Strings and AdS/CFT at finite density

Goykhman, M.

Author: Goykhman, Mikhail
Title: Strings and AdS/CFT at finite density
Issue Date: 2014-06-24

Strings and AdS/CFT at finite density

Proefschrift

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus
Prof. Mr. C.J.J.M. Stolker,
volgens besluit van het College voor Promoties
te verdedigen op dinsdag 24 juni 2014
klokke 10.00 uur

door

Mikhail Goykhman

geboren te Sumgait, Sovjet-Unie in 1989

Promotiecommissie

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Casimir PhD series Delft-Leiden 2014-16
ISBN 978-90-8593-191-1

Dit werk maakt deel uit van het onderzoekprogramma van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).

This work is part of the research programme of the Netherlands Organisation for Scientific Research (NWO).

To my parents

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Chapter 1

Introduction

1.1 Preface

This thesis is devoted to applications of string theoretic methods of holography to strongly coupled phases of quantum field theories. The general definition of holography states that there is an exact equivalence between a closed string theory on a manifold and a quantum field theory in the asymptotic region (on the boundary) of this manifold. Usually the manifold is an asymptotically Anti-de Sitter space, and the QFT lives on the boundary of this asymptotically AdS space. In the original example of the holographic correspondence, the QFT on the boundary is $\mathcal{N} = 4$ supersymmetric conformal Yang-Mills theory. Due to this example, the holographic duality is called the AdS/CFT correspondence.

If the gauge quantum field theory on the boundary of AdS is in a strongly coupled phase then the dual sting theory in the bulk of AdS can be approximated by its low-energy limit, supergravity. Moreover, when the number of colors in the QFT is taken to infinity, the dual supergravity in AdS is classical. Therefore the AdS/CFT correspondence facilitates a powerful approach to difficult questions of infra-red physics of quantum field theories, like confinement and chiral symmetry breaking in quantum chromodynamics. In practice there are still technical limitations, as well as principal restrictions, on the kinds of field theories which can be studied holographically. As for now there exists no holographic description of real-world QCD. However the systems which are qualitatively close to QCD can be successfully dealt with by the methods of AdS/CFT. In fact, in a large number of situations the AdS/CFT correspondence is the only available analytical tool.

Theoretical high-energy physics studies the structure of the world on the fundamental, microscopic level. At high energies the gravitational force becomes strongly coupled and quantum gravity effects cannot be neglected. The scale at which quantum gravity effects become important is known as the Planck scale. This is the scale at which the Standard Model of particle physics and General Relativity break down. The theory which incorporates SM and GR and provides a successful ultra-violet completion of these theories is known as string theory (although as this thesis is being written difficult phenomenological questions remain to be answered). The UV completion of the theories of fundamental interactions is an important and complicated research area.

A no less complicated set of problems exists in the opposite regime on an energy scale, infra-red phases of quantum field theories. Consider, for example, a QCD-like model. At high energies we have a system of weakly coupled quarks and gluons interacting by an exchange of gluons and self-interaction of gluons. It is described by a gauge-invariant matter action and a non-abelian Yang-Mills action. The processes involving scattering of quarks and gluons are described by Feynman diagrams, and since the coupling constant is small we can get accurate predictions perturbatively, by accounting for just the leading loop corrections. As we move towards low energies, due to the renormalization group flow the gauge coupling constant grows. In fact the IR phase can have a qualitatively different interaction of quarks than the UV phase, as the Coulomb force of the UV regime disappears and instead, in the IR, quarks are confined by the flux tube, with the force growing proportionally to the separation between quarks. The perturbative approach of Feynman diagrams is completely useless for the description of these phenomena. In fact it is not even correct to talk about quarks and gluons in the IR, the fundamental degrees of freedom are glueballs, mesons and baryons, within which gluons and quarks are confined.

This is where the tools of the AdS/CFT correspondence can become useful. The IR phases of QCD-like systems at strong coupling and large number of colors are dual to classical supergravity in an asymptotically AdS space. Systems of condensed matter physics can also be qualitatively described by the gravitational AdS physics. These are the kinds of models which we have studied holographically in this thesis.

Asymptotically AdS space is not the only possible bulk geometry of a holographic dual to a quantum field theory. Another kind of holographic correspondence which we have considered in this thesis is a duality between Little String Theory and a gauged Wess-Zumino-Witten model in a charged black brane background. The advantage of this holographic correspondence is that one does not have to resort to the limit of the supergravity approximation, because string theory in this background is exactly solvable. In this case, it means that the QFT can be taken at finite number of degrees of freedom.

Such an example of holographic duality can be ascribed to the realm of top-down holography. The term 'top-down' in general means that we know the string origin of the bulk degrees of freedom we are dealing with, and the their effective action appears as a low-energy limit of string theory. The original example of the AdS/CFT correspondence, the duality between type-IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory on the boundary of AdS_5, is another example of top-down holography. A different way to apply holography is known as the bottom-up approach, and it assumes a generic form of the action for some of the bulk fields. Its advantage is that it allows one to get a quick perspective on properties of the dual field theory. Its disadvantage is that such a model can turn out to be outside of the realm of string theory and therefore become inconsistent. Another disadvantage is that one does not know what the QFT degrees of freedom dual to the bulk fields described by a bottom-up action are. In this thesis we have used both top-down and bottom-up methods.

1.2 String theory

In this chapter we are going to review some basic string theory which will be useful for this thesis. The entire content of this chapter is a review of textbook material. String theoretic basics are presented for the purpose of assisting the understanding of chapter 4. We refer the reader to the references [1–4] for a more complete exposition of the topics discussed here.

1.2.1 The Polyakov action and two-dimensional conformal field theory

The basic object of string theory is an extended one-dimensional relativistic string. As a string moves it sweeps a two-dimensional surface, which is called a world-sheet. Let us parametrize a world-sheet by a time-like coordinate τ and a space-like coordinate σ. We are using the convention $\sigma \in [0, \pi]$. String can be open or closed. An open string has two ends, located at $\sigma = 0$ and $\sigma = \pi$. In the case of a closed string these two ends are identified.

Consider the field $X^\mu(\tau, \sigma)$ which describes an embedding of a string into a d-dimensional space-time, $\mu = 0, \ldots, d-1$. It is called a target space-time. The action for the field X^μ in a flat target space-time is the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \,, \tag{1.1}$$

where the index μ is lowered with the flat space-time metric $\eta_{\mu\nu}$. This is the action of a free bosonic string theory. In (1.1) we have $\alpha' = \ell_s^2/2$, where ℓ_s is the string length. In what follows we set $\ell_s = 1$. A world-sheet has a two-dimensional metric, $h_{\alpha\beta}$, $\alpha, \beta = \tau, \sigma$, with three independent components. We can set it to the flat metric, $\eta_{\alpha\beta}$, by two-dimensional diffeomorphism transformations, $\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta)$, and a Weyl rescaling, $h_{\alpha\beta} \to e^{2\omega(\tau,\sigma)} h_{\alpha\beta}$. Both are symmetries of the action (1.1). The choice $h_{\alpha\beta} = \eta_{\alpha\beta}$ for the world-sheet metric is called the conformal gauge.

Perform now a Wick rotation $\tau \to -i\tau$ and introduce a complex coordinate $z = e^{\tau + i\sigma}$ on an open string world-sheet, and $z = e^{2(\tau + i\sigma)}$ on a closed string world-sheet. The ends of an open string, $\sigma = 0, \pi$, are then parametrized by $z = e^\tau$ and $z = -e^\tau$. Now a world-sheet is a two-dimensional Riemann surface with Euclidean signature. The Polyakov action (1.1) in conformal gauge is then

$$S_P = \frac{1}{2\pi} \int d^2z \, \partial X^\mu \bar{\partial} X_\mu \,, \tag{1.2}$$

where ∂ denotes a derivative w.r.t. z and $\bar{\partial}$ denotes a derivative w.r.t. $\bar{z}$. The classical wave equations of motion for the fields X^μ are

$$\partial\bar{\partial} \, X^\mu(z, \bar{z}) = 0 \,, \tag{1.3}$$

and must be accompanied by boundary conditions. If the string is closed then the boundary conditions are satisfied due to the periodicity $X^\mu|_{\sigma=0} = X^\mu|_{\sigma=\pi}$. In the case of an open string we must impose the following boundary conditions at the ends of the string,

$$\delta X^\mu \, \left(\bar{z}\bar{\partial} - z\partial\right) X_\mu|_{z=\pm e^\tau} = 0 \,\,, \tag{1.4}$$

which can be resolved in two different ways. These are called Neumann and Dirichlet boundary conditions:

$$\begin{aligned} \textbf{Neumann}: &\qquad (\bar{z}\bar{\partial} - z\partial)\, X^\mu|_{z=\pm e^\tau} = 0\,, \\ \textbf{Dirichlet}: &\qquad \delta X^\mu|_{z=\pm e^\tau} = 0\,. \end{aligned} \tag{1.5}$$

Neumann boundary conditions do not break the translational symmetry of the flat target space-time: they are invariant under the replacement $X^\mu \to X^\mu + a^\mu$ with any constant a^μ. On the other hand, Dirichlet boundary conditions mean that the end of the string is held at a fixed, distinguished, point in a given space direction and it breaks translational symmetry in that direction. The physical reason for such a boundary condition is that the string ends on some object. This object, which is heavy and which is localized at a certain value of the x^μ coordinate, is called a D-brane. The dynamics of open strings determine the fluctuations of a D-branes. We will get back to D-branes later in this chapter.

Besides the equations of motion (1.3) and the boundary conditions (1.5) one has to impose Virasoro constraints. These constraints originate as follows. Recall that the action (1.2) is written in the conformal gauge, $h_{\alpha\beta} = \delta_{\alpha\beta}$. It is invariant under all two-dimensional coordinate transformations which keep a flat world-sheet metric conformally flat, $h_{\alpha\beta} = e^{2\omega(z,\bar{z})}\delta_{\alpha\beta}$, that is flat up to an overall factor $e^{2\omega(z,\bar{z})}$. Such transformations are called conformal transformations. They are generated by the stress-energy tensor. In two dimensions, the stress-energy tensor has three independent components. Due to conformal invariance, the stress-energy tensor is classically traceless, and therefore two independent components remain

$$T(z) = -\partial X^\mu(z)\partial X_\mu(z)\,, \qquad \tilde{T}(\bar{z}) = -\bar{\partial} X^\mu(\bar{z})\bar{\partial} X_\mu(\bar{z})\,. \tag{1.6}$$

String theory is a two-dimensional conformal field theory. The classical requirement of conformal invariance of states of a string boils down to demanding the vanishing of the stress-energy tensor, $T(z) = 0$, $\tilde{T}(\bar{z}) = 0$. These conditions are refined in the quantum theory, and are called Virasoro constraints.

Let us proceed to a first quantization of string theory. In chapter 4 we are going to consider a string moving in a space which is a direct product of flat space and coset space. In this section, for simplicity, let us refrain to string moving in a flat space-time. Due to the translational invariance of flat space-time we have left-moving and right-moving currents,

$$j^\mu(z) = i\sqrt{2}\,\partial X^\mu(z)\,, \quad \tilde{j}^\mu(\bar{z}) = i\sqrt{2}\,\bar{\partial} X^\mu(\bar{z})\,, \tag{1.7}$$

which due to the equations of motion (1.3) are conserved separately. For a classical string products like $j(z_1)j(z_2)$ are non-singular for any $z_1 - z_2$. If we quantize the string, these become singular as z_1 approaches z_2:

$$j^\mu(z_1)j^\nu(z_2) = :j^\mu(z_1)j^\nu(z_2): + \langle j^\mu(z_1)j^\nu(z_2)\rangle\,, \tag{1.8}$$

and similarly for $\tilde{j}^\mu(\bar{z}_1)\tilde{j}^\nu(\bar{z}_2)$. In (1.8), the $:j^\mu(z_1)j^\nu(z_2):$ is regular in the limit $z_1 \to z_2$, and it is called a normal-ordered product of the operators $j(z_1)$ and $j(z_2)$. If not specified otherwise, all the products of world-sheet operators in this chapter are normal-ordered. The correlation function $\langle j^\mu(z_1)j^\nu(z_2)\rangle$ is singular when $z_1 \to z_2$ and is defined by the Polyakov path integral

$$\langle j^\mu(z_1)j^\nu(z_2)\rangle = \int [dX]\,(i\sqrt{2}\,\partial X^\mu(z_1))(i\sqrt{2}\,\partial X^\nu(z_2))e^{-S_P[X]}\,. \tag{1.9}$$

We obtain

$$\langle j^\mu(z_1)j^\nu(z_2)\rangle = \eta^{\mu\nu}\frac{1}{(z_1-z_2)^2}\,, \quad \langle \tilde{j}^\mu(\bar{z}_1)\tilde{j}^\nu(\bar{z}_2)\rangle = \eta^{\mu\nu}\frac{1}{(\bar{z}_1-\bar{z}_2)^2}\,. \tag{1.10}$$

Let us focus on holomorphic fields; the conclusions are similar for anti-holomorphic fields. The stress-energy tensor in terms of the currents j^μ is

$$T(z) = \frac{1}{2}:j^\mu(z)j_\mu(z):\,. \tag{1.11}$$

We have the operator product expansion (OPE)

$$T(z_1)j^\mu(z_2) = \frac{j^\mu(z_2)}{(z_1-z_2)^2} + \frac{\partial j^\mu(z_2)}{z_1-z_2} + \ldots\,, \tag{1.12}$$

where dots represent normal-ordered terms, regular in the limit $z_1 \to z_2$. We also have the OPE

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial T(z_2)}{z_1-z_2} + \ldots\,. \tag{1.13}$$

Here c is a central charge, which is not present in the classical theory. For the stress-energy tensor (1.11), it is equal to the dimension of the target space-time, d, where each component of j^μ contributes central charge equal to one. In bosonic string theory the central charge is $c = d - 26$, where -26 comes from conformal ghost fields. In superstring theory (discussed below) $c = 3d/2 - 15$, where each one of d world-sheet fermions contributes $1/2$ and superconformal ghosts contribute -15. Non-vanishing central charge signifies a world-sheet conformal anomaly: quantum violation of the classical tracelessness of the stress-energy tensor.

The conformal anomaly in a consistent string theory should vanish. This is necessary because conformal invariance of string theory ensures that the negative-norm time-like component of the current j^0 (and time-like component of the fermion ψ^0, in the case of superstring theory, where conformal symmetry is extended to superconformal symmetry) decouples from the spectrum of physical operators. In this chapter we always assume that a bosonic string lives in 26-dimensional space-time, and a supersymmetric string lives in 10-dimensional space-time, so that the central charge vanishes.

Let us perform a Laurent series expansion

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}\,, \quad j^\mu(z) = \sum_n \frac{j_n^\mu}{z^{n+1}}\,, \tag{1.14}$$

where $L_{-n} = (L_n)^\dagger$, $j_{-n} = (j_n)^\dagger$. Therefore the OPE (1.10), and radial ordering of operators on a complex world-sheet plane (if one considers the operator product $\mathcal{O}(z_1)\mathcal{O}(z_2)$ and $z_1 \to z_2$, then the operator with larger $|z|$ is placed on the left of the operator with smaller $|z|$), give the commutator

$$[j_m^\mu\,, j_n^\nu] = m\, \eta^{\mu\nu}\, \delta_{m+n,0}\,. \tag{1.15}$$

We see that $j_{-n}^\mu\,,\;\; n > 0$ are the operators creating oscillatory string states and $j_n^\mu\,,\;\; n > 0$ are the operators annihilating string states.

The OPE (1.12) gives the commutator

$$[L_m,\, j_{-n}^\mu] = n j_{m-n}^\mu\,, \tag{1.16}$$

and the OPE (1.13) gives the Virasoro algebra commutation relation

$$[L_m,\, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\,. \tag{1.17}$$

We also obtain

$$L_n = \frac{1}{2} \sum_k \eta_{\mu\nu} j_{-k}^\mu j_{n+k}^\nu\,, \tag{1.18}$$

where the operators are normal-ordered: creation operators j^μ_{-m}, $m > 0$ are always on the left of annihilation operators j^μ_m, $m > 0$; due to (1.15) this subtlety only arises when $n = 0$.

The classical physical state conditions, which demand that the stress-energy tensor components (1.6) vanish, are replaced in the quantum theory by the Virasoro conditions

$$L_n|\text{phys}\rangle = 0\,, \quad n > 0\,, \quad (L_0 - a)|\text{phys}\rangle = 0\,, \tag{1.19}$$

where a is a normal ordering constant. For a bosonic string $a = 1$. We will derive the value of a for superstring later in this chapter.

A general string state is

$$|\text{state}\rangle \;=\; j^{\mu_1}_{-n_1} \dots j^{\mu_k}_{-n_k}|0;p\rangle\,, \tag{1.20}$$

where the vacuum $|0;p\rangle$ is annihilated by all j^μ_n with $n > 0$. The eigenvalues of j^μ_0 are values of the momentum p^μ of the center of mass of a string. If the state (1.20) satisfies the Virasoro constraints and has a non-vanishing norm, it is called physical.

The L_0 Virasoro constraint defines the mass-shell equation:

$$\frac{1}{2}M^2 \;=\; \sum_{n>0} \eta_{\mu\nu} j^\mu_{-n} j^\nu_n - a\,, \tag{1.21}$$

where we have used the equation $-j_0^2 = -p^2 = M^2$. The $-a$ term is therefore the zero-point energy of a string.

Due to (1.19), the negative-norm states which are created by the operators j^0_{-k}, $k > 0$ are decoupled from the physical spectrum. We are going to illustrate this now with a simple example. Recall that $|p;0\rangle$ is a string oscillatory vacuum state, with a center-of-mass momentum p^μ, satisfying $L_n|p;0\rangle = 0$ for $n > 0$. Consider the first excited string state, $|\psi\rangle = e_\mu j^\mu_{-1}|p;0\rangle$, where the polarization vector e_μ has d independent components. The only non-trivial Virasoro constraint (besides the mass-shell condition (1.21)) is $L_1|\psi\rangle = 0$, which due to (1.16), (1.18) gets re-written as $p^\mu e_\mu = 0$. The mass of this state, due to (1.21) and the fact that $a = 1$, is zero.

Due to (1.18) and (1.21) the state $|\chi\rangle = L_{-1}|p;0\rangle = p_\mu j^\mu_{-1}|p;0\rangle$ also has zero mass, as the state $|\psi\rangle$, and has a longitudinal polarization $e_\mu = p_\mu$. The state $|\chi\rangle$ is called spurious: it satisfies the Virasoro constraints but it is decoupled from any physical state $|\omega\rangle$, because $\langle\chi|\omega\rangle = \langle p;0|L_1|\omega\rangle =$

0. In particular its own norm is equal to zero. Any state $|\psi\rangle$ is therefore defined up to a state $t|\chi\rangle$, with an arbitrary parameter t.

Let us now choose $p^0 = \sqrt{M^2 + p^2} = p$, $p^{d-1} = p$, $p^i = 0$ for $i = 2, \ldots, d-2$. The index i labels polarizations of the string states transverse to the direction of motion of its center of mass. The L_1 Virasoro constraint is then

$$p\, e_0 + p\, e_{d-1} = 0\,, \tag{1.22}$$

and the spurious state is

$$t|\chi\rangle = t(-p\, j^0_{-1} + p\, j^{d-1}_{-1})|p;0\rangle\,. \tag{1.23}$$

Due to (1.22), (1.23) we can fix $e_0 = e_{d-1} = 0$, which means that the states with time-like and longitudinal polarizations are decoupled from the physical spectrum of a string.

The systematic application of such an approach to the construction of the physical spectrum of a first-quantized string is called covariant quantization of a string. We adopt this method in chapter 4. Another covariant (w.r.t. Lorentz symmetry in the target space-time) way to quantize a (super)string is the BRST method, which requires the introduction of (super)conformal ghosts, contributing to the central charge. The method which we are going to use below in this chapter is light-cone quantization, which breaks space-time Lorentz symmetry down to the rotation group $SO(d-2)$, acting only on transverse physical polarizations.

