supersymmetry. We perform a picture changing and consider vertex operators in zero-ghost picture. Due to noted in the previous subsection, we can actually study bosonic string and then make contact with the results of [16].

Without loss of generality let us focus on holomorphic vertex operators. The ground state vertex operator of the $R \times \frac{SL(2,R)}{U(1)}$ coset theory is

$$V_t = e^{ipX} e^{iNw} V_{jm} \,. \tag{4.45}$$

This state must be closed under the action of the null $U(1)$ BRST current,

$$J = j^3 - \sqrt{\hat{k}} \partial w \,, \tag{4.46}$$

which imposes the condition

$$iN = \frac{2m}{\sqrt{\hat{k}}} \,. \tag{4.47}$$

The most general holomorphic vertex operator of the massless state (which is a gauge field from the space-time point of view) on $R \times \frac{SL(2,R)}{U(1)}$ is

$$V^{\chi} = (a_+ j^+_{-1} j^-_0 + a_- j^-_{-1} j^+_0 + a_3 j^3 + a_w \partial w + a_X \partial X) V_t \,. \tag{4.48}$$

Mass-shell Virasoro constraint (see (4.42)) gives

$$L_0 V^{\chi} = V^{\chi} \quad \Rightarrow \quad -\frac{j(j+1)}{\hat{k}-2} + \frac{p^2 - N^2}{4} = 0 \tag{4.49}$$

Closeness of (4.48) w.r.t. J_1 (see (4.46)) reduces the number of parameters by one, giving the most general BRST-closed state

$$\begin{aligned} V^{\chi} = (a_+ j^+_{-1} j^-_0 + a_- j^-_{-1} j^+_0 + \frac{2}{\sqrt{\hat{k}}} (a_+ (m+j)(m-1-j) \\ - a_- (m-j)(m+1+j)) \partial w + AJ + a_X \partial X) V_t \,. \end{aligned} \tag{4.50}$$

Also V^{χ} is defined up to BRST-exact state JV_t, which makes one more parameter unphysical, leaving us with a gauge field in three dimensional $R \times \frac{SL(2,R)}{U(1)}$ with three polarization parameters.

Gauge field in three dimensions has one transverse physical d.o.f. Two of the three d.o.f. are eliminated in the following way. First, we impose

Virasoro constraint $L_1 V^\chi = 0$. Second, the state V^χ, which satisfies this constraint, is defined up to the null state $L_{-1}V_t$

$$L_{-1}V_t = \left(\frac{1}{\hat{k}-2}(j^+_{-1}j^-_0 + j^-_{-1}j^+_0 - 2mj^3) + iN\partial w + ip\partial X \right) V_t \,. \quad (4.51)$$

As a result we are left with one transverse d.o.f.

The $L_1 V^\chi = 0$ constraint gives

$$\begin{aligned} &a_+(m+j)(m-1-j) + a_-(m-j)(m+1+j) \\ &+ \frac{iN}{\sqrt{\hat{k}}}(a_+(m+j)(m-1-j) - a_-(m-j)(m+1+j)) = a_X \frac{ip}{2} \,. \end{aligned} \quad (4.52)$$

Let us parametrize the solution to this equation by two independent parameters a_X, a:

$$a_+ = \frac{a_X \frac{ip}{4} + a\left(1 - \frac{2m}{\hat{k}}\right)}{(m+j)(m-1-j)} \,, \quad a_- = \frac{a_X \frac{ip}{4} - a\left(1 + \frac{2m}{\hat{k}}\right)}{(m-j)(m+1+j)} \,. \quad (4.53)$$

Therefore the most general massless left-moving state, satisfying all the Virasoro and $U(1)$ gauge BRST constraints (and defined up to BRST-exact state $J_{-1}V_t$ and null Virasoro state $L_{-1}V_t$) is described by holomorphic vertex operator

$$\begin{aligned} V^\chi = &\left(\frac{a_X \frac{ip}{4} + a\left(1 - \frac{2m}{\hat{k}}\right)}{(m+j)(m-1-j)} j^+_{-1}j^-_0 + \frac{a_X \frac{ip}{4} - a\left(1 + \frac{2m}{\hat{k}}\right)}{(m-j)(m+1+j)} j^-_{-1}j^+_0 \right. \\ &\left. + \frac{4a}{\sqrt{\hat{k}}}\partial w + a_X \partial X \right) V_t \,. \end{aligned} \quad (4.54)$$

Now notice that for

$$a = m a_1 \,, \qquad a_X = -i(\hat{k}-2)p a_1 \quad (4.55)$$

we obtain that the state (4.54) is

$$V^\chi_0 = a_1(-2mJ_{-1} - (\hat{k}-2)L_{-1})V_t \,. \quad (4.56)$$

Such a state is a pure gauge (BRST-exact).

Therefore the most general physical state, which satisfies all the constraints and which is not a pure gauge, is a state for which

$$\frac{a}{a_X} \neq \frac{im}{(\hat{k}-2)p} \,. \quad (4.57)$$

Any such state is orthogonal to the V_0^χ state (4.56), due to Virasoro and BRST physical state conditions.

The two-point function of the most general physical state (4.54) is

$$\langle V^\chi(p,j,m)V^\chi(-p,j,-m)\rangle = \frac{(\hat{k}^2-\hat{k}p^2-4m^2)(ma_X+i(\hat{k}-2)pa)^2}{2\hat{k}^2(m^2-j^2)(m^2-(j+1)^2))} \langle V_t(p,j,m)V_t(-p,j,-m)\rangle\,. \tag{4.58}$$

When (4.57) is not satisfied, we are dealing with the null pure gauge state, which is a linear combination of timelike and longitudinal polarizations, that is for such a state

$$ma_X+i(\hat{k}-2)pa=0\,. \tag{4.59}$$

Finally let us make contact with the result of [16]. The two holomorphic supercurrents of $\mathcal{N}=2$ supersymmetric $SL(2,R)/U(1)$ gWZW theory are

$$G^+=\psi^+j^-\,,\quad G^-=\psi^-j^+\,. \tag{4.60}$$

Applying the picture-changing operator $G^+_{-1/2}+G^-_{-1/2}$ to the physical holomorphic vertex operator of [16] we obtain the vertex operator of the form

$$V^\chi=(\partial X+a_+j^+_{-1}j^-_0+a_-j^-_{-1}j^+_0+\text{fermions})V_t\,. \tag{4.61}$$

Here due to (4.54) we have

$$a_+=\frac{ip/4}{(m+j)(m-1-j)}\,,\quad a_-=\frac{ip/4}{(m-j)(m+1+j)}\,. \tag{4.62}$$

Due to (4.58) we obtain that the two-point function of this vertex operator has poles at $m=\pm j$. Bellow we discuss these poles in detail and show that actually only $m=-j$ pole is present, which after taking into account the mass-shell condition precisely reproduces the dispersion relation of the gapless low-energy mode, found in [16].

4.3.5 (Anti)holomorphic vertex operators of massless modes in the $R\times\frac{SL(2,R)\times U(1)}{U(1)}$ coset model

In this subsection we are going to construct (anti)holomorphic vertex operators, describing massless (right-)left-moving string excitations in the

$R\times\frac{SL(2,R)\times U(1)}{U(1)}$ model. The classical geometry of this model is a geometry of the 2d charged black hole in a direct product with a real line. We choose this real line as a direction of propagation of all the excitations, and parametrize it by the coordinate X. The momentum of propagation is p.

The vertex operator operators must satisfy BRST and Virasoro physical state conditions. Recall the null BRST currents (4.28):

$$J=\frac{2}{\sqrt{\hat{k}}}j^3\cos\psi+2\sin\psi\partial x+2\partial w \tag{4.63}$$

$$\tilde{J}=-\frac{2}{\sqrt{\hat{k}}}\tilde{j}^3+2\bar{\partial}\tilde{w}\,. \tag{4.64}$$

Notice that the anti-holomorphic sector is the same as for the model of the previous subsection: anti-holomorphic (right-moving) sector of the circle, $U(1)_{\tilde{x}}$, is disconnected from the rest of the geometry.

Consider holomorphic sector. Ground state vertex operator is

$$V_t=e^{ipX}e^{iNw}V_{jm}\,. \tag{4.65}$$

This state must be closed w.r.t. BRST current (4.63), which imposes the constraint

$$iN=-\frac{2m\cos\psi}{\sqrt{\hat{k}}}\,. \tag{4.66}$$

The most general massless holomorphic vertex operator on the $R\times\frac{SL(2,R)\times U(1)}{U(1)}$ is given by

$$V^{\chi}=(a_+j^+_{-1}j^-_0+a_-j^-_{-1}j^+_0+a_w\partial w+a_X\partial X+b_x\partial x+AJ)V_t\,. \tag{4.67}$$

It must be closed w.r.t. BRST current (4.63), which requires

$$a_w=b_x\sin\psi-\frac{2}{\sqrt{\hat{k}}}\cos\psi(a'_+-a'_-)\,. \tag{4.68}$$

where we have denoted for brevity

$$a'_+=a_+(m+j)(m-1-j)\,,\qquad a'_-=a_-(m-j)(m+1+j)\,. \tag{4.69}$$

The most general massless state, closed w.r.t. (4.63), is therefore described by the vertex operator

$$V^{\chi}=\Big(a_+j^+_{-1}j^-_0+a_-j^-_{-1}j^+_0+a_X\partial X+b_x\partial x \\ +\Big(b_x\sin\psi-\frac{2}{\sqrt{\hat{k}}}\cos\psi(a'_+-a'_-)\Big)\partial w+AJ\Big)V_t\,. \tag{4.70}$$

One d.o.f. in (4.70) is unphysical due to the fact that each state V^χ is defined up to BRST-exact state $J_{-1}V_t$. Therefore there remain four d.o.f. of the gauge field V^χ in four dimensional target space. Two of them are unphysical, and are eliminated due to Virasoro constraints, as we show bellow.

