# Strings and AdS/CFT at finite density

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# Chapter 1

# Introduction

## 1.1 Preface

This thesis is devoted to applications of string theoretic methods of holography to strongly coupled phases of quantum field theories. The general definition of holography states that there is an exact equivalence between a closed string theory on a manifold and a quantum field theory in the asymptotic region (on the boundary) of this manifold. Usually the manifold is an asymptotically Anti-de Sitter space, and the QFT lives on the boundary of this asymptotically AdS space. In the original example of the holographic correspondence, the QFT on the boundary is $\mathcal{N} = 4$ supersymmetric conformal Yang-Mills theory. Due to this example, the holographic duality is called the AdS/CFT correspondence.

If the gauge quantum field theory on the boundary of AdS is in a strongly coupled phase then the dual sting theory in the bulk of AdS can be approximated by its low-energy limit, supergravity. Moreover, when the number of colors in the QFT is taken to infinity, the dual supergravity in AdS is classical. Therefore the AdS/CFT correspondence facilitates a powerful approach to difficult questions of infra-red physics of quantum field theories, like confinement and chiral symmetry breaking in quantum chromodynamics. In practice there are still technical limitations, as well as principal restrictions, on the kinds of field theories which can be studied holographically. As for now there exists no holographic description of real-world QCD. However the systems which are qualitatively close to QCD can be successfully dealt with by the methods of AdS/CFT. In fact, in a large number of situations the AdS/CFT correspondence is the only available analytical tool.

Theoretical high-energy physics studies the structure of the world on the fundamental, microscopic level. At high energies the gravitational force becomes strongly coupled and quantum gravity effects cannot be neglected. The scale at which quantum gravity effects become important is known as the Planck scale. This is the scale at which the Standard Model of particle physics and General Relativity break down. The theory which incorporates SM and GR and provides a successful ultra-violet completion of these theories is known as string theory (although as this thesis is being written difficult phenomenological questions remain to be answered). The UV completion of the theories of fundamental interactions is an important and complicated research area.

A no less complicated set of problems exists in the opposite regime on an energy scale, infra-red phases of quantum field theories. Consider, for example, a QCD-like model. At high energies we have a system of weakly coupled quarks and gluons interacting by an exchange of gluons and self-interaction of gluons. It is described by a gauge-invariant matter action and a non-abelian Yang-Mills action. The processes involving scattering of quarks and gluons are described by Feynman diagrams, and since the coupling constant is small we can get accurate predictions perturbatively, by accounting for just the leading loop corrections. As we move towards low energies, due to the renormalization group flow the gauge coupling constant grows. In fact the IR phase can have a qualitatively different interaction of quarks than the UV phase, as the Coulomb force of the UV regime disappears and instead, in the IR, quarks are confined by the flux tube, with the force growing proportionally to the separation between quarks. The perturbative approach of Feynman diagrams is completely useless for the description of these phenomena. In fact it is not even correct to talk about quarks and gluons in the IR, the fundamental degrees of freedom are glueballs, mesons and baryons, within which gluons and quarks are confined.

This is where the tools of the AdS/CFT correspondence can become useful. The IR phases of QCD-like systems at strong coupling and large number of colors are dual to classical supergravity in an asymptotically AdS space. Systems of condensed matter physics can also be qualitatively described by the gravitational AdS physics. These are the kinds of models which we have studied holographically in this thesis.

Asymptotically AdS space is not the only possible bulk geometry of a holographic dual to a quantum field theory. Another kind of holographic correspondence which we have considered in this thesis is a duality between Little String Theory and a gauged Wess-Zumino-Witten model in a charged black brane background. The advantage of this holographic correspondence is that one does not have to resort to the limit of the supergravity approximation, because string theory in this background is exactly solvable. In this case, it means that the QFT can be taken at finite number of degrees of freedom.

Such an example of holographic duality can be ascribed to the realm of top-down holography. The term 'top-down' in general means that we know the string origin of the bulk degrees of freedom we are dealing with, and the their effective action appears as a low-energy limit of string theory. The original example of the AdS/CFT correspondence, the duality between type-IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory on the boundary of $AdS_5$, is another example of top-down holography. A different way to apply holography is known as the bottom-up approach, and it assumes a generic form of the action for some of the bulk fields. Its advantage is that it allows one to get a quick perspective on properties of the dual field theory. Its disadvantage is that such a model can turn out to be outside of the realm of string theory and therefore become inconsistent. Another disadvantage is that one does not know what the QFT degrees of freedom dual to the bulk fields described by a bottom-up action are. In this thesis we have used both top-down and bottom-up methods.

## 1.2 String theory

In this chapter we are going to review some basic string theory which will be useful for this thesis. The entire content of this chapter is a review of textbook material. String theoretic basics are presented for the purpose of assisting the understanding of chapter 4. We refer the reader to the references [1–4] for a more complete exposition of the topics discussed here.

### 1.2.1 The Polyakov action and two-dimensional conformal field theory

The basic object of string theory is an extended one-dimensional relativistic string. As a string moves it sweeps a two-dimensional surface, which is called a world-sheet. Let us parametrize a world-sheet by a time-like coordinate $\tau$ and a space-like coordinate $\sigma$. We are using the convention $\sigma \in [0, \pi]$. String can be open or closed. An open string has two ends, located at $\sigma = 0$ and $\sigma = \pi$. In the case of a closed string these two ends are identified.

Consider the field $X^\mu(\tau, \sigma)$ which describes an embedding of a string into a $d$-dimensional space-time, $\mu = 0, \ldots, d-1$. It is called a target space-time. The action for the field $X^\mu$ in a flat target space-time is the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \,, \tag{1.1}$$

where the index $\mu$ is lowered with the flat space-time metric $\eta_{\mu\nu}$. This is the action of a free bosonic string theory. In (1.1) we have $\alpha' = \ell_s^2/2$, where $\ell_s$ is the string length. In what follows we set $\ell_s = 1$. A world-sheet has a two-dimensional metric, $h_{\alpha\beta}$, $\alpha, \beta = \tau, \sigma$, with three independent components. We can set it to the flat metric, $\eta_{\alpha\beta}$, by two-dimensional diffeomorphism transformations, $\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta)$, and a Weyl rescaling, $h_{\alpha\beta} \to e^{2\omega(\tau,\sigma)} h_{\alpha\beta}$. Both are symmetries of the action (1.1). The choice $h_{\alpha\beta} = \eta_{\alpha\beta}$ for the world-sheet metric is called the conformal gauge.

Perform now a Wick rotation $\tau \to -i\tau$ and introduce a complex coordinate $z = e^{\tau + i\sigma}$ on an open string world-sheet, and $z = e^{2(\tau+i\sigma)}$ on a closed string world-sheet. The ends of an open string, $\sigma = 0, \pi$, are then parametrized by $z = e^\tau$ and $z = -e^\tau$. Now a world-sheet is a two-dimensional Riemann surface with Euclidean signature. The Polyakov action (1.1) in conformal gauge is then

$$S_P = \frac{1}{2\pi} \int d^2z \, \partial X^\mu \bar\partial X_\mu \,, \tag{1.2}$$

where $\partial$ denotes a derivative w.r.t. $z$ and $\bar\partial$ denotes a derivative w.r.t. $\bar z$. The classical wave equations of motion for the fields $X^\mu$ are

$$\partial\bar\partial \, X^\mu(z, \bar z) = 0 \,, \tag{1.3}$$

and must be accompanied by boundary conditions. If the string is closed then the boundary conditions are satisfied due to the periodicity $X^\mu|_{\sigma=0} = X^\mu|_{\sigma=\pi}$. In the case of an open string we must impose the following boundary conditions at the ends of the string,

$$\delta X^\mu \; (\bar z \bar\partial - z\partial) \, X_\mu|_{z=\pm e^\tau} = 0 \;\,, \tag{1.4}$$

which can be resolved in two different ways. These are called Neumann and Dirichlet boundary conditions:

$$\begin{aligned} \textbf{Neumann :} \qquad & (\bar{z}\bar{\partial} - z\partial)\, X^\mu|_{z=\pm e^\tau} = 0\,, \\ \textbf{Dirichlet :} \qquad & \delta X^\mu|_{z=\pm e^\tau} = 0\,. \end{aligned} \tag{1.5}$$

Neumann boundary conditions do not break the translational symmetry of the flat target space-time: they are invariant under the replacement $X^\mu \to X^\mu + a^\mu$ with any constant $a^\mu$. On the other hand, Dirichlet boundary conditions mean that the end of the string is held at a fixed, distinguished, point in a given space direction and it breaks translational symmetry in that direction. The physical reason for such a boundary condition is that the string ends on some object. This object, which is heavy and which is localized at a certain value of the $x^\mu$ coordinate, is called a $D$-brane. The dynamics of open strings determine the fluctuations of a $D$-branes. We will get back to $D$-branes later in this chapter.

Besides the equations of motion (1.3) and the boundary conditions (1.5) one has to impose Virasoro constraints. These constraints originate as follows. Recall that the action (1.2) is written in the conformal gauge, $h_{\alpha\beta} = \delta_{\alpha\beta}$. It is invariant under all two-dimensional coordinate transformations which keep a flat world-sheet metric conformally flat, $h_{\alpha\beta} = e^{2\omega(z,\bar{z})}\delta_{\alpha\beta}$, that is flat up to an overall factor $e^{2\omega(z,\bar{z})}$. Such transformations are called conformal transformations. They are generated by the stress-energy tensor. In two dimensions, the stress-energy tensor has three independent components. Due to conformal invariance, the stress-energy tensor is classically traceless, and therefore two independent components remain

$$T(z) = -\partial X^\mu(z)\partial X_\mu(z)\,, \qquad \tilde{T}(\bar{z}) = -\bar{\partial} X^\mu(\bar{z})\bar{\partial} X_\mu(\bar{z})\,. \tag{1.6}$$

String theory is a two-dimensional conformal field theory. The classical requirement of conformal invariance of states of a string boils down to demanding the vanishing of the stress-energy tensor, $T(z) = 0$, $\tilde{T}(\bar{z}) = 0$. These conditions are refined in the quantum theory, and are called Virasoro constraints.

Let us proceed to a first quantization of string theory. In chapter 4 we are going to consider a string moving in a space which is a direct product of flat space and coset space. In this section, for simplicity, let us refrain to string moving in a flat space-time. Due to the translational invariance of flat space-time we have left-moving and right-moving currents,

$$j^\mu(z) = i\sqrt{2}\,\partial X^\mu(z)\,, \quad \tilde{j}^\mu(\bar{z}) = i\sqrt{2}\,\bar{\partial} X^\mu(\bar{z})\,, \tag{1.7}$$

which due to the equations of motion (1.3) are conserved separately. For a classical string products like $j(z_1)j(z_2)$ are non-singular for any $z_1 - z_2$. If we quantize the string, these become singular as $z_1$ approaches $z_2$:

$$j^\mu(z_1)j^\nu(z_2) = : j^\mu(z_1)j^\nu(z_2) : + \langle j^\mu(z_1)j^\nu(z_2)\rangle\,, \tag{1.8}$$

and similarly for $\tilde{j}^\mu(\bar{z}_1)\tilde{j}^\nu(\bar{z}_2)$. In (1.8), the $: j^\mu(z_1)j^\nu(z_2) :$ is regular in the limit $z_1 \to z_2$, and it is called a normal-ordered product of the operators $j(z_1)$ and $j(z_2)$. If not specified otherwise, all the products of world-sheet operators in this chapter are normal-ordered. The correlation function $\langle j^\mu(z_1)j^\nu(z_2)\rangle$ is singular when $z_1 \to z_2$ and is defined by the Polyakov path integral

$$\langle j^\mu(z_1)j^\nu(z_2)\rangle = \int [dX]\,(i\sqrt{2}\,\partial X^\mu(z_1))(i\sqrt{2}\,\partial X^\nu(z_2))e^{-S_P[X]}\,. \tag{1.9}$$

We obtain

$$\langle j^\mu(z_1)j^\nu(z_2)\rangle = \eta^{\mu\nu}\frac{1}{(z_1-z_2)^2}\,, \quad \langle \tilde{j}^\mu(\bar{z}_1)\tilde{j}^\nu(\bar{z}_2)\rangle = \eta^{\mu\nu}\frac{1}{(\bar{z}_1-\bar{z}_2)^2}\,. \tag{1.10}$$

Let us focus on holomorphic fields; the conclusions are similar for anti-holomorphic fields. The stress-energy tensor in terms of the currents $j^\mu$ is

$$T(z) = \frac{1}{2} : j^\mu(z)j_\mu(z) : \,. \tag{1.11}$$

We have the operator product expansion (OPE)

$$T(z_1)j^\mu(z_2) = \frac{j^\mu(z_2)}{(z_1-z_2)^2} + \frac{\partial j^\mu(z_2)}{z_1-z_2} + \ldots\,, \tag{1.12}$$

where dots represent normal-ordered terms, regular in the limit $z_1 \to z_2$. We also have the OPE

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial T(z_2)}{z_1-z_2} + \ldots\,. \tag{1.13}$$

Here $c$ is a central charge, which is not present in the classical theory. For the stress-energy tensor (1.11), it is equal to the dimension of the target space-time, $d$, where each component of $j^\mu$ contributes central charge equal to one. In bosonic string theory the central charge is $c = d - 26$, where $-26$ comes from conformal ghost fields. In superstring theory (discussed below) $c = 3d/2 - 15$, where each one of $d$ world-sheet fermions contributes $1/2$ and superconformal ghosts contribute $-15$. Non-vanishing central charge signifies a world-sheet conformal anomaly: quantum violation of the classical tracelessness of the stress-energy tensor.