1.2.2 Ramond-Neveu-Schwarz superstring

The Ramond-Neveu-Schwarz (RNS) superstring is one way to formulate superstring theory. In the RNS superstring the dynamical fields on the world-sheet are the bosons $X^\mu(z,\bar{z})$ and the fermions $\Psi^\mu(z,\bar{z})$. Both of these fields have a target space-time vector index μ. The RNS superstring action is a sum of the Polyakov action (1.2) and Dirac terms for the world-sheet fermions,

$$S = \frac{1}{2\pi}\int d^2z\, \left(\partial X^\mu \bar{\partial} X_\mu + \frac{1}{2}\psi^\mu \bar{\partial}\psi_\mu + \frac{1}{2}\tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right)\,. \tag{1.24}$$

Here ψ and $\tilde{\psi}$ are Majorana-Weyl one-component two-dimensional spinors, with the two-component Majorana spinor being $\Psi = (\psi\,,\ \tilde{\psi})$. The action (1.24) possesses two-dimensional supersymmetry. The fermions ψ and $\tilde{\psi}$ are respectively the left-moving and the right-moving superpartners of the currents j and $\tilde{j}$ defined by eq. (1.7). The equations of motion following from the action (1.24) are

$$\bar{\partial}\partial X^\mu = 0\,, \quad \bar{\partial}\psi^\mu = 0\,, \quad \partial\tilde{\psi}^\mu = 0\,. \tag{1.25}$$

These equations must be accompanied by boundary conditions, which for bosons X^μ are the same as in bosonic string theory, and for fermions are to be chosen so that

$$\int d\tau \left[\psi^\mu \delta\psi_\mu - \tilde{\psi}^\mu \delta\tilde{\psi}_\mu\right]_{\sigma=0} - \int d\tau \left[\psi^\mu \delta\psi_\mu - \tilde{\psi}^\mu \delta\tilde{\psi}_\mu\right]_{\sigma=\pi} = 0\,. \tag{1.26}$$

Consider an open string. Suppose the bosons X^μ satisfy Neumann boundary conditions. On the l.h.s. of (1.26) there are two square brackets, each corresponding to one end of a string. We have to satisfy the boundary conditions independently at each end, which means that we have to impose $\psi = \pm\tilde{\psi}$ at $\sigma = 0, \pi$. As a matter of convention we put $\psi = \tilde{\psi}$ at $\sigma = 0$. The two options at the other end ($\sigma = \pi$) define two sectors of an open string, the Neveu-Schwarz sector and the Ramond sector:

$$\begin{aligned} \mathbf{NS}: \quad & \psi^\mu|_{\sigma=\pi} = -\tilde{\psi}^\mu|_{\sigma=\pi}\,, \\ \mathbf{R}: \quad & \psi^\mu|_{\sigma=\pi} = \tilde{\psi}^\mu|_{\sigma=\pi}\,. \end{aligned} \tag{1.27}$$

The Laurent expansions of a general open string solution to (1.25) for fermions with boundary conditions (1.27) are

$$\begin{aligned} \mathbf{NS}: \quad & \psi^\mu(z) = \sum_{r\in Z+1/2} \frac{b_r^\mu}{z^{r+1/2}}\,, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in Z+1/2} \frac{b_r^\mu}{\bar{z}^{r+1/2}}\,, \\ \mathbf{R}: \quad & \psi^\mu(z) = \sum_{n\in Z} \frac{d_n^\mu}{z^{n+1/2}}\,, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{n\in Z} \frac{d_n^\mu}{\bar{z}^{n+1/2}}\,, \end{aligned} \tag{1.28}$$

where n is integer-valued and r is half-integer-valued.

In the case of a closed string we have to impose (anti)periodic boundary conditions separately for left- and right-moving states: $\psi|_{\sigma=0} = \pm\psi|_{\sigma=\pi}$ and $\tilde{\psi}|_{\sigma=0} = \pm\tilde{\psi}|_{\sigma=\pi}$. Each state belongs either to the NS sector and is expanded in half-integer modes, or the R sector and is expanded in integer modes. In total we can form four different combinations of left- and right-movers.

The stress-energy tensor corresponding to the RNS action (1.24) is given by

$$T(z) = -\partial X^\mu(z)\partial X_\mu(z) - \frac{1}{2}\psi^\mu(z)\partial\psi_\mu(z)\,, \tag{1.29}$$

and similarly for the anti-holomorphic (right-moving) component $\tilde{T}(\bar{z})$. Furthermore, the action (1.24) is invariant under $\mathcal{N}=(1,1)$ two-dimensional supersymmetry transformations, generated by a supercurrent with the components

$$\mathcal{J}(z)=\psi(z)j(z)\,, \qquad \tilde{\mathcal{J}}(\bar{z})=\tilde{\psi}(\bar{z})\tilde{j}(\bar{z})\,. \tag{1.30}$$

The stress-energy tensor $T(z)$ and the supercurrent $\mathcal{J}(z)$ form a holomorphic superconformal world-sheet current algebra. The operators $\tilde{T}(\bar{z})$ and $\tilde{\mathcal{J}}(\bar{z})$ form an anti-holomorphic copy of this superconformal algebra.

In the NS sector the supercurrent is expanded in half-integer modes, and in the R sector it is expanded in integer modes

$$\begin{aligned}\mathcal{J}_{NS}(z)&=\sum_{r\in Z+1/2}\frac{G_r}{z^{r+3/2}}\,, \quad G_r=\sum_{s\in Z+1/2}\eta_{\mu\nu}b^{\mu}_s j^{\nu}_{r-s}\,,\\ \mathcal{J}_{R}(z)&=\sum_{m\in Z}\frac{F_m}{z^{m+3/2}}\,, \quad F_m=\sum_{n\in Z}\eta_{\mu\nu}d^{\mu}_n j^{\nu}_{m-n}\,.\end{aligned} \tag{1.31}$$

and similarly for the anti-holomorphic sector. Notice that in (1.31) the indices $r-s$, $m-n$ of the modes of the current j^{μ} are integer-valued. This is a consequence of the Neumann boundary conditions for bosons X^{μ}. Below we generalize our consideration to the case of Dirichlet boundary condition, with the bosons expanded in half-integer modes, j^{μ}_s , $s\in Z+1/2$. Supersymmetry requires the coefficients G_r to have a half-integer-valued index r, and the coefficients F_m to have an integer-valued index m. Therefore, due to (1.31), the corresponding NS fermions must be expanded in integer-valued modes, b^{μ}_n , $n\in Z$, while the R fermionic modes, d^{μ}_s must have half-integer valued indices, $s\in Z+1/2$.

As we first-quantize the theory with the action (1.24) we get the correlation functions (1.10) for the bosonic fields $j^{\mu}(z)$, $\tilde{j}^{\mu}(\bar{z})$, and the correlation functions

$$\langle\psi^{\mu}(z_1)\psi^{\mu}(z_2)\rangle=\eta^{\mu\nu}\frac{1}{z_1-z_2}\,, \quad \langle\tilde{\psi}^{\mu}(\bar{z}_1)\tilde{\psi}^{\mu}(\bar{z}_2)\rangle=\eta^{\mu\nu}\frac{1}{\bar{z}_1-\bar{z}_2} \tag{1.32}$$

for the fermions. Using (1.10) and (1.32) we find the operator product expansions of the operators of the superconformal algebra (only terms singular in the $z_1 \to z_2$ limit are written down)

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{T(z_2)}{z_1-z_2} + \dots\,, \tag{1.33}$$

$$T(z_1)\mathcal{J}(z_2) = \frac{(3/2)\mathcal{J}(z_2)}{(z_1-z_2)^2} + \frac{\partial\mathcal{J}(z_2)}{z_1-z_2} + \dots\,, \tag{1.34}$$

$$\mathcal{J}(z_1)\mathcal{J}(z_2) = \frac{2c/3}{(z_1-z_2)^3} + \frac{2T(z_2)}{z_1-z_2} + \dots\,. \tag{1.35}$$

The supersymmetry transformations are defined by the OPEs

$$\mathcal{J}(z_1)j(z_2) = \frac{\psi(z_2)}{(z_1-z_2)^2} + \frac{\partial\psi(z_2)}{z_1-z_2} + \dots\,, \tag{1.36}$$

$$\mathcal{J}(z_1)\psi(z_2) = \frac{j(z_2)}{z_1-z_2} + \dots\,. \tag{1.37}$$

From (1.28), (1.32) we obtain the anti-commutation relations

$$\{b^\mu_r,\, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}\,, \quad \{d^\mu_n,\, d^\nu_m\} = \eta^{\mu\nu}\delta_{n+m,0}\,. \tag{1.38}$$

Therefore the operators b^μ_{-r}, $r > 0$ and d^μ_{-n}, $n > 0$ create string states and the operators b^μ_r, $r > 0$ and d^μ_n, $n > 0$ annihilate string states. We also have creation and annihilation operators, respectively, j^μ_{-n}, $n > 0$, and j^μ_n, $n > 0$, as in bosonic string theory. The string vacuum state $|0\rangle$ vanishes when we act on it with any annihilation operator. The negative-norm states, created by the time-like polarized operators j^0_{-n}, d^0_{-m} and b^0_{-r} $(m,n,r > 0)$ are decoupled from the physical spectrum due to the super-Virasoro constraints:

$$\begin{aligned} &\mathbf{NS}:\ (L_0 - a_{NS})|\mathrm{phys}\rangle = 0\,,\ \ L_n|\mathrm{phys}\rangle = 0\,,\ \ G_r|\mathrm{phys}\rangle = 0\,,\ \ n,\ r > 0\,,\\ &\mathbf{R}:\ (L_0 - a_R)|\mathrm{phys}\rangle = 0\,,\ L_n|\mathrm{phys}\rangle = 0\,,\ F_m|\mathrm{phys}\rangle = 0\,,\ n > 0, m \geq 0\,. \end{aligned} \tag{1.39}$$

In the next subsection we will prove that if we impose Neumann boundary condition for all polarizations then $a_{NS} = 1/2$. One can show that $L_0 = F_0^2$ in the R sector. Therefore, due to the supersymmetry constraint $F_0 = 0$, we have to put $a_R = 0$.

In the R sector we have operators d^μ_0, which, due to (1.38), form a Dirac algebra in d dimensions:

$$\{\Gamma^\mu\,,\ \Gamma^\nu\} = 2\,\eta^{\mu\nu}\,, \quad \Gamma^\mu = \sqrt{2}\,d^\mu_0\,. \tag{1.40}$$

If the state $|0\rangle_R$ is the vacuum of the R sector then the state $d_0^\mu|0\rangle_R$ is also a vacuum of the R sector. Consequently any state in the R sector lives in spinor representation of the Dirac algebra $\mathcal{C}_{d-1,1}$, and therefore is a target space-time spinor. This should be compared to the fact that any state in the NS sector is a space-time boson.

1.2.3 Open RNS superstring and zero-point energy

As was explained above, the end of an open string can be free so that the bosons X^μ on its world-sheet satisfy Neumann boundary conditions, or it can terminate on a D-brane giving rise to Dirichlet boundary conditions. In total there are four possibilities for the two ends of an open string. We denote them as

$$(\mathbf{NN})\,, \quad (\mathbf{ND})\,, \quad (\mathbf{DN})\,, \quad (\mathbf{DD})\,. \tag{1.41}$$

The first letter specifies the boundary condition at the $\sigma = 0$ ($z = e^\tau$) end, the second letter does so for the $\sigma = \pi$ ($z = -e^\tau$) end.

For the possible boundary conditions (1.41), we have the following solutions to the equation (1.3):

$$\begin{aligned}
&\mathbf{NN}: \ X^\mu(z,\bar z) = x^\mu - \frac{ip^\mu}{\sqrt{2}}\log\left(z\bar z\right) + \frac{i}{\sqrt{2}}\sum_{m\neq 0}\frac{1}{m}j_m^\mu\left(z^{-m}+\bar z^{-m}\right),\\
&\mathbf{DD}: \ X^\mu(z,\bar z) = x^\mu - \frac{i\tilde p^\mu}{\sqrt{2}}\log\left(\frac{z}{\bar z}\right) + \frac{i}{\sqrt{2}}\sum_{m\neq 0}\frac{1}{m}j_m^\mu(z^{-m}-\bar z^{-m}),\\
&\mathbf{DN}: \ X^\mu(z,\bar z) = x^\mu + \frac{i}{\sqrt{2}}\sum_{r\neq 0}\frac{1}{r}j_r^\mu(z^{-r}-\bar z^{-r}),\\
&\mathbf{ND}: \ X^\mu(z,\bar z) = x^\mu + \frac{i}{\sqrt{2}}\sum_{r\neq 0}\frac{1}{r}j_r^\mu(z^{-r}+\bar z^{-r}).
\end{aligned} \tag{1.42}$$

Here the index m is integer-valued and the index r is half-integer-valued.

Let us solve the Virasoro constraints explicitly, so that we are left with only $d-2$ transverse polarizations in the target space-time. This method is called light-cone quantization, and it is convenient for our current purposes. The physical operators are b_r^i, d_n^i, j_n^i, $i = 1,\ldots,d-2$, and we do not have to worry about the super-Virasoro constraints.

Assume first that the bosons X^μ satisfy either NN or DD boundary conditions, and therefore are expanded in integer modes. Using (1.29) we derive

$$\begin{aligned}
\mathbf{R}:\quad L_0 &= \frac{1}{2}\sum_{n\in Z}\sum_{i=1}^{d-2} j^i_{-n} j^i_n + \frac{1}{2}\sum_{n\in Z}\sum_{i=1}^{d-2} n d^i_{-n} d^i_n\,, \\
\mathbf{NS}:\quad L_0 &= \frac{1}{2}\sum_{n\in Z}\sum_{i=1}^{d-2} j^i_{-n} j^i_n + \frac{1}{2}\sum_{r\in Z+1/2}\sum_{i=1}^{d-2} r b^i_{-r} b^i_r\,.
\end{aligned} \tag{1.43}$$

Let us perform normal ordering of the creation and annihilation operators in (1.43), placing annihilation operators to the right of creation operators. In the R sector we obtain

$$\begin{aligned}
L_0 &= \sum_{n>0}\sum_{i=1}^{d-2}(j^i_{-n} j^i_n + n d^i_{-n} d^i_n) + \frac{1}{2} j_0^2 + \frac{1}{2}\sum_{n>0}\sum_{i=1}^{d-2}([j^i_n,\, j^i_{-n}] - n\{d^i_n,\, d^i_{-n}\}) \\
&= \sum_{n>0}\sum_{i=1}^{d-2}(j^i_{-n} j^i_n + n d^i_{-n} d^i_n) + \frac{1}{2} j_0^2\,,
\end{aligned} \tag{1.44}$$

where we have used the commutators (1.15) and (1.38). We see from (1.44) that the zero-point energy in the R sector is zero, $a_R = 0$, as we concluded at the end of the previous subsection from the point of view of the super-Virasoro constraints. Similarly in the NS sector we obtain

$$\begin{aligned}
L_0 = &\sum_{n>0}\sum_{i=1}^{d-2} j^i_{-n} j^i_n + \sum_{r>0}\sum_{i=1}^{d-2} r b^i_{-r} b^i_r + \frac{1}{2} j_0^2 \\
&+ \frac{1}{2}\sum_{n>0}\sum_{i=1}^{d-2}[j^i_n,\, j^i_{-n}] - \frac{1}{2}\sum_{r>0}\sum_{i=1}^{d-2} r\{b^i_r,\, b^i_{-r}\}\,.
\end{aligned} \tag{1.45}$$

Therefore due to (1.15) and (1.38), the zero-point energy in the NS sector is given by

$$-a_{NS} = \frac{d-2}{2}\left(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right) = -\frac{d-2}{16}\,. \tag{1.46}$$

Inserting the superstring value $d = 10$ we obtain $a_{NS} = \frac{1}{2}$. In the last equality of (1.46) we used zeta-function regularization. We know that the zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.47}$$

can be analytically continued so that

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}\,. \tag{1.48}$$

Now, introduce the sum of even numbers, $S_{even} = \sum_{n>0}(2n)$, and the sum of odd numbers $S_{odd} = \sum_{n>0}(2n+1)$. We have $S_{even} = 2\zeta(-1) = -\frac{1}{6}$, and $S_{odd} = \zeta(-1) - S_{even} = \frac{1}{12}$. Therefore in (1.46) we have $\sum_{r>0} r = \frac{1}{24}$. There exist independent ways to derive the same results for zero-point energies.

The bottom line is that for NN and DD bosons we have $a_{NS} = 1/2$, and $a_R = 0$, and therefore the mass formulae are,

$$\mathbf{R}: \quad \frac{1}{2}M^2 = \sum_{n>0}\sum_{i=1}^{d-2} n d^i_{-n} d^i_n + \sum_{n>0}\sum_{i=1}^{d-2} j^i_{-n} j^i_n\,, \tag{1.49}$$

$$\mathbf{NS}: \quad \frac{1}{2}M^2 = \sum_{r>0}\sum_{i=1}^{d-2} r b^i_{-r} b^i_r + \sum_{n>0}\sum_{i=1}^{d-2} j^i_{-n} j^i_n - \frac{1}{2}\,. \tag{1.50}$$

Consider an open string with NN or DD boundary conditions. The vacuum $|0\rangle_{NS}$ in the NS sector is defined by the conditions $b^i_r|0\rangle_{NS} = 0$, $j^i_n|0\rangle_{NS} = 0$, for $r, n > 0$. This state is a tachyon with $M^2 = -1$, as follows from (1.50). The procedure called Gliozzi-Scherk-Olive (GSO) projection eliminates $|0\rangle_{NS}$ from the spectrum of the NS sector of the RNS superstring. The lowest NS string state, which survives GSO projection, is the $d-2$-component massless vector $b^i_{-1/2}|0\rangle_{NS}$.

The lowest state in the R sector, the vacuum $|0\rangle_R$ with zero mass, survives GSO projection. As we have noticed above, this state is a space-time fermion. In $d = 10$ dimensions it is a 16-component Majorana-Weyl massless fermion with eight physical d.o.f. (A general spinor in ten dimensions has 32 complex-valued components; each condition of being a Majorana and Weyl decreases the number of independent components by a factor of two, and the super-Virasoro constraint $F_0 = 0$ further reduces the number of independent components by a factor of two.) Therefore the lowest state of an open RNS superstring consists of eight bosonic and eight fermionic massless degrees of freedom, which is a vector supermultiplet field content of $\mathcal{N} = 1$, $d = 10$ supersymmetric Yang-Mills theory.

This result has a deep reason behind it: the RNS superstring, which by construction has two-dimensional world-sheet supersymmetry, is actually space-time supersymmetric. Superstring theory which contains open strings has $\mathcal{N} = 1$ space-time supersymmetry (16 supercharges), superstring theory without open strings has $\mathcal{N} = 2$ space-time supersymmetry (32 supercharges). A formulation of superstring theory with explicit space-time supersymmetry is called the Green-Schwarz (GS) superstring.

Now let us consider the DN and ND boundary conditions. In general we find the following contributions to the normal ordering constant a from half-integer or integer bosonic and fermionic modes:

$$a_{b\,i} = \frac{1}{24}\,, \quad a_{b\,h} = -\frac{1}{48}\,, \quad a_{f\,i} = -\frac{1}{24}\,, \quad a_{f\,h} = \frac{1}{48}\,, \tag{1.51}$$

where b and f stand for bosons and fermions, and i and h stand for integer and half-integer, respectively.

One requirement that should always be satisfied is that the zero-point energy in the R sector is zero. Suppose therefore that among the $d-2$ transverse polarizations, ν of them are either DN or ND, with half-integer bosonic current modes. Therefore the contribution of the bosons to the zero-point energy is $-\frac{\nu}{48} + \frac{d-2-\nu}{24} = \frac{2(d-2)-3\nu}{48}$. Therefore fermions in the R sector should contribute $\frac{3\nu-2(d-2)}{48}$. The fermions which are polarized along $d-2-\nu$ NN or DD directions have integer modes and contribute $-\frac{d-2-\nu}{24}$. The fermions which are polarized along ν DN or ND directions therefore must have half-integer modes and contribute $\frac{\nu}{48}$, adding up to a required quantity $\frac{3\nu-2(d-2)}{48}$.