Imposing Virasoro constraint $L_1 V^\chi = 0$, with account to (4.66), we obtain condition

$$a'_+ + a'_- + \frac{2m\cos^2\psi}{\hat{k}}(a'_+ - a'_-) = a_X\frac{ip}{2} + b_x\frac{m\sin 2\psi}{2\sqrt{\hat{k}}}\,. \tag{4.71}$$

We parametrize the solution to this equation as

$$a_+ = \frac{a_X\frac{ip}{4} + a\left(1 - \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m+j)(m-1-j)} \tag{4.72}$$

$$a_- = \frac{a_X\frac{ip}{4} - a\left(1 + \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m-j)(m+1+j)}\,. \tag{4.73}$$

Using the mass-shell Virasoro condition (see (4.42))

$$L_0 V^\chi = V^\chi \quad \Rightarrow \quad -\frac{j(j+1)}{\hat{k}-2} + \frac{p^2 - N^2}{4} = 0\,. \tag{4.74}$$

we can re-write

$$(m+j)(m-1-j) = -\frac{\hat{k}-2}{4}p^2 - m\left(1 - \frac{2m\cos^2\psi}{\hat{k}}\right) + m^2\sin^2\psi\,, \tag{4.75}$$

$$(m-j)(m+1+j) = -\frac{\hat{k}-2}{4}p^2 + m\left(1 + \frac{2m\cos^2\psi}{\hat{k}}\right) + m^2\sin^2\psi\,. \tag{4.76}$$

These expressions are useful for computations, described bellow.

To summarize, the most general massless physical state V^χ on the $R\times\frac{SL(2,R)\times U(1)}{U(1)}$, satisfying all the physical constraints, is

$$\begin{aligned} V^\chi = \Bigg(&\frac{a_X\frac{ip}{4} + a\left(1 - \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m+j)(m-1-j)} j^+_{-1} j^-_0 \\ &+ \frac{a_X\frac{ip}{4} - a\left(1 + \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m-j)(m+1+j)} j^-_{-1} j^+_0 \\ &+ a_X\partial X + b_x\partial x + \left(b_x\sin\psi - \frac{4a\cos\psi}{\sqrt{\hat{k}}}\right)\partial w\Bigg) V_t\,. \end{aligned} \tag{4.77}$$

This state is defined up to the null Virasoro state (recall $n_{L,R} = 0$ due to Kaluza-Klein reduction of the x-circle)

$$L_{-1}V'_{jm} = \left(\frac{1}{\hat{k}-2}(j^+_{-1}j^-_0 + j^-_{-1}j^+_0 - 2mj^3) + iN\partial w + ip\partial X\right)V_t\,. \quad (4.78)$$

Using (4.75) and (4.76) one then demonstrates that for

$$a_X = -i(\hat{k}-2)pa_1\,, \quad a = ma_1\,, \quad b_x = -2m\sqrt{\hat{k}}\tan\psi a_1 \quad (4.79)$$

the state (4.77) is non-physical (it is the sum of the BRST-exact and the null Virasoro states)

$$V_0^\chi = -a_1\left((\hat{k}-2)L_{-1} + \frac{m\sqrt{\hat{k}}}{\cos\psi}j\right)V_t\,. \quad (4.80)$$

The two-point function of the vertex operator (4.77) is given by

$$\begin{aligned}\langle V^\chi V^\chi\rangle =& (m^2-j^2)^{-1}(m-(j+1)^2)^{-1}(c_1(i(\hat{k}-2)pa+ma_X)^2 \quad (4.81)\\ &+c_2(i(\hat{k}-2)pb_x-2m\sqrt{\hat{k}}\tan\psi a_X)^2+c_3(b_x+2a\sqrt{\hat{k}}\tan\psi)^2)\langle V_tV_t\rangle\,,\end{aligned}$$

where

$$\begin{aligned}c_1 =& \frac{1}{4\hat{k}^2(\hat{k}-2)}(\hat{k}(\hat{k}-2)^2\cos^2\psi p^2 - 8(\hat{k}-2)\cos^4\psi m^2\\ &-\hat{k}^2(\hat{k}-2)(p^2-2) + 2\hat{k}(\sin^2 2\psi + 2\hat{k}\sin^4\psi)m^2)\,, \quad (4.82)\end{aligned}$$

$$c_2 = \frac{\cos^2\psi}{32\hat{k}^2(\hat{k}-2)}(2((\hat{k}-2)^2\cos 2\psi + 4 - 4\hat{k} - \hat{k}^2)m^2 + (\hat{k}-2)^2\hat{k}p^2)\,, \quad (4.83)$$

$$\begin{aligned}c_3 =& \frac{\cot\psi}{32\hat{k}^2}((8m^2(\hat{k}^2-2m^2) + 2(\hat{k}^2-4)m^2p^2 - (\hat{k}-2)^2\hat{k}p^4)\sin 2\psi\\ &-m^2(8m^2 + (\hat{k}-2)^2p^2)\sin 4\psi)\,. \quad (4.84)\end{aligned}$$

When the (4.79) is satisfied, we are dealing with the null state V_0^χ with zero norm.

Like in the previous section, where we derived the vertex operator (4.61), we now proceed to writing down the vertex operators $V^x = (\partial x +$

...)V_t and $V^X = (\partial X + ...)V_t$, where dots denote contribution from $j^{\pm}_{-1}$ currents:

$$V^X = \left(\partial X + \frac{ip/4}{(m+j)(m-1-j)} j^+_{-1} j^-_0 + \frac{ip/4}{(m-j)(m+1+j)} j^-_{-1} j^+_0\right) V_t \quad (4.85)$$

$$V^x = \left(\partial x + \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m+j)(m-1-j)} j^+_{-1} j^-_0 - \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m-j)(m+1+j)} j^-_{-1} j^+_0\right) V_t\,. \quad (4.86)$$

4.4 Vertex operators of massless NS-NS states, and correlation functions

In the previous section we constructed holomorphic and anti-holomorphic vertex operators, describing respectively left-moving and right-moving massless excitations of the string on the $\frac{SL(2,R)\times U(1)_x}{U(1)}$ coset. The state of the closed string is described by the vertex operator which is a direct product of holomorphic and anti-holomorphic vertex operators. In this section we will construct the vertex operators for graviton and antisymmetric tensor field, which are massless NS-NS states of type-II gravity. Kaluza-Klein reduction on $U(1)_x$, applied to graviton and antisymmetric tensor field vertex operators, gives vertex operators for gauge fields. We will split the vertex operators into two decoupled from each other groups, and find correlation functions for vertex operators within each group.

Denote $M = a, X, x$, and $\mu = a, X$, where a labels non-compactified directions, transverse to the direction X of propagation of all the excitations, and x is a coordinate of the compactified circle. Then, $V^M = j^M V_t$ are holomorphic physical vertex operators and $\tilde{V}^M = \tilde{j}^M V_t$ are anti-holomorphic physical vertex operators of the massless left-moving and right-moving states.

Here $j^a = \partial x^a$ and $\tilde{j}^a = \bar{\partial} x^a$. Due to (4.85) and (4.86) the j^x and j^X are elements of two different BRST and Virasoro cohomology classes, and are defined by

$$j^X = \partial X + \frac{ip/4}{(m+j)(m-1-j)} j^+_{-1} j^-_0 + \frac{ip/4}{(m-j)(m+1+j)} j^-_{-1} j^+_0 \quad (4.87)$$

$$j^x = \partial x + \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m+j)(m-1-j)} j^+_{-1} j^-_0 - \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m-j)(m+1+j)} j^-_{-1} j^+_0\,. \quad (4.88)$$

Due to anti-holomorphic version of (4.61),

$$\tilde{j}^X = \bar{\partial} X + \frac{ip/4}{(\bar{m}+j)(\bar{m}-1-j)} \tilde{j}^+_{-1} \tilde{j}^-_0 + \frac{ip/4}{(\bar{m}-j)(\bar{m}+1+j)} \tilde{j}^-_{-1} \tilde{j}^+_0 \,. \tag{4.89}$$

Finally, $\tilde{j}^x = \bar{\partial} x$.

Notice that due to (4.81) the normalized two-point functions $\langle j^x j^x \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$, $\langle j^x j^X \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$, $\langle j^X j^X \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$ have simple poles at $m = \pm j$, and due to (4.58) (the anti-holomorphic version of it) the two-point function $\langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$ has simple poles at $\bar{m} = \pm j$.

The two-point function for ground state of the $SL(2,R)$ model is given by (see e.g. [14, 15] for a recent discussion)

$$\langle V_{j,m,\bar{m}} V_{j,-m,-\bar{m}} \rangle = \nu \frac{\Gamma\left(1 - \frac{2j+1}{\hat{k}-2}\right) \Gamma(-2j-1) \Gamma(1+j+m) \Gamma(1+j-\bar{m})}{\Gamma\left(1 + \frac{2j+1}{\hat{k}-2}\right) \Gamma(2j+1) \Gamma(-j+m) \Gamma(-\bar{m}-j)} \tag{4.90}$$

where ν is some number. Notice that due to factors of $\Gamma(-j+m)$ and $\Gamma(-\bar{m}-j)$ in the denominator, the (4.90) has simple zeroth are $j = m$ and $j = -\bar{m}$. Therefore the two-point functions $\langle j^x j^x \rangle_{jm\bar{m}}$, $\langle j^x j^X \rangle_{jm\bar{m}}$, $\langle j^X j^X \rangle_{jm\bar{m}}$ have simple pole at $m = -j$, while the simple pole at $m = j$ is canceled, and the two-point function $\langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}}$ has simple pole at $\bar{m} = j$, while the pole at $\bar{m} = -j$ is canceled.

4.4.1 Vertex operators and their correlation functions

Graviton vertex operator is

$$G^{MN} = (j^M \tilde{j}^N + j^N \tilde{j}^M) V_t \,. \tag{4.91}$$

Antisymmetric tensor field vertex operator is

$$B^{MN} = (j^M \tilde{j}^N - j^N \tilde{j}^M) V_t \,. \tag{4.92}$$

Gauge field vertex operators are:

$$A^\mu = G^{x\mu} = (j^x \tilde{j}^\mu + \bar{\partial} x j^\mu) V_t \tag{4.93}$$

$$B^\mu = B^{x\mu} = (j^x \tilde{j}^\mu - \bar{\partial} x j^\mu) V_t \,. \tag{4.94}$$

We have the following groups of vertex operators defined by the spin w.r.t. to the rotations in the transverse non-compactified space (for which the coordinates are labeled by small Latin indices).[5] In the sound channel the spin is zero, and one considers the fields G^{XX}, A^X B^X. In the shear channel the spin is one, and one considers the fields G^{Xa}, B^{Xa}, A^a, B^a. In the scalar channel the spin is two, and one considers the fields G^{ab} and B^{ab}. Due to the rotational symmetry in the transverse space, vertex operators from different groups are decoupled from each other.