The conformal anomaly in a consistent string theory should vanish. This is necessary because conformal invariance of string theory ensures that the negative-norm time-like component of the current $j^0$ (and time-like component of the fermion $\psi^0$, in the case of superstring theory, where conformal symmetry is extended to superconformal symmetry) decouples from the spectrum of physical operators. In this chapter we always assume that a bosonic string lives in 26-dimensional space-time, and a supersymmetric string lives in 10-dimensional space-time, so that the central charge vanishes.

Let us perform a Laurent series expansion

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}\,, \quad j^\mu(z) = \sum_n \frac{j_n^\mu}{z^{n+1}}\,, \tag{1.14}$$

where $L_{-n} = (L_n)^\dagger$, $j_{-n} = (j_n)^\dagger$. Therefore the OPE (1.10), and radial ordering of operators on a complex world-sheet plane (if one considers the operator product $\mathcal{O}(z_1)\mathcal{O}(z_2)$ and $z_1 \to z_2$, then the operator with larger $|z|$ is placed on the left of the operator with smaller $|z|$), give the commutator

$$[j_m^\mu\,, j_n^\nu] = m\,\eta^{\mu\nu}\,\delta_{m+n,0}\,. \tag{1.15}$$

We see that $j_{-n}^\mu\,,\ \ n > 0$ are the operators creating oscillatory string states and $j_n^\mu\,,\ \ n > 0$ are the operators annihilating string states.

The OPE (1.12) gives the commutator

$$[L_m,\, j_{-n}^\mu] = n j_{m-n}^\mu\,, \tag{1.16}$$

and the OPE (1.13) gives the Virasoro algebra commutation relation

$$[L_m,\, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\,. \tag{1.17}$$

We also obtain

$$L_n = \frac{1}{2}\sum_k \eta_{\mu\nu} j_{-k}^\mu j_{n+k}^\nu\,, \tag{1.18}$$

where the operators are normal-ordered: creation operators $j^\mu_{-m}$, $m > 0$ are always on the left of annihilation operators $j^\mu_m$, $m > 0$; due to (1.15) this subtlety only arises when $n = 0$.

The classical physical state conditions, which demand that the stress-energy tensor components (1.6) vanish, are replaced in the quantum theory by the Virasoro conditions

$$L_n|\text{phys}\rangle = 0\,, \quad n > 0\,, \quad (L_0 - a)|\text{phys}\rangle = 0\,, \tag{1.19}$$

where $a$ is a normal ordering constant. For a bosonic string $a = 1$. We will derive the value of $a$ for superstring later in this chapter.

A general string state is

$$|\text{state}\rangle \;=\; j^{\mu_1}_{-n_1} \ldots j^{\mu_k}_{-n_k}|0;p\rangle\,, \tag{1.20}$$

where the vacuum $|0;p\rangle$ is annihilated by all $j^\mu_n$ with $n > 0$. The eigenvalues of $j^\mu_0$ are values of the momentum $p^\mu$ of the center of mass of a string. If the state (1.20) satisfies the Virasoro constraints and has a non-vanishing norm, it is called physical.

The $L_0$ Virasoro constraint defines the mass-shell equation:

$$\frac{1}{2}M^2 \;=\; \sum_{n>0} \eta_{\mu\nu} j^\mu_{-n} j^\nu_n - a\,, \tag{1.21}$$

where we have used the equation $-j_0^2 = -p^2 = M^2$. The $-a$ term is therefore the zero-point energy of a string.

Due to (1.19), the negative-norm states which are created by the operators $j^0_{-k}$, $k > 0$ are decoupled from the physical spectrum. We are going to illustrate this now with a simple example. Recall that $|p;0\rangle$ is a string oscillatory vacuum state, with a center-of-mass momentum $p^\mu$, satisfying $L_n|p;0\rangle = 0$ for $n > 0$. Consider the first excited string state, $|\psi\rangle = e_\mu j^\mu_{-1}|p;0\rangle$, where the polarization vector $e_\mu$ has $d$ independent components. The only non-trivial Virasoro constraint (besides the mass-shell condition (1.21)) is $L_1|\psi\rangle = 0$, which due to (1.16), (1.18) gets re-written as $p^\mu e_\mu = 0$. The mass of this state, due to (1.21) and the fact that $a = 1$, is zero.

Due to (1.18) and (1.21) the state $|\chi\rangle = L_{-1}|p;0\rangle = p_\mu j^\mu_{-1}|p;0\rangle$ also has zero mass, as the state $|\psi\rangle$, and has a longitudinal polarization $e_\mu = p_\mu$. The state $|\chi\rangle$ is called spurious: it satisfies the Virasoro constraints but it is decoupled from any physical state $|\omega\rangle$, because $\langle\chi|\omega\rangle = \langle p;0|L_1|\omega\rangle =$

0. In particular its own norm is equal to zero. Any state $|\psi\rangle$ is therefore defined up to a state $t|\chi\rangle$, with an arbitrary parameter $t$.

Let us now choose $p^0 = \sqrt{M^2+p^2} = p$, $p^{d-1} = p$, $p^i = 0$ for $i = 2,\ldots,d-2$. The index $i$ labels polarizations of the string states transverse to the direction of motion of its center of mass. The $L_1$ Virasoro constraint is then

$$p\,e_0 + p\,e_{d-1} = 0\,, \tag{1.22}$$

and the spurious state is

$$t|\chi\rangle = t(-p\,j^0_{-1} + p\,j^{d-1}_{-1})|p;0\rangle\,. \tag{1.23}$$

Due to (1.22), (1.23) we can fix $e_0 = e_{d-1} = 0$, which means that the states with time-like and longitudinal polarizations are decoupled from the physical spectrum of a string.

The systematic application of such an approach to the construction of the physical spectrum of a first-quantized string is called covariant quantization of a string. We adopt this method in chapter 4. Another covariant (w.r.t. Lorentz symmetry in the target space-time) way to quantize a (super)string is the BRST method, which requires the introduction of (super)conformal ghosts, contributing to the central charge. The method which we are going to use below in this chapter is light-cone quantization, which breaks space-time Lorentz symmetry down to the rotation group $SO(d-2)$, acting only on transverse physical polarizations.

### 1.2.2 Ramond-Neveu-Schwarz superstring

The Ramond-Neveu-Schwarz (RNS) superstring is one way to formulate superstring theory. In the RNS superstring the dynamical fields on the world-sheet are the bosons $X^\mu(z,\bar z)$ and the fermions $\Psi^\mu(z,\bar z)$. Both of these fields have a target space-time vector index $\mu$. The RNS superstring action is a sum of the Polyakov action (1.2) and Dirac terms for the world-sheet fermions,

$$S = \frac{1}{2\pi}\int d^2z\,\left(\partial X^\mu \bar\partial X_\mu + \frac{1}{2}\psi^\mu\bar\partial\psi_\mu + \frac{1}{2}\tilde\psi^\mu\partial\tilde\psi_\mu\right)\,. \tag{1.24}$$

Here $\psi$ and $\tilde\psi$ are Majorana-Weyl one-component two-dimensional spinors, with the two-component Majorana spinor being $\Psi = (\psi\,,\ \tilde\psi)$. The action (1.24) possesses two-dimensional supersymmetry. The fermions $\psi$ and $\tilde\psi$ are respectively the left-moving and the right-moving superpartners of the currents $j$ and $\tilde{j}$ defined by eq. (1.7). The equations of motion following from the action (1.24) are

$$\bar{\partial}\partial X^\mu = 0\,, \quad \bar{\partial}\psi^\mu = 0\,, \quad \partial\tilde{\psi}^\mu = 0\,. \tag{1.25}$$

These equations must be accompanied by boundary conditions, which for bosons $X^\mu$ are the same as in bosonic string theory, and for fermions are to be chosen so that

$$\int d\tau \left[\psi^\mu \delta\psi_\mu - \tilde{\psi}^\mu \delta\tilde{\psi}_\mu\right]_{\sigma=0} - \int d\tau \left[\psi^\mu \delta\psi_\mu - \tilde{\psi}^\mu \delta\tilde{\psi}_\mu\right]_{\sigma=\pi} = 0\,. \tag{1.26}$$

Consider an open string. Suppose the bosons $X^\mu$ satisfy Neumann boundary conditions. On the l.h.s. of (1.26) there are two square brackets, each corresponding to one end of a string. We have to satisfy the boundary conditions independently at each end, which means that we have to impose $\psi = \pm\tilde{\psi}$ at $\sigma = 0, \pi$. As a matter of convention we put $\psi = \tilde{\psi}$ at $\sigma = 0$. The two options at the other end ($\sigma = \pi$) define two sectors of an open string, the Neveu-Schwarz sector and the Ramond sector:

$$\begin{aligned} \textbf{NS}: \quad & \psi^\mu|_{\sigma=\pi} = -\tilde{\psi}^\mu|_{\sigma=\pi}\,, \\ \textbf{R}: \quad & \psi^\mu|_{\sigma=\pi} = \tilde{\psi}^\mu|_{\sigma=\pi}\,. \end{aligned} \tag{1.27}$$

The Laurent expansions of a general open string solution to (1.25) for fermions with boundary conditions (1.27) are

$$\begin{aligned} \textbf{NS}: \quad & \psi^\mu(z) = \sum_{r\in Z+1/2} \frac{b^\mu_r}{z^{r+1/2}}\,, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in Z+1/2} \frac{b^\mu_r}{\bar{z}^{r+1/2}}\,, \\ \textbf{R}: \quad & \psi^\mu(z) = \sum_{n\in Z} \frac{d^\mu_n}{z^{n+1/2}}\,, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{n\in Z} \frac{d^\mu_n}{\bar{z}^{n+1/2}}\,, \end{aligned} \tag{1.28}$$

where $n$ is integer-valued and $r$ is half-integer-valued.

In the case of a closed string we have to impose (anti)periodic boundary conditions separately for left- and right-moving states: $\psi|_{\sigma=0} = \pm\psi|_{\sigma=\pi}$ and $\tilde{\psi}|_{\sigma=0} = \pm\tilde{\psi}|_{\sigma=\pi}$. Each state belongs either to the NS sector and is expanded in half-integer modes, or the R sector and is expanded in integer modes. In total we can form four different combinations of left- and right-movers.