Now we are ready to compute the zero-point energy in the NS sector. The NS fermions have opposite kind of modes to those of the R fermions. Therefore the fermions in $d-2-\nu$ NN or DD directions have half-integer modes and contribute $\frac{d-2-\nu}{48}$, and fermions in ν DN or ND directions have integer modes and contribute $-\frac{\nu}{24}$. The contribution from the bosons is of course the same as in the R sector and is equal to $\frac{2(d-2)-3\nu}{48}$. We conclude that

$$a_{NS} = \frac{d-2}{16} - \frac{\nu}{8} = \frac{1}{2} - \frac{\nu}{8}\,. \tag{1.52}$$

Due to the fact that $a_R = 0$, the ground state of the R sector of an open string is always a massless fermion. In the case of $\nu = 4$ we get $a_{NS} = 0$, and the mass of the lowest NS state is therefore $M^2 = -2a_{NS} = 0$. It is not projected out by GSO, so that we have an equal number of massless bosons and fermions, furnishing a vector supermultiplet.

1.2.4 Closed RNS superstring and Ramond-Ramond fields

In the case of the closed string, to construct the spectrum of excitations we have to build physical states of the holomorphic and anti-holomorphic sectors separately, and then take a direct product of these states. As was already mentioned above, the boundary conditions which must be imposed on fermions of the closed string are $\psi|_{\sigma=0} = \pm\psi|_{\sigma=\pi}$, $\tilde{\psi}|_{\sigma=0} = \pm\tilde{\psi}|_{\sigma=\pi}$, giving rise to the NS and R sectors of left-movers (holomorphic states) and right-movers (anti-holomorphic states). The fermions of the NS sector are expanded in half-integer modes, b^μ_r, $\tilde{b}^\mu_r$, and the fermions of the R sector are expanded in integer modes, d^μ_n, $\tilde{d}^\mu_n$. After GSO projection, the massless (anti)holomorphic states in the light-cone quantization are

$$\begin{aligned} \textbf{left}: &\quad b^i_{-1/2}|0\rangle_{NS}\,, \quad |0\rangle_{R\,\mathrm{left}}\,, \\ \textbf{right}: &\quad \tilde{b}^i_{-1/2}|0\rangle_{NS}\,, \quad |0\rangle_{R\,\mathrm{right}}\,. \end{aligned} \tag{1.53}$$

Here $|0\rangle_R$ is a space-time fermion with eight independent real-valued components and definite chirality. Let us specify its chirality by introducing the notation $|+\rangle$ and $|-\rangle$.

By forming the direct product of left-moving and right-moving states (1.53) we obtain 128 bosonic states, NS-NS and R-R, and 128 fermionic states, NS-R and R-NS. The equality of number of bosonic and fermionic degrees of freedom is not accidental: as was mentioned above, the RNS superstring is actually space-time supersymmetric. In fact, 128+128 is the on-shell field content of $d = 10$ $\mathcal{N} = 2$ supergravity.

In the case when both left- and right-moving R fermions $|0\rangle_R$ have opposite chirality we get type-IIA supergravity. The corresponding string theory is type-IIA superstring theory. In this case the 64 of R-R degrees of freedom $|+\rangle \otimes |-\rangle$ are expanded in irreducible representations of $SO(8)$ as $C_1 \oplus C_3$, where C_1 is a 1-form with 8 d.o.f. and C_3 is a 3-form with 56 d.o.f. Similarly the R-R fields $|+\rangle \otimes |+\rangle$ of type-IIB supergravity are p-forms C_0, C_2 and C_4, with 1, 28 and 35 d.o.f. respectively (there is a subtlety with C_4, requiring that $F_5 = dC_4$ is Hodge self-dual in ten dimensions, $F_5 = \star F_5$, which reduces the number of d.o.f. of the C_4 by a factor of two).

1.2.5 *D*-branes

A *D*-brane is an extended object on which an open string can end. For example, if all of the coordinates but X^1 satisfy Neumann boundary conditions, it means that there is $D8$-brane located at some point $x^1 = x_0^1$ and extended in the $x^{2,3,\ldots,9}$ directions. Similarly to a string having a two-dimensional world-sheet, a $D8$-brane sweeps a nine-dimensional world-volume as it moves in a space-time. If all coordinates of an open string satisfy Neumann boundary conditions then we actually have a space-time filling $D9$-brane, with its ten-dimensional world-volume being the entire space-time.

In type-IIA superstring theory we can have Dp-branes with even-valued p, and in type IIB superstring theory p must be odd-valued. The reason for this selection originates in the stability of a Dp-brane, and is tightly connected to the fact that a Dp-brane embedded in $\mathcal{N} = 2$ superstring theory is a Bogomol'nyi-Prasad-Sommerfield (BPS) object, preserving 16 of the original 32 supercharges. We discuss this in more detail in the next subsection.

Now, recall that a BPS object satisfies the condition $M = |Z|$, with M being a mass, and Z being (an appropriately defined) conserved charge. It turns out that in the case of a Dp-brane, the role of the charge Z is played by the charge w.r.t. the R-R field. The coupling of the Dp-brane to the R-R field is described by the Chern-Simons action

$$S_{CS} = T_{Dp} \int C_{p+1}\,, \tag{1.54}$$

where integration of the R-R $p+1$-form C_{p+1} is performed over the $p+1$-dimensional world-volume of the Dp-brane. In (1.54) the $T_{Dp} \simeq \frac{1}{\ell_s^{p+1} g_s}$ is the tension of the Dp-brane, g_s is the closed string coupling constant. Notice that in the perturbative regime of small g_s, a D-brane is very heavy.

A single Dp-brane is described by a supersymmetric theory on a $p+1$-dimensional world-volume, with 16 conserved supercharges. The number of on-shell fermionic d.o.f. is equal to eight. Due to supersymmetry, the number of dynamical massless bosons on the world-volume must also add up to eight.

An embedding of a Dp-brane into ten-dimensional target space-time is described by ten fields $X^\mu(\sigma^a)$, $\mu = 0, 1, \ldots, 10$, where $\sigma^a\,,\ a = 0, 1, \ldots, p$ are the world-volume coordinates. However, due to $p+1$-dimensional diffeomorphism symmetry, $\sigma^a \to \sigma'^a(\sigma^b)$, we have only $9-p$ independent bosonic d.o.f. Supersymmetry therefore requires the addition of $p-1$ bosonic d.o.f. These come about as transverse polarizations of the $U(1)$ gauge field A_μ on the Dp-brane world-volume. In fact, the origin of this massless vector supermultiplet on a Dp-brane world-volume is simple: its fields are the lowest modes of an open string attached to this Dp-brane by both ends.

To summarize, the eight massless dynamical bosonic d.o.f. on a Dp-brane world-volume are split into a gauge field A_μ with $p-1$ physical polarizations, and $9-p$ scalars Φ^I, $I = p+1, \ldots 10$, describing the embedding of the Dp-brane into 10-dimensional target space-time.

Let us study the low-energy dynamics of a Dp-brane. Clasically we can set the fermionic gaugino field to zero. Suppose the Dp-brane is embedded in a space-time with a metric $G_{\mu\nu}$. It induces a metric g_{ab}, $a, b = 1, \ldots, p+1$, on the Dp-brane world-volume, such that

$$g_{ab} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} G_{\mu\nu} \,. \tag{1.55}$$

The total low-energy effective action of a Dp-brane consists of two terms. One of them is a generalization of the Chern-Simons term (1.54),

$$S_{CS} = T_{Dp} \int \left[\sum_p C_p e^{B+F} \right]_{p+1} , \tag{1.56}$$

where, on the r.h.s. of (1.56), B is the NS-NS two-form, and we introduced the field strength $F = dA$, then we took the $p+1$-form part. The other term, describing the low-energy Dp-brane dynamics, is the Dirac-Born-Infeld (DBI) term,

$$S_{DBI} = -T_{Dp} \int d^{p+1}x \sqrt{-\det\left(g + B + F + G_{IJ} \partial \Phi^I \partial \Phi^J\right)} \,, \tag{1.57}$$

where, on the r.h.s of (1.57), one takes the determinant of the matrix

$$||g_{ab} + B_{ab} + F_{ab} + G_{IJ} \partial_a \Phi^I \partial_b \Phi^J|| \,. \tag{1.58}$$

By varying the total action $S_{tot} = S_{DBI} + S_{CS}$ of the Dp-brane, one obtains the equations of motion determining the embedding of the Dp-brane into the target space-time, and the dynamics of the gauge field A_μ on its world-volume.

1.2.6 T-duality and D-brane intersections

We have already discussed that a single Dp-brane embedded into a space-time of type-II string theory breaks half of the original $\mathcal{N} = 2$ supersymmetry. Putting more Dp-branes which span various spatial directions generally breaks more supersymmetry, and in particular can result in a non-supersymmetric theory. In this subsection we are going to discuss how to count the number of supersymmetries which are preserved by the given configuration of Dp-branes.

Let us start with type-IIB closed string theory. It has two 16-component Majorana-Weyl conserved supercharges, Q_1 and Q_2, of the same chirality. Now suppose we want to put an open string with Neumann boundary conditions into the space-time. Open string boundary conditions are not compatible with the $\mathcal{N} = 2$ supersymmetry. The supersymmetry in fact gets broken by a factor of two, the remaining conserved supercharge is given by $Q = Q_1 + Q_2$. As we discussed above, the presence of an open string with Neumann boundary conditions means the presence of a space-time filling $D9$-brane. Therefore a $D9$-brane in type-IIB string theory breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ supersymmetry with the supercharge $Q = Q_1 + Q_2$.

The next question to ask is how many supersymmetries are preserved by Dp-branes with $p < 9$. To answer it we are going to use T-duality of superstring theory. Consider type-IIB superstring theory in a space-time with the x^9 coordinate compactified on a circle of radius R. It turn out that the spectrum of superstring theory is left invariant if we perform the T-duality transformation in the x^9 direction, which amounts to replacements of the compactification radius $R \to 1/R$ and the anti-holomorphic boson $\tilde{X}^9 \to -\tilde{X}^9$. As can be seen from (1.42), T-duality transformation interchanges the NN and DD boundary conditions in the x^9 direction. Therefore if in the type-IIB theory which we started with we had a space-time filling $D9$-brane, which wrapped the x^9 circle, then in the T-dual theory we have a $D8$-brane which is localized at a certain point on the x^9 circle, making the open string boundary condition along the x^9 direction be Dirichlet.

However we know that type-IIB string theory can only have Dp-branes with odd-valued p. Therefore the T-dual string theory with a $D8$-brane described above must be type-IIA string theory. Let us see what happens. Due to the world-sheet supersymmetry the transformation of the boson $\tilde{X}^9 \to -\tilde{X}^9$ demands the transformation of the fermion $\tilde{\psi}^9 \to -\tilde{\psi}^9$. In particular the zero mode $\tilde{d}_0^9$ of the anti-holomorphic fermion in the R sector reverses its sign. Consequently the Γ^9 element of the *anti-holomorphic* copy of the space-time Dirac algebra reverses its sign as well, and therefore so does the chirality operator $\Gamma^{11} = \Gamma^0\Gamma^1 \dots \Gamma^9$. We conclude that the space-time supercharge Q_2 changes chirality. In fact, supersymmetry implies the T-duality transformation $Q_2 \to \Gamma^9 Q_2$. The supercharge preserved by the $D8$-brane localized at some point in the x^9 direction is therefore $Q = Q_1 + \Gamma^9 Q_2$.

We can generalize the procedure described above: an arbitrary Dp-brane preserves a supercharge

$$Q = Q_1 + \Gamma^{k_1} \ldots \Gamma^{k_{9-p}} Q_2 \,, \tag{1.59}$$

where indices $k_1, \ldots, k_{9-p}$ label $9-p$ directions, orthogonal to the Dp-brane. In the case of type-IIA(-IIB) string theory p is even (odd), Q_1 and Q_2 have opposite (the same) chiralities, and Q_1 and $\Gamma^{k_1} \ldots \Gamma^{k_{9-p}} Q_2$ have the same chirality, and can be added to one another. Because $\Gamma^{11} Q_2 = \pm Q_2$ and due to the Dirac algebra anti-commutation relations, we can re-write (1.59) as

$$Q = Q_1 + \Gamma^0 \ldots \Gamma^p Q_2 \,, \tag{1.60}$$

fixing an overall sign in front of the second term in the r.h.s. of (1.60) to be plus as a matter of convention.

Now, as we know which supercharge is preserved by a single Dp-brane, we can find the supercharge preserved by a configuration of several Dp-branes. A simple observation is that several Dp-branes spanning the same space directions but, in general, located at different points in the transverse space, preserve the same supercharge as just one Dp-brane from this set.

If we have several Dp-branes of various dimensions p and spanning different space directions, the preserved supercharge, if any, is the one which is preserved by each Dp-brane from the set. For concreteness and following the needs of chapter 2, we are going to deal with the $D3-Dp$ system in type-IIB string theory. We are allowed to take any odd-valued p. But we want the Dp-brane to have an intersection with the $D3$-brane in three or four space-time dimensions. Therefore we can choose $p = 3, 5, 7, 9$. The cases $p = 3$ and $p = 9$ are relatively trivial and not interesting physically for the purposes of chapter 2, so we do not discuss them any more.

We denote the directions spanned by the $D3$-brane as $x^{0,1,2,3}$. The corresponding conserved supercharge is $Q_{D3} = Q_1 + \Gamma^0\Gamma^1\Gamma^2\Gamma^3 Q_2$. This can be reformulated in the following way. Suppose $\epsilon_{1,2}$ are sixteen-component spinor parameters of the supersymmetry transformation generated by the operator $\epsilon_1^T Q_1 + \epsilon_2^T Q_2$. From the expression for Q_{D3} we conclude that the

$D3$-brane is only supersymmetric under transformations with arbitrary ϵ_2 and ϵ_1 completely determined by ϵ_2 via the equation $\epsilon_1 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\epsilon_2$. We have used the facts that $\epsilon_{1,2}$ are Majorana-Weyl real-valued spinors, and the Γ^μ matrices are written in a Majorana real-valued representation, with $(\Gamma^a)^T = \Gamma^a$ for $a \neq 0$, and $(\Gamma^0)^T = -\Gamma^0$.

The Dp-brane is assumed to span the directions $x^{0,1,2}$ or $x^{0,1,2,3}$ along with $p-2$ or $p-3$ directions in the six-dimensional space transverse to the $D3$-brane. Let us be specific. We are considering the cases of $p = 5$ and $p = 7$. Consider first a $D5$-brane which spans the directions $x^{0,1,2,3,4,5}$, therefore intersecting the $D3$-brane in the $x^{0,1,2,3}$ directions. Such a $D5$-brane preserves the supercharge $Q_{D5} = Q_1 + \Gamma^0\Gamma^1 \dots \Gamma^5 Q_2$, that is by itself invariant under supersymmetry transformations with $\epsilon_1 = \Gamma^0\Gamma^1 \dots \Gamma^5\epsilon_2$. Therefore, to make sure that the $D3-D5$ system is invariant under supersymmetry transformations, we must satisfy the constraint $\Gamma^4\Gamma^5\epsilon_2 = \epsilon_2$. Recall now that chirality in the $(4,5)$ plane is defined as an eigenvalue of the operator $S^3 = i\Gamma^4\Gamma^5/2$, which is equal to $\pm 1/2$. Therefore $\Gamma^4\Gamma^5\epsilon_2 = \pm i\epsilon_2$, and the constraint $\Gamma^4\Gamma^5\epsilon_2 = \epsilon_2$ can never be satisfied.

Consider now the $D3-D5$ system with a three-dimensional intersection, that is consider a $D5$-brane, which spans directions $x^{0,1,2,4,5,6}$. We call it a $D5'$-brane, where prime is introduced as a short-hand notation for this specific $D5$-brane. Supersymmetry is preserved by the $D5'$-brane alone if the transformation parameters satisfy the constraint $\epsilon_1 = \Gamma^0\Gamma^1\Gamma^2\Gamma^4\Gamma^5\Gamma^6\epsilon_2$. The $D3-D5'$ system therefore preserves supersymmetry with parameter ϵ_2, which satisfies $\Gamma^3\Gamma^4\Gamma^5\Gamma^6\epsilon_2 = \epsilon_2$. This condition means that the chiralities of the spinor ϵ_2 in, say, the $(3,4)$ and $(5,6)$ planes are the same, which reduces the number of independent components of ϵ_2 by a factor of two. The $D3-D5'$ system therefore preserves one-quarter of the original 32 supersymmetries.

In exactly the same way one can prove that the $D3-D7$ system with a four-dimensional intersection is invariant under the action of eight supercharges, while the $D3-D7'$ system with a three-dimensional intersection breaks all the supersymmetries.

Notice that the fact that $D3-D5$ with a three-dimensional intersection, and $D3-D7$ with a four-dimensional intersection, are supersymmetric is in agrement with the equation (1.52) (with $\nu = 4$) for zero-point energy, which gives $a_{NS} = 0$ for both of these intersections. In both of these cases we have a massless bosonic field in the spectrum of an open string stretched between $D3$-brane and Dp-brane.

For completeness it is worth underlying that one can also consider an open string which starts and ends on the $D3$-brane and an open string which starts and ends on the Dp-brane of the $D3-Dp$ system. The massless modes of these strings comprise vector supermultiplets on the world-volumes of the $D3$-brane and Dp-brane respectively.

1.2.7 Strings in a background; supergravity and supersymmetric Yang-Mills theories

In the previous subsections we focused on strings and branes propagating in a flat space-time, with no background fields turned on. We have also found out that the massless modes of closed and open strings are fields of supergravity and supersymmetric Yang-Mills, respectively. Therefore one can consider a setup of strings creating a background with curved metric and various non-vanishing fields, and other strings and branes moving in this background as probe objects.

We are going to focus on classical string theory, with only bosonic fields present. The bosonic field content of type-II supergravity consists of NS-NS fields and R-R fields. The NS-NS fields are the same for type-IIA and type-IIB supergravities, as well as for bosonic gravity. For simplicity we will consider bosonic gravity. The field content is graviton $G_{\mu\nu}$, anti-symmetric tensor $B_{\mu\nu}$, and dilaton Φ.

One can derive equations of motion for these fields from string theory in the following way. Instead of the Polyakov action (1.2) we now have the action

$$S_P=\frac{1}{4\pi\alpha'}\int d^2z\left(G_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu+\epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu+\alpha'\Phi R^{(2)}\right)\,, \quad (1.61)$$

where $R^{(2)}$ denotes Ricci scalar on the world-sheet, $\epsilon^{z\bar{z}}=-\epsilon^{\bar{z}z}=1$, and $\alpha,\beta=z,\bar{z}$ are world-sheet vector indices.

Let us perform a first quantization of a string described by the action (1.61). It is defined by the Polyakov path integral, as in the case of a string moving in the Minkowski background. However now due to non-trivial background fields the situation is more complicated: the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ in the action (1.61) are themselves functions of the fields X^μ. The way to proceed is to use string perturbation theory. In (1.61) we restored the parameter $\alpha'=\ell_s^2/2$. In string perturbation theory one assumes that the string length ℓ_s is small (compared to a characteristic length scale in a target space-time), and performs a perturbative expansion in α'.

The methodology is the same as in the case of usual perturbation theory in quantum field theory. A path integral in interacting quantum field theory accounts for high-energy modes in the low-energy effective Lagrangian by renormalizing the coupling constants and scaling dimensions. Renormalization is described by beta-functions. String perturbation theory also has beta-functions (with $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ in (1.61) playing the role of coupling constants), and these beta-functions take into account string loop corrections in the Polyakov path integral. In this thesis we only consider classical string theory. To avoid possible confusion which may be caused by the word 'classical' we remind the reader that in quantum string field theory one also has to take into account vertices associated with the string coupling constant g_s, and loops of virtual particles between these vertices.