Shear channel

In the shear channel we have vertex operators

$$G^{Xa} = (j^X \bar{\partial} x^a + \tilde{j}^X \partial x^a) V_t \tag{4.95}$$

$$B^{Xa} = (j^X \bar{\partial} x^a - \tilde{j}^X \partial x^a) V_t \tag{4.96}$$

$$A^a = G^{xa} = (j^x \bar{\partial} x^a + \bar{\partial} x \partial x^a) V_t \tag{4.97}$$

$$B^a = B^{xa} = (j^x \bar{\partial} x^a - \bar{\partial} x \partial x^a) V_t \tag{4.98}$$

Notice that all these vertex operators are coupled to each other. We can consider instead two groups of operators:

the first group is

$$S^{Xa} = \frac{1}{2}(G^{Xa} + B^{Xa}) = j^X \bar{\partial} x^a V_t \tag{4.99}$$

$$W^a = \frac{1}{2}(A^a + B^a) = j^x \bar{\partial} x^a V_t \tag{4.100}$$

and the second group is

$$R^{Xa} = \frac{1}{2}(G^{Xa} - B^{Xa}) = \tilde{j}^X \partial x^a V_t \tag{4.101}$$

$$U^a = \frac{1}{2}(A^a - B^a) = \bar{\partial} x \partial x^a V_t \,. \tag{4.102}$$

We call the operators from the first group S-system and the operators from the second group R-system. The S-system is decoupled from the R-system. For the vertex operators of the S-system the two-point functions are

$$\langle S^{Xa} S^{Xb} \rangle = -\frac{1}{2} \delta^{ab} \langle j^X j^X \rangle_{jm\bar{m}} \tag{4.103}$$

[5] See e.g. [31] for a recent discussion in the holographic context.

$$\langle W^a W^b \rangle = -\frac{1}{2}\delta^{ab}\langle j^x j^x \rangle_{jm\bar{m}} \tag{4.104}$$

$$\langle S^{Xa} W^b \rangle = -\frac{1}{2}\delta^{ab}\langle j^x j^X \rangle_{jm\bar{m}} \tag{4.105}$$

These correlation functions have a simple pole at $j = -m$.

For the vertex operators of the R-system the two-point functions are

$$\langle R^{Xa} R^{Xb} \rangle = -\frac{1}{2}\delta^{ab}\langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} \tag{4.106}$$

$$\langle U^a U^b \rangle = \frac{1}{4}\delta^{ab} \tag{4.107}$$

$$\langle R^{Xa} U^b \rangle = 0\,. \tag{4.108}$$

These correlation functions have a simple pole at $j = \bar{m}$.

Due to holographic correspondence we obtain correlation functions of the shear components of the stress-energy tensor of the dual field theory:[6]

$$\langle G^{Xa} G^{Xb} \rangle = \langle T^{Xa} T^{Xb} \rangle = -\frac{1}{2}\delta^{ab}\left(\langle j^X j^X \rangle_{jm\bar{m}} + \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}}\right) \tag{4.109}$$

The correlation functions for the transverse components of the charge current are

$$\langle J^a J^b \rangle = \langle A^a A^b \rangle = -\frac{1}{2}\delta^{ab}\left(\langle j^x j^x \rangle_{jm\bar{m}} - \frac{1}{2}\right) \tag{4.110}$$

Finally,

$$\langle J^a T^{Xb} \rangle = \langle A^a G^{Xb} \rangle = -\frac{1}{2}\delta^{ab}\langle j^x j^X \rangle_{jm\bar{m}}\,. \tag{4.111}$$

We conclude that in the shear/transverse diffusion channel we have modes with the dispersion relations $m = -j$ and $\bar{m} = j$.

Sound channel

In the sound channel we have vertex operators

$$G^{XX} = j^X \tilde{j}^X V_t \tag{4.112}$$

$$A^X = G^{xX} = (j^x \tilde{j}^X + \bar{\partial} x j^X) V_t \tag{4.113}$$

---

[6] One also may be interested in computing correlation functions of the operator, dual to B_{MN}-field. See [32, 33], where the primary operator in $\mathcal{N} = 4$ SYM, holographically dual to the B-field in $AdS_5 \times S^5$, was found.

$$B^X = B^{xX} = (j^x \tilde{j}^X - \bar{\partial} x j^X) V_t \,. \tag{4.114}$$

Notice that A^X and B^X are coupled. Consider instead decoupled gauge fields vertex operators

$$W^X = \frac{1}{2}(A^X + B^X) = j^x \tilde{j}^X V_t \tag{4.115}$$

$$U^X = \frac{1}{2}(A^X - B^X) = \bar{\partial} x j^X V_t \,. \tag{4.116}$$

Correlation functions are

$$\langle G^{XX} G^{XX} \rangle = \langle j^X j^X \rangle_{jm\bar{m}} \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} \tag{4.117}$$

$$\langle G^{XX} W^X \rangle = \langle j^x j^X \rangle_{jm\bar{m}} \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} \tag{4.118}$$

$$\langle G^{XX} U^X \rangle = 0 \tag{4.119}$$

$$\langle W^X W^X \rangle = \langle j^x j^x \rangle_{jm\bar{m}} \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} \tag{4.120}$$

$$\langle U^X U^X \rangle = -\frac{1}{2} \langle j^X j^X \rangle_{jm\bar{m}} \,. \tag{4.121}$$

The correlation functions (4.117), (4.118) and (4.120) have simple poles at $j = -m$ and $j = \bar{m}$ and the correlation function (4.121) has simple pole at $j = -m$.

Due to holographic correspondence we obtain correlation functions of the longitudinal component of the stress-energy tensor of the dual field theory:

$$\langle T^{XX} T^{XX} \rangle = \langle G^{XX} G^{XX} \rangle = \langle j^X j^X \rangle_{jm\bar{m}} \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} \,. \tag{4.122}$$

The correlation function of the longitudinal component of the charge current is

$$\langle J^X J^X \rangle = \langle A^X A^X \rangle = \langle j^x j^x \rangle_{jm\bar{m}} \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} - \frac{1}{2} \langle j^X j^X \rangle_{jm\bar{m}} \,. \tag{4.123}$$

Finally,

$$\langle J^X T^{XX} \rangle = \langle A^X G^{XX} \rangle = \langle j^x j^X \rangle_{jm\bar{m}} \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} \,. \tag{4.124}$$

Therefore in the sound channel we have modes with the dispersion relations $m = -j$ and $\bar{m} = j$.

Scalar channel

In the scalar channel one considers G^{ab} and B^{ab}, correlation functions for which do not have poles at $j = -m$ and $j = \bar{m}$.

4.4.2 Low-energy modes

In the previous subsection we concluded that there are modes with the dispersion relations $m = -j$ and $\bar{m} = j$ in the shear and sound channels of the quantum field theory holographically dual to the charged black brane. Now we are going to show, considering small ω and q, that these modes are actually gapless modes.

The frequency is determined by the asymptotic behavior of the tachyon vertex operator $V_t \sim e^{i\omega t}$ (see [25]), and is given by (for $\psi \neq \pi/2$, that is in non-extremal case)

$$\omega = \frac{(1 - \tan^2(\psi/2))(m - \bar{m})}{\sqrt{\hat{k}}} \tag{4.125}$$

Due to the gauge physical state condition (4.40) we have $\bar{m} = -m\cos\psi$, and therefore (also after Wick rotation $m \to im$)

$$\omega = \frac{2im\cos\psi}{\sqrt{\hat{k}}}\,. \tag{4.126}$$

Due to the mass-shell condition

$$-\frac{j(j+1)}{\hat{k}} + \frac{p^2 - N^2}{4} = 0 \tag{4.127}$$

where $N \sim m \sim \omega$, for $\omega \sim p^2$ and $p \ll 1$ we obtain

$$j = \frac{\hat{k}}{4}p^2\,. \tag{4.128}$$

Therefore the S-system possesses the low-energy excitation mode with the dispersion relation

$$\omega = -i\frac{\sqrt{\hat{k}}}{2}\cos\psi\, p^2 \tag{4.129}$$

while for the R-system we obtain the mode with the dispersion relation

$$\omega = -i\frac{\sqrt{\hat{k}}}{2}\,p^2\,. \tag{4.130}$$

In the extremal case $\psi = \pi/2$ the two-point functions in the S-system, due to (4.129), behave as $\langle SS \rangle \sim 1/\omega$, indicating local criticality, while the dispersion relation (4.130) of the R-system stays unaffected.

At zero density $q = 0$ we have $\psi = 0$, and the mode (4.129) coincides with the mode (4.130). In this case $U(1)_x$ completely decouples from $SL(2,R)/U(1)$, and we recover the results of [16] for the model on the $SL(2,R)/U(1)$. Due to (4.4) we obtain

$$\omega = -i\frac{1}{4\pi T}p^2 \tag{4.131}$$

Comparing it with the shear mode dispersion relation at zero density

$$\omega = -i\frac{\eta}{sT}p^2 \tag{4.132}$$

we recover the result of [16]

$$\frac{\eta}{s} = \frac{1}{4\pi}\,. \tag{4.133}$$

4.4.3 "$2k_F$" singularity

Expression (4.90) for the groundstate two-point function contains the factor of $\Gamma\left(1 - \frac{2j+1}{\hat{k}-2}\right)$. Due to the physical state mass-shell condition (4.127) we obtain

$$j = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + (\hat{k}-2)(p^2 - N^2)}\,. \tag{4.134}$$

At zero frequency $N = 0$. Therefore equation $2j + 1 = \hat{k} - 2$, which defines singularity of $\Gamma\left(1 - \frac{2j+1}{\hat{k}-2}\right)$, has a zero frequency and finite momentum solution. The value of the momentum is given by

$$p_\star^2 = \frac{1}{\ell_s^2}\left(\hat{k} - 2 - \frac{1}{\hat{k}-2}\right)\,. \tag{4.135}$$

Note that p_* is independent of the chemical potential $\mu = \tan\frac{\psi}{2}$. The singular behavior of $\langle V_{jm\bar{m}} V_{jm\bar{m}} \rangle$ at $\omega = 0$ and $p = p_*$ was compared by Polchinski and Silverstein [8] with "$2k_F$" singularities in current correlation functions of condensed matter systems (see e.g. [34]).