The stress-energy tensor corresponding to the RNS action (1.24) is given by

$$T(z) = -\partial X^\mu(z)\partial X_\mu(z) - \frac{1}{2}\psi^\mu(z)\partial\psi_\mu(z)\,, \tag{1.29}$$

and similarly for the anti-holomorphic (right-moving) component $\tilde{T}(\bar{z})$. Furthermore, the action (1.24) is invariant under $\mathcal{N}=(1,1)$ two-dimensional supersymmetry transformations, generated by a supercurrent with the components

$$\mathcal{J}(z)=\psi(z)j(z)\,, \qquad \tilde{\mathcal{J}}(\bar{z})=\tilde{\psi}(\bar{z})\tilde{j}(\bar{z})\,. \tag{1.30}$$

The stress-energy tensor $T(z)$ and the supercurrent $\mathcal{J}(z)$ form a holomorphic superconformal world-sheet current algebra. The operators $\tilde{T}(\bar{z})$ and $\tilde{\mathcal{J}}(\bar{z})$ form an anti-holomorphic copy of this superconformal algebra.

In the NS sector the supercurrent is expanded in half-integer modes, and in the R sector it is expanded in integer modes

$$\begin{aligned}
\mathcal{J}_{NS}(z) &= \sum_{r\in Z+1/2}\frac{G_r}{z^{r+3/2}}\,, \qquad G_r=\sum_{s\in Z+1/2}\eta_{\mu\nu}b_s^{\mu}j_{r-s}^{\nu}\,,\\
\mathcal{J}_{R}(z) &= \sum_{m\in Z}\frac{F_m}{z^{m+3/2}}\,, \qquad F_m=\sum_{n\in Z}\eta_{\mu\nu}d_n^{\mu}j_{m-n}^{\nu}\,.
\end{aligned} \tag{1.31}$$

and similarly for the anti-holomorphic sector. Notice that in (1.31) the indices $r-s$, $m-n$ of the modes of the current $j^\mu$ are integer-valued. This is a consequence of the Neumann boundary conditions for bosons $X^\mu$. Below we generalize our consideration to the case of Dirichlet boundary condition, with the bosons expanded in half-integer modes, $j_s^\mu\,,\ s\in Z+1/2$. Supersymmetry requires the coefficients $G_r$ to have a half-integer-valued index $r$, and the coefficients $F_m$ to have an integer-valued index $m$. Therefore, due to (1.31), the corresponding NS fermions must be expanded in integer-valued modes, $b_n^\mu\,,\ n\in Z$, while the R fermionic modes, $d_s^\mu$ must have half-integer valued indices, $s\in Z+1/2$.

As we first-quantize the theory with the action (1.24) we get the correlation functions (1.10) for the bosonic fields $j^\mu(z)$, $\tilde{j}^\mu(\bar{z})$, and the correlation functions

$$\langle\psi^\mu(z_1)\psi^\mu(z_2)\rangle=\eta^{\mu\nu}\frac{1}{z_1-z_2}\,, \quad \langle\tilde{\psi}^\mu(\bar{z}_1)\tilde{\psi}^\mu(\bar{z}_2)\rangle=\eta^{\mu\nu}\frac{1}{\bar{z}_1-\bar{z}_2} \tag{1.32}$$

for the fermions. Using (1.10) and (1.32) we find the operator product expansions of the operators of the superconformal algebra (only terms singular in the $z_1 \to z_2$ limit are written down)

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{T(z_2)}{z_1-z_2} + \dots\,, \tag{1.33}$$

$$T(z_1)\mathcal{J}(z_2) = \frac{(3/2)\mathcal{J}(z_2)}{(z_1-z_2)^2} + \frac{\partial\mathcal{J}(z_2)}{z_1-z_2} + \dots\,, \tag{1.34}$$

$$\mathcal{J}(z_1)\mathcal{J}(z_2) = \frac{2c/3}{(z_1-z_2)^3} + \frac{2T(z_2)}{z_1-z_2} + \dots\,. \tag{1.35}$$

The supersymmetry transformations are defined by the OPEs

$$\mathcal{J}(z_1)j(z_2) = \frac{\psi(z_2)}{(z_1-z_2)^2} + \frac{\partial\psi(z_2)}{z_1-z_2} + \dots\,, \tag{1.36}$$

$$\mathcal{J}(z_1)\psi(z_2) = \frac{j(z_2)}{z_1-z_2} + \dots\,. \tag{1.37}$$

From (1.28), (1.32) we obtain the anti-commutation relations

$$\{b^\mu_r,\, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}\,, \quad \{d^\mu_n,\, d^\nu_m\} = \eta^{\mu\nu}\delta_{n+m,0}\,. \tag{1.38}$$

Therefore the operators $b^\mu_{-r}$, $r > 0$ and $d^\mu_{-n}$, $n > 0$ create string states and the operators $b^\mu_r$, $r > 0$ and $d^\mu_n$, $n > 0$ annihilate string states. We also have creation and annihilation operators, respectively, $j^\mu_{-n}$, $n > 0$, and $j^\mu_n$, $n > 0$, as in bosonic string theory. The string vacuum state $|0\rangle$ vanishes when we act on it with any annihilation operator. The negative-norm states, created by the time-like polarized operators $j^0_{-n}$, $d^0_{-m}$ and $b^0_{-r}$ $(m, n, r > 0)$ are decoupled from the physical spectrum due to the super-Virasoro constraints:

$$\begin{aligned} &\mathbf{NS}: \ (L_0 - a_{NS})|\text{phys}\rangle = 0\,, \ \ L_n|\text{phys}\rangle = 0\,, \ \ G_r|\text{phys}\rangle = 0\,, \ \ n,\ r > 0\,, \\ &\mathbf{R}: (L_0 - a_R)|\text{phys}\rangle = 0\,, \ L_n|\text{phys}\rangle = 0\,, \ F_m|\text{phys}\rangle = 0\,, \ n > 0, m \geq 0\,. \end{aligned} \tag{1.39}$$

In the next subsection we will prove that if we impose Neumann boundary condition for all polarizations then $a_{NS} = 1/2$. One can show that $L_0 = F_0^2$ in the R sector. Therefore, due to the supersymmetry constraint $F_0 = 0$, we have to put $a_R = 0$.

In the R sector we have operators $d^\mu_0$, which, due to (1.38), form a Dirac algebra in $d$ dimensions:

$$\{\Gamma^\mu\,,\, \Gamma^\nu\} = 2\,\eta^{\mu\nu}\,, \quad \Gamma^\mu = \sqrt{2}\,d^\mu_0\,. \tag{1.40}$$

If the state $|0\rangle_R$ is the vacuum of the R sector then the state $d_0^\mu|0\rangle_R$ is also a vacuum of the R sector. Consequently any state in the R sector lives in spinor representation of the Dirac algebra $\mathcal{C}_{d-1,1}$, and therefore is a target space-time spinor. This should be compared to the fact that any state in the NS sector is a space-time boson.

### 1.2.3 Open RNS superstring and zero-point energy

As was explained above, the end of an open string can be free so that the bosons $X^\mu$ on its world-sheet satisfy Neumann boundary conditions, or it can terminate on a $D$-brane giving rise to Dirichlet boundary conditions. In total there are four possibilities for the two ends of an open string. We denote them as

$$(\mathbf{NN})\,, \quad (\mathbf{ND})\,, \quad (\mathbf{DN})\,, \quad (\mathbf{DD})\,. \tag{1.41}$$

The first letter specifies the boundary condition at the $\sigma = 0$ ($z = e^\tau$) end, the second letter does so for the $\sigma = \pi$ ($z = -e^\tau$) end.

For the possible boundary conditions (1.41), we have the following solutions to the equation (1.3):

$$\begin{aligned}
\mathbf{NN}:\ X^\mu(z,\bar z) &= x^\mu - \frac{ip^\mu}{\sqrt{2}}\log(z\bar z) + \frac{i}{\sqrt{2}}\sum_{m\neq 0}\frac{1}{m}j^\mu_m\left(z^{-m}+\bar z^{-m}\right),\\
\mathbf{DD}:\ X^\mu(z,\bar z) &= x^\mu - \frac{i\tilde p^\mu}{\sqrt{2}}\log\left(\frac{z}{\bar z}\right) + \frac{i}{\sqrt{2}}\sum_{m\neq 0}\frac{1}{m}j^\mu_m\left(z^{-m}-\bar z^{-m}\right),\\
\mathbf{DN}:\ X^\mu(z,\bar z) &= x^\mu + \frac{i}{\sqrt{2}}\sum_{r\neq 0}\frac{1}{r}j^\mu_r\left(z^{-r}-\bar z^{-r}\right),\\
\mathbf{ND}:\ X^\mu(z,\bar z) &= x^\mu + \frac{i}{\sqrt{2}}\sum_{r\neq 0}\frac{1}{r}j^\mu_r\left(z^{-r}+\bar z^{-r}\right).
\end{aligned} \tag{1.42}$$

Here the index $m$ is integer-valued and the index $r$ is half-integer-valued.

Let us solve the Virasoro constraints explicitly, so that we are left with only $d-2$ transverse polarizations in the target space-time. This method is called light-cone quantization, and it is convenient for our current purposes. The physical operators are $b^i_r$, $d^i_n$, $j^i_n$, $i = 1,\ldots,d-2$, and we do not have to worry about the super-Virasoro constraints.

Assume first that the bosons $X^\mu$ satisfy either NN or DD boundary conditions, and therefore are expanded in integer modes. Using (1.29) we derive

$$
\begin{aligned}
\mathbf{R}: \quad L_0 &= \frac{1}{2}\sum_{n\in Z}\sum_{i=1}^{d-2} j^i_{-n} j^i_n + \frac{1}{2}\sum_{n\in Z}\sum_{i=1}^{d-2} n d^i_{-n} d^i_n\,, \\
\mathbf{NS}: \quad L_0 &= \frac{1}{2}\sum_{n\in Z}\sum_{i=1}^{d-2} j^i_{-n} j^i_n + \frac{1}{2}\sum_{r\in Z+1/2}\sum_{i=1}^{d-2} r b^i_{-r} b^i_r\,.
\end{aligned}
\tag{1.43}
$$

Let us perform normal ordering of the creation and annihilation operators in (1.43), placing annihilation operators to the right of creation operators. In the R sector we obtain

$$
\begin{aligned}
L_0 &= \sum_{n>0}\sum_{i=1}^{d-2}(j^i_{-n} j^i_n + n d^i_{-n} d^i_n) + \frac{1}{2} j_0^2 + \frac{1}{2}\sum_{n>0}\sum_{i=1}^{d-2}([j^i_n,\, j^i_{-n}] - n\{d^i_n,\, d^i_{-n}\}) \\
&= \sum_{n>0}\sum_{i=1}^{d-2}(j^i_{-n} j^i_n + n d^i_{-n} d^i_n) + \frac{1}{2} j_0^2\,,
\end{aligned}
\tag{1.44}
$$

where we have used the commutators (1.15) and (1.38). We see from (1.44) that the zero-point energy in the R sector is zero, $a_R = 0$, as we concluded at the end of the previous subsection from the point of view of the super-Virasoro constraints. Similarly in the NS sector we obtain

$$
\begin{aligned}
L_0 = &\sum_{n>0}\sum_{i=1}^{d-2} j^i_{-n} j^i_n + \sum_{r>0}\sum_{i=1}^{d-2} r b^i_{-r} b^i_r + \frac{1}{2} j_0^2 \\
&+ \frac{1}{2}\sum_{n>0}\sum_{i=1}^{d-2}[j^i_n,\, j^i_{-n}] - \frac{1}{2}\sum_{r>0}\sum_{i=1}^{d-2} r\{b^i_r,\, b^i_{-r}\}\,.
\end{aligned}
\tag{1.45}
$$

Therefore due to (1.15) and (1.38), the zero-point energy in the NS sector is given by

$$
-a_{NS} = \frac{d-2}{2}\left(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right) = -\frac{d-2}{16}\,. \tag{1.46}
$$

Inserting the superstring value $d = 10$ we obtain $a_{NS} = \frac{1}{2}$. In the last equality of (1.46) we used zeta-function regularization. We know that the zeta-function

$$
\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \tag{1.47}
$$

can be analytically continued so that

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}\,. \tag{1.48}$$

Now, introduce the sum of even numbers, $S_{even} = \sum_{n>0}(2n)$, and the sum of odd numbers $S_{odd} = \sum_{n>0}(2n+1)$. We have $S_{even} = 2\zeta(-1) = -\frac{1}{6}$, and $S_{odd} = \zeta(-1) - S_{even} = \frac{1}{12}$. Therefore in (1.46) we have $\sum_{r>0} r = \frac{1}{24}$. There exist independent ways to derive the same results for zero-point energies.