The beta-functions for the 'coupling constants' $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ in (1.61) are given by

$$\begin{aligned} \beta^G_{\mu\nu} &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \partial_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + \mathcal{O}(\alpha'^2)\,, \\ \beta^B_{\mu\nu} &= -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi \, H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2)\,, \\ \beta^\Phi &= \frac{d-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\lambda \Phi \nabla^\lambda \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2)\,, \end{aligned} \tag{1.62}$$

where $H = dB$ is the field strength of the two-form field $B_{\mu\nu}$. We have seen above that for a bosonic string in a flat space-time the requirement of conformal invariance on quantum level, which is a requirement of absence of Weyl anomaly, is a restriction $d = 26$ on the target space-time dimension. Now, as the background fields are turned on, the requirement of conformal invariance is vanishing of the beta-functions (1.62). In particular, when $G = \eta$, $B = \Phi = 0$, it reduces to the $d = 26$ constraint.

We have outlined the derivation of equations of motion of bosonic gravity. Up to the $d - 26$ central charge term in the last line of (1.62) these are the same as the equations of motion for the NS-NS fields of type-II supergravity (with gravitino, dilatino and R-R fields set to zero). These equations do not provide a UV-complete description of gravity since the derivation was made under the assumption of a smallness of the string length $\ell_s = \sqrt{2\alpha'}$. At the string scale higher-order effects in the α' expansion become essential, and the gravity approximation (1.62) becomes completely unreliable.

In a similar way one can derive the equations of supersymmetric Yang-Mills theory from open string theory or from heterotic string theory. In principle, systematically accounting for higher order corrections in α', one can derive precisely the higher-derivative terms which one needs to add to the effective low-energy Einstein, Yang-Mills, etc., actions.

1.2.8 Wess-Zumino-Witten model

A particular case of a string moving in a non-trivial background is given by a Wess-Zumino-Witten (WZW) model. It describes a string moving on a group manifold or on a coset space; in the latter case it is called a gauged Wess-Zumino-Witten (gWZW) model. Such a model is a central element of chapter 4, so here we give an introduction to it.

Consider a string moving on the manifold of the group G with dimension $\dim G$. Suppose g is an element of G. An embedding of the string into the manifold G is then described by the field $g(z, \bar{z})$. The action for the bosonic string is the WZW action,

$$S_{WZW} = \frac{k}{4\pi} \left[\int d^2 z \, \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial} g) - \frac{1}{3} \int_B \mathrm{Tr}(g^{-1}dg)^3 \right] , \qquad (1.63)$$

where the second term on the r.h.s. of (1.63), which is called the Wess-Zumino (WZ) term, is an integral over a ball B , the boundary of which is a world-sheet. Here k is called the level of the WZW model; for the WZ term to be unambiguous the level k must be integer-valued. In the case of a superstring one adds to this action the Dirac terms for $\dim G$ free left-moving and right-moving Majorana-Weyl world-sheet fermions. For simplicity we consider just a bosonic string in this subsection.

The Polyakov action (1.2) is conformally invariant. For the WZW action (1.63) to describe a string, it must also be conformally invariant. In the previous subsection we obtained that, for a string in background fields to be conformally invariant, the background fields must satisfy effective equations of motion in the target space-time. In the case of a WZW model it turns out that an interplay between two terms in (1.63) is such that the action S_{WZW} is conformally invariant, and one only has to make sure that the total central charge c of the theory vanishes.

Recall that a bosonic string in a flat space-time, described by the Polyakov action (1.2), has conserved holomorphic (left-moving) and anti-holomorphic (right-moving) currents (1.7). These currents originate as

Noether charges from the translational invariance of a flat space-time (as well as from the chirality of the Polyakov action, giving rise to two independently conserved currents). For the action (1.63) to describe a string it must also give rise to two independently conserved currents. The corresponding symmetry transformation of the action (1.63) is the Kac-Moody (KM) symmetry transformation, given by

$$g(z,\bar z) \to g'(z,\bar z) = \Omega(z) g(z,\bar z) \tilde\Omega^{-1}(\bar z)\,. \tag{1.64}$$

Correspondingly, for the WZW model at level k we have the conserved currents

$$\begin{aligned} j(z) &= j_A t^A = -\frac{k}{2}\partial g g^{-1} = \sum_n \frac{j_n^A}{z^{n+1}}\,,\\ \tilde j(\bar z) &= \tilde j_A t^A = \frac{k}{2} g^{-1}\bar\partial g = \sum_n \frac{\tilde j_n^A}{\bar z^{n+1}}\,. \end{aligned} \tag{1.65}$$

The zero modes of the currents are generators of the algebra **g** of the group G,

$$t^A = j_0^A\,, \qquad [t^A, t^B] = i f^{ABC} t^C\,. \tag{1.66}$$

From the expression for currents (1.65), using (1.64) one can derive the Kac-Moody transformations of the currents. For example, under a holomorphic infinitesimal transformation $\Omega(z) = I - \omega(z)$, we obtain

$$\delta j(z) = -[\omega(z), j(z)] + \frac{k}{2}\partial\omega(z)\,, \tag{1.67}$$

that is

$$\delta j^A(z) = -i f^{ABC}\omega_B(z) j_C(z) + \frac{k}{2}\partial\omega^A(z)\,. \tag{1.68}$$

On the other hand, the KM of j with an infinitesimal parameter ω is realized by the KM current itself:

$$\delta j^A(z) = \frac{1}{2\pi i}\oint_z dw\, \omega^B(w) j_B(w) j^A(z)\,, \tag{1.69}$$

where the integral is taken over the contour around $w = z$. Matching these two expressions, we obtain the current algebra OPE

$$j^A(z) j^B(w) = \frac{\frac{k}{2}\eta^{AB}}{(z-w)^2} + \frac{i f^{ABC}}{z-w} j_C(w) + \ldots\,. \tag{1.70}$$

Similar expressions are true for the anti-holomorphic current algebra.

The holomorphic component of the stress-energy tensor is given by the Sugawara expression

$$T(z) = \frac{1}{\kappa} : j^A(z) j_A(z) := \frac{1}{\kappa} \frac{1}{2\pi i} \oint_z \frac{dx}{x-z} j^A(x) j_A(z) \,, \tag{1.71}$$

where the index in the adjoint representation of the corresponding algebra is defined as

$$c_V \delta^{AB} = f^{ACD} f^B{}_{CD} \,. \tag{1.72}$$

In (1.71) we have used a simple expression for normal ordering. The normalization constant κ will be fixed below.

The stress-energy tensor (1.71) satisfies the Virasoro OPE (1.13), with the central charge given by

$$c = \frac{k \dim G}{k + c_V} \,. \tag{1.73}$$

For completeness notice that in the supersymmetric WZW model we also have $\dim G$ free world-sheet fermions, each fermion contributes central charge $1/2$.

The infinitesimal conformal transformation $\delta z = \epsilon(z)$ acts on the current as

$$\delta j^A(z) = \partial \epsilon(z) j^A(z) + \epsilon(z) \partial j^A(z) \,. \tag{1.74}$$

On the other hand, this transformation is generated by the stress-energy tensor (1.71) as

$$\delta j^A(z) = \frac{1}{2\pi i} \oint_z dw \, \epsilon(w) T(w) j^A(z) \,. \tag{1.75}$$

After some algebra, using (1.71), one finds the OPE

$$T(z) j^A(w) = \frac{k + c_V}{\kappa} \left(\frac{j^A(w)}{(z-w)^2} + \frac{\partial j^A(w)}{z-w} \right) \,, \tag{1.76}$$

which therefore implies that the normalization constant is equal to

$$\kappa = k + c_V \,. \tag{1.77}$$

1.2.9 D-branes and black branes

One can perform a consistent truncation of type-II supergravity: set most of the fields to zero in such a way that the equations of motion are satisfied. Let us consider the following consistent truncation of type-IIB supergravity: the only non-vanishing fields are the metric $g_{\mu\nu}$, the R-R field C_4 with field strength $F_5 = dC_4$, and the dilaton ϕ. The action is

$$S = \frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{-g} \left[e^{-2\phi}(R + 4(\partial\phi)^2) - \frac{1}{4}|F_5|^2 \right] , \qquad (1.78)$$

where $|F_5|^2 = \frac{1}{5!} F_{\mu_1 \dots \mu_5} F^{\mu_1 \dots \mu_5}$.

Notice that the action is written for the string frame metric $g_{\mu\nu}$, which is related to the Einstein frame metric $G_{\mu\nu}$ by $g_{\mu\nu} = e^{\phi/2} G_{\mu\nu}$. The Einstein frame is defined so that the Ricci scalar term in the action is not multiplied by an exponent of the dilaton, $L = \sqrt{-G} R_G + \dots$. Suppose the dilaton is constant. It defines a closed string coupling constant, $g_s = e^{\phi}$. In the string frame the dimensions are measured in units of the string length, ℓ_s, and in the Einstein frame these are measured in the units of the Planck length, ℓ_p. We conclude that $1/\ell_s^2 = g_s^{1/2}/\ell_p^2$, and therefore

$$\ell_p = g_s^{1/4} \ell_s \, . \qquad (1.79)$$

The supergravity equations of motion following from the action (1.78) admit the 3-brane solution,

$$\begin{aligned} ds^2 &= -D_+(r) D_-(r)^{-1/2} dt^2 + D_-(r)^{1/2} (dx^2 + dy^2 + dz^2) \\ &\quad + \frac{dr^2}{D_+(r) D_-(r)} + r^2 d\Omega_5^2 \, , \\ D_\pm(r) &= 1 - \left(\frac{r_\pm}{r}\right)^4 , \quad e^{\phi} = g_s \, , \quad F_5 = Q(\omega_5 + \star\omega_5) \, , \end{aligned} \qquad (1.80)$$

where g_s is a constant, and ω_5 is the volume form of the unit five-sphere S^5.

The metric (1.80) is a generalization of a black hole metric to a higher-dimensional space. It describes a black brane, extended in R^3 space with (x, y, z) coordinates. This is generally a non-extremal black 3-brane, characterized by two radii parameters $r_\pm$. The condition of absence of a naked singularity at $r = 0$ demands $r_+ \geq r_-$.

We have two kinds of 3-branes now: the black 3-brane (1.80) and the $D3$-brane. They both are coupled to the R-R field C_4. The $D3$-brane is a supersymmetric BPS object: both its tension and R-R charge are equal to T_{D3}. While the black 3-brane (1.80) is generally not supersymmetric. When the black 3-brane is supersymmetric it becomes equivalent to the $D3$-brane. Let us see how the supersymmetry constraint on the black 3-brane comes about. The black 3-brane metric is supported by a flux of F_5 through the five-sphere surrounding the 3-brane in ten space-time dimensions. We conclude that the 3-brane carries N units of charge of the R-R field C_4, with $N = Q\,\mathrm{vol}(S^5)$. The mass and the R-R charge of the black brane (1.80) per unit volume of R^3 are given by

$$T_3 = \frac{1}{4(2\pi)^4 g_s^2 \ell_s^8 d_3}(5r_+^4 - r_-^4)\,, \quad N = \frac{(r_+ r_-)^2}{d_3 g_s \ell_s^4}\,, \tag{1.81}$$

where d_3 is a numerical factor. The condition $r_+ \geq r_-$ therefore becomes

$$T_3 \geq N T_{D3}\,, \quad T_{D3} = \frac{1}{(2\pi)^3 g_s \ell_s^4}\,, \tag{1.82}$$

where T_{D3} is the tension of the $D3$-brane. Equation (1.82) is precisely a supersymmetry BPS constraint: a single 3-brane (with $N = 1$) is supersymmetric if its tension T_3 is equal to its R-R charge T_{D3}.

We start the next section with a discussion of the extremal 3-brane metric and its near-horizon limit. We will also return to the consideration of the non-extremal 3-brane in the context of configurations with finite temperature.

1.3 Holographic correspondence

In this section we are going to review the holographic AdS/CFT correspondence: the equivalence between string theory on Anti-de Sitter (AdS) space and gauge field theory on the boundary of AdS space. We are also going to review the holographic correspondence between Little String Theory and closed string theory in the 'cigar' geometry.

1.3.1 The near-horizon limit

At the end of the previous section we derived the black three-brane solution (1.80) of type-IIB supergravity. Its metric has two horizons, and in the case when the horizons coincide we obtain an extremal black three-brane equivalent to the $D3$-brane of type-IIB superstring theory.

Let us re-write the metric of the $D3$-brane in the following form,

$$ds^2 = H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(dr^2 + r^2 d\Omega_5^2)\,, \quad H(r) = 1 + \left(\frac{R}{r}\right)^4\,, \tag{1.83}$$

with the horizon located at $r = 0$, and the scale parameter defined as

$$R^4 = 4\pi g_s N \alpha'^2\,, \tag{1.84}$$

where N is the R-R charge of the $D3$-brane, which is actually the number of coincident $D3$-branes. In the near-horizon limit, $r/R \ll 1$, we obtain

$$ds^2 = \left(\frac{r}{R}\right)^2 \eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_5^2\,. \tag{1.85}$$

This is the metric of the $AdS_5 \times S^5$ geometry.

1.3.2 AdS space and its symmetries

The AdS_5 space can be described as a surface in a six-dimensional flat space (with coordinates t_μ), with $(-,-,+,+,+,+)$ signature. The embedding is given by the equation

$$-t_1^2 - t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 = -R^2\,. \tag{1.86}$$

The parameter R is called the AdS scale. The group of transformations which leaves the surface (1.86) invariant is $SO(2,4)$.

Combining it with the symmetry group $SO(6)$ of the five-sphere we conclude that the subgroup of ten-dimensional diffeomorphisms which leaves the $AdS_5 \times S^5$ invariant is $SO(2,4) \times SO(6)$. Type-IIB superstring theory in the $AdS_5 \times S^5$ geometry is also invariant under 32 supersymmetries, the same amount as in ten-dimensional Minkowski space-time. The conserved supercharges split into $(\mathbf{4},\mathbf{4}) \oplus (\bar{\mathbf{4}},\bar{\mathbf{4}})$ under the covering bosonic symmetry group $SU(2,2) \times SU(4)$. The total supersymmetry group is therefore $PSU(2,2|4)$.

Let us now focus more on AdS space. Consider AdS_{d+1} space embedded into $d+2$-dimensional space,

$$-t_1^2 - t_2^2 + \sum_{i=1}^{d} y_i^2 = -R^2\,. \tag{1.87}$$

We can solve the equation (1.87) by $d+1$ independent coordinates, (x_0, x_a, z),

$$
\begin{aligned}
t_2 &= R\frac{x_0}{z}\,, \qquad t_1 = \frac{z}{2}\left(\frac{R^2}{z^2} + 1 + \frac{x_a^2 - x_0^2}{z^2}\right)\,, \\
y_a &= R\frac{x_a}{z}\,, \qquad y_d = \frac{z}{2}\left(\frac{R^2}{z^2} - 1 - \frac{x_a^2 - x_0^2}{z^2}\right)\,.
\end{aligned}
\tag{1.88}
$$

The coordinates (x_0, x_a, z) parametrize half of the AdS_{d+1} space, known as the Poincare patch. The boundary of AdS is located at $z = 0$. The metric is given by

$$
ds^2 = R^2 \frac{-dx_0^2 + dx_1^2 + \cdots + dx_{d-1}^2 + dz^2}{z^2}\,. \tag{1.89}
$$

Notice the presence of the horizon, $g_{tt}|_{z=\infty} = 0$, at $z = \infty$. This is the Poincare horizon originating from the choice of coordinates (1.88). To compare with (1.85) we make a change of the radial coordinate, $z = R^2/r$.

1.3.3 $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

The $AdS_5 \times S^5$ geometry appeared as the near-horizon geometry of N coincident $D3$-branes. Let us now look at the world-volume theory of the $D3$-branes. In the previous section it was explained that the low-energy theory of a Dp-brane is given by the DBI and CS actions for the $U(1)$ gauge field, scalars and fermions on the $p+1$-dimensional world-volume. Keeping only the lowest order terms in the α' expansion, we obtain the Lagrangian of supersymmetric Yang-Mills theory with $U(1)$ gauge group. For the N coincident Dp-branes we obtain non-abelian supersymmetric Yang-Mills theory with gauge group $U(N)$.

In the case of $D3$-branes there are six scalars Φ^I, $I = 1, \ldots, 6$, describing the fluctuations of the $D3$-branes in the six-dimensional transverse space. Together with the two transverse polarizations of the vector field on the four-dimensional world-volume, the total number of physical bosonic d.o.f. is therefore equal to eight and matches the number of physical d.o.f. of four fermions ψ^i, $i = 1, 2, 3, 4$ (recall that Weyl fermion in 4d has four independent components). All the fields live in the adjoint representation of the gauge group $U(N)$.

The $U(1)$ subgroup decouples from the rest of the $U(N)$ group. The resulting low-energy theory on the world-volume of $D3$-branes is $\mathcal{N} = 4$

supersymmetric $SU(N)$ Yang-Mills theory (SYM), with the Lagrangian schematically given by

$$L_{SYM}=-\frac{1}{4g_{YM}^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}+D_\mu\Phi^I D^\mu\Phi^I+\bar{\psi}^i\gamma^\mu D_\mu\psi^i+[\Phi^I,\Phi^J]^2+\dots\right), \tag{1.90}$$

where dots denote all extra interaction terms required by supersymmetry. This is a maximally supersymmetric four-dimensional gauge theory.

In four dimensions gauge theory is classically conformally invariant. Quantum corrections generally spoil conformal invariance creating renormalization group flow. However in the case of $\mathcal{N}=4$ SYM theory this turns out not to be the case; the Lagrangian (1.90) is exactly conformally invariant at quantum level. The conformal symmetry group in four dimensional space-time is $SO(2,4)$.

The R-symmetry subgroup of the $\mathcal{N}=4$ supersymmetry group is $SU(4)$. The fermions ψ^i live in the fundamental representation **4** of $SU(4)$, and the six scalars $\mathbf{\Phi}^I$ are rotated by $SO(6)\simeq SU(4)$. Therefore the covering bosonic symmetry group of $\mathcal{N}=4$ SYM is $SU(2,2)\times SU(4)$. The fermionic symmetry generators consist of four supersymmetry generators and four super-conformal generators. The latter appear in commutators of supersymmetry generators with special conformal generators. In total there are 32 conserved fermionic charges. The supersymmetry group is $PSU(2,2|4)$.

1.3.4 Large N limit

It turns out that when the number of colors N is sent to infinity, the $SU(N)$ gauge theory simplifies. We need to consider the 't Hooft coupling $\lambda=g_{YM}^2 N$ instead of the Yang-Mills coupling g_{YM}: as N is varied g_{YM} is varied accordingly so that λ remains unchanged. The reason for such rearrangement is that we want to obtain a sensible large N limit of Feynman diagrams. Consider for example the gluon one-loop correction to the gluon propagator. Each of the two three-gluon vertices contributes the factor of g_{YM}, and the loop contributes the factor of N. The diagram is therefore proportional to $g_{YM}^2 N$. It describes the lowest order term in the renormalization group flow of the YM coupling,

$$\frac{dg_{YM}}{d\log M}=b_0 g_{YM}^3 N+\dots \quad\Rightarrow\quad \frac{d\lambda}{d\log M}=2b_0\lambda^2+\dots. \tag{1.91}$$

and it has a smooth large N limit if we keep $\lambda = g_{YM}^2 N$ fixed as N is sent to infinity.

In the large N limit, non-planar Feynman diagrams, that is the diagrams which cannot be drawn on a plane (in the double-line notation), become sub-leading [5]. We refer the reader to chapter 2 of [6] for a nice review of the large-N limit of gauge theories.