4.5 Type-II gravity approximation

In this section we will compute the two-point functions for graviton, antisymmetric tensor field and gauge fields in the background of the 2d charged black hole (in a direct product with a flat space). Our purpose is to verify the dispersion relations (4.129-(4.130).

One-loop beta-functions for the NS-NS fields of type-II gravity are given by (see e.g. [35],Polchinski:1998rr4)

$$\beta^G_{MN} = R_{MN} + 2\nabla_M \partial_N \Phi - \frac{1}{4} H_M{}^{LS} H_{NLS}\,, \tag{4.136}$$

$$\beta^{\Phi} = c + \frac{1}{16\pi^2}\left(4(\partial\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2\right)\,, \tag{4.137}$$

$$\beta^B_{MN} = \nabla_L H^L{}_{MN} - 2(\partial_L \Phi) H^L{}_{MN}\,. \tag{4.138}$$

Corresponding equations of motion are $\beta^{G,\Phi,B} = 0$.

Here the field strength of antisymmetric tensor B_{MN} is given by

$$H_{MNL} = \partial_M B_{NL} + \partial_N B_{LM} + \partial_L B_{MN}\,. \tag{4.139}$$

The beta-functions (4.216)-(4.218) are invariant w.r.t. the gauge symmetry

$$\delta B_{MN} = \partial_M \Lambda_N - \partial_N \Lambda_M\,. \tag{4.140}$$

Requirement of the world-sheet conformal invariance gives the equations of motion $\beta^{G,B,\Phi} = 0$. These equations have a black brane solution, which is a direct product of two-dimensional charged black hole (CBH) and flat space, $CBH \times R^{d-1}$:

$$\begin{aligned} g_{MN} &= \mathrm{diag}\{-f(r), 1/f(r), 1, ..., 1\}\,, \\ f(r) &= 1 - 2Me^{-Qr} + q^2 e^{-2Qr}\,, \end{aligned} \tag{4.141}$$

$$\Phi = \Phi_0 - \frac{Qr}{2}\,, \qquad F_{tr} = F(r) = Qqe^{-Qr}\,. \tag{4.142}$$

where[7]

$$g_{tx} = B_{xt} = -B_{tx} = A_t\,. \tag{4.143}$$

The string theory solution, described in the previous section, implies $Q = 2/\sqrt{\hat{k}}$.

[7] We thank A. Giveon for pointing out to us the role of this equation in the 2d charged black hole solution of type-II superstring theory.

Consider fluctuations h_{MN}, b_{MN} and φ around this solution. Use the diffeomorphism invariance to fix $h_{Mr} = 0$. Use gauge invariance (4.140) to fix $b_{Mr} = 0$. Among $d+1$ space-time coordinates, denoted by capital Latin indices, we have t, r coordinates of the charged black hole and $d-1$ flat coordinates. Let us consider $CBH \times R^3$. Choose X to be the R^3 direction of propagation of excitations (with momentum p) and choose Y to be the R^3 direction, transverse to propagation of excitations. Finally x is the direction of R^3 which we are going to Kaluza-Klein reduce. Fluctuations depend on t, r, X. The dependence on t and X in momentum representation boils down to the factor $e^{-i\omega t + ipX}$.

Perform Kaluza-Klein reduction of the x coordinate. Small Greek indices are used for non-reduced coordinates, $M = \mu, x$. It is convenient, as we did in the world-sheet consideration, to consider fluctuations of the fields

$$S_{MN} = \frac{1}{2}(h_{MN} + b_{MN}) \tag{4.144}$$

$$R_{MN} = \frac{1}{2}(h_{MN} - b_{MN})\,. \tag{4.145}$$

The fields (4.144) belong to the S-system and the fields (4.145) belong to the R-system, we are using the same terminology as in the previous section.

Let us consider shear fluctuations in the reduced space $CBH \times R^2$: R_{tY}, R_{XY}, S_{tY}, S_{XY} and transverse components of gauge fields w_Y and u_Y (see bellow). The ansatz for graviton and two-form field in the non-reduced space $CBH \times R^3$ in terms of the fields on the reduced space $CBH \times R^2$ is

$$G = \begin{pmatrix} A_t^2 - f & 0 & 0 & R_{tY}+S_{tY}+A_t(u_Y+w_Y) & A_t \\ 0 & 1/f & 0 & 0 & 0 \\ 0 & 0 & 1 & R_{XY}+S_{XY} & 0 \\ R_{tY}+S_{tY}+A_t(u_Y+w_Y) & 0 & R_{XY}+S_{XY} & 1 & u_Y+w_Y \\ A_t & 0 & 0 & u_Y+w_Y & 1 \end{pmatrix} \tag{4.146}$$

$$B = \tag{4.147}$$

$$\begin{pmatrix} 0 & 0 & 0 & S_{tY}-R_{tY}+A_t(w_Y-u_Y) & -A_t \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{XY}-R_{XY} & 0 \\ -(S_{tY}-R_{tY})-A_t(w_Y-u_Y) & 0 & -(S_{XY}-R_{XY}) & 0 & -(w_Y-u_Y) \\ A_t & 0 & 0 & w_Y-u_Y & 0 \end{pmatrix}.$$

Before proceeding, rescale

$$\mathbf{r} = rQ\,, \quad \mathbf{w} = \omega/Q\,, \quad \mathbf{p} = p/Q\,, \tag{4.148}$$

which eliminates Q dependence from the equations of motion.

Due to string theory result we know that the R-system fields R and u are decoupled from the S-system fields S and w. We will find out that this decoupling is true in gravity computations as well. To find equations of motion for the fields $R_{\mu\nu}$, $S_{\mu\nu}$, w_μ and u_μ we compute the beta functions $\beta^R_{MN} = \beta^G_{MN} - \beta^B_{MN}$ and $\beta^S_{MN} = \beta^G_{MN} + \beta^B_{MN}$ for $(MN) = (tY),\ (rY),\ (XY),\ (xY)$.

Consider first equations of motion in the R-system.

β^R_{tY}:

$$pf(pR_{tY} + \omega R_{XY}) + f^2(2\Phi' R'_{tY} - R''_{tY}) - A_t(f^2 u''_Y + f(f' - 2f\Phi')u'_Y + (\omega^2 - p^2 f + (f^2/A_t)(A''_t - 2A'_t\Phi')u_Y) = 0\,. \tag{4.149}$$

β^R_{rY}:

$$\omega R'_{tY} + pfR'_{XY} = 0\,. \tag{4.150}$$

β^R_{XY}:

$$\omega(pR_{tY} + \omega R_{XY}) + f(fR''_{XY} + (f' - 2f\Phi')R'_{XY}) = 0\,. \tag{4.151}$$

β^R_{xY}:

$$f^2 u''_Y + f(f' - 2f\Phi')u'_Y + (\omega^2 - p^2 f)u_Y = 0\,. \tag{4.152}$$

Notice that for the CBH background $\Phi' = -1/2$ and $A''_t = -A'_t$. Therefore one sees that u_Y contribution to R_{tY} equation (4.149) vanishes due to u_Y equation (4.152). Therefore we see that $R_{\mu Y}$ and u_Y fluctuations decouple, as expected from the string theory computations (4.108).

Introduce diff-invariant quantity

$$Z = pR_{tY} + \omega R_{XY}\,. \tag{4.153}$$

Solving following from this definition equation

$$Z' = pR'_{tY} + \omega R'_{XY} \tag{4.154}$$

together with R_{rY} equation (4.150) one obtains

$$R'_{tY} = -\frac{pfZ'}{\omega^2 - p^2 f}\,, \qquad R'_{XY} = \frac{\omega Z'}{\omega^2 - p^2 f}\,. \tag{4.155}$$

Plugging expressions (4.155) into R_{tY} equation (4.149) one obtains equation (the same equation is obtained if one plugs (4.155) into R_{XY} equation (4.151))

$$Z'' + \left(\frac{\omega^2 f'}{f(\omega^2 - p^2 f)} - 2\Phi' \right) Z' + \frac{\omega^2 - p^2 f}{f^2} Z = 0\,. \tag{4.156}$$

Together with decoupled from it transverse gauge field equation (4.152) for u_Y these are fluctuation equations for shear components of R-system.

Consider now S and w fluctuation equations of the S-system.

β^S_{tY}:

$$\begin{aligned} &-\omega^2 A_t w_Y + f(p(pS_{tY} + \omega S_{XY}) + A_t(w_Y(p^2 - 2A_t'^2) - 2A_t' S_{tY}' \\ &- f' w_Y')) - f^2(-2\Phi'(S_{tY}' + A_t w_Y') + 2A_t'(w_Y' - \Phi' w_Y) + A_t'' w_Y \\ &+ S_{tY}'' + A_t w_Y'') = 0\,. \end{aligned} \tag{4.157}$$

β^S_{rY}:

$$2\omega A_t' w_Y + \omega S_{tY}' + pf S_{XY}' = 0\,. \tag{4.158}$$

β^S_{XY}:

$$\omega(pS_{tY} + \omega S_{XY}) + f(S_{XY}'(f' - 2\Phi' f) + f S_{XY}'') = 0\,. \tag{4.159}$$

β^S_{xY}:

$$(\omega^2 - f(p^2 - 2A_t'^2))w_Y + f(2A_t' S_{tY}' + (f' - 2\Phi' f)w_Y' + f w_Y'') = 0\,. \tag{4.160}$$

Introduce diff-invariant quantity

$$V = pS_{tY} + \omega S_{XY}\,. \tag{4.161}$$

Then solving equation

$$V' = pS_{tY}' + \omega S_{XY}'\,. \tag{4.162}$$

together with β^S_{rY} equation (4.158) we obtain

$$S_{XY}' = \frac{\omega(2pA_t' w_Y + V')}{\omega^2 - p^2 f}\,, \quad S_{tY}' = -\frac{2\omega^2 A_t' w_Y + pfV'}{\omega^2 - p^2 f}\,. \tag{4.163}$$

Plugging into β^S_{tY} equation (4.157) the expressions (4.163) together with w''_Y, expressed from w_Y equation (4.160), we arrive at (the same result is obtained by plugging (4.163) into β^S_{XY} equation (4.159)) [8]

$$V'' + \left(\frac{\omega^2 f'}{f(\omega^2 - p^2 f)} - 2\Phi'\right) V' + \frac{\omega^2 - p^2 f}{f^2} V + \frac{2pA'_t}{f}\left(f w'_Y + \frac{\omega^2 f'}{\omega^2 - p^2 f} w_Y\right) = 0\,. \tag{4.164}$$

Finally, using S'_{tY}, expressed in (4.163), in w_Y equation (4.160) we obtain

$$w''_Y + \frac{f' - 2f\Phi'}{f} w'_Y + \left(\frac{\omega^2 - p^2 f}{f^2} + \frac{2A'^2_t}{f}\left(1 - \frac{2\omega^2}{\omega^2 - p^2 f}\right)\right) w_Y - \frac{2pA'_t}{\omega^2 - p^2 f} V' = 0\,. \tag{4.165}$$

We see that in the S-system tensor field shear components are coupled to gauge field transverse component, which agrees with string computation (4.105).