The bottom line is that for NN and DD bosons we have $a_{NS} = 1/2$, and $a_R = 0$, and therefore the mass formulae are,

$$\mathbf{R}: \quad \frac{1}{2}M^2 = \sum_{n>0}\sum_{i=1}^{d-2} n d^i_{-n} d^i_n + \sum_{n>0}\sum_{i=1}^{d-2} j^i_{-n} j^i_n\,, \tag{1.49}$$

$$\mathbf{NS}: \quad \frac{1}{2}M^2 = \sum_{r>0}\sum_{i=1}^{d-2} r b^i_{-r} b^i_r + \sum_{n>0}\sum_{i=1}^{d-2} j^i_{-n} j^i_n - \frac{1}{2}\,. \tag{1.50}$$

Consider an open string with NN or DD boundary conditions. The vacuum $|0\rangle_{NS}$ in the NS sector is defined by the conditions $b^i_r|0\rangle_{NS} = 0$, $j^i_n|0\rangle_{NS} = 0$, for $r, n > 0$. This state is a tachyon with $M^2 = -1$, as follows from (1.50). The procedure called Gliozzi-Scherk-Olive (GSO) projection eliminates $|0\rangle_{NS}$ from the spectrum of the NS sector of the RNS superstring. The lowest NS string state, which survives GSO projection, is the $d-2$-component massless vector $b^i_{-1/2}|0\rangle_{NS}$.

The lowest state in the R sector, the vacuum $|0\rangle_R$ with zero mass, survives GSO projection. As we have noticed above, this state is a space-time fermion. In $d = 10$ dimensions it is a 16-component Majorana-Weyl massless fermion with eight physical d.o.f. (A general spinor in ten dimensions has 32 complex-valued components; each condition of being a Majorana and Weyl decreases the number of independent components by a factor of two, and the super-Virasoro constraint $F_0 = 0$ further reduces the number of independent components by a factor of two.) Therefore the lowest state of an open RNS superstring consists of eight bosonic and eight fermionic massless degrees of freedom, which is a vector supermultiplet field content of $\mathcal{N} = 1$, $d = 10$ supersymmetric Yang-Mills theory.

This result has a deep reason behind it: the RNS superstring, which by construction has two-dimensional world-sheet supersymmetry, is actually space-time supersymmetric. Superstring theory which contains open strings has $\mathcal{N}=1$ space-time supersymmetry (16 supercharges), superstring theory without open strings has $\mathcal{N}=2$ space-time supersymmetry (32 supercharges). A formulation of superstring theory with explicit space-time supersymmetry is called the Green-Schwarz (GS) superstring.

Now let us consider the DN and ND boundary conditions. In general we find the following contributions to the normal ordering constant $a$ from half-integer or integer bosonic and fermionic modes:

$$a_{b\,i}=\frac{1}{24}\,,\quad a_{b\,h}=-\frac{1}{48}\,,\quad a_{f\,i}=-\frac{1}{24}\,,\quad a_{f\,h}=\frac{1}{48}\,, \tag{1.51}$$

where $b$ and $f$ stand for bosons and fermions, and $i$ and $h$ stand for integer and half-integer, respectively.

One requirement that should always be satisfied is that the zero-point energy in the R sector is zero. Suppose therefore that among the $d-2$ transverse polarizations, $\nu$ of them are either DN or ND, with half-integer bosonic current modes. Therefore the contribution of the bosons to the zero-point energy is $-\frac{\nu}{48}+\frac{d-2-\nu}{24}=\frac{2(d-2)-3\nu}{48}$. Therefore fermions in the R sector should contribute $\frac{3\nu-2(d-2)}{48}$. The fermions which are polarized along $d-2-\nu$ NN or DD directions have integer modes and contribute $-\frac{d-2-\nu}{24}$. The fermions which are polarized along $\nu$ DN or ND directions therefore must have half-integer modes and contribute $\frac{\nu}{48}$, adding up to a required quantity $\frac{3\nu-2(d-2)}{48}$.

Now we are ready to compute the zero-point energy in the NS sector. The NS fermions have opposite kind of modes to those of the R fermions. Therefore the fermions in $d-2-\nu$ NN or DD directions have half-integer modes and contribute $\frac{d-2-\nu}{48}$, and fermions in $\nu$ DN or ND directions have integer modes and contribute $-\frac{\nu}{24}$. The contribution from the bosons is of course the same as in the R sector and is equal to $\frac{2(d-2)-3\nu}{48}$. We conclude that

$$a_{NS}=\frac{d-2}{16}-\frac{\nu}{8}=\frac{1}{2}-\frac{\nu}{8}\,. \tag{1.52}$$

Due to the fact that $a_R=0$, the ground state of the R sector of an open string is always a massless fermion. In the case of $\nu=4$ we get $a_{NS}=0$, and the mass of the lowest NS state is therefore $M^2=-2a_{NS}=0$. It is not projected out by GSO, so that we have an equal number of massless bosons and fermions, furnishing a vector supermultiplet.

### 1.2.4 Closed RNS superstring and Ramond-Ramond fields

In the case of the closed string, to construct the spectrum of excitations we have to build physical states of the holomorphic and anti-holomorphic sectors separately, and then take a direct product of these states. As was already mentioned above, the boundary conditions which must be imposed on fermions of the closed string are $\psi|_{\sigma=0} = \pm\psi|_{\sigma=\pi}$, $\tilde{\psi}|_{\sigma=0} = \pm\tilde{\psi}|_{\sigma=\pi}$, giving rise to the NS and R sectors of left-movers (holomorphic states) and right-movers (anti-holomorphic states). The fermions of the NS sector are expanded in half-integer modes, $b^\mu_r$, $\tilde{b}^\mu_r$, and the fermions of the R sector are expanded in integer modes, $d^\mu_n$, $\tilde{d}^\mu_n$. After GSO projection, the massless (anti)holomorphic states in the light-cone quantization are

$$\begin{aligned} \textbf{left}: &\quad b^i_{-1/2}|0\rangle_{NS}\,, \quad |0\rangle_{R\,\text{left}}\,, \\ \textbf{right}: &\quad \tilde{b}^i_{-1/2}|0\rangle_{NS}\,, \quad |0\rangle_{R\,\text{right}}\,. \end{aligned} \tag{1.53}$$

Here $|0\rangle_R$ is a space-time fermion with eight independent real-valued components and definite chirality. Let us specify its chirality by introducing the notation $|+\rangle$ and $|-\rangle$.

By forming the direct product of left-moving and right-moving states (1.53) we obtain 128 bosonic states, NS-NS and R-R, and 128 fermionic states, NS-R and R-NS. The equality of number of bosonic and fermionic degrees of freedom is not accidental: as was mentioned above, the RNS superstring is actually space-time supersymmetric. In fact, 128+128 is the on-shell field content of $d = 10$ $\mathcal{N} = 2$ supergravity.

In the case when both left- and right-moving R fermions $|0\rangle_R$ have opposite chirality we get type-IIA supergravity. The corresponding string theory is type-IIA superstring theory. In this case the 64 of R-R degrees of freedom $|+\rangle \otimes |-\rangle$ are expanded in irreducible representations of $SO(8)$ as $C_1 \oplus C_3$, where $C_1$ is a 1-form with 8 d.o.f. and $C_3$ is a 3-form with 56 d.o.f. Similarly the R-R fields $|+\rangle \otimes |+\rangle$ of type-IIB supergravity are $p$-forms $C_0$, $C_2$ and $C_4$, with 1, 28 and 35 d.o.f. respectively (there is a subtlety with $C_4$, requiring that $F_5 = dC_4$ is Hodge self-dual in ten dimensions, $F_5 = \star F_5$, which reduces the number of d.o.f. of the $C_4$ by a factor of two).

### 1.2.5 *D*-branes

A $D$-brane is an extended object on which an open string can end. For example, if all of the coordinates but $X^1$ satisfy Neumann boundary conditions, it means that there is $D8$-brane located at some point $x^1 = x^1_0$ and extended in the $x^{2,3,\ldots,9}$ directions. Similarly to a string having a two-dimensional world-sheet, a $D8$-brane sweeps a nine-dimensional world-volume as it moves in a space-time. If all coordinates of an open string satisfy Neumann boundary conditions then we actually have a space-time filling $D9$-brane, with its ten-dimensional world-volume being the entire space-time.

In type-IIA superstring theory we can have $Dp$-branes with even-valued $p$, and in type IIB superstring theory $p$ must be odd-valued. The reason for this selection originates in the stability of a $Dp$-brane, and is tightly connected to the fact that a $Dp$-brane embedded in $\mathcal{N} = 2$ superstring theory is a Bogomol'nyi-Prasad-Sommerfield (BPS) object, preserving 16 of the original 32 supercharges. We discuss this in more detail in the next subsection.

Now, recall that a BPS object satisfies the condition $M = |Z|$, with $M$ being a mass, and $Z$ being (an appropriately defined) conserved charge. It turns out that in the case of a $Dp$-brane, the role of the charge $Z$ is played by the charge w.r.t. the R-R field. The coupling of the $Dp$-brane to the R-R field is described by the Chern-Simons action

$$S_{CS} = T_{Dp} \int C_{p+1}\,, \tag{1.54}$$

where integration of the R-R $p+1$-form $C_{p+1}$ is performed over the $p+1$-dimensional world-volume of the $Dp$-brane. In (1.54) the $T_{Dp} \simeq \frac{1}{\ell_s^{p+1} g_s}$ is the tension of the $Dp$-brane, $g_s$ is the closed string coupling constant. Notice that in the perturbative regime of small $g_s$, a $D$-brane is very heavy.

A single $Dp$-brane is described by a supersymmetric theory on a $p+1$-dimensional world-volume, with 16 conserved supercharges. The number of on-shell fermionic d.o.f. is equal to eight. Due to supersymmetry, the number of dynamical massless bosons on the world-volume must also add up to eight.

An embedding of a $Dp$-brane into ten-dimensional target space-time is described by ten fields $X^\mu(\sigma^a)$, $\mu = 0, 1, \ldots, 10$, where $\sigma^a\,,\ a = 0, 1, \ldots, p$ are the world-volume coordinates. However, due to $p+1$-dimensional diffeomorphism symmetry, $\sigma^a \to \sigma'^a(\sigma^b)$, we have only $9-p$ independent bosonic d.o.f. Supersymmetry therefore requires the addition of $p-1$ bosonic d.o.f. These come about as transverse polarizations of the $U(1)$ gauge field $A_\mu$ on the $Dp$-brane world-volume. In fact, the origin of this massless vector supermultiplet on a $Dp$-brane world-volume is simple: its fields are the lowest modes of an open string attached to this $Dp$-brane by both ends.

To summarize, the eight massless dynamical bosonic d.o.f. on a $Dp$-brane world-volume are split into a gauge field $A_\mu$ with $p-1$ physical polarizations, and $9-p$ scalars $\Phi^I$, $I = p+1, \ldots 10$, describing the embedding of the $Dp$-brane into 10-dimensional target space-time.

Let us study the low-energy dynamics of a $Dp$-brane. Clasically we can set the fermionic gaugino field to zero. Suppose the $Dp$-brane is embedded in a space-time with a metric $G_{\mu\nu}$. It induces a metric $g_{ab}$, $a, b = 1, \ldots, p+1$, on the $Dp$-brane world-volume, such that

$$g_{ab} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} G_{\mu\nu} \,. \tag{1.55}$$

The total low-energy effective action of a $Dp$-brane consists of two terms. One of them is a generalization of the Chern-Simons term (1.54),

$$S_{CS} = T_{Dp} \int \left[ \sum_p C_p e^{B+F} \right]_{p+1} \,, \tag{1.56}$$

where, on the r.h.s. of (1.56), $B$ is the NS-NS two-form, and we introduced the field strength $F = dA$, then we took the $p+1$-form part. The other term, describing the low-energy $Dp$-brane dynamics, is the Dirac-Born-Infeld (DBI) term,

$$S_{DBI} = -T_{Dp} \int d^{p+1}x \sqrt{-\det\left(g + B + F + G_{IJ}\partial\Phi^I \partial\Phi^J\right)} \,, \tag{1.57}$$

where, on the r.h.s of (1.57), one takes the determinant of the matrix

$$||g_{ab} + B_{ab} + F_{ab} + G_{IJ}\partial_a \Phi^I \partial_b \Phi^J|| \,. \tag{1.58}$$

By varying the total action $S_{tot} = S_{DBI} + S_{CS}$ of the $Dp$-brane, one obtains the equations of motion determining the embedding of the $Dp$-brane into the target space-time, and the dynamics of the gauge field $A_\mu$ on its world-volume.