In the large N limit, the low-energy gauge-singlet degrees of freedom are decoupled from each other. This is usually referred to as large-N factorization. The diagrams with two single-trace vertices can be either connected, when the vertices are linked to each other by internal lines, or disconnected, when the vertices are closed up on themselves. The latter diagrams are leading in the large-N limit. Nevertheless the theory is still non-trivial. For instance in the large-N QCD at low energies, one has free mesons. But the spectrum of mesons as well as the scaling dimensions of the meson operators are unknown, and the conventional derivation of these quantities by QFT means dealing with strongly coupled (and confining) dynamics of quarks within mesons. The spectrum of mesons can be read off from the poles of the two-point functions of the baryon current operators. In the large N limit the computation of such two-point functions requires summation of planar diagrams, which is an ill-defined procedure at strong coupling.

1.3.5 AdS/CFT correspondence

We have demonstrated in the previous subsections that the symmetry group $PSU(2,2|4)$ of type-IIB superstring theory on $AdS_5 \times S^5$ is the same as the symmetry group of $\mathcal{N} = 4$ SYM theory. It turns out that this matching is not accidental. The $\mathcal{N} = 4$ SYM theory is the low-energy theory of massless modes of open strings attached with both ends to the $D3$-branes. According to the AdS/CFT correspondence this theory is exactly equivalent to the type-IIB superstring theory in the near-horizon $AdS_5 \times S^5$ background created by the $D3$-branes [7]. One can in fact perform a reduction on the five-sphere S^5. This correspondence is holographic: the $\mathcal{N} = 4$ SYM theory lives in the four-dimensional Minkowski space-time, while closed type-IIB strings live in the five-dimensional bulk space, AdS_5. The field theory can be referred to as living on the four-dimensional boundary of the AdS_5 space.

The bulk side of the duality is gravitational, since gravity fields are the lowest (massless) modes of the closed string theory; the boundary side is non-gravitational, because gravity is not a part of the open string spectrum. The gauge coupling constant, g_{YM}, is the open string coupling constant, related to the closed string coupling constant, g_s, by the equation

$$g_{YM}^2 = 4\pi g_s \,. \tag{1.92}$$

Combining this equation with the equation (1.84) for the AdS scale we obtain (in this section string length is defined as $\ell_s = \sqrt{\alpha'}$)

$$\frac{\ell_s}{R} = \frac{1}{\lambda^{1/4}} \,. \tag{1.93}$$

The closed string excitations in the $AdS_5 \times S^5$ bulk can be neglected if

$$\frac{\ell_s}{R} \ll 1 \quad \Rightarrow \quad \lambda \gg 1 \,. \tag{1.94}$$

If the condition (1.94) is satisfied then the the bulk dynamics is well approximated by type-IIB supergravity.

Due to the equation (1.79) for the relation between the string scale and the Planck scale, we conclude that quantum gravity effects are negligible provided

$$\frac{\ell_p}{R} = \frac{1}{(4\pi N)^{1/4}} \ll 1 \quad \Rightarrow \quad N \gg 1 \,. \tag{1.95}$$

This is a consequence of the fact that at fixed λ and large N the closed string coupling constant g_s is small, and the bulk theory is classical.

We conclude that the large-N limit of strongly coupled $\mathcal{N} = 4$ SYM theory is dual to classical supergravity theory in the $AdS_5 \times S^5$ space.

1.3.6 Less supersymmetry, non-conformal field theories

We have reviewed the holographic duality between $\mathcal{N} = 4$ SYM theory and type-IIB string theory on $AdS_5 \times S^5$. The set of holographic dualities is not exhausted by this example. One can break a fraction or all of supersymmetries. One can consider holographic descriptions of non-conformal field theories. For example, one can turn on a finite temperature and/or chemical potential in the field theory, breaking supersymmetry and conformal invariance. The dual bulk geometry in this particular example is a charged black hole in AdS.

In the following subsections of this chapter we are going to review the holographic correspondence in the most general way, with a QFT on the boundary of asymptotically AdS space dual to a classical gravitational theory in the bulk. Then we proceed to the holographic description of Little String Theory.

1.3.7 Gubser-Klebanov-Polyakov-Witten formula

Consider quantum gauge field theory with the gauge-singlet operators $\mathcal{O}^I$. For example, one can deal with a charge current, fermionic bi-linear operator, baryon operator, glueball, stress-energy tensor, etc. Holographic duality maps these boundary QFT operators to the fields ϕ^I defined in the bulk of AdS space. One can ask a QFT question: what is the n-point function $\langle \mathcal{O}^{I_1} \dots \mathcal{O}^{I_n} \rangle$ equal to? In a strongly interacting system this question is generally impossible to find an answer to by conventional QFT means. One generally looks for a generating functional, $W[J^I]$, which depends on the sources J^I. In the Euclidean set-up it is defined as

$$e^{-W[J_I]} = \langle e^{J_I \mathcal{O}^I} \rangle_{QFT} \,, \tag{1.96}$$

where on the r.h.s. of (1.96) we have a path integral of the QFT with the operators $\mathcal{O}^I$ sourced by the external currents J^I. The n-point function is then

$$\langle \mathcal{O}^{I_1} \dots \mathcal{O}^{I_n} \rangle = \frac{\partial^n}{\partial J_{I_1} \dots \partial J_{I_n}} e^{-W[J_I]} \,. \tag{1.97}$$

The Gubser-Klebanov-Polyakov-Witten formula [8, 9] gives the holographic prescription for computation of the generating functional W,

$$e^{-W[J_I]} = Z_{string}|_{\phi_I(z=0)=J_I} \,, \tag{1.98}$$

where Z_{string} on the r.h.s. of (1.98) is the string partition function in the bulk, with the bulk fields ϕ_I fixed at the boundary $z = 0$ to the values of the sources J_I of the dual QFT operators $\mathcal{O}^I$. When the QFT is strongly coupled, due to (1.94), the string partition function can be approximated by the supergravity partition function. If the number of colors N is large supergravity is classical, see (1.95), and one can use a saddle point approximation,

$$Z_{string}{=}Z_{sugra}{=}e^{-S_{sugra}} \quad \Rightarrow \quad W[J_I] = S_{sugra}[\phi_I(z{=}0){=}J_I] \,. \tag{1.99}$$

In (1.99) the supergravity action is evaluated on the classical solution to the bulk equations of motion.

1.3.8 Probe branes and flavor

Let us add matter fields living in the fundamental representation of the gauge group. Ultimately it would be useful to apply holographic methods to QCD-like models, and this is why we have to know how to add quarks to the system. Quarks have a color and a flavor. These are realized as open strings attached with one end to N coincident color $D3$-branes and with the other end to F coincident flavor Dp-branes [10]. The lowest excitation modes of such strings are quarks, and they live in the fundamental representation of the $U(N)$ color group and the $U(F)$ flavor group. Taking $F/N \ll 1$ one can consider flavor branes as probes in the $AdS_5 \times S^5$ background. World-volume $U(F)$ degrees of freedom on the probe Dp-branes decouple from the $U(N)$ adjoint gauge fields and fundamental matter fields on the $D3$-branes.

Now we have a global flavor $U(F)$ symmetry on the field theory side of the duality. According to the AdS/CFT correspondence, a global symmetry in the QFT is mapped to a local symmetry in the bulk. In the case at hand the conserved $U(F)$ Noether currents are mapped to the $U(F)$ gauge d.o.f. on the Dp-brane world-volume.

In subsection 1.2.6 we discussed the system of intersecting branes and concluded that the interesting non-trivial cases are $D3/Dp$-brane systems with $p = 5, 7$, with three or four dimensional intersections. We also discussed the conditions for the non-broken supersymmetry in such systems. We are going to use these results in chapter 2, where we study probe brane matter at finite baryon density, strongly coupled to the $\mathcal{N} = 4$ gauge d.o.f. In that case the global (baryon) symmetry is represented holographically by the $U(1)$ gauge field on the world-volume of the probe brane. A finite density of bound states of strongly coupled quarks is dual to a non-trivial background $U(1)$ gauge field. Fluctuations of density are represented by fluctuations of this gauge field.

1.3.9 Finite temperature and chemical potential, thermodynamics

Suppose we have a four-dimensional gauge theory. In the IR it is described by the effective action, W, obtained in the Wilsonian framework by path integration over the high-energy modes of the fields. In subsection 1.3.7 we described the prescription of the AdS/CFT correspondence for computation of the effective action.

Let us turn on a temperature, T. The effective action is then replaced by the free energy, $\mathcal{F} = E - T\mathcal{S}$, where E is the energy and $\mathcal{S}$ is the entropy. We proceed further and turn on chemical potentials for some of the conserved charges. For example, we can consider a finite density of baryon matter in the IR. Then the free energy is replaced by the grand potential, $\Omega = E - T\mathcal{S} - \mu N$, where N is the number of baryons and μ is the chemical potential. In this subsection we are going to focus on this general case.

The AdS/CFT prescription for the grand potential of the field theory at finite temperature and chemical potential is a straightforward generalization of the GKPW formula (1.98). First of all, the effective action W is replaced by the grand potential Ω. Now, as the temperature in the CFT is turned on, the dual AdS_5 geometry gets replaced with the Schwarzshild black hole in AdS_5 space [11] (which is a dominant solution when the temperature is large enough). A finite chemical potential of a conserved charge is described holographically by a non-trivial background profile of the corresponding gauge field in the bulk. As a result we obtain a charged black hole in AdS space.

Suppose the boundary field theory is strongly coupled and the number of colors is large. We can use a saddle point approximation equating the grand potential with the bulk on-shell classical regularized action S:

$$T S_{on-shell} = E - T\mathcal{S} - \mu N \,. \tag{1.100}$$

1.3.10 Holographic description of Little String Theory

One way to introduce Little String Theory (LST) [12, 13] is to consider the low-energy theory on the world-volume of N coincident $NS5$-branes at a fixed energy scale and vanishing string coupling. Recall that $NS5$-branes arise both in type-IIA and type-IIB superstring theory as electro-magnetic duals of the superstring. A superstring couples electrically to the massless NS-NS two-form field $B_{\mu\nu}$, see (1.61). In nine spatial dimensions, a string is surrounded by seven-sphere. The charge of a string w.r.t. to the B field is equal to the flux of the Hodge dual of the field strength $H = dB$ through the seven-sphere. As a consistency check notice that the Hodge dual $\star H$ in ten dimensions is a seven-form.

Similarly, recall that a Dp-brane couples electrically to the C_{p+1} R-R field and magnetically to the C_{7-p} field. The object which couples magnetically to the B field is an NS5-brane. It is surrounded by a three-sphere, and its magnetic charge w.r.t. to the B-field is equal to the flux of H through this three-sphere. In type-IIB superstring theory, fluctuations of an $NS5$-brane are determined by a $D1$-brane attached to it by both ends, in type-IIA superstring theory a $D2$-brane attaches to an $NS5$-brane. S-duality between a string and a $D1$-brane is consistent with S-duality between an $NS5$-brane and a $D5$-brane. The S-dual of a string attached to a $D5$-brane is therefore a $D1$-brane attached to an $NS5$-brane.

The low-energy d.o.f. on an $NS5$-brane form a $U(N)$ gauge supermultiplet on a six-dimensional world-volume. The coupling constant is given by $g = \ell_s$; the theory is formulated at a fixed energy scale . Under the RG flow, the coupling constant of the six-dimensional theory grows in the UV. Of course in the UV more of the string d.o.f. should be added to the theory.

One can study LST holographically. We refer the reader to [14] for an extensive exposition of the subject and restrict here to a general outline. The background created by N coincident $NS5$-branes has the geometry $R^{5,1} \times R^{\phi} \times S^3$. Here $R^{5,1}$ is the world-volume of an $NS5$-brane, R^{ϕ} is the radial (bulk) direction and S^3 is the sphere surrounding the $NS5$-branes, with N units of the B-field flux threading through it. The radius of the three-sphere is $R = \sqrt{N}\ell_s$, and therefore string excitations in the bulk are suppressed when N is large. String theory on S^3 is given by the $SU(2)$ WZW model at level N. There is a background dilaton field depending linearly on the bulk radial coordinate ϕ.

The statement is that LST is holographically dual to closed string theory in the background of N coincident $NS5$-branes. In the double scaling limit this background is $R^5 \times \frac{SL(2,R)_N}{U(1)} \times SU(2)_N$, where $SL(2,R)_N/U(1)$ is a two-dimensional 'cigar' geometry with the linear dilaton [15]. The time coordinate is periodic, the corresponding temperature is $T = (2\pi\sqrt{N})^{-1}$. A string on $SL(2,R)_N/U(1)$ is described by the gauged WZW model on this coset space.

In chapter 4 we generalize this set-up to the situation of a non-vanishing charge density in the LST. To be precise, we do not know what is the field theory side of the holographic duality presented in chapter 4. Instead we study the bulk side of the duality, which is the gauged WZW model on $\frac{SL(2,R)_N \times U(1)}{U(1)}$. The classical geometry is a two-dimensional charged black hole, which is therefore dual to a field theory at finite charge density.

1.4 This thesis

Chapters 2, 3 and 4 of this thesis are based on the original research papers [16], [17] and [18] respectively, which I have written in collaboration with Dr. Andrei Parnachev and Prof. Dr. Jan Zaanen.

1.4.1 Chapter 2

In this chapter we study correlators of the global $U(1)$ currents in holographic models which involve $\mathcal{N} = 4$ SYM coupled to finite density matter in the probe brane sector. We find the spectral density associated with the longitudinal response to be exhausted by the zero sound pole and argue that this could be consistent with the behavior of a Fermi liquid with vanishing Fermi velocity. However the transversal response shows an unusual momentum independent behavior. Inclusion of magnetic field leads to a gap in the dispersion relation for the zero sound mode propagating in the plane of magnetic field. For small values of the magnetic field B, the gap in the spectrum scales linearly with B, which is consistent with Kohn's theorem for nonrelativistic fermions with pairwise interaction. We do not find signatures of multiple Landau levels expected in Landau Fermi liquid theory. We also consider the influence of generic higher derivative corrections on the form of the spectral function.

1.4.2 Chapter 3

In this chapter we investigate some phenomenological aspects of the holographic models based on the tachyon Dirac-Born-Infeld action in the AdS space-time. These holographic theories model strongly interacting fermions and feature dynamical mass generation and symmetry breaking. We show that they can be viewed as models of holographic walking technicolor and compute the Peskin-Takeuchi S-parameter and the masses of the lightest technimesons for a variety of tachyon potentials. We also investigate the phase structure at finite temperature and charge density. Finally, we comment on the holographic Wilsonian RG in the context of holographic tachyon DBI models.

1.4.3 Chapter 4

In this chapter we consider an exactly solvable worldsheet string theory in the background of a black brane with a gauge field flux. Holographically, such a system can be interpreted as a field theory with a finite number of degrees of freedom at finite temperature and density. This is to be contrasted with more conventional holographic models which involve gravity in the bulk and possess infinite number of degrees of freedom and mean field critical exponents. We construct closed string vertex operators which holographically represent the $U(1)$ gauge field and the stress energy tensor and compute their two-point functions. At finite temperature and vanishing charge density the low energy excitations are described by hydrodynamics. As the density is raised, the system behaves like a sum of two noninteracting fluids. We find low-energy excitations in the shear and sound channels of each fluid.

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Chapter 2

Fluctuations in finite density holographic quantum liquids

2.1 Introduction and summary

Perhaps the deepest open problem in condensed matter physics is the classification of compressible quantum liquids. This refers to stable states of zero temperature quantum matter that do not break any symmetry and support massless excitations. This question cannot be easily addressed within the confines of standard field theory. The issue arises when fermions are considered at finite density and the culprit is known as the "fermion sign" problem. Dealing with time-reversal symmetric finite density bosonic matter the methods of equilibrium statistical physics give a full control and invariably one finds that the ground states break symmetry. Dealing with incompressible quantum fluids like the fractional quantum Hall states the mass gap is quite instrumental to control the theory, revealing the profound non-classical phenomenon of topological order. The hardship is with the compressible quantum fluids: the only example which is fully understood is the Fermi-liquid.

The ease of the mathematical description of the Fermi-liquid as the adiabatic continuation of the Fermi-gas is in a way deceptive. Compared to classical fluids its low energy spectrum of non-charged excitations is amazingly rich. In addition to the zero sound, there is a continuum of volume conserving "shape fluctuations" of the Fermi-surface, corresponding with the particle-hole excitations (Lindhard continuum) of the conventional perturbative lore. Although serious doubts exist regarding the mathematical consistency and their relevance towards real physics, the "fractionalized (spin) liquids" that were constructed in condensed matter physics appear to be still controlled by the presence of a Fermi-surface while these are not Landau Fermi-liquids in the strict sense. This inspired Sachdev to put forward the interesting conjecture that the Fermi-surface might be ubiquitous for all compressible quantum liquids [1].

The gauge-gravity duality or AdS/CFT correspondence provides a unique framework to deal with these matters in a controlled way (see [1–6] for recent reviews). Although it addresses field theories that are at first sight very remote from the interacting electrons of condensed matter, there are reasons to believe that it reveals generic emergence phenomena associated with strongly interacting quantum systems. Field theories whose understanding is plagued by the "fermion sign" problem appear to be quite tractable in the dual gravitational description. With regard to unconventional Fermion physics, perhaps the most important achievement has been the discovery of the "AdS_2 metal" [7, 8], dual to the asymptotically AdS Reissner-Nordstrom black hole. On the field theory side this describes a local (purely temporal) quantum critical state that was not expected on basis of conventional field theoretic means. Although quite promising regarding the intermediate temperature physics (the "strange" normal states) in high Tc superconductors and so forth, this AdS_2 metal is probably not a stable state, given its zero temperature entropy. Much of recent activity has been devoted to the study of the instability of this metal towards bosonic symmetry breaking (holographic superconductivity [9], "stripe" instabilities [10]) and towards the stable Fermi liquid [11–13].

The top-down constructions might become quite instrumental in facilitating the search for truly new quantum liquids. An important category are the Dp/Dq brane intersections; the $p = 3$ case provides us with a set of especially tractable examples. The dynamics of the low energy degrees of freedom of the D3-Dp strings can be studied in the probe approximation where the back-reaction to the $AdS_5 \times S^5$ geometry can be neglected [14]. In this chapter we will consider D3 and Dp branes intersecting along 2+1 dimensions, where p=5 (p=7) corresponds to the (non)-supersymmetric system. As emphasized in [15] the nonsupersymmetric system can be viewed as a model of graphene: the brane intersection fermions are like the Dirac fermions moving on the 2+1D graphene backbone, (tunable to finite density by gating), interacting strongly through the gauge fields living in 3+1 dimensions. We will present a number of results for the longitudinal- and transversal *dynamical* charge susceptibilities (at finite frequency $\mathbf{w}$ and momenta $\mathbf{q}$)[1], in the absence and presence of a magnetic field, for both the supersymmetric and non-supersymmetric D3/Dp systems at finite density. We find very similar results in both the supersymmetric- and fermionic set ups, showing that these outcomes at strong 't Hooft coupling are not caused by the difference in the Lagrangians. We find suggestive indications for the presence of an entirely new form of quantum liquid, but we cannot be entirely conclusive. Our observations cannot entirely rule out the existence of a Fermi liquid with vanishing Fermi velocity.

In fact, the first study of these systems at finite density already produced evidence that some odd state is created. In ref. [16] it was observed that the density-dependent part of the heat capacity in the D3/Dp systems with $2+1$ dimensional intersection behaves like T^4. This is in contrast to the result for the Fermi-liquids which is set by the Sommerfeld law of the specific heat $C = \gamma T$, where the Sommerfeld coefficient γ is proportional to the quasiparticle mass. This behavior remains to be understood: for example, it is conceivable that the linear term in the heat capacity exists, but is parametrically suppressed in the holographic model. On a side, it is worth noting that in the context of pnictide superconductivity a rogue signal has been detected that refuses to disappear: this indicates that the electronic specific heat of the metal state $\sim T^3$ [17].