Let us look for poles of the correlation functions in R-system and in S-system. Notice that R-system is just S-system at vanishing background flux $A'_t = 0$; compare equation (4.164) with equation (4.229) and equation (4.165) with equation (4.152) to see that. Therefore it is sufficient to study S-system.

Introduce new radial coordinate $u = e^{\mathbf{r}}$. Then inner and outer horizons are located at

$$u_\pm = M \pm \sqrt{M^2 - q^2}\,. \tag{4.166}$$

The equations of motion (4.164) and (4.165) become (also take into account $q = \sqrt{u_+ u_-}$)

$$\frac{d^2 w_Y}{du^2} + \left(\frac{1}{u - u_-} + \frac{1}{u - u_+}\right)\frac{dw_Y}{du} + \frac{1}{(u - u_-)(u - u_+)} \times \left(\frac{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)}{(u - u_-)(u - u_+)} - \frac{2u_+ u_- \mathbf{w}^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)} + \frac{2u_+ u_-}{u^2}\right) w_Y + \frac{2\mathbf{p}\sqrt{u_+ u_-}}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)}\frac{d\mathbf{V}}{du} = 0\,. \tag{4.167}$$

[8] Also take into account $\Phi' = -1/2$ and $A''_t = -A'_t$.

$$
\begin{aligned}
&\frac{d^2\mathbf{V}}{du^2}+\frac{1}{u}\left(2+\frac{\mathbf{w}^2u^2}{\mathbf{w}^2u^2-\mathbf{p}^2(u-u_-)(u-u_+)}\left(\frac{u_+}{u-u_+}+\frac{u_-}{u-u_-}\right)\right)\frac{d\mathbf{V}}{du}\\
&-\frac{2\mathbf{p}\sqrt{u_+u_-}}{u^2}\frac{dw_Y}{du}+\frac{\mathbf{w}^2u^2-\mathbf{p}^2(u-u_-)(u-u_+)}{(u-u_-)^2(u-u_+)^2}\mathbf{V} \qquad (4.168)\\
&-\frac{2\mathbf{p}\sqrt{u_+u_-}}{u}\frac{\mathbf{w}^2}{\mathbf{w}^2u^2-\mathbf{p}^2(u-u_-)(u-u_+)}\left(\frac{u_+}{u-u_+}+\frac{u_-}{u-u_-}\right)w_Y=0\,.
\end{aligned}
$$

In the near horizon limit $v=u-u_+\ll 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2w_Y}{dv^2}+\frac{1}{v}\frac{dw_Y}{dv}+\frac{\mathbf{w}^2u_+^2}{(u_+-u_-)^2v^2}w_Y=0 \qquad (4.169)$$

$$\frac{d^2\mathbf{V}}{dv^2}+\frac{1}{v}\frac{d\mathbf{V}}{dv}+\frac{\mathbf{w}^2u_+^2}{(u_+-u_-)^2v^2}\mathbf{V}=0\,. \qquad (4.170)$$

The incoming-wave solutions are

$$w_Y(u)=C_1(u-u_+)^{\frac{-i\mathbf{w}u_+}{u_+-u_-}}\,,\quad \mathbf{V}(u)=C_2(u-u_+)^{\frac{-i\mathbf{w}u_+}{u_+-u_-}}\,. \qquad (4.171)$$

In the asymptotic region $u\gg 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2w_Y}{du^2}+\frac{2}{u}\frac{dw_Y}{du}+\frac{\mathbf{w}^2-\mathbf{p}^2}{u^2}w_Y=0 \qquad (4.172)$$

$$\frac{d^2\mathbf{V}}{du^2}+\frac{2}{u}\frac{d\mathbf{V}}{du}+\frac{\mathbf{w}^2-\mathbf{p}^2}{u^2}\mathbf{V}=0 \qquad (4.173)$$

with the solution

$$w_Y=\mathcal{A}_w u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}+\mathcal{B}_w u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})} \qquad (4.174)$$

$$\mathbf{V}=\mathcal{A}_V u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}+\mathcal{B}_V u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}\,. \qquad (4.175)$$

We solve numerically the equations (4.235), (4.236) with boundary conditions (4.239) and find two linearly-independent solutions $(w_Y^{(1)},\mathbf{V}^{(1)})$ and $(w_Y^{(2)},\mathbf{V}^{(2)})$ (for two independent choices of $C_{1,2}$). The correlation matrix is given by [37] $\mathcal{G}\simeq\mathcal{B}\mathcal{A}^{-1}$, where the matrices of leading and subleading coefficients are determined by (4.242) and (4.243):

$$\mathcal{A}=\begin{pmatrix}\mathcal{A}_V^{(1)} & \mathcal{A}\mathcal{A}_V^{(2)}\\ \mathcal{A}_w^{(1)} & \mathcal{A}_w^{(2)}\end{pmatrix}\,,\quad \mathcal{B}=\begin{pmatrix}\mathcal{B}_V^{(1)} & \mathcal{B}_V^{(2)}\\ \mathcal{B}_w^{(1)} & \mathcal{B}_w^{(2)}\end{pmatrix}\,. \qquad (4.176)$$

Zeroes of the determinant of the matrix of the leading behavior coefficients [9]

$$\mathcal{A}_V^{(1)}\mathcal{A}_w^{(2)}-\mathcal{A}_V^{(2)}\mathcal{A}_w^{(1)} \tag{4.177}$$

define the dispersion relation of low-energy mode, which is given by

$$\mathbf{w}=-i\mathbf{p}^2\cos\psi\,. \tag{4.178}$$

Due to (4.232) and $Q=2/\sqrt{\hat{k}}$ the dispersion relation (4.245) coincides with the dispersion relation (4.129), obtained for the S-system by the world-sheet computation.

From the S-system result (4.245) we conclude that in the R-system $\langle ZZ\rangle$ correlation function has pole at

$$\mathbf{w}=-i\mathbf{p}^2\,, \tag{4.179}$$

which coincides with the pole (4.130), obtained by the world-sheet computation. Notice that as in [16] the supergravity result does not receive stringy corrections.

4.6 Discussion

In this chapter we have used the holographically dual string theory to study quantum field theory at finite temperature and chemical potential. The string theory was defined by the gWZW model on the $\frac{SL(2,R)\times U(1)_x}{U(1)}\times R^{d-1}$, with the $U(1)$ gauged asymmetrically [25] coset and the covariant quantization of the string, we have constructed vertex operators, representing massless NS-NS states of the string. The gauge fields vertex operators were obtained by the Kaluza-Klein reduction of the graviton and the two-form field vertex operators on the $U(1)_x$.

We have found that these vertex operators split into two decoupled systems. This implies that the boundary low energy theory splits into two decoupled models, as far as the two-point functions are concerned. At low energies the Green's functions of stress energy tensor and global $U(1)$ current exhibit two gapless poles. Corresponding dispersion relations are (4.129) and (4.130) in the shear and sound channels. The dispersion relation (4.130) does not depend on the charge to mass ratio of the charged

[9] See e.g. [38] where computation of correlation matrix in the different system of two coupled differential equations is explained in detail.

black hole background. When the charge density is zero, the dispersion relation (4.129) coincides with the dispersion relation (4.130). We have verified these results by computations in type-II supergravity; the supergravity results exactly coincide with superstring results. We speculate that the system is described at low energies by a decoupled sum of two non-interacting fluids. It would be interesting to make this picture more precise.

The current and stress-energy tensor two-point correlation functions, which we have computed, also possess finite-momentum zero-frequency singularity. As in [8] it originates from the two-point function of the vertex operator of the ground state of the WZW model on $SL(2,R)$. This "$2k_F$" singularity is a purely stringy effect [8], absent in the supergravity approximation: from (4.135) it follows, that the momentum p_*, measured in units of inverse curvature radius, scales as $(Rp_*)^2 \simeq \hat{k}\ell_s^2 p_*^2 \sim \hat{k}^2$ when $\hat{k}$ is large. Therefore in supergravtiy approximation p_* is parametrically large.

We have also studied the shear channel in heterotic gravity (see Appendix B), and found one low-energy mode. Matching its dispersion relation to the one obtained from the thermodynamics of the 2d charged black hole, we have derived $\eta/s = 1/(4\pi)$ for any charge to mass ratio. It would be interesting to obtain this result from heterotic string theory as well. However naive construction of the heterotic string theory, based on the $\frac{SL(2,R)\times U(1)_x}{U(1)}$ coset model (where $U(1)_x$ is holomorphic, that is a part of internal space from purely bosonic left-moving sector of heterotic string theory), appears to contain $U(1)$ chiral anomaly. Indeed, naively, to construct heterotic string, based on the coset model used in this chapter, one takes the gWZW action (4.11) and adds to it the Dirac term $S_f \simeq \int d^2z \, \mathrm{Tr}\, \tilde{\Psi}(\partial + A)\tilde{\Psi}$, where anti-holomorphic (right-moving) fermions $\tilde{\Psi} \in sl(2,R) \ominus u(1)$ are superpartners of the anti-holomorphic bosonic currents on $SL(2,R)/U(1)$, and A is the $U(1)$ gauge field. Due to such chiral interaction, on the quantum level the anomaly appears, and the theory becomes inconsistent.