### 1.2.6 T-duality and $D$-brane intersections

We have already discussed that a single $Dp$-brane embedded into a space-time of type-II string theory breaks half of the original $\mathcal{N} = 2$ supersymmetry. Putting more $Dp$-branes which span various spatial directions generally breaks more supersymmetry, and in particular can result in a non-supersymmetric theory. In this subsection we are going to discuss how to count the number of supersymmetries which are preserved by the given configuration of $Dp$-branes.

Let us start with type-IIB closed string theory. It has two 16-component Majorana-Weyl conserved supercharges, $Q_1$ and $Q_2$, of the same chirality. Now suppose we want to put an open string with Neumann boundary conditions into the space-time. Open string boundary conditions are not compatible with the $\mathcal{N} = 2$ supersymmetry. The supersymmetry in fact gets broken by a factor of two, the remaining conserved supercharge is given by $Q = Q_1 + Q_2$. As we discussed above, the presence of an open string with Neumann boundary conditions means the presence of a space-time filling $D9$-brane. Therefore a $D9$-brane in type-IIB string theory breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ supersymmetry with the supercharge $Q = Q_1 + Q_2$.

The next question to ask is how many supersymmetries are preserved by $Dp$-branes with $p < 9$. To answer it we are going to use T-duality of superstring theory. Consider type-IIB superstring theory in a space-time with the $x^9$ coordinate compactified on a circle of radius $R$. It turn out that the spectrum of superstring theory is left invariant if we perform the T-duality transformation in the $x^9$ direction, which amounts to replacements of the compactification radius $R \to 1/R$ and the anti-holomorphic boson $\tilde{X}^9 \to -\tilde{X}^9$. As can be seen from (1.42), T-duality transformation interchanges the NN and DD boundary conditions in the $x^9$ direction. Therefore if in the type-IIB theory which we started with we had a space-time filling $D9$-brane, which wrapped the $x^9$ circle, then in the T-dual theory we have a $D8$-brane which is localized at a certain point on the $x^9$ circle, making the open string boundary condition along the $x^9$ direction be Dirichlet.

However we know that type-IIB string theory can only have $Dp$-branes with odd-valued $p$. Therefore the T-dual string theory with a $D8$-brane described above must be type-IIA string theory. Let us see what happens. Due to the world-sheet supersymmetry the transformation of the boson $\tilde{X}^9 \to -\tilde{X}^9$ demands the transformation of the fermion $\tilde{\psi}^9 \to -\tilde{\psi}^9$. In particular the zero mode $\tilde{d}_0^9$ of the anti-holomorphic fermion in the R sector reverses its sign. Consequently the $\Gamma^9$ element of the *anti-holomorphic* copy of the space-time Dirac algebra reverses its sign as well, and therefore so does the chirality operator $\Gamma^{11} = \Gamma^0\Gamma^1 \ldots \Gamma^9$. We conclude that the space-time supercharge $Q_2$ changes chirality. In fact, supersymmetry implies the T-duality transformation $Q_2 \to \Gamma^9 Q_2$. The supercharge preserved by the $D8$-brane localized at some point in the $x^9$ direction is therefore $Q = Q_1 + \Gamma^9 Q_2$.

We can generalize the procedure described above: an arbitrary $Dp$-brane preserves a supercharge

$$Q = Q_1 + \Gamma^{k_1} \dots \Gamma^{k_{9-p}} Q_2 \,, \tag{1.59}$$

where indices $k_1, \dots, k_{9-p}$ label $9-p$ directions, orthogonal to the $Dp$-brane. In the case of type-IIA(-IIB) string theory $p$ is even (odd), $Q_1$ and $Q_2$ have opposite (the same) chiralities, and $Q_1$ and $\Gamma^{k_1} \dots \Gamma^{k_{9-p}} Q_2$ have the same chirality, and can be added to one another. Because $\Gamma^{11} Q_2 = \pm Q_2$ and due to the Dirac algebra anti-commutation relations, we can re-write (1.59) as

$$Q = Q_1 + \Gamma^0 \dots \Gamma^p Q_2 \,, \tag{1.60}$$

fixing an overall sign in front of the second term in the r.h.s. of (1.60) to be plus as a matter of convention.

Now, as we know which supercharge is preserved by a single $Dp$-brane, we can find the supercharge preserved by a configuration of several $Dp$-branes. A simple observation is that several $Dp$-branes spanning the same space directions but, in general, located at different points in the transverse space, preserve the same supercharge as just one $Dp$-brane from this set.

If we have several $Dp$-branes of various dimensions $p$ and spanning different space directions, the preserved supercharge, if any, is the one which is preserved by each $Dp$-brane from the set. For concreteness and following the needs of chapter 2, we are going to deal with the $D3-Dp$ system in type-IIB string theory. We are allowed to take any odd-valued $p$. But we want the $Dp$-brane to have an intersection with the $D3$-brane in three or four space-time dimensions. Therefore we can choose $p = 3, 5, 7, 9$. The cases $p = 3$ and $p = 9$ are relatively trivial and not interesting physically for the purposes of chapter 2, so we do not discuss them any more.

We denote the directions spanned by the $D3$-brane as $x^{0,1,2,3}$. The corresponding conserved supercharge is $Q_{D3} = Q_1 + \Gamma^0\Gamma^1\Gamma^2\Gamma^3 Q_2$. This can be reformulated in the following way. Suppose $\epsilon_{1,2}$ are sixteen-component spinor parameters of the supersymmetry transformation generated by the operator $\epsilon_1^T Q_1 + \epsilon_2^T Q_2$. From the expression for $Q_{D3}$ we conclude that the

$D3$-brane is only supersymmetric under transformations with arbitrary $\epsilon_2$ and $\epsilon_1$ completely determined by $\epsilon_2$ via the equation $\epsilon_1 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\epsilon_2$. We have used the facts that $\epsilon_{1,2}$ are Majorana-Weyl real-valued spinors, and the $\Gamma^\mu$ matrices are written in a Majorana real-valued representation, with $(\Gamma^a)^T = \Gamma^a$ for $a \neq 0$, and $(\Gamma^0)^T = -\Gamma^0$.

The $Dp$-brane is assumed to span the directions $x^{0,1,2}$ or $x^{0,1,2,3}$ along with $p-2$ or $p-3$ directions in the six-dimensional space transverse to the $D3$-brane. Let us be specific. We are considering the cases of $p = 5$ and $p = 7$. Consider first a $D5$-brane which spans the directions $x^{0,1,2,3,4,5}$, therefore intersecting the $D3$-brane in the $x^{0,1,2,3}$ directions. Such a $D5$-brane preserves the supercharge $Q_{D5} = Q_1 + \Gamma^0\Gamma^1\ldots\Gamma^5 Q_2$, that is by itself invariant under supersymmetry transformations with $\epsilon_1 = \Gamma^0\Gamma^1\ldots\Gamma^5\epsilon_2$. Therefore, to make sure that the $D3-D5$ system is invariant under supersymmetry transformations, we must satisfy the constraint $\Gamma^4\Gamma^5\epsilon_2 = \epsilon_2$. Recall now that chirality in the $(4,5)$ plane is defined as an eigenvalue of the operator $S^3 = i\Gamma^4\Gamma^5/2$, which is equal to $\pm 1/2$. Therefore $\Gamma^4\Gamma^5\epsilon_2 = \pm i\epsilon_2$, and the constraint $\Gamma^4\Gamma^5\epsilon_2 = \epsilon_2$ can never be satisfied.

Consider now the $D3-D5$ system with a three-dimensional intersection, that is consider a $D5$-brane, which spans directions $x^{0,1,2,4,5,6}$. We call it a $D5'$-brane, where prime is introduced as a short-hand notation for this specific $D5$-brane. Supersymmetry is preserved by the $D5'$-brane alone if the transformation parameters satisfy the constraint $\epsilon_1 = \Gamma^0\Gamma^1\Gamma^2\Gamma^4\Gamma^5\Gamma^6\epsilon_2$. The $D3-D5'$ system therefore preserves supersymmetry with parameter $\epsilon_2$, which satisfies $\Gamma^3\Gamma^4\Gamma^5\Gamma^6\epsilon_2 = \epsilon_2$. This condition means that the chiralities of the spinor $\epsilon_2$ in, say, the $(3,4)$ and $(5,6)$ planes are the same, which reduces the number of independent components of $\epsilon_2$ by a factor of two. The $D3-D5'$ system therefore preserves one-quarter of the original 32 supersymmetries.

In exactly the same way one can prove that the $D3-D7$ system with a four-dimensional intersection is invariant under the action of eight supercharges, while the $D3-D7'$ system with a three-dimensional intersection breaks all the supersymmetries.

Notice that the fact that $D3-D5$ with a three-dimensional intersection, and $D3-D7$ with a four-dimensional intersection, are supersymmetric is in agrement with the equation (1.52) (with $\nu = 4$) for zero-point energy, which gives $a_{NS} = 0$ for both of these intersections. In both of these cases we have a massless bosonic field in the spectrum of an open string stretched between $D3$-brane and $Dp$-brane.

For completeness it is worth underlying that one can also consider an open string which starts and ends on the $D3$-brane and an open string which starts and ends on the $Dp$-brane of the $D3-Dp$ system. The massless modes of these strings comprise vector supermultiplets on the world-volumes of the $D3$-brane and $Dp$-brane respectively.

### 1.2.7 Strings in a background; supergravity and supersymmetric Yang-Mills theories

In the previous subsections we focused on strings and branes propagating in a flat space-time, with no background fields turned on. We have also found out that the massless modes of closed and open strings are fields of supergravity and supersymmetric Yang-Mills, respectively. Therefore one can consider a setup of strings creating a background with curved metric and various non-vanishing fields, and other strings and branes moving in this background as probe objects.

We are going to focus on classical string theory, with only bosonic fields present. The bosonic field content of type-II supergravity consists of NS-NS fields and R-R fields. The NS-NS fields are the same for type-IIA and type-IIB supergravities, as well as for bosonic gravity. For simplicity we will consider bosonic gravity. The field content is graviton $G_{\mu\nu}$, anti-symmetric tensor $B_{\mu\nu}$, and dilaton $\Phi$.

One can derive equations of motion for these fields from string theory in the following way. Instead of the Polyakov action (1.2) we now have the action

$$S_P = \frac{1}{4\pi\alpha'} \int d^2z \left( G_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + \epsilon^{\alpha\beta} B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' \Phi R^{(2)} \right) , \quad (1.61)$$

where $R^{(2)}$ denotes Ricci scalar on the world-sheet, $\epsilon^{z\bar{z}} = -\epsilon^{\bar{z}z} = 1$, and $\alpha, \beta = z, \bar{z}$ are world-sheet vector indices.

Let us perform a first quantization of a string described by the action (1.61). It is defined by the Polyakov path integral, as in the case of a string moving in the Minkowski background. However now due to non-trivial background fields the situation is more complicated: the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ in the action (1.61) are themselves functions of the fields $X^\mu$. The way to proceed is to use string perturbation theory. In (1.61) we restored the parameter $\alpha' = \ell_s^2/2$. In string perturbation theory one assumes that the string length $\ell_s$ is small (compared to a characteristic length scale in a target space-time), and performs a perturbative expansion in $\alpha'$.