As mentioned above, besides the Lindhard continuum an interacting Fermi liquid will carry a single propagating mode called zero sound. Unlike the usual sound at finite temperature, translational invariance alone is not sufficient for establishing the existence of the zero sound mode. The discovery of zero sound associated with the brane intersection matter [16] is therefore significant. The fate of the holographic zero sound was further studied in [18–25] (see also [26, 27] for closely related work). At very low temperature the attenuation (damping) of this zero sound behaves like the ("collisionless") Fermi liquid zero sound, in the sense that it increases like the square of its momentum. In [24] it was found also that upon increasing temperature the zero sound velocity decreases while the attenuation increases, turning into a purely diffusive pole at high temperatures. This is different from the crossover from zero sound to ordinary sound as

[1] In this chapter we denote the values of frequency and momentum by bold letters. The usual letters, defined below, are reserved for dimensionless variables.

function of temperature in a single component Fermi-system like ^3He. In the brane intersection systems momentum is shared between the superconformal strongly coupled uncharged sector and the material system on the intersection, and the latter does not support hydrodynamical sound in isolation. Somehow, upon lowering temperature the momentum of the brane intersection matter becomes separately conserved, facilitating the emergence of the zero sound in the low temperature limit.

Given that zero sound is rather ubiquitous, one would like to obtain more direct information regarding the density fluctuations of the quantum liquid. These are expected to be contained in the fully dynamical, momentum and energy dependent charge susceptibility/density-density propagator associated with the conserved charge on the brane-intersection. One strategy is to look for the momentum dependence of the reactive response (real part) at zero frequency: one expects a singularity at twice the Fermi momentum, $2\mathbf{q}_F$ where the Luttinger's theorem implies that $\mathbf{q}_F$ is set by the bare chemical potential, $\mathbf{q}_F \sim \mu$. A number of papers has been devoted to the search of such singular behavior in the framework of AdS/CFT. In [19] the $\langle J^0 J^0 \rangle$ correlator has been computed in the holographic setup where the only charged degrees of freedom are four-dimensional fermions. The resulting function was completely smooth. In [28–31] the two-point function for global currents was computed for various systems and again the tree-level computation in the bulk did not show any nonanalytic behavior. Very recently it has been argued that a singularity can be observed in the systems where an exact result to all orders in α' is available [32].

Searching for the singularity at $2\mathbf{q}_F$ is in principle a tricky procedure because these "Friedel oscillation" singularities are strongly weakened by the self energy effects in the strongly interacting Fermi-liquid. Another way to probe for the signatures of the Fermi liquid is to compute the imaginary part of the dynamical density susceptibility in a large kinematical window because this spectral function shows directly the density excitations of the system. The result is well known in the weakly interacting Fermi liquid, see Fig. 1: besides the zero sound pole one finds the Lindhard continuum of particle hole excitations. It is worth noting that as the value of the Landau parameter F_0 increases, the spectral weight in the density response is increasingly concentrated in the zero sound poles, "hiding" the Lindhard continuum. In this regard the *transversal* density propagator is quite informative: since in this channel no collective modes are expected to form, this is the place to look for the incoherent Fermi-surface fluctuations. Unfortunately technical issues prevent us from accessing the regime of parametrically small Fermi velocity. Our holographic computations of the longitudinal and transversal dynamical charge susceptibilities are limited to a kinematical window where $\mathbf{w} \sim |\mathbf{q}|$.

Despite this caveat, the holographic density propagators that we compute reveal very interesting information. We find that the longitudinal density propagator is within our numerical resolution completely exhausted by the zero sound pole (Fig. 4). Regardless the precise nature of the underlying state this signals very strong density/density correlations in this liquid. The transversal charge propagator shows that sound is not the whole story. The "other stuff", albeit very unlike a Lindhard continuum, signals the presence of a sector of highly collective, deep IR density fluctuations: the imaginary part of the transversal propagator behaves like $\chi_t^{(i)}(\mathbf{q}, \mathbf{w}) \sim \mathbf{w}$. This response is surprisingly momentum independent and suggests local quantum criticality, which was instrumental in the "AdS_2 metal" setup. All of this seems to imply that we are indeed dealing with some entirely new quantum liquid.

To probe some of the features of this quantum liquid, we introduce an external magnetic field which is a valuable "experimental tool". This induces the gap in the spectrum that is visible in the holographic calculations. Dealing with a 2+1D Fermi-liquid one would expect the signatures of Landau levels also in the density response. In the strongly interacting system, the longitudinal response should reveal the "magneto-roton", the left over of zero sound in the system with a magnetic field which is well known from (fractional) quantum Hall systems [33] [2]. According to Kohn's theorem [35], the density spectrum should show a gap equal to the cyclotron frequency at zero momentum. Note that this theorem is very generic and only assumes that degrees of freedom, charged under the magnetic field, interact pairwise. Our holographic calculation reveals that: i) at small values of the magnetic field B the value of the gap[3] scales linearly with B, which is consistent with Kohn's theorem for the nonrelativistic fermions and ii) there are no signatures of Landau levels associated with incoherent particle-hole excitations (Fig. 2).

The remainder of this chapter is organized as follows. The next sec-

[2]See [34] for related work in the context of holography.

[3]This is also consistent with the observations made in [23, 36] where the same D3/D7 system, modified by the inclusion of flux through the internal cycles, is considered.

tion is devoted to the review of Landau Fermi liquid theory including the random phase approximation (RPA) for the dynamical response. In particular, we review the appearance of the zero sound mode in the RPA calculation of the density-density correlator. As the value of the interaction strength increases, the Lindhard continuum gets separated from the zero sound pole (Fig. 1) and gradually disappears. In the extreme limit of vanishing Fermi velocity, the spectral density is completely exhausted by the zero sound mode. We also review the RPA expectations for the 2+1 dimensional fermion system in the presence of magnetic field. There we expect Landau levels to contribute to the spectral density (Fig. 2).

In Section III we review the holographic description of the D3/Dp brane systems. The subject of our interest is the fermion matter, which is formed (at finite chemical potential for the fermion number) in the low energy theory living on intersection of the N_c $D3$ branes and N_f Dp branes. We consider the case of $N_c \gg N_f \sim 1$ and strong 't Hooft coupling λ, where the holographic description is applicable.

In Section IV we focus on the zero sound mode and show that it develops a gap in the presence of magnetic field. In the case of vanishing magnetic field, $B = 0$, we observe a zero sound mode whose speed is the same as that of the first sound. As long as the value of the magnetic field B is small compared to $\mathbf{w}^2, \mathbf{q}^2$ (in appropriate units), the sound mode peak in the spectral function is not significantly affected. On the other hand, the presence of the nonvanishing magnetic field leads to a gap in the dispersion relation for zero sound. (The effective action proposed by Nickel and Son [37] in the presence of the magnetic field gives vanishing sound velocity). In the regime of small magnetic field we derive the scaling behavior of the gap in the spectrum $\mathbf{w}_c$ as a function of magnetic field. The result, $\mathbf{w}_c \sim B$ is consistent with fermions acquiring an effective mass.

In Section V we investigate the current-current correlator at nonvanishing frequency $\mathbf{w}$ and momentum $\mathbf{q}$. We observe that in the longitudinal channel, the only nontrivial structure both in the real and in imaginary parts of the correlators is provided by the zero sound. There is no nontrivial structure in the transverse correlators when $B = 0$. We discuss our results in Section VI.

In Appendix we consider higher derivative corrections and show that when they are added to the DBI the correlators are not significantly modified.

2.2 Fermi liquid and the random phase approximation

In this section we review the application of the random phase approximation (RPA) for the computation of the density-density response function $\langle J_0(\mathbf{w}, \mathbf{q}) J_0(-\mathbf{w}, -\mathbf{q})\rangle$ in Landau Fermi liquid theory. We consider the 2+1 dimensional theory for both cases of vanishing and non-vanishing magnetic field.

Due to the interaction of quasiparticles, the variation of quasiparticle energy due to small perturbation of the distribution function, is given by (see, e.g, [38])

$$\delta\varepsilon(\mathbf{q}) = \int d\mathbf{q}' f(\mathbf{q}, \mathbf{q}')\delta n(\mathbf{q}') \tag{2.1}$$

Because the small changes of quasiparticle density occur in the vicinity of a Fermi surface, one considers the function $f(\mathbf{q}, \mathbf{q}')$ to be dependent on the momenta on the Fermi surface, and therefore it boils down to a function of the angle between $\mathbf{q}$ and $\mathbf{q}'$:

$$\frac{m^*}{\pi} f(\theta) = 2F(\theta)\,. \tag{2.2}$$

where, as usual, the effective mass at the Fermi surface is defined via

$$m^* = \frac{\mathbf{q}_F}{v_F}, \qquad v_F = \frac{\partial\epsilon(\mathbf{q})}{\partial\mathbf{q}}|_{\mathbf{q}=\mathbf{q}_F} \tag{2.3}$$

Landau parameters F_l are the coefficients of the expansion of $F(\theta)$ in Legendre polynomials:

$$F(\theta) = \sum_l (2l+1) F_l P_l(\cos\theta) \tag{2.4}$$

The Fermi liquid has a collective excitation at vanishing temperature called zero sound. In the case of $F_l = 0,\ l > 0$, the speed of zero sound u_0 can be determined from

$$\frac{s}{2}\log\frac{s+1}{s-1} - 1 = \frac{1}{F_0}, \qquad s = \frac{u_0}{v_F} \tag{2.5}$$

which, in the limit $F_0 \gg 1$ gives $s \sim \sqrt{F_0}$.

To compute the dynamical collective responses of a Fermi liquid, one evaluates the time dependent mean field (random phase approximation) obtained by summing up the quasiparticle “bubble” diagrams. Assuming for simplicity only the presence of a contact interaction, with effective coupling constant $V \simeq F_0$, the nth diagram is equal to $V^{n-1}(\chi_0(\mathbf{q},\mathbf{w}))^n$. The susceptibility in the RPA is then given by the sum of a geometric progression:

$$\chi(\mathbf{q},\mathbf{w}) = \frac{\chi_0(\mathbf{q},\mathbf{w})}{1 - V\chi_0(\mathbf{q},\mathbf{w})}\,, \tag{2.6}$$

Express $\chi = \chi^{(r)} + i\chi^{(i)}$, hence

$$\chi^{(i)}(\mathbf{q},\mathbf{w}) = \frac{\chi_0^{(i)}(\mathbf{q},\mathbf{w})}{(1 - V\chi_0^{(r)}(\mathbf{q},\mathbf{w}))^2 + (\chi_0^{(i)}(\mathbf{q},\mathbf{w}))^2}\,. \tag{2.7}$$

Then we study density of excitations by plotting $\chi^{(i)}(\mathbf{q},\mathbf{w})$. The result for vanishing magnetic field is presented in Fig. 3.1, where we plot the susceptibility (for $q_F = 0.2$) at strong and weak coupling V. In the case

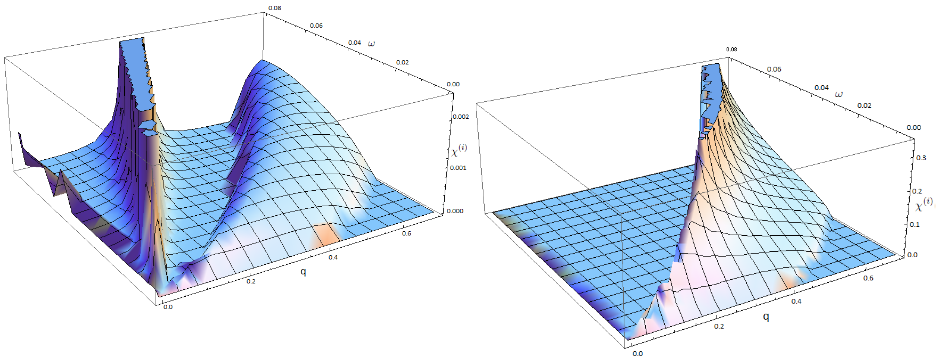

Figure 2.1. Spectral density $\chi^{(i)}(\mathbf{q},\mathbf{w})$ at strong coupling ($V = 50$, left graph) and weak coupling ($V = 3$, right graph) in the random phase approximation, at vanishing magnetic field. Fermi momentum is put to $q_F = 0.2$. Note that at strong coupling zero sound is well separated from the particle-hole continuum, while at weak coupling zero sound merges with the left edge of the particle-hole continuum. At small frequencies particle-hole continuum sharply ends at $\mathbf{q} = 2\mathbf{q}_F$.

of strong coupling there is a finite gap, separating the zero sound collective mode, and the band of the particle-hole excitations. For given small frequency $\mathbf{w}$ the width of the gap is given by $\delta\mathbf{q} \simeq \frac{\mathbf{w}}{u_0}(s-1)$. Note the non-analytic step behavior at $\mathbf{q} = 2\mathbf{q}_F$, originating from the free response function $\chi_0^{(i)}(\mathbf{q},\mathbf{w})$. In the case of weak coupling the zero sound mode merges with the left edge of particle-hole band.

The location of zero sound pole is determined as a solution to equations $\chi_0^{(i)}(\mathbf{q},\mathbf{w}) = 0$, $\chi_0^{(r)}(\mathbf{q},\mathbf{w}) = 1/V$. The real part $\chi_0^{(r)}(\mathbf{q},\mathbf{w})$ of Lindhard function for 2D Fermi gas is given by (see, e.g., [39]):

$$\chi_0^{(r)}(\mathbf{q},\mathbf{w}) = -\left(1 + \frac{\mathbf{q}_F}{\mathbf{q}}\left[\mathrm{sign}(\nu_-)\theta(|\nu_-|-1)\sqrt{\nu_-^2-1}\right.\right.$$
$$\left.\left. - \ \mathrm{sign}(\nu_+)\theta(|\nu_+|-1)\sqrt{\nu_+^2-1}\right]\right), \tag{2.8}$$

where $\nu_\pm = \frac{\mathbf{w}\pm\varepsilon_\mathbf{q}}{\mathbf{q}v_F}$. For large $\frac{\mathbf{w}}{\mathbf{q}v_F} = s \gg 1$ one may expand

$$\chi_0^{(r)}(\mathbf{q},\mathbf{w}) \simeq \frac{\mathbf{q}^2 v_F^2}{2\mathbf{w}^2}. \tag{2.9}$$

Therefore, for the speed of zero sound one obtains $s = \sqrt{V/2}$, exactly as it follows at large F_0 from the equation (2.5).

Suppose now that besides F_0 there is also non-vanishing "mass" Landau parameter F_1. In the relativistic case, the value of m^* is related to the value of the chemical potential [40],

$$m^* = \mu\left(1 + \frac{F_1}{3}\right) \tag{2.10}$$

The speed of zero sound u_0 then satisfies equation

$$\frac{s}{2}\log\frac{s+1}{s-1} - 1 = \frac{1+F_1/3}{F_0 + F_0F_1/3 + F_1 s^2}, \qquad s = \frac{u_0}{v_F}. \tag{2.11}$$

For free fermions in a magnetic field B, the Lindhard function is equal to (see, e.g., [39])

$$\chi_0(\mathbf{q},\mathbf{w}) = \frac{1}{2\pi\ell^2}\sum_{n,n'}\frac{f(\varepsilon_n) - f(\varepsilon_{n'})}{\mathbf{w} + (n-n')\mathbf{w}_c + i\eta}|F_{n',n}(\mathbf{q})|^2, \tag{2.12}$$

where

$$F_{n',n}(\mathbf{q}) = \sqrt{\frac{n!}{n'!}}\left(\frac{(\mathbf{q}_y - i\mathbf{q}_x)\ell}{\sqrt{2}}\right)^{n'-n} e^{-\mathbf{q}^2\ell^2/4} L_n^{n'-n}\left(\frac{\mathbf{q}^2\ell^2}{2}\right), \tag{2.13}$$

for $n' \geq n$. Here we have introduced the cyclotron frequency $\mathbf{w}_c = B/m^\star$ and the magnetic length $\ell = \frac{1}{\sqrt{B}}$. The functions $L_n^{n'-n}$ are Laguerre polynomials, and $f(\varepsilon_n)$ is an occupation number for the *nth* Landau level.

We would like to compute the effect of the magnetic field on the density-density response function of the interacting fermions. Let us write the quasiparticle interaction Hamiltonian

$$H_{int} = \sum_{\mathbf{q}} V_{\mathbf{q}} n_{\mathbf{q}} n_{-\mathbf{q}} \tag{2.14}$$

in the basis of Landau levels wavefunctions. The corresponding matrix elements of the density fluctuation operator $n_{\mathbf{q}} = \sum_{\mathbf{k}} c_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{q}}$ are given by

$$\langle n'\mathbf{k}'_y | n_{\mathbf{q}} | n\mathbf{k}_y \rangle = \exp\left(-i \frac{\mathbf{q}_x(\mathbf{k}_y + \mathbf{k}'_y)\ell^2}{2} \right) F_{n'n}(\mathbf{q}) \delta_{\mathbf{k}_y - \mathbf{k}'_y, \mathbf{q}_y} \,. \tag{2.15}$$

The density fluctuation operator in the basis of Landau level wavefunctions is then given by

$$n_{\mathbf{q}} = \sum_{n, \mathbf{k}_y,\, n', \mathbf{k}'_y} \langle n'\mathbf{k}'_y | n_{\mathbf{q}} | n\mathbf{k}_y \rangle c_{n\mathbf{k}_y} c^\dagger_{n'\mathbf{k}'_y} \,. \tag{2.16}$$

Note that

$$\Big(\langle n\mathbf{k}_y | n_{\mathbf{q}} | n'\mathbf{k}'_y \rangle \Big)^\star = \langle n'\mathbf{k}'_y | n_{-\mathbf{q}} | n\mathbf{k}_y \rangle \tag{2.17}$$

implies $(n_{\mathbf{q}})^\dagger = n_{-\mathbf{q}}$. Substituting (2.16) into the interaction Hamiltonian (2.14), assuming again only a contact interaction of plane waves $V_{\mathbf{q}} \equiv V \simeq F_0$, and considering all quasiparticles in the same Landau level n, one obtains

$$\begin{aligned} H_{int} = V \sum_{\mathbf{q}, \mathbf{k}_y, \mathbf{k}'_y} & c_{n\mathbf{k}_y} c^\dagger_{n\mathbf{k}_y - \mathbf{q}_y} c_{n\mathbf{k}'_y} c^\dagger_{n\mathbf{k}'_y + \mathbf{q}_y} \\ & \exp\left(-i\ell^2 \mathbf{q}_x (\mathbf{k}_y - \mathbf{k}'_y - \mathbf{q}_y) - \frac{\mathbf{q}^2 \ell^2}{2} \right) [L_n^0(\mathbf{q}^2\ell^2/2)]^2 \,. \end{aligned} \tag{2.18}$$

Let us choose the momentum to be in y-direction, then

$$H_{int} = \sum_{\mathbf{q}_y, \mathbf{k}_y, \mathbf{k}'_y} V_{\mathbf{q}_y} c_{n\mathbf{k}_y} c^\dagger_{n\mathbf{k}_y - \mathbf{q}_y} c_{n\mathbf{k}'_y} c^\dagger_{n\mathbf{k}'_y + \mathbf{q}_y} \,, \tag{2.19}$$

where $V_{\mathbf{q}_y} = [L_n^0(\mathbf{q}^2\ell^2/2)]^2 \exp\left(-\frac{\mathbf{q}^2\ell^2}{2} \right) V$.