This issue was actually resolved in a different heterotic coset stringy realization of the 2d charged black hole [24]. As it was observed there, the chiral anomaly due to fermions should be compensated by the classical anomaly of gWZW action for bosons [39]. In fact, bosonization of the fermions results in the chiral anomaly due to fermions appearing on the classical level, just as in the anomalous gWZW action [40]. Therefore separately the bosonic and fermionic parts of the action are not invariant under $U(1)$ gauge transformation, while their sum is invariant. It is not clear however how these ideas can be directly applied to the model, based on the bosonic action (4.11), which was constructed [25] to be anomaly-free on its own.

4.7 Appendix A: Conventions and review of the gWZW model on the $\frac{SL(2,R)\times U(1)}{U(1)}$

4.7.1 Conventions

Put the string length equal to one, $\alpha' \equiv \frac{\ell_s^2}{2} = \frac{1}{2}$. The contribution to the world-sheet stress-energy tensor, coming from the coordinates $X^\mu(z,\bar{z})$ of the flat subspace of the target space-time, is given by

$$T_{flat}(z) = -\partial X^\mu(z)\partial X_\mu(z)\,, \tag{4.180}$$

and similarly for the anti-holomorphic part $\tilde{T}(\bar{z})$. The Polyakov action is

$$S_P = \frac{1}{2\pi}\int d^2z\,\partial X\bar{\partial}X\,. \tag{4.181}$$

The two-point function is

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\frac{1}{2}\eta^{\mu\nu}\left(\log(z-w)+\log(\bar{z}-\bar{w})\right)\,. \tag{4.182}$$

The Kac-Moody holomorphic (left-moving) and anti-holomorphic (right-moving) currents of the WZW model at level $\hat{k}$ are given by

$$j(z) = j_A t^A = -\frac{\hat{k}}{2}\partial g g^{-1}\,, \qquad \tilde{j}(\bar{z}) = \tilde{j}_A t^A = \frac{\hat{k}}{2}g^{-1}\bar{\partial}g\,. \tag{4.183}$$

Here hermitean generators of a gauge algebra are

$$t^A = j_0^A\,, \qquad [t^A,t^B] = if^{ABC}t^C\,. \tag{4.184}$$

For $SL(2,R)$, which is the group we are interested in, the following expressions in terms of Pauli matrices take place:

$$j_0^A = \frac{1}{2}\sigma^A\,, \qquad f^{ABC} = \epsilon^{ABC}\,, \tag{4.185}$$

and indices are raised and lowered with the help of $\eta_{AB} = \mathrm{diag}\{1, 1, -1\}$. In Euclidean realization of $SL(2,R)$ we put $\eta^{AB} = \delta^{AB}$.

The holomorphic currents of Kac-Moody algebra satisfy the following OPE

$$j^A(z) j^B(w) = \frac{\frac{\hat{k}}{2}\eta^{AB}}{(z-w)^2} + \frac{i f^{ABC}}{z-w} j_C(w) \,, \tag{4.186}$$

and similarly for the anti-holomorphic currents.

The holomorphic component of the stress-energy is given by the Sugawara expression

$$T(z) = \frac{1}{\kappa} \eta_{AB} j^A(z) j^B(z) \,, \tag{4.187}$$

similar expression is true for the anti-holomorphic component. Here

$$\kappa = \hat{k} + c_V \,. \tag{4.188}$$

For $SU(2)$ (and for Euclidean $SL(2,R)$) the index of the adjoint representation is $c_V = 2$ and for $SL(2,R)$ it is $c_V = -2$.

The groundstate representation space of the $SL(2,R)$ currents is formed by the primary fields $V_j(x,\bar{x};w,\bar{w})$, characterized by the index j. This index determines the value of Casimir operator of $SL(2,R)$. The $(x,\bar{x})$ coordinates can be regarded as the boundary coordinates of the $SL(2,R)$ target space-time, and $(w,\bar{w})$ are world-sheet coordinates. One can replace the boundary coordinates with the numbers $(m,\bar{m})$, defined via transformation

$$V_{j;m,\bar{m}}(w,\bar{w}) = \int d^2x x^{j+m} \bar{x}^{j+\bar{m}} V_j(x,\bar{x};w,\bar{w}) \,. \tag{4.189}$$

OPE of $SL(2,R)$ currents and $SL(2,R)$ primaries are [10]

$$\begin{aligned} J^3(z) V_{j;m,\bar{m}}(w,\bar{w}) &= \frac{m}{z-w} V_{j;m,\bar{m}}(w,\bar{w}) + \cdots \,, \\ J^{\pm}(z) V_{j;m,\bar{m}}(w,\bar{w}) &= \frac{m \mp j}{z-w} V_{j;m\pm 1,\bar{m}}(w,\bar{w}) + \cdots \,. \end{aligned} \tag{4.192}$$

[10] For Euclidean $SL(2,R)$,

$$J^3(z) V_{j;m,\bar{m}}(w,\bar{w}) = \frac{im}{z-w} V_{j;m,\bar{m}}(w,\bar{w}) + \cdots \,. \tag{4.190}$$

Therefore

$$\eta_{33} (J^3)^2(z) V_{j;m,\bar{m}}(w,\bar{w}) = -\frac{m^2}{z-w} V_{j;m,\bar{m}}(w,\bar{w}) \tag{4.191}$$

is true for both Euclidean and Minkowski signatures.

From this one finds how the $SL(2,R)$ currents act on the primaries:

$$J_0^3 \cdot V_{j;m,\bar{m}}(w,\bar{w}) = m V_{j;m,\bar{m}}(w,\bar{w})\,, \qquad (4.193)$$

$$J_0^{\pm} \cdot V_{j;m,\bar{m}}(w,\bar{w}) = (m \mp j) V_{j;m\pm 1,\bar{m}}(w,\bar{w})\,, \qquad (4.194)$$

with all other $J_n^A \cdot V_{j;m,\bar{m}}(w,\bar{w}) = 0$, $n \geq 1$.

Second order $SL(2,R)$ Casimir operator is given by

$$C_2 = \eta_{AB} J_0^A J_0^B \equiv -(J_0^3)^2 + \frac{1}{2}\{J_0^+, J_0^-\}\,. \qquad (4.195)$$

Here

$$J_0^1 = \frac{1}{2}(J_0^+ + J_0^-)\,, \qquad J_0^2 = \frac{i}{2}(J_0^- - J_0^+)\,. \qquad (4.196)$$

It takes place

$$C_2 \cdot V_j(w,\bar{w}) = -j(j+1) V_j(w,\bar{w})\,. \qquad (4.197)$$

This expression is also true for Euclidean $SL(2,R)$, due to (4.190). Then clearly for $SL(2,R)$ algebra with currents of weight $\hat{k}$,

$$L_0 \cdot V_j(w,\bar{w}) = -\frac{j(j+1)}{\hat{k}-2} V_j(w,\bar{w})\,, \qquad (4.198)$$

which gives the conformal dimension of V_j

$$\Delta_j = -\frac{j(j+1)}{\hat{k}-2}\,. \qquad (4.199)$$

In the superstring theory one considers the total bosonic currents J^a, which include contributions from world-sheet fermions, dual to $SL(2,R)$ currents. The level of total $SL(2,R)$ currents is equal to $\hat{k}+2$, if $\hat{k}$ denotes the level of purely bosonic currentt, and therefore the conformal dimension of the $V_{jm\bar{m}}$ is equal to $\Delta_j = -\frac{j(j+1)}{\hat{k}}$.

4.7.2 Gauged WZW model on the $\frac{SL(2,R)\times U(1)}{U(1)}$

Let us review the derivation [25] of the gWZW action on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset.

Perform the following asymmetric gauging of the $U(1)$ subgroup of $SL(2,R) \times U(1)$ group with the parameter τ:

$$(g,\ x_L,\ x_R) \sim \left(e^{\tau \cos\psi \sigma_3/\sqrt{\hat{k}}} g e^{\tau\sigma_3/\sqrt{\hat{k}}},\ x_L + \tau \sin\psi,\ x_R\right)\,. \qquad (4.200)$$

The condition that the gauge transformation leaves the action invariant is

$$\mathrm{Tr}\left(T_L^2 - T_R^2\right) = 0\,, \tag{4.201}$$

where T_L and T_R are the generators of left-moving and right-moving sectors of the gauged $U(1)$ group.

Let us write the element of the $SL(2,R)\times U(1)$ group as

$$G = \begin{pmatrix} g & 0 \\ 0 & \exp\left(\sqrt{\frac{2}{\hat{k}}}\,x\right)\end{pmatrix}. \tag{4.202}$$

Then G is a field of the $SL(2,R)\times U(1)$ WZW model at level $\hat{k}$:

$$\begin{aligned} S[G] &= \frac{\hat{k}}{4\pi}[\int d^2z \mathrm{Tr}(G^{-1}\partial G G^{-1}\bar{\partial}G) - \frac{1}{3}\int_B \mathrm{Tr}(G^{-1}dG)^3] \\ &= \frac{\hat{k}}{4\pi}[\int d^2z \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial}g) - \frac{1}{3}\int_B \mathrm{Tr}(g^{-1}dg)^3] \\ &+ \frac{1}{2\pi}\int d^2z\partial x\bar{\partial}x\,. \end{aligned} \tag{4.203}$$

The gauge transformation (4.200) acts on G-field as

$$G \to e^{T_L\tau} G e^{T_R\tau}\,, \tag{4.204}$$

where the generators of left and right sectors of the $u(1)$ algebra are

$$T_L = \begin{pmatrix} \frac{1}{\sqrt{\hat{k}}}\cos\psi\sigma^3 & 0 \\ 0 & \sqrt{\frac{2}{\hat{k}}}\sin\psi \end{pmatrix}, \quad T_R = \begin{pmatrix} \frac{1}{\sqrt{\hat{k}}}\sigma^3 & 0 \\ 0 & 0\end{pmatrix}. \tag{4.205}$$

These generators satisfy anomaly-free condition (4.201. Because of this condition is satisfied we can make gauge fields non-dynamical, as it is shown bellow.