The methodology is the same as in the case of usual perturbation theory in quantum field theory. A path integral in interacting quantum field theory accounts for high-energy modes in the low-energy effective Lagrangian by renormalizing the coupling constants and scaling dimensions. Renormalization is described by beta-functions. String perturbation theory also has beta-functions (with $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ in (1.61) playing the role of coupling constants), and these beta-functions take into account string loop corrections in the Polyakov path integral. In this thesis we only consider classical string theory. To avoid possible confusion which may be caused by the word 'classical' we remind the reader that in quantum string field theory one also has to take into account vertices associated with the string coupling constant $g_s$, and loops of virtual particles between these vertices.

The beta-functions for the 'coupling constants' $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ in (1.61) are given by

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \partial_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + \mathcal{O}(\alpha'^2)\,, \\
\beta^B_{\mu\nu} &= -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi\, H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2)\,, \\
\beta^\Phi &= \frac{d-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha' \nabla_\lambda \Phi \nabla^\lambda \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2)\,,
\end{aligned} \tag{1.62}$$

where $H = dB$ is the field strength of the two-form field $B_{\mu\nu}$. We have seen above that for a bosonic string in a flat space-time the requirement of conformal invariance on quantum level, which is a requirement of absence of Weyl anomaly, is a restriction $d = 26$ on the target space-time dimension. Now, as the background fields are turned on, the requirement of conformal invariance is vanishing of the beta-functions (1.62). In particular, when $G = \eta$, $B = \Phi = 0$, it reduces to the $d = 26$ constraint.

We have outlined the derivation of equations of motion of bosonic gravity. Up to the $d-26$ central charge term in the last line of (1.62) these are the same as the equations of motion for the NS-NS fields of type-II supergravity (with gravitino, dilatino and R-R fields set to zero). These equations do not provide a UV-complete description of gravity since the derivation was made under the assumption of a smallness of the string length $\ell_s = \sqrt{2\alpha'}$. At the string scale higher-order effects in the $\alpha'$ expansion become essential, and the gravity approximation (1.62) becomes completely unreliable.

In a similar way one can derive the equations of supersymmetric Yang-Mills theory from open string theory or from heterotic string theory. In principle, systematically accounting for higher order corrections in $\alpha'$, one can derive precisely the higher-derivative terms which one needs to add to the effective low-energy Einstein, Yang-Mills, etc., actions.

### 1.2.8 Wess-Zumino-Witten model

A particular case of a string moving in a non-trivial background is given by a Wess-Zumino-Witten (WZW) model. It describes a string moving on a group manifold or on a coset space; in the latter case it is called a gauged Wess-Zumino-Witten (gWZW) model. Such a model is a central element of chapter 4, so here we give an introduction to it.

Consider a string moving on the manifold of the group $G$ with dimension $\dim G$. Suppose $g$ is an element of $G$. An embedding of the string into the manifold $G$ is then described by the field $g(z,\bar{z})$. The action for the bosonic string is the WZW action,

$$S_{WZW} = \frac{k}{4\pi}\left[\int d^2z\,\mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial} g) - \frac{1}{3}\int_B \mathrm{Tr}(g^{-1}dg)^3\right], \qquad (1.63)$$

where the second term on the r.h.s. of (1.63), which is called the Wess-Zumino (WZ) term, is an integral over a ball $B$ , the boundary of which is a world-sheet. Here $k$ is called the level of the WZW model; for the WZ term to be unambiguous the level $k$ must be integer-valued. In the case of a superstring one adds to this action the Dirac terms for $\dim G$ free left-moving and right-moving Majorana-Weyl world-sheet fermions. For simplicity we consider just a bosonic string in this subsection.

The Polyakov action (1.2) is conformally invariant. For the WZW action (1.63) to describe a string, it must also be conformally invariant. In the previous subsection we obtained that, for a string in background fields to be conformally invariant, the background fields must satisfy effective equations of motion in the target space-time. In the case of a WZW model it turns out that an interplay between two terms in (1.63) is such that the action $S_{WZW}$ is conformally invariant, and one only has to make sure that the total central charge $c$ of the theory vanishes.

Recall that a bosonic string in a flat space-time, described by the Polyakov action (1.2), has conserved holomorphic (left-moving) and anti-holomorphic (right-moving) currents (1.7). These currents originate as

Noether charges from the translational invariance of a flat space-time (as well as from the chirality of the Polyakov action, giving rise to two independently conserved currents). For the action (1.63) to describe a string it must also give rise to two independently conserved currents. The corresponding symmetry transformation of the action (1.63) is the Kac-Moody (KM) symmetry transformation, given by

$$g(z,\bar{z}) \to g'(z,\bar{z}) = \Omega(z) g(z,\bar{z}) \tilde{\Omega}^{-1}(\bar{z})\,. \tag{1.64}$$

Correspondingly, for the WZW model at level $k$ we have the conserved currents

$$\begin{aligned} j(z) &= j_A t^A = -\frac{k}{2}\partial g g^{-1} = \sum_n \frac{j_n^A}{z^{n+1}}\,, \\ \tilde{j}(\bar{z}) &= \tilde{j}_A t^A = \frac{k}{2} g^{-1}\bar{\partial} g = \sum_n \frac{\tilde{j}_n^A}{\bar{z}^{n+1}}\,. \end{aligned} \tag{1.65}$$

The zero modes of the currents are generators of the algebra $\mathbf{g}$ of the group $G$,

$$t^A = j_0^A\,, \qquad [t^A, t^B] = i f^{ABC} t^C\,. \tag{1.66}$$

From the expression for currents (1.65), using (1.64) one can derive the Kac-Moody transformations of the currents. For example, under a holomorphic infinitesimal transformation $\Omega(z) = I - \omega(z)$, we obtain

$$\delta j(z) = -[\omega(z), j(z)] + \frac{k}{2}\partial\omega(z)\,, \tag{1.67}$$

that is

$$\delta j^A(z) = -i f^{ABC}\omega_B(z) j_C(z) + \frac{k}{2}\partial\omega^A(z)\,. \tag{1.68}$$

On the other hand, the KM of $j$ with an infinitesimal parameter $\omega$ is realized by the KM current itself:

$$\delta j^A(z) = \frac{1}{2\pi i}\oint_z dw\, \omega^B(w) j_B(w) j^A(z)\,, \tag{1.69}$$

where the integral is taken over the contour around $w = z$. Matching these two expressions, we obtain the current algebra OPE

$$j^A(z) j^B(w) = \frac{\frac{k}{2}\eta^{AB}}{(z-w)^2} + \frac{i f^{ABC}}{z-w} j_C(w) + \ldots\,. \tag{1.70}$$

Similar expressions are true for the anti-holomorphic current algebra.

The holomorphic component of the stress-energy tensor is given by the Sugawara expression

$$T(z) = \frac{1}{\kappa} : j^A(z) j_A(z) := \frac{1}{\kappa} \frac{1}{2\pi i} \oint_z \frac{dx}{x-z} j^A(x) j_A(z) \,, \tag{1.71}$$

where the index in the adjoint representation of the corresponding algebra is defined as

$$c_V \delta^{AB} = f^{ACD} f^B{}_{CD} \,. \tag{1.72}$$

In (1.71) we have used a simple expression for normal ordering. The normalization constant $\kappa$ will be fixed below.

The stress-energy tensor (1.71) satisfies the Virasoro OPE (1.13), with the central charge given by

$$c = \frac{k \dim G}{k + c_V} \,. \tag{1.73}$$

For completeness notice that in the supersymmetric WZW model we also have $\dim G$ free world-sheet fermions, each fermion contributes central charge $1/2$.

The infinitesimal conformal transformation $\delta z = \epsilon(z)$ acts on the current as

$$\delta j^A(z) = \partial\epsilon(z) j^A(z) + \epsilon(z) \partial j^A(z) \,. \tag{1.74}$$

On the other hand, this transformation is generated by the stress-energy tensor (1.71) as

$$\delta j^A(z) = \frac{1}{2\pi i} \oint_z dw \, \epsilon(w) T(w) j^A(z) \,. \tag{1.75}$$

After some algebra, using (1.71), one finds the OPE

$$T(z) j^A(w) = \frac{k + c_V}{\kappa} \left( \frac{j^A(w)}{(z-w)^2} + \frac{\partial j^A(w)}{z-w} \right) , \tag{1.76}$$

which therefore implies that the normalization constant is equal to

$$\kappa = k + c_V \,. \tag{1.77}$$

### 1.2.9 $D$-branes and black branes

One can perform a consistent truncation of type-II supergravity: set most of the fields to zero in such a way that the equations of motion are satisfied. Let us consider the following consistent truncation of type-IIB supergravity: the only non-vanishing fields are the metric $g_{\mu\nu}$, the R-R field $C_4$ with field strength $F_5 = dC_4$, and the dilaton $\phi$. The action is

$$S = \frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{-g} \left[ e^{-2\phi}(R + 4(\partial\phi)^2) - \frac{1}{4}|F_5|^2 \right] , \tag{1.78}$$

where $|F_5|^2 = \frac{1}{5!} F_{\mu_1 \ldots \mu_5} F^{\mu_1 \ldots \mu_5}$.

Notice that the action is written for the string frame metric $g_{\mu\nu}$, which is related to the Einstein frame metric $G_{\mu\nu}$ by $g_{\mu\nu} = e^{\phi/2} G_{\mu\nu}$. The Einstein frame is defined so that the Ricci scalar term in the action is not multiplied by an exponent of the dilaton, $L = \sqrt{-G} R_G + \ldots$. Suppose the dilaton is constant. It defines a closed string coupling constant, $g_s = e^{\phi}$. In the string frame the dimensions are measured in units of the string length, $\ell_s$, and in the Einstein frame these are measured in the units of the Planck length, $\ell_p$. We conclude that $1/\ell_s^2 = g_s^{1/2}/\ell_p^2$, and therefore

$$\ell_p = g_s^{1/4} \ell_s \, . \tag{1.79}$$

The supergravity equations of motion following from the action (1.78) admit the 3-brane solution,

$$\begin{aligned} ds^2 &= -D_+(r) D_-(r)^{-1/2} dt^2 + D_-(r)^{1/2}(dx^2 + dy^2 + dz^2) \\ &\quad + \frac{dr^2}{D_+(r) D_-(r)} + r^2 d\Omega_5^2 \, , \\ D_\pm(r) &= 1 - \left(\frac{r_\pm}{r}\right)^4 , \quad e^{\phi} = g_s \, , \quad F_5 = Q(\omega_5 + \star\omega_5) \, , \end{aligned} \tag{1.80}$$

where $g_s$ is a constant, and $\omega_5$ is the volume form of the unit five-sphere $S^5$.

The metric (1.80) is a generalization of a black hole metric to a higher-dimensional space. It describes a black brane, extended in $R^3$ space with $(x, y, z)$ coordinates. This is generally a non-extremal black 3-brane, characterized by two radii parameters $r_\pm$. The condition of absence of a naked singularity at $r = 0$ demands $r_+ \geq r_-$.

We have two kinds of 3-branes now: the black 3-brane (1.80) and the $D3$-brane. They both are coupled to the R-R field $C_4$. The $D3$-brane is a supersymmetric BPS object: both its tension and R-R charge are equal to $T_{D3}$. While the black 3-brane (1.80) is generally not supersymmetric. When the black 3-brane is supersymmetric it becomes equivalent to the $D3$-brane. Let us see how the supersymmetry constraint on the black 3-brane comes about. The black 3-brane metric is supported by a flux of $F_5$ through the five-sphere surrounding the 3-brane in ten space-time dimensions. We conclude that the 3-brane carries $N$ units of charge of the R-R field $C_4$, with $N = Q\,\mathrm{vol}(S^5)$. The mass and the R-R charge of the black brane (1.80) per unit volume of $R^3$ are given by

$$T_3 = \frac{1}{4(2\pi)^4 g_s^2 \ell_s^8 d_3}(5r_+^4 - r_-^4)\,, \quad N = \frac{(r_+ r_-)^2}{d_3 g_s \ell_s^4}\,, \tag{1.81}$$

where $d_3$ is a numerical factor. The condition $r_+ \geq r_-$ therefore becomes

$$T_3 \geq N T_{D3}\,, \quad T_{D3} = \frac{1}{(2\pi)^3 g_s \ell_s^4}\,, \tag{1.82}$$

where $T_{D3}$ is the tension of the $D3$-brane. Equation (1.82) is precisely a supersymmetry BPS constraint: a single 3-brane (with $N = 1$) is supersymmetric if its tension $T_3$ is equal to its R-R charge $T_{D3}$.