We can explicitly demonstrate that the zero sound mode is gapped in the magnetic field, with the gap being equal to $\mathbf{w}_c$, in agreement with the Kohn's theorem [35]. For this aim we are to solve equation $\chi_0^{(r)}(\mathbf{q},\mathbf{w}) = 1/V_{\mathbf{q}}$ again. From (2.12), (2.13) one may obtain the following expression for $\chi_0^{(r)}$:

$$\chi_0^{(r)}(\mathbf{q},\mathbf{w}) = \frac{e^{-\mathbf{q}^2\ell^2/2}}{2\pi\hbar\ell^2}\sum_{k=1}^{\infty}\sum_{j}{}' \frac{j!}{(j+k)!}\left(\frac{\mathbf{q}^2\ell^2}{2}\right)^k \left[L_j^k\left(\frac{\mathbf{q}^2\ell^2}{2}\right)\right]^2 \frac{2k\mathbf{w}_c}{\mathbf{w}^2-(k\mathbf{w}_c)^2}\,, \tag{2.20}$$

where the prime denotes summation in the range $\max(0,\nu-k)\le j\le\nu$, and ν is the number of occupied Landau levels. Following [39], we consider this equation for small $\mathbf{q}$ and $\mathbf{w}\simeq\mathbf{w}_c$. Then the main contribution in the sum over k comes from the term with $k=1$, and we obtain equation:

$$\mathrm{const}\frac{\mathbf{q}^2}{\mathbf{w}^2-\mathbf{w}_c^2}\simeq\frac{1}{V}\,, \tag{2.21}$$

and therefore the zero sound dispersion relation is given by

$$\mathbf{w}=\sqrt{\mathbf{w}_c^2+c\mathbf{q}^2}\,, \tag{2.22}$$

where $c\sim V\mathbf{w}_c$ is a constant. Similarly, for any integer M, there is a mode with dispersion relation

$$\mathbf{w}=\sqrt{(M\mathbf{w}_c)^2+c'\mathbf{q}^{2M}}\,. \tag{2.23}$$

We plot RPA computations of two-point function, for $\omega_c=0.25$, restricting to the first two first branches, in Fig. 2.2.

2.3 *Dp* brane in $AdS_5\times S^5$ background

We study strongly interacting massless fermions at zero temperature and finite density. A good field theoretical model of such a system is $\mathcal{N}=4$ SYM theory with gauge group $SU(N_c)$, coupled to matter in the fundamental representation. A convenient way to study strongly coupled theories is provided by holography where one considers a dual gravitational theory, taking the limit of large 't Hooft coupling $\lambda=g_{YM}^2N_c$, and

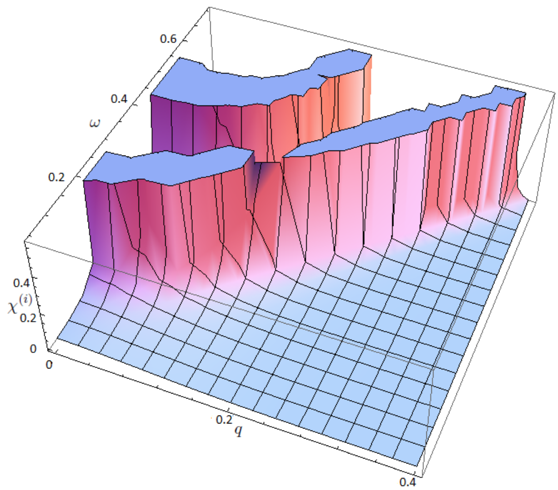

Figure 2.2. Spectral density in the random phase approximation of the $2+1$ dimensional Fermi liquid in the plane of the magnetic field, with cyclotron frequency $\omega_c = 0.25$. First two of infinitely many collective excitation branches are shown. Each branch starts at $(q = 0,\, \omega = M\omega_c)$, where M is an integer.

the limit of large N_c. The dual gravitational background is created by $N_c \gg 1$ $D3$ branes, and has an $AdS_5 \times S^5$ geometry. The coupling to fundamental matter is realized by considering an embedding of a probe Dp brane in the $AdS_5 \times S^5$ background [14]. We will consider $D3/Dp$ configurations with $d = 2+1$ dimensional intersections.

Let us now provide a more detailed description of the bulk gravitational theory set-up. Consider $AdS_5 \times S^5$ geometry, with the metric

$$ds^2 = L^2 \left(r^2(-dt^2 + dx_\alpha dx^\alpha) + \frac{dr^2}{r^2} + d\Omega_5^2 \right) . \tag{2.24}$$

Here L is the radius of S^5 and scale of curvature of AdS_5. We will study the probe Dp brane, embedded in the geometry described by (2.24). We represent the metric on S^5 as

$$d\Omega_5^2 = d\Omega_n^2 + \sin^2\tilde{\theta} d\Omega_{5-n}^2 = d\theta^2 + \sin^2\theta d\Omega_{n-1}^2 + \cos^2\theta d\Omega_{5-n}^2 \,,$$

where $n = p + 1 - d$. Then we define coordinates $\rho\,, f$ via the relation

$$\rho = r\sin\theta\,, \qquad f = r\cos\theta\,, \quad r^2 = \rho^2 + f^2\,, \tag{2.25}$$

and write

$$d\theta^2 = \frac{(f - \rho\,\partial_\rho f)^2}{r^4} d\rho^2\,, \quad dr^2 = \frac{(\rho + f\,\partial_\rho f)^2}{r^2} d\rho^2\,, \tag{2.26}$$

which gives the following induced Dp brane world-volume metric

$$ds^2_{Dp} = L^2 \left(r^2(-dt^2 + dx_i dx^i) + \frac{1}{r^2}\left(1 + (\partial_\rho f)^2\right) d\rho^2 + \frac{\rho^2}{r^2} d\Omega^2_{n-1} \right) . \tag{2.27}$$

The coordinate $f(\rho)$ defines an embedding of the Dp brane in the AdS background (2.24). In the case of the trivial embedding $f(\rho) \equiv 0$, which is what we are going to deal with in this chapter, Dp brane crosses the Poincaré horizon of the AdS space. In the case of $d = 3$ $p = 7$ such a configuration becomes stable only for sufficiently large values of chemical potential $\bar{\mu}_{ch}$ in the dual field theory [41]. (See also [42] for the phase structure of the similar model in the presence of the magnetic field.) Note that holographically computed correlators do not depend on the dimensionality of the probe brane; in particular our results apply in the case of stable supersymmetric D3/D5 defect theory.

Subsequently we add a gauge field A_μ on the world-volume of the probe $D7$ brane. In general we are interested in non-vanishing magnetic field B. So we consider the following components of the field strength:

$$F_{12} = B\,, \qquad F_{0\rho} = -\partial_\rho A_0(\rho)\,. \tag{2.28}$$

Consequently the DBI action for the Dp brane is given by [4]

$$S_{DBI} \simeq \frac{N_c}{L^4} \int d^{p+1}x \sqrt{-\det(G + F)} = \int d\Omega_{n-1} \int d^d x\, S\,, \tag{2.29}$$

where we have denoted

$$S \simeq N_c L^{p-5} \int d\rho \rho^{d-3} \sqrt{(L^4\rho^4 + B^2)(1 - (\partial_\rho A_0)^2 L^{-4})}\,. \tag{2.30}$$

Now rescale gauge field on the world-volume as

$$\bar{A}_\mu = \frac{A_\mu}{L^2}\,, \tag{2.31}$$

[4] We adopt the convention $2\pi\alpha' = 1$. For our purposes we are ignoring the total numerical coefficient, which leaves us with an overall normalization of the action proportional to $\frac{1}{g_s} \sim \frac{N_c}{\lambda} \sim \frac{N_c}{L^4}$.

which yields the DBI action in the form,

$$S \simeq N_c L^{p-3} \int d\rho \rho^{d-3} \sqrt{(\rho^4 + \bar{B}^2)(1 - (\partial_\rho \bar{A}_0)^2)}\,, \qquad (2.32)$$

where $\bar{B} = B/L^2$.

In the case of a non-vanishing magnetic field there is also a Chern-Simons term in the total action for the Dp brane. It can be shown that this term vanishes in the case of $f \equiv 0$ embedding.

The boundary value of $\bar{A}_0$ is equal to the chemical potential of the dual field theory: $\bar{A}_0(\rho = \infty) = \bar{\mu}_{ch}$. Due to $f(\rho = 0) = 0$ and the initial condition $\bar{A}_0(r = 0) = 0$ (imposed to ensure that chemical potential vanishes when the charge density is zero) we obtain $\bar{A}_0(\rho = 0) = 0$, and therefore the chemical potential may be expressed as

$$\bar{\mu}_{ch} = \int_0^\infty d\rho\, \partial_\rho \bar{A}_0\,. \qquad (2.33)$$

Introducing a constant of integration $\hat{d}$, the solution of the equation of motion for $\partial_\rho \bar{A}_0$ field strength becomes,

$$\partial_\rho \bar{A}_0 = \frac{\hat{d}^2}{\sqrt{\hat{d}^4 + \rho^4 + \bar{B}^2}}\,. \qquad (2.34)$$

Using this expression and eq. (2.33), we obtain the value of the chemical potential

$$\bar{\mu}_{ch} = \int_0^\infty d\rho\, \partial_\rho \bar{A}_0 = \frac{4\Gamma(5/4)^2}{\sqrt{\pi}} \frac{\hat{d}^2}{(\hat{d}^4 + \bar{B}^2)^{1/4}}\,. \qquad (2.35)$$

2.4 Holographic zero sound

In this and the next sections we study $D3/Dp$ system with $d = 2 + 1$ dimensional intersection, described by trivial $f(\rho) \equiv 0$ embedding of the probe Dp brane in the $AdS_5 \times S^5$ background. We consider the gauge field on the Dp brane world-volume, solve its classical equations of motion and use AdS/CFT to find the two-point functions of the $U(1)$ current in the dual field theory. In this section we show the existence of holographic zero sound in the $D3/Dp$ configuration, to observe that it develops a gap as the magnetic field is turned on. In the next section we will study the current-current correlation function numerically.

2.4.1 Zero sound in the $D3/Dp$ system with $d = 2+1$ dimensional intersection

Equation (2.34) is the expression for the background field strength $\partial_\rho \bar{A}_0$. Let us turn on small fluctuations $\bar{a}_0$, $\bar{a}_1$, $\bar{a}_2$, dependent on coordinates x^0, x^2, ρ. In addition let us fix the gauge $\bar{a}_\rho = 0$. The longitudinal response is described holographically by the $\bar{a}_0$ and $\bar{a}_2$ components of the gauge field, and the transverse response is described by the $\bar{a}_1$ component. The DBI action, expanded up to the second order in fluctuations, then takes the form [5]

$$S{=}\int d\rho \left(\sqrt{\frac{\rho^4{+}\bar{B}^2}{1{-}(\partial_\rho \bar{A}_0)^2}}\left(-\frac{(\partial_\rho \bar{a}_0)^2}{1{-}(\partial_\rho \bar{A}_0)^2}{+}\frac{\rho^4(\partial_\rho \bar{a}_2)^2{-}(\partial_0\bar{a}_2{-}\partial_2\bar{a}_0)^2}{\rho^4{+}\bar{B}^2}\right){+}\right.$$
$$+\sqrt{\frac{1-(\partial_\rho \bar{A}_0)^2}{\rho^4+\bar{B}^2}}\left(\frac{\rho^4(\partial_2\bar{a}_1)^2}{\rho^4+\bar{B}^2}+\frac{\rho^4(\partial_\rho\bar{a}_1)^2-(\partial_0\bar{a}_1)^2}{1-(\partial_\rho\bar{A}_0)^2}\right)+ \tag{2.36}$$
$$\left.+\frac{2\bar{B}\partial_\rho\bar{A}_0}{\sqrt{(\rho^4{+}\bar{B}^2)(1{-}(\partial_\rho\bar{A}_0)^2)}}(\partial_2\bar{a}_1\partial_\rho\bar{a}_0{-}\partial_0\bar{a}_1\partial_\rho\bar{a}_2{+}(\partial_0\bar{a}_2{-}\partial_2\bar{a}_0)\partial_\rho\bar{a}_1)\right)$$

Note that the last line in (2.36) describes a coupling of the transverse and longitudinal gauge potential components. Bellow we will consider Fourier transform of the gauge field

$$\bar{a}_\mu(\rho, x^0, x^2) = \int \frac{d\mathbf{w}d\mathbf{q}}{(2\pi)^2} e^{-i\mathbf{w}x_0+i\mathbf{q}x_2}\tilde{a}_\mu(\rho, \mathbf{w}, \mathbf{q}) \tag{2.37}$$

Now we substitute eq. (2.34) into the action (2.36), define $b^2 = \bar{B}^2/\hat{d}^4$, and introduce a new variable $z = \frac{\hat{d}}{\rho}$, so that $z = 0$ is a boundary and $z = \infty$ is a Poincaré horizon of AdS_5. In addition, we make the quantities $\mathbf{w}$, $\mathbf{q}$ dimensionless, by measuring these in units of $\hat{d}$: $\mathbf{w} = \omega\hat{d}$, $\mathbf{q} = q\hat{d}$. We also denote for shortness of notation

$$\zeta = 1 + (1+b^2)z^4 \tag{2.38}$$

Then the action (2.36) becomes written as

$$S{=}\int \frac{dz}{1{+}b^2z^4}\left(-\zeta^{3/2}a_0'^2{+}\zeta^{1/2}a_2'^2{-}\zeta^{1/2}(\partial_0 a_2{-}\partial_2 a_0)^2{+}\zeta^{-1/2}(\partial_2 a_1)^2\right.$$
$$\left. -\,\zeta^{1/2}(\partial_0 a_1)^2{+}\zeta^{1/2}a_1'^2{-}2bz^4(\partial_2 a_1 a_0'{-}\partial_0 a_1 a_2'{+}(\partial_0 a_2{-}\partial_2 a_0)a_1')\right), \tag{2.39}$$

[5]We thank J. Shock for comments on this action.

where we have omitted bars for simplicity of notation, and prime denotes differentiation w.r.t. z. In momentum representation

$$
\begin{aligned}
S = \int \frac{dz}{1+b^2z^4} \Big(& -\zeta^{3/2} a_0'(\omega,\, q) a_0'(-\omega,\, -q) + \zeta^{1/2} a_2'(\omega,\, q) a_2'(-\omega,\, -q)] \\
& + \zeta^{1/2} E(\omega,\, q) E(-\omega,\, -q) + \zeta^{-1/2} q^2 a_1(\omega,\, q) a_1(-\omega,\, -q) \\
& - \zeta^{1/2} \omega^2 a_1(\omega,\, q) a_1(-\omega,\, -q) + \zeta^{1/2} a_1'(\omega,\, q) a_1'(-\omega,\, -q) \\
& + 2ibz^4 (q a_1(-\omega,\, -q) a_0'(\omega,\, q) + \omega a_1(-\omega,\, -q) a_2'(\omega,\, q) \\
& + E(\omega,\, q) a_1'(-\omega,\, -q))\Big) \;,
\end{aligned}
\tag{2.40}
$$

where we have omitted tildes for simplicity of notation and introduced the gauge-invariant electric field strength [43],

$$
E(\omega,\, q) = \omega a_2(\omega,\, q) + q a_0(\omega,\, q)\,. \tag{2.41}
$$

In addition we have Gauss's law [6]

$$
\omega \zeta^{3/2} a_0'(\omega,\, q) + q \zeta^{1/2} a_2'(\omega,\, q) = 0 \tag{2.42}
$$

Together with

$$
E'(\omega,\, q) = \omega a_2'(\omega,\, q) + q a_0'(\omega,\, q)\,, \tag{2.43}
$$

eq. (2.42) gives

$$
a_0'(\omega,\, q) = \frac{q}{q^2 - \zeta\omega^2} E'\,, \tag{2.44}
$$

$$
a_2'(\omega,\, q) = \frac{\omega\zeta}{\omega^2\zeta - q^2} E'\,. \tag{2.45}
$$

Plugging these expressions into the action (2.40), we obtain

$$
\begin{aligned}
S = \int \frac{dz}{1+b^2z^4} \bigg(& \frac{q^2 - \zeta\omega^2}{\zeta^{1/2}} a_1^2 - \zeta^{3/2} \frac{E'^2}{\zeta\omega^2 - q^2} + \zeta^{1/2} E^2 + \zeta^{1/2} a_1'^2 \\
& + 2ibz^4 (E a_1)' \bigg) \;.
\end{aligned}
\tag{2.46}
$$

[6] This is an equation of motion for a_z. To derive it replace

$$
a_2' \to a_2' - \partial_2 a_z\,, \quad a_0' \to a_0' - \partial_0 a_z
$$

in the Lagrangian (2.39) and leave only terms linear in derivatives of a_z, because only these will survive when we consider the equation of motion for a_z in the $a_z = 0$ gauge. Then use the Fourier transform (2.37).

Corresponding fluctuation equations are

$$E''+\left(\omega^2-\frac{q^2}{1+(1+b^2)z^4}\right)E-\frac{4ibz^3(\omega^2(1+(1+b^2)z^4)-q^2)a_1}{(1+b^2z^4)(1+(1+b^2)z^4)^{3/2}} \tag{2.47}$$
$$+\frac{2}{z}\left(\frac{1}{1+((1+b^2)z^4)^{-1}}+2\left(\frac{1}{1+b^2z^4}-\frac{1-(q/\omega)^2(1+(1+b^2)z^4)^{-2}}{1-(q/\omega)^2(1+(1+b^2)z^4)^{-1}}\right)\right)E'=0\,,$$

$$a_1''+2z^3\left(\frac{1+b^2}{1+(1+b^2)z^4}-\frac{2b^2}{1+b^2z^4}\right)a_1'+\left(\omega^2-\frac{q^2}{1+(1+b^2)z^4}\right)a_1$$
$$+\frac{4ibz^3E}{(1+b^2z^4)(1+(1+b^2)z^4)^{1/2}}=0\,. \tag{2.48}$$

Vanishing magnetic field

In this subsection we set the magnetic field to zero. Fluctuations of E and a_1 fields then decouple, and we can consider separately transverse and longitudinal responses,

$$E''+\frac{2}{z}\left(\frac{1}{1+z^{-4}}+2\left(1-\frac{1-(q/\omega)^2(1+z^4)^{-2}}{1-(q/\omega)^2(1+z^4)^{-1}}\right)\right)E'$$
$$+(\omega^2-q^2(1+z^4)^{-1})E=0\,, \tag{2.49}$$

$$a_1''+\frac{2z^3}{1+z^4}a_1'+\left(\omega^2-\frac{q^2}{1+z^4}\right)a_1=0\,. \tag{2.50}$$

Let us first study the longitudinal response. In the near-horizon $z\gg 1$ region eq. (2.49) becomes:

$$E''+\frac{2}{z}E'+\omega^2E=0\,, \tag{2.51}$$

The general solution of (2.51) is a linear combination of $e^{\pm i\omega z}/z$. We choose the solution with the incoming near-horizon behavior, since it corresponds to retarded propagator in the dual field theory [44]:

$$E=C\frac{e^{i\omega z}}{z}\,. \tag{2.52}$$

The constant C is undetermined, because the fluctuation equation is linear. When $\omega z\ll 1$, we obtain

$$E=C\left(\frac{1}{z}+i\omega\right)\,. \tag{2.53}$$

Condition (2.52) together with the boundary condition $E(0) = 0$ (imposed to get normalizable solutions) defines an eigenvalue problem for the fluctuation equation (2.49). In the limit $\omega z, qz \ll 1$, (2.49) reduces to,

$$E'' + \frac{2}{z}\left(\frac{1}{1+z^{-4}} + 2\left(1 - \frac{1-(q/\omega)^2(1+z^4)^{-2}}{1-(q/\omega)^2(1+z^4)^{-1}}\right)\right)E' = 0\,, \qquad (2.54)$$

having as general solution,

$$E(z) = C_1 + C_2(q^2 - 2\omega^2)\sqrt{i}F\left(i\sinh^{-1}(\sqrt{i}z)|-1\right) - \frac{C_2 q^2 z}{\sqrt{1+z^4}}\,, \quad (2.55)$$

where $F(z)$ is an elliptic integral of the first kind. In the limit $z \to \infty$ it has an expansion

$$\sqrt{i}F\left(i\sinh^{-1}(\sqrt{i}z)|-1\right) \to -K(1/2) + \frac{1}{z} + O\left(\frac{1}{z^5}\right)\,, \qquad (2.56)$$

where $K(z)$ is the complete elliptic integral of the first kind. The solution (2.55) becomes in this limit

$$E(z) = C_1 - C_2 K(1/2)(q^2 - 2\omega^2) - \frac{2C_2}{z}\omega^2. \qquad (2.57)$$

Now we compare (2.53) and (2.57), and obtain as result

$$C_1 = \left(i\omega - \frac{(q^2-2\omega^2)K(1/2)}{2\omega^2}\right)C\,, \qquad C_2 = -\frac{C}{2\omega^2} \qquad (2.58)$$

Recalling the boundary condition $E(0) = 0$, we deduce from (2.55) that $C_1 = 0$, and consequently

$$\left(1 + \frac{i\omega}{K(1/2)}\right)^{-1} = \frac{2\omega^2}{q^2}\,, \qquad (2.59)$$

which in the limit of small q, ω is solved by the considering leading orders in momentum q,

$$\omega = \pm\frac{q}{\sqrt{2}} - \frac{iq^2}{4K(1/2)}\,. \qquad (2.60)$$

This excitation has been identified before, and is called [16] holographic zero sound. In the $d = 2+1$ dimensional system this mode has been observed in [23]. Note that the speed of sound does not depend on dimensionality p of a probe brane and for any value of p is equal to the speed of the usual sound in the hydrodynamic regime [45]. In Section IV we will study current-current two-point functions, and the peak in the spectral function, corresponding to zero sound mode, will also be observed in the numerics.