Consider compensator fields (gauge field ‘prepotentials’):

$$U = \exp\left(-uT_L\right)\,, \qquad V = \exp\left(-vT_R\right)\,. \tag{4.206}$$

Define gauge transformation of compensator fields as

$$u \to u + \tau\,, \qquad v \to v + \tau\,. \tag{4.207}$$

The combination UGV is clearly invariant under gauge transformations (4.200), and therefore the WZW-action $S[UGV]$ is gauge-invariant.

But it contains terms which are quadratic in derivatives of compensator field u and quadratic in derivatives of compensator field v. Such terms make the compensator (gauge) d.o.f. dynamical, and therefore the theory with the action $S[UGV]$ instead of gauging some degrees of freedom away adds more degrees of freedom.

Therefore let us consider instead the gWZW action

$$S_g = S[UGV] - \frac{1}{2\pi}\int d^2z \partial w \bar{\partial} w \,, \tag{4.208}$$

where we have introduced gauge-invariant field

$$w = u - v \,. \tag{4.209}$$

Due to the Polyakov-Wiegmann indentity

$$S[UGV] = S[G] + S[U] + S[V] \tag{4.210}$$
$$+\frac{\hat{k}}{2\pi}\int d^2z \mathrm{Tr}\left[G^{-1}\bar{\partial}G\partial V V^{-1} + U^{-1}\bar{\partial}U\partial G G^{-1} + U^{-1}\bar{\partial}U G\partial V V^{-1}G^{-1}\right] .$$

Here

$$S[U] = \frac{1}{2\pi}\int d^2z \partial u \bar{\partial} u \,, \quad S[V] = \frac{1}{2\pi}\int d^2z \partial v \bar{\partial} v \,, \tag{4.211}$$

and therefore

$$S[U] + S[V] - \frac{1}{2\pi}\int d^2z \partial w \bar{\partial} w = \frac{1}{2\pi}\int d^2z (\partial v \bar{\partial} u + \partial u \bar{\partial} v) \tag{4.212}$$
$$= \frac{1}{\pi}\int d^2z A\bar{A} \,, \tag{4.213}$$

where

$$A = -\partial v \,, \qquad \bar{A} = -\bar{\partial} u \,. \tag{4.214}$$

The action term (4.212) is non-dynamical, as it is expected in gWZW model with asymmetric gauging, satisfying anomaly-free condition (4.201). As a result, the gWZW action on the $\frac{SL(2,R)\times U(1)}{U(1)}$ is given by

$$S_g = S[g] + \frac{1}{2\pi}\int d^2z \partial x \bar{\partial} x \tag{4.215}$$
$$+\frac{1}{2\pi}\int d^2z \left[A\sqrt{\hat{k}}\mathrm{Tr}\,(g^{-1}\bar{\partial}g\sigma^3) + \bar{A}\left(\sqrt{\hat{k}}\mathrm{Tr}(\partial g g^{-1}\sigma^3)\cos\psi + 2\sin\psi\partial x\right) + \right.$$
$$\left. + \, A\bar{A}\left(2 + \mathrm{Tr}(g^{-1}\sigma^3 g\sigma^3)\cos\psi\right)\right] .$$

4.8 Appendix B: Heterotic gravity approximation

In the type-II supergravity, considered in the section 5, two gauge fields appear as $G_{x\mu}$ and $B_{x\mu}$ components after Kaluza-Klein reduction of the compact x coordinate. The two-dimensional charged black hole is also a solution [23] of heterotic supergravity equations of motion. In this section we compute graviton and gauge field two-point functions in the two-dimensional charged black hole background in heterotic supergravity. In this case there is just one background gauge field. We solve fluctuation equations of motion for the shear components of graviton and the transverse component of the gauge potential and find one hydrodynamic mode. Matching the obtained dispersion relation with the result obtained in the study of thermodynamics of the 2d charged black hole we derive shear viscosity to entropy ratio for any value of ψ.

The two-loop beta-functions of bosonic fields in heterotic string theory are [41, 23]

$$\beta^G_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu\partial_\nu\Phi - \frac{1}{2}g^{\lambda\rho}F_{\mu\rho}F_{\nu\lambda}\,, \tag{4.216}$$

$$\beta^\Phi = \frac{1}{4}F^2 - R + c + 4(\partial\Phi)^2 - 4\nabla^2\Phi\,, \tag{4.217}$$

$$\beta^A_\nu = g^{\mu\lambda}(\nabla_\mu F_{\nu\lambda} - 2F_{\nu\lambda}\partial_\mu\Phi)\,. \tag{4.218}$$

Corresponding equations of motion, $\beta^{G,B,\Phi} = 0$, have the $CBH \times R^{d-1}$ solution,

$$g_{\mu\nu} = \mathrm{diag}\{-f(r), 1/f(r), 1, ..., 1\}\,, \qquad f(r) = 1 - 2Me^{-Qr} + q^2e^{-2Qr}\,, \tag{4.219}$$

$$\Phi = \Phi_0 - \frac{Qr}{2}\,, \qquad F_{tr} = F(r) = \sqrt{2}Qqe^{-Qr}\,. \tag{4.220}$$

Here $Q = 2/\sqrt{k}$.

Consider fluctuations $h_{\mu\nu}$, a_μ and φ around this solution. Use the diffeomorphism invariance to fix $h_{\mu r} = 0$. Among $d+1$ space-time coordinates we have t, r coordinates of CBH and $d-1$ flat coordinates. Let us consider $CBH \times R^2$. Choose X to be the R^2 direction of propagation of excitations (with momentum p) and choose Y to be the R^2 direction, transverse to the direction of propagation of excitations. Fluctuations depend on t, r, X. The dependence on t and X in momentum representation boils down to the factor $e^{-i\omega t + ipX}$.

Plugging $g_{\mu\nu} + h_{\mu\nu}$, $A_\mu + a_\mu$ and $\Phi + \varphi$ with the most general fluctuations, we obtain shear channel expressions (prime denotes differentiation w.r.t. r)

$$\beta^G_{rY} = \frac{ie^{-i\omega t+ipX}}{2f}\left(\omega h'_{tY} + pfh'_{XY} - \omega F a_Y\right), \tag{4.221}$$

$$\beta^G_{XY} = e^{-i\omega t+ipX}\left(-\frac{1}{2f}(\omega p h_{tY} + \omega^2 h_{XY} + ff'h'_{XY} + f^2 h''_{XY}) + f\Phi' h'_{XY}\right), \tag{4.222}$$

$$\beta^G_{tY} = \frac{e^{-i\omega t+ipX}}{2}\left(p^2 h_{tY} + \omega p h_{XY} - f h''_{tY} + 2f\Phi' h'_{tY} + fFa'_Y\right). \tag{4.223}$$

The equations of motion in shear channel are therefore

$$\omega h'_{tY} + pfh'_{XY} - \omega F a_Y = 0, \tag{4.224}$$

$$\omega p h_{tY} + \omega^2 h_{XY} + ff'h'_{XY} + f^2 h''_{XY} - 2f^2\Phi' h'_{XY} = 0, \tag{4.225}$$

$$p^2 h_{tY} + \omega p h_{XY} - f h''_{tY} + 2f\Phi' h'_{tY} + fFa'_Y = 0. \tag{4.226}$$

Consider diff-invariant field

$$Z = \omega h_{XY} + p h_{tY}. \tag{4.227}$$

Using (4.227) and (4.224) express

$$h'_{tY} = \frac{\omega^2 F a_Y - pfZ'}{\omega^2 - p^2 f}, \quad h'_{XY} = \omega\frac{Z' - pFa_Y}{\omega^2 - p^2 f}. \tag{4.228}$$

The equations (4.225) and (4.226) after one substitutes (4.228) into them, both give rise to the same equation (due to $F' - 2F\Phi' = 0$)

$$Z'' + \left(\frac{\omega^2 f'}{f(\omega^2 - p^2 f)} - 2\Phi'\right)Z' + \frac{\omega^2 - p^2 f}{f^2}Z - \frac{pF}{f}\left(fa'_Y + \frac{\omega^2 f'}{\omega^2 - p^2 f}a_Y\right) = 0. \tag{4.229}$$

Compute the beta-function for gauge field fluctuation a_Y (choose the gauge $a_r = 0$)

$$\beta^A_Y = -\, e^{-i\omega t + ipX} f \left(a''_Y + \frac{f' - 2f\Phi'}{f} a'_Y + \frac{\omega^2 - p^2 f}{f^2} a_Y - \frac{F h'_{tY}}{f} \right) . \tag{4.230}$$

Express h'_{tY} using (4.228). The equation on a_Y is then

$$a''_Y + \frac{f' - 2f\Phi'}{f} a'_Y + \left(\frac{\omega^2 - p^2 f}{f^2} - \frac{\omega^2 F^2}{f(\omega^2 - p^2 f)} \right) a_Y + \frac{pFZ'}{\omega^2 - p^2 f} = 0 \,. \tag{4.231}$$

Before proceeding, rescale

$$\mathbf{r} = rQ \,, \quad \mathbf{w} = \omega/Q \,, \quad \mathbf{p} = p/Q \,, \quad \mathbf{Z} = Z/Q \,. \tag{4.232}$$

The dependence on Q disappears from both fluctuation equations, and due to (4.220) we obtain

$$f = 1 - 2Me^{-\mathbf{r}} + q^2 e^{-2\mathbf{r}} \,, \quad \Phi = \Phi_0 - \frac{\mathbf{r}}{2} \,. \tag{4.233}$$