We start the next section with a discussion of the extremal 3-brane metric and its near-horizon limit. We will also return to the consideration of the non-extremal 3-brane in the context of configurations with finite temperature.

## 1.3 Holographic correspondence

In this section we are going to review the holographic AdS/CFT correspondence: the equivalence between string theory on Anti-de Sitter (AdS) space and gauge field theory on the boundary of AdS space. We are also going to review the holographic correspondence between Little String Theory and closed string theory in the 'cigar' geometry.

### 1.3.1 The near-horizon limit

At the end of the previous section we derived the black three-brane solution (1.80) of type-IIB supergravity. Its metric has two horizons, and in the case when the horizons coincide we obtain an extremal black three-brane equivalent to the $D3$-brane of type-IIB superstring theory.

Let us re-write the metric of the $D3$-brane in the following form,

$$ds^2 = H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(dr^2 + r^2 d\Omega_5^2)\,, \quad H(r) = 1 + \left(\frac{R}{r}\right)^4\,, \tag{1.83}$$

with the horizon located at $r = 0$, and the scale parameter defined as

$$R^4 = 4\pi g_s N \alpha'^2\,, \tag{1.84}$$

where $N$ is the R-R charge of the $D3$-brane, which is actually the number of coincident $D3$-branes. In the near-horizon limit, $r/R \ll 1$, we obtain

$$ds^2 = \left(\frac{r}{R}\right)^2 \eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_5^2\,. \tag{1.85}$$

This is the metric of the $AdS_5 \times S^5$ geometry.

### 1.3.2 AdS space and its symmetries

The $AdS_5$ space can be described as a surface in a six-dimensional flat space (with coordinates $t_\mu$), with $(-,-,+,+,+,+)$ signature. The embedding is given by the equation

$$-t_1^2 - t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 = -R^2\,. \tag{1.86}$$

The parameter $R$ is called the AdS scale. The group of transformations which leaves the surface (1.86) invariant is $SO(2,4)$.

Combining it with the symmetry group $SO(6)$ of the five-sphere we conclude that the subgroup of ten-dimensional diffeomorphisms which leaves the $AdS_5 \times S^5$ invariant is $SO(2,4) \times SO(6)$. Type-IIB superstring theory in the $AdS_5 \times S^5$ geometry is also invariant under 32 supersymmetries, the same amount as in ten-dimensional Minkowski space-time. The conserved supercharges split into $(\mathbf{4},\mathbf{4}) \oplus (\bar{\mathbf{4}},\bar{\mathbf{4}})$ under the covering bosonic symmetry group $SU(2,2) \times SU(4)$. The total supersymmetry group is therefore $PSU(2,2|4)$.

Let us now focus more on AdS space. Consider $AdS_{d+1}$ space embedded into $d+2$-dimensional space,

$$-t_1^2 - t_2^2 + \sum_{i=1}^{d} y_i^2 = -R^2\,. \tag{1.87}$$

We can solve the equation (1.87) by $d+1$ independent coordinates, $(x_0, x_a, z)$,

$$t_2 = R\frac{x_0}{z}\,, \qquad t_1 = \frac{z}{2}\left(\frac{R^2}{z^2} + 1 + \frac{x_a^2 - x_0^2}{z^2}\right)\,,$$
$$y_a = R\frac{x_a}{z}\,, \qquad y_d = \frac{z}{2}\left(\frac{R^2}{z^2} - 1 - \frac{x_a^2 - x_0^2}{z^2}\right)\,. \tag{1.88}$$

The coordinates $(x_0, x_a, z)$ parametrize half of the $AdS_{d+1}$ space, known as the Poincare patch. The boundary of AdS is located at $z = 0$. The metric is given by

$$ds^2 = R^2 \frac{-dx_0^2 + dx_1^2 + \cdots + dx_{d-1}^2 + dz^2}{z^2}\,. \tag{1.89}$$

Notice the presence of the horizon, $g_{tt}|_{z=\infty} = 0$, at $z = \infty$. This is the Poincare horizon originating from the choice of coordinates (1.88). To compare with (1.85) we make a change of the radial coordinate, $z = R^2/r$.

### 1.3.3 $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

The $AdS_5 \times S^5$ geometry appeared as the near-horizon geometry of $N$ coincident $D3$-branes. Let us now look at the world-volume theory of the $D3$-branes. In the previous section it was explained that the low-energy theory of a $Dp$-brane is given by the DBI and CS actions for the $U(1)$ gauge field, scalars and fermions on the $p+1$-dimensional world-volume. Keeping only the lowest order terms in the $\alpha'$ expansion, we obtain the Lagrangian of supersymmetric Yang-Mills theory with $U(1)$ gauge group. For the $N$ coincident $Dp$-branes we obtain non-abelian supersymmetric Yang-Mills theory with gauge group $U(N)$.

In the case of $D3$-branes there are six scalars $\Phi^I$, $I = 1, \ldots, 6$, describing the fluctuations of the $D3$-branes in the six-dimensional transverse space. Together with the two transverse polarizations of the vector field on the four-dimensional world-volume, the total number of physical bosonic d.o.f. is therefore equal to eight and matches the number of physical d.o.f. of four fermions $\psi^i$, $i = 1, 2, 3, 4$ (recall that Weyl fermion in 4d has four independent components). All the fields live in the adjoint representation of the gauge group $U(N)$.

The $U(1)$ subgroup decouples from the rest of the $U(N)$ group. The resulting low-energy theory on the world-volume of $D3$-branes is $\mathcal{N} = 4$

supersymmetric $SU(N)$ Yang-Mills theory (SYM), with the Lagrangian schematically given by

$$L_{SYM} = -\frac{1}{4g_{YM}^2} \mathrm{Tr} \left( F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^I D^\mu \Phi^I + \bar{\psi}^i \gamma^\mu D_\mu \psi^i + [\Phi^I, \Phi^J]^2 + \dots \right) , \tag{1.90}$$

where dots denote all extra interaction terms required by supersymmetry. This is a maximally supersymmetric four-dimensional gauge theory.

In four dimensions gauge theory is classically conformally invariant. Quantum corrections generally spoil conformal invariance creating renormalization group flow. However in the case of $\mathcal{N} = 4$ SYM theory this turns out not to be the case; the Lagrangian (1.90) is exactly conformally invariant at quantum level. The conformal symmetry group in four dimensional space-time is $SO(2,4)$.

The R-symmetry subgroup of the $\mathcal{N} = 4$ supersymmetry group is $SU(4)$. The fermions $\psi^i$ live in the fundamental representation $\mathbf{4}$ of $SU(4)$, and the six scalars $\Phi^I$ are rotated by $SO(6) \simeq SU(4)$. Therefore the covering bosonic symmetry group of $\mathcal{N} = 4$ SYM is $SU(2,2) \times SU(4)$. The fermionic symmetry generators consist of four supersymmetry generators and four super-conformal generators. The latter appear in commutators of supersymmetry generators with special conformal generators. In total there are 32 conserved fermionic charges. The supersymmetry group is $PSU(2,2|4)$.

### 1.3.4 Large $N$ limit

It turns out that when the number of colors $N$ is sent to infinity, the $SU(N)$ gauge theory simplifies. We need to consider the 't Hooft coupling $\lambda = g_{YM}^2 N$ instead of the Yang-Mills coupling $g_{YM}$: as $N$ is varied $g_{YM}$ is varied accordingly so that $\lambda$ remains unchanged. The reason for such rearrangement is that we want to obtain a sensible large $N$ limit of Feynman diagrams. Consider for example the gluon one-loop correction to the gluon propagator. Each of the two three-gluon vertices contributes the factor of $g_{YM}$, and the loop contributes the factor of $N$. The diagram is therefore proportional to $g_{YM}^2 N$. It describes the lowest order term in the renormalization group flow of the YM coupling,

$$\frac{dg_{YM}}{d\log M} = b_0 g_{YM}^3 N + \dots \quad \Rightarrow \quad \frac{d\lambda}{d\log M} = 2b_0\lambda^2 + \dots . \tag{1.91}$$

and it has a smooth large $N$ limit if we keep $\lambda = g_{YM}^2 N$ fixed as $N$ is sent to infinity.

In the large $N$ limit, non-planar Feynman diagrams, that is the diagrams which cannot be drawn on a plane (in the double-line notation), become sub-leading [5]. We refer the reader to chapter 2 of [6] for a nice review of the large-$N$ limit of gauge theories.

In the large $N$ limit, the low-energy gauge-singlet degrees of freedom are decoupled from each other. This is usually referred to as large-$N$ factorization. The diagrams with two single-trace vertices can be either connected, when the vertices are linked to each other by internal lines, or disconnected, when the vertices are closed up on themselves. The latter diagrams are leading in the large-$N$ limit. Nevertheless the theory is still non-trivial. For instance in the large-$N$ QCD at low energies, one has free mesons. But the spectrum of mesons as well as the scaling dimensions of the meson operators are unknown, and the conventional derivation of these quantities by QFT means dealing with strongly coupled (and confining) dynamics of quarks within mesons. The spectrum of mesons can be read off from the poles of the two-point functions of the baryon current operators. In the large $N$ limit the computation of such two-point functions requires summation of planar diagrams, which is an ill-defined procedure at strong coupling.

### 1.3.5 AdS/CFT correspondence

We have demonstrated in the previous subsections that the symmetry group $PSU(2,2|4)$ of type-IIB superstring theory on $AdS_5 \times S^5$ is the same as the symmetry group of $\mathcal{N} = 4$ SYM theory. It turns out that this matching is not accidental. The $\mathcal{N} = 4$ SYM theory is the low-energy theory of massless modes of open strings attached with both ends to the $D3$-branes. According to the AdS/CFT correspondence this theory is exactly equivalent to the type-IIB superstring theory in the near-horizon $AdS_5 \times S^5$ background created by the $D3$-branes [7]. One can in fact perform a reduction on the five-sphere $S^5$. This correspondence is holographic: the $\mathcal{N} = 4$ SYM theory lives in the four-dimensional Minkowski space-time, while closed type-IIB strings live in the five-dimensional bulk space, $AdS_5$. The field theory can be referred to as living on the four-dimensional boundary of the $AdS_5$ space.

The bulk side of the duality is gravitational, since gravity fields are the lowest (massless) modes of the closed string theory; the boundary side is non-gravitational, because gravity is not a part of the open string spectrum. The gauge coupling constant, $g_{YM}$, is the open string coupling constant, related to the closed string coupling constant, $g_s$, by the equation

$$g_{YM}^2 = 4\pi g_s\,. \tag{1.92}$$

Combining this equation with the equation (1.84) for the AdS scale we obtain (in this section string length is defined as $\ell_s = \sqrt{\alpha'}$)

$$\frac{\ell_s}{R} = \frac{1}{\lambda^{1/4}}\,. \tag{1.93}$$

The closed string excitations in the $AdS_5 \times S^5$ bulk can be neglected if

$$\frac{\ell_s}{R} \ll 1 \quad \Rightarrow \quad \lambda \gg 1\,. \tag{1.94}$$

If the condition (1.94) is satisfied then the the bulk dynamics is well approximated by type-IIB supergravity.

Due to the equation (1.79) for the relation between the string scale and the Planck scale, we conclude that quantum gravity effects are negligible provided

$$\frac{\ell_p}{R} = \frac{1}{(4\pi N)^{1/4}} \ll 1 \quad \Rightarrow \quad N \gg 1\,. \tag{1.95}$$

This is a consequence of the fact that at fixed $\lambda$ and large $N$ the closed string coupling constant $g_s$ is small, and the bulk theory is classical.

We conclude that the large-$N$ limit of strongly coupled $\mathcal{N} = 4$ SYM theory is dual to classical supergravity theory in the $AdS_5 \times S^5$ space.

### 1.3.6 Less supersymmetry, non-conformal field theories

We have reviewed the holographic duality between $\mathcal{N} = 4$ SYM theory and type-IIB string theory on $AdS_5 \times S^5$. The set of holographic dualities is not exhausted by this example. One can break a fraction or all of supersymmetries. One can consider holographic descriptions of non-conformal field theories. For example, one can turn on a finite temperature and/or chemical potential in the field theory, breaking supersymmetry and conformal invariance. The dual bulk geometry in this particular example is a charged black hole in AdS.