Introduce new radial coordinate $u = e^{\mathbf{r}}$. Then inner and outer horizons are located at

$$u_\pm = M \pm \sqrt{M^2 - q^2} \,. \tag{4.234}$$

The equations of motion become (substitute $q = \sqrt{u_+ u_-}$)

$$\frac{d^2 a}{du^2} + \left(\frac{1}{u - u_-} + \frac{1}{u - u_+} \right) \frac{da}{du} + \frac{1}{(u - u_-)(u - u_+)} \times \left(\frac{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)}{(u - u_-)(u - u_+)} - \frac{2u_+ u_- \mathbf{w}^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)} \right) a + \frac{\mathbf{p}\sqrt{2u_+ u_-}}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)} \frac{d\mathbf{Z}}{du} = 0 \,, \tag{4.235}$$

$$\frac{d^2 \mathbf{Z}}{du^2} + \frac{1}{u} \left(2 + \frac{\mathbf{w}^2 u^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)} \left(\frac{u_+}{u - u_+} + \frac{u_-}{u - u_-} \right) \right) \frac{d\mathbf{Z}}{du} - \frac{\mathbf{p}\sqrt{2u_+ u_-}}{u^2} \frac{da}{du} + \frac{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)}{(u - u_-)^2 (u - u_+)^2} \mathbf{Z} - \frac{\mathbf{p}\sqrt{2u_+ u_-}}{u} \frac{\mathbf{w}^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)} \left(\frac{u_+}{u - u_+} + \frac{u_-}{u - u_-} \right) a = 0 \,. \tag{4.236}$$

In the near horizon limit $v = u - u_+ \ll 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2 a}{dv^2} + \frac{1}{v}\frac{da}{dv} + \frac{\mathbf{w}^2 u_+^2}{(u_+ - u_-)^2 v^2} a = 0\,, \tag{4.237}$$

$$\frac{d^2 \mathbf{Z}}{dv^2} + \frac{1}{v}\frac{d\mathbf{Z}}{dv} + \frac{\mathbf{w}^2 u_+^2}{(u_+ - u_-)^2 v^2} \mathbf{Z} = 0\,. \tag{4.238}$$

The incoming-wave solutions are

$$a_Y(u) = C_1 (u - u_+)^{\frac{-i\mathbf{w}u_+}{u_+ - u_-}}\,, \quad \mathbf{Z}(u) = C_2 (u - u_+)^{\frac{-i\mathbf{w}u_+}{u_+ - u_-}}\,. \tag{4.239}$$

In the asymptotic region $u \gg 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2 a}{du^2} + \frac{2}{u}\frac{da}{du} + \frac{\mathbf{w}^2 - \mathbf{p}^2}{u^2} a = 0\,, \tag{4.240}$$

$$\frac{d^2 \mathbf{Z}}{du^2} + \frac{2}{u}\frac{d\mathbf{Z}}{du} + \frac{\mathbf{w}^2 - \mathbf{p}^2}{u^2} \mathbf{Z} = 0\,, \tag{4.241}$$

with the solution

$$a_Y = \mathring{A}_a u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})} + \mathcal{B}_a u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}\,, \tag{4.242}$$

$$\mathbf{Z} = \mathring{A}_Z u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})} + \mathcal{B}_Z u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}\,. \tag{4.243}$$

We solve numerically the equations (4.235), (4.236) with boundary conditions (4.239). Zeroes of determinant of the leading behavior coefficients matrix

$$\mathring{A}_Z^{(1)} \mathring{A}_a^{(2)} - \mathring{A}_Z^{(2)} \mathring{A}_a^{(1)} \tag{4.244}$$

are located at

$$\mathbf{w} = -i\mathbf{p}^2 \cos^2(\psi/2)\,. \tag{4.245}$$

Due to (4.232) and $Q = 2/\sqrt{\hat{k}}$ from (4.245) it follows

$$\omega = -i\frac{\sqrt{\hat{k}}\cos^2(\psi/2)}{2} p^2\,. \tag{4.246}$$

Finally, matching the dispersion relation (4.246) to the dispersion relation (4.10), obtained in Section 2 from the consideration of thermodynamics of the 2d charged black hole, we conclude

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{4.247}$$

is valid for any value of ψ, and due to $q = M \sin\psi$ it is valid for any charge density.

Bibliography

[1] N. Seiberg, “New theories in six-dimensions and matrix description of M theory on T**5 and T**5 / Z(2),” Phys. Lett. B **408** (1997) 98 [hep-th/9705221].

[2] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, “Linear dilatons, NS five-branes and holography,” JHEP **9810** (1998) 004 [hep-th/9808149].

[3] D. Kutasov, “Introduction to little string theory,”

[4] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” Phys. Rev. D **58** (1998) 046004 [hep-th/9802042].

[5] H. J. Boonstra, K. Skenderis and P. K. Townsend, “The domain wall / QFT correspondence,” JHEP **9901** (1999) 003 [hep-th/9807137].

[6] E. Witten, “On string theory and black holes,” Phys. Rev. D **44** (1991) 314.

[7] R. Dijkgraaf, H. L. Verlinde and E. P. Verlinde, “String propagation in a black hole geometry,” Nucl. Phys. B **371** (1992) 269.

[8] J. Polchinski and E. Silverstein, “Large-density field theory, viscosity, and '$2k_F$' singularities from string duals,” Class. Quant. Grav. **29** (2012) 194008 [arXiv:1203.1015 [hep-th]].

[9] J. Teschner, “On structure constants and fusion rules in the SL(2,C) / SU(2) WZNW model,” Nucl. Phys. B **546** (1999) 390 [hep-th/9712256].

[10] J. Teschner, “Operator product expansion and factorization in the H+(3) WZNW model,” Nucl. Phys. B **571** (2000) 555 [hep-th/9906215].

[11] A. Giveon, D. Kutasov and N. Seiberg, “Comments on string theory on AdS(3),” Adv. Theor. Math. Phys. **2** (1998) 733 [hep-th/9806194].

[12] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, “String theory on AdS(3),” JHEP **9812** (1998) 026 [hep-th/9812046].

[13] D. Kutasov and N. Seiberg, “More comments on string theory on AdS(3),” JHEP **9904** (1999) 008 [hep-th/9903219].

[14] A. Giveon and D. Kutasov, “Little string theory in a double scaling limit,” JHEP **9910** (1999) 034 [hep-th/9909110].

[15] A. Giveon and D. Kutasov, “Comments on double scaled little string theory,” JHEP **0001** (2000) 023 [hep-th/9911039].

[16] A. Parnachev and A. Starinets, “The Silence of the little strings,” JHEP **0510** (2005) 027 [hep-th/0506144].

[17] O. Aharony and T. Banks, “Note on the quantum mechanics of M theory,” JHEP **9903** (1999) 016 [hep-th/9812237].

[18] S. Minwalla and N. Seiberg, “Comments on the IIA (NS)five-brane,” JHEP **9906** (1999) 007 [hep-th/9904142].

[19] O. Aharony and M. Berkooz, “IR dynamics of D = 2, N=(4,4) gauge theories and DLCQ of 'little string theories',” JHEP **9910** (1999) 030 [hep-th/9909101].

[20] K. Narayan and M. Rangamani, “Hot little string correlators: A View from supergravity,” JHEP **0108** (2001) 054 [hep-th/0107111].

[21] P. A. DeBoer and M. Rozali, “Thermal correlators in little string theory,” Phys. Rev. D **67** (2003) 086009 [hep-th/0301059].

[22] O. Aharony, A. Giveon and D. Kutasov, “LSZ in LST,” Nucl. Phys. B **691** (2004) 3 [hep-th/0404016].

[23] M. D. McGuigan, C. R. Nappi and S. A. Yost, “Charged black holes in two-dimensional string theory,” Nucl. Phys. B **375** (1992) 421 [hep-th/9111038].

[24] C. V. Johnson, “Exact models of extremal dyonic 4-D black hole solutions of heterotic string theory,” Phys. Rev. D **50** (1994) 4032 [hep-th/9403192, hep-th/9403192].

[25] A. Giveon, E. Rabinovici and A. Sever, “Beyond the singularity of the 2-D charged black hole,” JHEP **0307** (2003) 055 [hep-th/0305140].

[26] A. Giveon and A. Sever, “Strings in a 2-d extremal black hole,” JHEP **0502** (2005) 065 [hep-th/0412294].

[27] A. Giveon and D. Kutasov, “The Charged black hole/string transition,” JHEP **0601** (2006) 120 [hep-th/0510211].

[28] D. Karabali and H. J. Schnitzer, “BRST Quantization of the Gauged WZW Action and Coset Conformal Field Theories,” Nucl. Phys. B **329** (1990) 649.

[29] Y. Kazama and H. Suzuki, “New N=2 Superconformal Field Theories and Superstring Compactification,” Nucl. Phys. B **321** (1989) 232.

[30] Y. Kazama and H. Suzuki, “Characterization of N=2 Superconformal Models Generated by Coset Space Method,” Phys. Lett. B **216** (1989) 112.

[31] P. K. Kovtun and A. O. Starinets, “Quasinormal modes and holography,” Phys. Rev. D **72** (2005) 086009 [hep-th/0506184].

[32] S. R. Das and S. P. Trivedi, “Three-brane action and the correspondence between N=4 Yang-Mills theory and anti-De Sitter space,” Phys. Lett. B **445** (1998) 142 [hep-th/9804149].

[33] S. Ferrara, M. A. Lledo and A. Zaffaroni, “Born-Infeld corrections to D3-brane action in AdS(5) x S(5) and N=4, d = 4 primary superfields,” Phys. Rev. D **58** (1998) 105029 [hep-th/9805082].

[34] G.F. Giuliani and G. Vignale, "Quantum theory of the electron liquid," Cambridge University Press, 2005.

[35] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,” Cambridge, UK: Univ. Pr. (1998) 402 p

[36] J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” Cambridge, UK: Univ. Pr. (1998) 531 p

[37] M. Kaminski, K. Landsteiner, J. Mas, J. P. Shock and J. Tarrio, “Holographic Operator Mixing and Quasinormal Modes on the Brane,” JHEP **1002** (2010) 021 [arXiv:0911.3610 [hep-th]].

[38] M. Goykhman, A. Parnachev and J. Zaanen, “Fluctuations in finite density holographic quantum liquids,” JHEP **1210** (2012) 045 [arXiv:1204.6232 [hep-th]].

[39] E. Witten, “On Holomorphic factorization of WZW and coset models,” Commun. Math. Phys. **144** (1992) 189.

[40] C. V. Johnson, “Heterotic Coset Models,” Mod. Phys. Lett. A **10** (1995) 549 [hep-th/9409062].

[41] C. G. Callan, Jr., E. J. Martinec, M. J. Perry and D. Friedan, “Strings in Background Fields,” Nucl. Phys. B **262** (1985) 593.