In the following subsections of this chapter we are going to review the holographic correspondence in the most general way, with a QFT on the boundary of asymptotically AdS space dual to a classical gravitational theory in the bulk. Then we proceed to the holographic description of Little String Theory.

### 1.3.7 Gubser-Klebanov-Polyakov-Witten formula

Consider quantum gauge field theory with the gauge-singlet operators $\mathcal{O}^I$. For example, one can deal with a charge current, fermionic bi-linear operator, baryon operator, glueball, stress-energy tensor, etc. Holographic duality maps these boundary QFT operators to the fields $\phi^I$ defined in the bulk of AdS space. One can ask a QFT question: what is the $n$-point function $\langle \mathcal{O}^{I_1} \ldots \mathcal{O}^{I_n} \rangle$ equal to? In a strongly interacting system this question is generally impossible to find an answer to by conventional QFT means. One generally looks for a generating functional, $W[J^I]$, which depends on the sources $J^I$. In the Euclidean set-up it is defined as

$$e^{-W[J_I]} = \langle e^{J_I \mathcal{O}^I} \rangle_{QFT} \,, \tag{1.96}$$

where on the r.h.s. of (1.96) we have a path integral of the QFT with the operators $\mathcal{O}^I$ sourced by the external currents $J^I$. The $n$-point function is then

$$\langle \mathcal{O}^{I_1} \ldots \mathcal{O}^{I_n} \rangle = \frac{\partial^n}{\partial J_{I_1} \ldots \partial J_{I_n}} e^{-W[J_I]} \,. \tag{1.97}$$

The Gubser-Klebanov-Polyakov-Witten formula [8, 9] gives the holographic prescription for computation of the generating functional $W$,

$$e^{-W[J_I]} = Z_{string}|_{\phi_I(z=0)=J_I} \,, \tag{1.98}$$

where $Z_{string}$ on the r.h.s. of (1.98) is the string partition function in the bulk, with the bulk fields $\phi_I$ fixed at the boundary $z = 0$ to the values of the sources $J_I$ of the dual QFT operators $\mathcal{O}^I$. When the QFT is strongly coupled, due to (1.94), the string partition function can be approximated by the supergravity partition function. If the number of colors $N$ is large supergravity is classical, see (1.95), and one can use a saddle point approximation,

$$Z_{string}{=}Z_{sugra}{=}e^{-S_{sugra}} \quad \Rightarrow \quad W[J_I] = S_{sugra}[\phi_I(z{=}0){=}J_I] \,. \tag{1.99}$$

In (1.99) the supergravity action is evaluated on the classical solution to the bulk equations of motion.

### 1.3.8 Probe branes and flavor

Let us add matter fields living in the fundamental representation of the gauge group. Ultimately it would be useful to apply holographic methods to QCD-like models, and this is why we have to know how to add quarks to the system. Quarks have a color and a flavor. These are realized as open strings attached with one end to $N$ coincident color $D3$-branes and with the other end to $F$ coincident flavor $Dp$-branes [10]. The lowest excitation modes of such strings are quarks, and they live in the fundamental representation of the $U(N)$ color group and the $U(F)$ flavor group. Taking $F/N \ll 1$ one can consider flavor branes as probes in the $AdS_5 \times S^5$ background. World-volume $U(F)$ degrees of freedom on the probe $Dp$-branes decouple from the $U(N)$ adjoint gauge fields and fundamental matter fields on the $D3$-branes.

Now we have a global flavor $U(F)$ symmetry on the field theory side of the duality. According to the AdS/CFT correspondence, a global symmetry in the QFT is mapped to a local symmetry in the bulk. In the case at hand the conserved $U(F)$ Noether currents are mapped to the $U(F)$ gauge d.o.f. on the $Dp$-brane world-volume.

In subsection 1.2.6 we discussed the system of intersecting branes and concluded that the interesting non-trivial cases are $D3/Dp$-brane systems with $p = 5, 7$, with three or four dimensional intersections. We also discussed the conditions for the non-broken supersymmetry in such systems. We are going to use these results in chapter 2, where we study probe brane matter at finite baryon density, strongly coupled to the $\mathcal{N} = 4$ gauge d.o.f. In that case the global (baryon) symmetry is represented holographically by the $U(1)$ gauge field on the world-volume of the probe brane. A finite density of bound states of strongly coupled quarks is dual to a non-trivial background $U(1)$ gauge field. Fluctuations of density are represented by fluctuations of this gauge field.

### 1.3.9 Finite temperature and chemical potential, thermodynamics

Suppose we have a four-dimensional gauge theory. In the IR it is described by the effective action, $W$, obtained in the Wilsonian framework by path integration over the high-energy modes of the fields. In subsection 1.3.7 we described the prescription of the AdS/CFT correspondence for computation of the effective action.

Let us turn on a temperature, $T$. The effective action is then replaced by the free energy, $\mathcal{F} = E - T\mathcal{S}$, where $E$ is the energy and $\mathcal{S}$ is the entropy. We proceed further and turn on chemical potentials for some of the conserved charges. For example, we can consider a finite density of baryon matter in the IR. Then the free energy is replaced by the grand potential, $\Omega = E - T\mathcal{S} - \mu N$, where $N$ is the number of baryons and $\mu$ is the chemical potential. In this subsection we are going to focus on this general case.

The AdS/CFT prescription for the grand potential of the field theory at finite temperature and chemical potential is a straightforward generalization of the GKPW formula (1.98). First of all, the effective action $W$ is replaced by the grand potential $\Omega$. Now, as the temperature in the CFT is turned on, the dual $AdS_5$ geometry gets replaced with the Schwarzshild black hole in $AdS_5$ space [11] (which is a dominant solution when the temperature is large enough). A finite chemical potential of a conserved charge is described holographically by a non-trivial background profile of the corresponding gauge field in the bulk. As a result we obtain a charged black hole in AdS space.

Suppose the boundary field theory is strongly coupled and the number of colors is large. We can use a saddle point approximation equating the grand potential with the bulk on-shell classical regularized action $S$:

$$TS_{on-shell} = E - T\mathcal{S} - \mu N \,. \tag{1.100}$$

### 1.3.10 Holographic description of Little String Theory

One way to introduce Little String Theory (LST) [12, 13] is to consider the low-energy theory on the world-volume of $N$ coincident $NS5$-branes at a fixed energy scale and vanishing string coupling. Recall that $NS5$-branes arise both in type-IIA and type-IIB superstring theory as electro-magnetic duals of the superstring. A superstring couples electrically to the massless NS-NS two-form field $B_{\mu\nu}$, see (1.61). In nine spatial dimensions, a string is surrounded by seven-sphere. The charge of a string w.r.t. to the $B$ field is equal to the flux of the Hodge dual of the field strength $H = dB$ through the seven-sphere. As a consistency check notice that the Hodge dual $\star H$ in ten dimensions is a seven-form.

Similarly, recall that a $Dp$-brane couples electrically to the $C_{p+1}$ R-R field and magnetically to the $C_{7-p}$ field. The object which couples magnetically to the $B$ field is an NS5-brane. It is surrounded by a three-sphere, and its magnetic charge w.r.t. to the $B$-field is equal to the flux of $H$ through this three-sphere. In type-IIB superstring theory, fluctuations of an $NS5$-brane are determined by a $D1$-brane attached to it by both ends, in type-IIA superstring theory a $D2$-brane attaches to an $NS5$-brane. S-duality between a string and a $D1$-brane is consistent with S-duality between an $NS5$-brane and a $D5$-brane. The S-dual of a string attached to a $D5$-brane is therefore a $D1$-brane attached to an $NS5$-brane.

The low-energy d.o.f. on an $NS5$-brane form a $U(N)$ gauge supermultiplet on a six-dimensional world-volume. The coupling constant is given by $g = \ell_s$; the theory is formulated at a fixed energy scale . Under the RG flow, the coupling constant of the six-dimensional theory grows in the UV. Of course in the UV more of the string d.o.f. should be added to the theory.

One can study LST holographically. We refer the reader to [14] for an extensive exposition of the subject and restrict here to a general outline. The background created by $N$ coincident $NS5$-branes has the geometry $R^{5,1} \times R^{\phi} \times S^3$. Here $R^{5,1}$ is the world-volume of an $NS5$-brane, $R^{\phi}$ is the radial (bulk) direction and $S^3$ is the sphere surrounding the $NS5$-branes, with $N$ units of the $B$-field flux threading through it. The radius of the three-sphere is $R = \sqrt{N}\ell_s$, and therefore string excitations in the bulk are suppressed when $N$ is large. String theory on $S^3$ is given by the $SU(2)$ WZW model at level $N$. There is a background dilaton field depending linearly on the bulk radial coordinate $\phi$.

The statement is that LST is holographically dual to closed string theory in the background of $N$ coincident $NS5$-branes. In the double scaling limit this background is $R^5 \times \frac{SL(2,R)_N}{U(1)} \times SU(2)_N$, where $SL(2,R)_N/U(1)$ is a two-dimensional 'cigar' geometry with the linear dilaton [15]. The time coordinate is periodic, the corresponding temperature is $T = (2\pi\sqrt{N})^{-1}$. A string on $SL(2,R)_N/U(1)$ is described by the gauged WZW model on this coset space.

In chapter 4 we generalize this set-up to the situation of a non-vanishing charge density in the LST. To be precise, we do not know what is the field theory side of the holographic duality presented in chapter 4. Instead we study the bulk side of the duality, which is the gauged WZW model on $\frac{SL(2,R)_N \times U(1)}{U(1)}$. The classical geometry is a two-dimensional charged black hole, which is therefore dual to a field theory at finite charge density.

## 1.4 This thesis

Chapters 2, 3 and 4 of this thesis are based on the original research papers [16], [17] and [18] respectively, which I have written in collaboration with Dr. Andrei Parnachev and Prof. Dr. Jan Zaanen.

### 1.4.1 Chapter 2

In this chapter we study correlators of the global $U(1)$ currents in holographic models which involve $\mathcal{N} = 4$ SYM coupled to finite density matter in the probe brane sector. We find the spectral density associated with the longitudinal response to be exhausted by the zero sound pole and argue that this could be consistent with the behavior of a Fermi liquid with vanishing Fermi velocity. However the transversal response shows an unusual momentum independent behavior. Inclusion of magnetic field leads to a gap in the dispersion relation for the zero sound mode propagating in the plane of magnetic field. For small values of the magnetic field $B$, the gap in the spectrum scales linearly with $B$, which is consistent with Kohn's theorem for nonrelativistic fermions with pairwise interaction. We do not find signatures of multiple Landau levels expected in Landau Fermi liquid theory. We also consider the influence of generic higher derivative corrections on the form of the spectral function.

### 1.4.2 Chapter 3

In this chapter we investigate some phenomenological aspects of the holographic models based on the tachyon Dirac-Born-Infeld action in the AdS space-time. These holographic theories model strongly interacting fermions and feature dynamical mass generation and symmetry breaking. We show that they can be viewed as models of holographic walking technicolor and compute the Peskin-Takeuchi S-parameter and the masses of the lightest technimesons for a variety of tachyon potentials. We also investigate the phase structure at finite temperature and charge density. Finally, we comment on the holographic Wilsonian RG in the context of holographic tachyon DBI models.

### 1.4.3 Chapter 4

In this chapter we consider an exactly solvable worldsheet string theory in the background of a black brane with a gauge field flux. Holographically, such a system can be interpreted as a field theory with a finite number of degrees of freedom at finite temperature and density. This is to be contrasted with more conventional holographic models which involve gravity in the bulk and possess infinite number of degrees of freedom and mean field critical exponents. We construct closed string vertex operators which holographically represent the $U(1)$ gauge field and the stress energy tensor and compute their two-point functions. At finite temperature and vanishing charge density the low energy excitations are described by hydrodynamics. As the density is raised, the system behaves like a sum of two noninteracting fluids. We find low-energy excitations in the shear and sound channels of each fluid.

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