Now, let us consider the fluctuation equation (2.50) for the transverse gauge field component, in the limit ω, $q \ll 1$. Then eq. (2.50) becomes

$$a_1'' + \frac{2z^3}{1+z^4} a_1' = 0 \,, \tag{2.61}$$

with an exact solution being

$$a_1(z) = C_1 + C_2\sqrt{i}F\left(i \sinh^{-1}(\sqrt{i}z)| - 1\right) \,. \tag{2.62}$$

In the near-horizon $z \to \infty$ limit it is expanded as

$$a_1(z) \simeq C_1 - C_2 K(1/2) + C_2/z \,. \tag{2.63}$$

Comparing it with the incoming-wave solution (2.114), one obtains

$$C_1 = (i\omega + K(1/2))\, C \,, \qquad C_2 = C \,. \tag{2.64}$$

Then, near-boundary $z \ll 1$ expansion of (2.62) is given by

$$a_1(z) \simeq A + Bz \,, \tag{2.65}$$

where

$$A = (i\omega + K(1/2))\, C \,, \qquad B = -C \,. \tag{2.66}$$

Therefore one may find the current two-point function $\langle J^1 J^1 \rangle = \frac{B}{A}$. In particular, its imaginary part is given by

$$\mathrm{Im}\langle J^1 J^1 \rangle \simeq \frac{\omega}{[K(1/2)]^2} \,. \tag{2.67}$$

We provide numerical results for the transverse fluctuations in Section IV.

Non-vanishing magnetic field

In this subsection we are going to study the case of small magnetic field, $b \ll 1$, which will allow us to achieve some simplifications. Let us rewrite the action (2.46) as

$$S = \int dz \left(\mathcal{G}_E E'^2 + \mathcal{U}_E E^2 + \mathcal{G}_a a_1'^2 + \mathcal{U}_a a_1^2 + \mathcal{C}^{(1)} (E a_1)' \right) , \tag{2.68}$$

where we have denoted

$$\mathcal{G}_E = -\frac{(1+(1+b^2)z^4)^{1/2}}{(1+b^2 z^4)\left(\omega^2 - \frac{q^2}{1+(1+b^2)z^4}\right)} , \quad \mathcal{U}_E = \frac{(1+(1+b^2)z^4)^{1/2}}{1+b^2 z^4} , \tag{2.69}$$

$$\mathcal{C}^{(1)} = \frac{2ibz^4}{1+b^2z^4} , \quad \mathcal{U}_a = -\frac{(1+(1+b^2)z^4)^{1/2}\left(\omega^2 - \frac{q^2}{1+(1+b^2)z^4}\right)}{1+b^2 z^4} , \tag{2.70}$$

$$\mathcal{G}_a = \frac{(1+(1+b^2)z^4)^{1/2}}{1+b^2 z^4} . \tag{2.71}$$

In the case of $b \ll 1$, we can approximate

$$\mathcal{G}_E = -\frac{(1+z^4)^{1/2}}{(1+b^2 z^4)\left(\omega^2 - \frac{q^2}{1+z^4}\right)} , \quad \mathcal{U}_E = \frac{(1+z^4)^{1/2}}{1+b^2 z^4} , \tag{2.72}$$

$$\mathcal{G}_a = \frac{(1+z^4)^{1/2}}{1+b^2 z^4} , \quad \mathcal{U}_a = -\frac{(1+z^4)^{1/2}\left(\omega^2 - \frac{q^2}{1+z^4}\right)}{1+b^2 z^4} , \tag{2.73}$$

$$\mathcal{C}^{(1)} = \frac{2ibz^4}{1+b^2 z^4} . \tag{2.74}$$

In the near-horizon limit, for $\omega > 0$, integrating the $\mathcal{C}^{(1)}$ term by parts, we arrive at

$$S = \int \frac{dz\, z^2}{1+b^2 z^4} \left(-\frac{E'^2}{\omega^2} + E^2 + a_1'^2 - (\omega^2 - b^2 q^2) a_1^2 - 8ibEa_1 \frac{z}{1+b^2 z^4} \right) . \tag{2.75}$$

Moreover, for $z \gg 1/\sqrt{b}$ and $z^3 \gg 1/(b(\omega^2 - b^2 q^2)^{1/2})$, we actually obtain decoupled system of equations

$$E'' - \frac{2}{z} E' + \omega^2 E = 0 \quad \Rightarrow \quad E = \tilde{C}_1 (1 - i\omega z) e^{i\omega z} , \tag{2.76}$$

$$a_1''-\frac{2}{z}a_1'+(\omega^2-b^2q^2)a_1=0 \;\Rightarrow\; a_1=\tilde{C}_2(1-i\sqrt{\omega^2-b^2q^2}z)e^{i\sqrt{\omega^2-b^2q^2}z}\,. \tag{2.77}$$

Now, assume that $\omega^2 \gg b^2q^2$, and perform the linear transformation in (2.75)

$$E = i\omega(\chi_1\tilde{E}+\chi_2\tilde{a}_1)\,, \qquad a_1 = \chi_1\tilde{E}-\chi_2\tilde{a}_1\,, \tag{2.78}$$

with arbitrary constant coefficients χ_1, χ_2, which brings the action to the form

$$S = \int \frac{2dz\,z^2}{1+b^2z^4}\left(\chi_1^2\left(\tilde{E}'^2-\omega^2\left(1-\frac{4bz}{\omega(1+b^2z^4)}\right)\tilde{E}^2\right)\right.$$
$$\left. +\,\chi_2^2\left(\tilde{a}_1'^2-\omega^2\left(1+\frac{4bz}{\omega(1+b^2z^4)}\right)\tilde{a}_1^2\right)\right)\,. \tag{2.79}$$

Corresponding equations of motion are

$$\tilde{E}''+\frac{2}{z}\frac{1-b^2z^4}{1+b^2z^4}\tilde{E}'+\omega^2\left(1-\frac{4bz}{\omega(1+b^2z^4)}\right)\tilde{E}=0\,, \tag{2.80}$$

$$\tilde{a}_1''+\frac{2}{z}\frac{1-b^2z^4}{1+b^2z^4}\tilde{a}_1'+\omega^2\left(1+\frac{4bz}{\omega(1+b^2z^4)}\right)\tilde{a}_1=0\,. \tag{2.81}$$

The solutions are

$$\tilde{E} = \frac{e^{\pm i\omega z}}{z}+\frac{b}{\omega}(1\mp i\omega z)e^{\pm i\omega z}\,, \tag{2.82}$$

$$\tilde{a}_1 = \frac{e^{\pm i\omega z}}{z}-\frac{b}{\omega}(1\mp i\omega z)e^{\pm i\omega z}\,. \tag{2.83}$$

We impose the incoming-wave behavior,

$$E = i\omega\left(\frac{\chi_1+\chi_2}{z}+(\chi_1-\chi_2)\frac{b}{\omega}(1-i\omega z)\right)e^{i\omega z}\,, \tag{2.84}$$

$$a_1 = \left(\frac{\chi_1-\chi_2}{z}+(\chi_1+\chi_2)\frac{b}{\omega}(1-i\omega z)\right)e^{i\omega z}\,, \tag{2.85}$$

which leaves us with two constant of integration $\chi_1 \pm \chi_2$.

When $\omega \sim q \ll 1$, we can consider fluctuation equations (2.49), (2.50), as for the case of vanishing magnetic field. Then we perform computations along the lines of the previous subsection, using now near-horizon boundary conditions (2.84) and (2.85).

First, we match (2.84) in $\omega z \ll 1$ limit,

$$E = i(b(\chi_1 - \chi_2) + i\omega^2(\chi_1 + \chi_2)) + i\omega \frac{\chi_1 + \chi_2}{z} \tag{2.86}$$

with eq. (2.57). Requiring that $C_1 = 0$, we arrive at

$$q^2 - 2\omega^2 - \frac{2}{K(1/2)} \left(\omega b \frac{\chi_1 - \chi_2}{\chi_1 + \chi_2} + i\omega^3 \right) = 0 \,. \tag{2.87}$$

Then, we match (2.85) in $\omega z \ll 1$ limit,

$$a_1 = i\omega(\chi_1 - \chi_2) + (\chi_1 + \chi_2)\frac{b}{\omega} + \frac{\chi_1 - \chi_2}{z} \tag{2.88}$$

with eq. (2.63). Again, imposing normalizability condition $C_1 = 0$, we obtain

$$\omega + \frac{1}{K(1/2)} \left(b \frac{\chi_1 + \chi_2}{\chi_1 - \chi_2} + i\omega^2 \right) = 0 \,. \tag{2.89}$$

Solving (2.87) together with (2.89), we get [7]

$$q^2 - 2\omega^2 + \frac{2b^2}{[K(1/2)]^2} + \frac{i\omega}{K(1/2)}(q^2 - 4\omega^2) = 0 \,. \tag{2.90}$$

We see that in the presence of a magnetic field b zero sound mode develops a gap ω_c in the spectrum,

$$\omega_c = \frac{b}{K(1/2)} \,. \tag{2.91}$$

2.4.2 Effective theory for the sound mode

Zero sound may also be studied in the framework of Ref. [37]. First, one introduces a hypersurface $z = z_\Lambda$ in the bulk, integrating out degrees of freedom in the UV region $0 \le z \le z_\Lambda$. The UV physics is then effectively encoded in the action by,

$$S = \frac{1}{2} \int d^3x (f_0^2(\partial_0\phi - W_0 + w_0)^2 - f_2^2(\partial_2\phi - W_2 + w_2)^2) \,, \tag{2.92}$$

[7] Equivalently, we can obtain this result requiring that (2.87) and (2.89) have a non-trivial solution for $\chi_1 \pm \chi_2$.

where $W_\mu = a_\mu(z=0)$, $w_\mu = a_\mu(z=z_\Lambda)$, and the "Godstone boson" ϕ corresponds to breaking of the $U(1)$ symmetry with a gauge field $W_\mu - w_\mu$. The zero sound mode may be interpreted in such a framework as a mode coming from an excitation of the field ϕ, and therefore the speed of zero sound is given by the expression $v = f_2/f_0$. Let us now compare the effective field theory action for the UV degrees of freedom with the bulk DBI action. To render the relation between bulk and boundary to be precise, we specify the zero boundary condition $W_\mu = 0$, putting the Goldstone boson ϕ to zero:

$$S = \frac{1}{2}\int d^3x (f_0^2 w_0^2 - f_2^2 w_2^2) \tag{2.93}$$

Let us consider all fields to be only z-dependent, in which case transverse fluctuations decouple, and we can put these to zero. Then we can rewrite the bulk theory action (2.39) in a form

$$S \simeq \frac{1}{2}\int d^3x \frac{dz}{1+b^2z^4}(h^3(z)\tilde{a}_0'^2 - h(z)\tilde{a}_2'^2)\,, \tag{2.94}$$

where we have defined $h(z) = \sqrt{1+(1+b^2)z^4}$. The solutions of the equations of motion on $\tilde{a}_0$, $\tilde{a}_2$, satisfying zero boundary condition at the AdS boundary, while being defined on the hypersurface $z = z_\Lambda$, are now given by:

$$w_0 = C_0 \int_0^{z_\Lambda} \frac{dz(1+b^2z^4)}{h^3(z)}\,, \qquad w_2 = C_2 \int_0^{z_\Lambda} \frac{dz(1+b^2z^4)}{h(z)}\,. \tag{2.95}$$

To match the bulk action and the boundary theory (2.93), we evaluate the action (2.94) on the solution of the EOM, which leaves us with the boundary terms at $z = z_\Lambda$ only

$$S \simeq \frac{1}{2}\int d^3x (C_0 w_0 - C_2 w_2)\,, \tag{2.96}$$

which in turn with the help of (2.95), may be rewritten as (2.93) with

$$f_0^{-2} = \int_0^{z_\Lambda} \frac{dz(1+b^2z^4)}{h^3(z)}\,, \qquad f_2^{-2} = \int_0^{z_\Lambda} \frac{dz(1+b^2z^4)}{h(z)}\,. \tag{2.97}$$

Therefore the speed of zero sound is given by

$$u_0^2 = \int_0^{z_\Lambda} \frac{dz(1+b^2z^4)}{h^3(z)} \left(\int_0^{z_\Lambda} \frac{dz(1+b^2z^4)}{h(z)}\right)^{-1}\,. \tag{2.98}$$

When $b \ll 1$, one obtains

$$u_0^2 \simeq \frac{1}{2 + \frac{8\pi^{1/2}}{3\Gamma[1/4]^2} b^2 z_\Lambda^3}, \tag{2.99}$$

and therefore for $b^2 z_\Lambda^3 \ll 1$ one recovers the value of the speed of zero sound in vanishing b-field, $u_0 = 1/\sqrt{2}$, while for $b^2 z_\Lambda^3 \gg 1$ the speed of zero sound approaches zero. In this regime the description of the low energy physics by the effective action (2.92) presumably breaks down; it would be interesting to write the low energy description that would account for the gap in the spectrum.

2.4.3 Thermodynamic properties of trivial embeddings

We will study the thermodynamics of the trivial Dp brane embedding, to obtain as a result the value of the speed of the usual first (hydrodynamic) sound. We consider here the $D3/Dp$ system with a $2+1$ dimensional intersection, and in the Appendix we will study the supersymmetric $D3/D7$ system with a $3+1$ dimensional intersection, in the presence of a non-vanishing magnetic field.

The total prefactor of the action is irrelevant for the computation of the speed of first sound. The grand canonical potential is given by the equation

$$\Xi = -S = \int d\rho(\rho^4 + \bar{B}^2)(\rho^4 + \bar{B}^2 + \hat{d}^4)^{-1/2} = a\frac{2\bar{B}^2 - \hat{d}^4}{(\bar{B}^2 + \hat{d}^4)^{1/4}}, \tag{2.100}$$

where $a = \Gamma(1/4)^2/(12\sqrt{\pi})$. Using (2.35) one may calculate the charge density as,

$$\hat{\rho} = -\frac{\partial \Xi}{\partial \bar{\mu}_{ch}}, \tag{2.101}$$

to find the energy density, being at zero temperature equal to the free energy,

$$\epsilon = \Xi + \bar{\mu}_{ch}\hat{\rho} = 2a(\bar{B}^2 + \hat{d}^4)^{3/4}. \tag{2.102}$$

Consequently, the speed of sound is given by

$$u^2 = \frac{\partial P}{\partial \epsilon} = -\frac{\partial \Xi}{\partial \epsilon} = \frac{1}{2}\frac{1 + 2b^2}{1 + b^2}. \tag{2.103}$$

Notice that this result is independent of p, which agrees with [45]. Observe that when the magnetic field vanishes we retrieve the value $u^2 = 1/2$, which we observed before in the dispersion relation (2.60).

Notice also that all the steps performed in the above may be combined into one expression (use $\partial S/\partial\bar{\mu}_{ch} = \hat{d}^2$):

$$u^2 = \frac{\partial S/\partial\bar{\mu}_{ch}}{\bar{\mu}_{ch}\partial^2 S/\partial\bar{\mu}_{ch}^2} = \frac{1}{2}\frac{\partial\log\bar{\mu}_{ch}}{\partial\log\hat{d}}\,. \tag{2.104}$$

2.5 Holographic current-current correlators at finite frequency and momentum

In the previous section we have shown that a propagating mode (zero sound) develops a gap in the presence of the magnetic field. In this section we compute numerically the two-point function of the $U(1)$ currents. First we set magnetic field to zero. We identify the holographic zero sound as a peak in the spectral function. We start by computing the density-density correlator $\langle J^0 J^0\rangle$ using the linearized DBI action. We then proceed to computing the transverse correlator $\langle J^1 J^1\rangle$. After that we proceed to the case of non-vanishing magnetic field and show that the gap in the zero sound spectrum shows itself on the numeric graphs.

2.5.1 Fluctuations of electric field strength E

Consider the fluctuation equation (2.47), near the boundary $z = 0$ for any value of magnetic field:

$$E'' - (q^2 - \omega^2)E = 0\,. \tag{2.105}$$

Its general solution is of the form,

$$E = \mathcal{A}_E F_I + \mathcal{B}_E F_{II}\,, \tag{2.106}$$

where we have denoted the two independent solutions as

$$F_I = 1 + \frac{q^2-\omega^2}{2}z^2 + \frac{(q^2-\omega^2)^2}{24}z^4 + \cdots\,, \tag{2.107}$$

$$F_{II} = z + \frac{q^2-\omega^2}{6}z^3 + \cdots\,. \tag{2.108}$$

The on-shell action is therefore given by

$$S_{on-shell} \simeq \lim_{\varepsilon\to 0} \int d\omega dq \mathcal{A}_E(\omega, q)\mathcal{A}_E(-\omega, -q)\frac{1}{q^2-\omega^2}\frac{\mathcal{B}_E(\omega, q)}{\mathcal{A}_E(\omega, q)}|_{z=\varepsilon} \,. \tag{2.109}$$

Non-vanishing Green functions are

$$\begin{aligned}
\langle J^0(\omega, q)J^0(-\omega, -q)\rangle &= \lim_{\varepsilon\to 0}\frac{\delta^2 S_{on-shell}}{\delta a_0(z=\varepsilon, \omega, q)\delta a_0(z=\varepsilon, -\omega, -q)} \\
&= \lim_{\varepsilon\to 0}\frac{\delta^2 S_{on-shell}}{\delta E(z=\varepsilon, \omega, q)\delta E(z=\varepsilon, -\omega, -q)} \\
&\times \frac{\delta E(\omega, q, z)}{\delta a_0(\omega, q, z)}\frac{\delta E(-\omega, -q, z)}{\delta a_0(-\omega, -q, z)} = \\
&= -\frac{q^2}{q^2-\omega^2}\frac{\mathcal{B}_E(\omega, q)}{\mathcal{A}_E(\omega, q)} \,,
\end{aligned} \tag{2.110}$$

$$\begin{aligned}
\langle J^2(\omega, q)J^2(-\omega, -q)\rangle &= \lim_{\varepsilon\to 0}\frac{\delta^2 S_{on-shell}}{\delta a_2(z=\varepsilon, \omega, q)\delta a_2(z=\varepsilon, -\omega, -q)} = \\
&= \lim_{\varepsilon\to 0}\frac{\delta^2 S_{on-shell}}{\delta E(z=\varepsilon, \omega, q)\delta E(z=\varepsilon, -\omega, -q)} \\
&\times \frac{\delta E(\omega, q, z)}{\delta a_2(\omega, q, z)}\frac{\delta E(-\omega, -q, z)}{\delta a_2(-\omega, -q, z)} = \\
&= -\frac{\omega^2}{q^2-\omega^2}\frac{\mathcal{B}_E(\omega, q)}{\mathcal{A}_E(\omega, q)} \,,
\end{aligned} \tag{2.111}$$

$$\begin{aligned}
\langle J^0(\omega, q)J^2(-\omega, -q)\rangle &= \lim_{\varepsilon\to 0}\frac{\delta^2 S_{on-shell}}{\delta a_2(z=\varepsilon, \omega, q)\delta a_0(z=\varepsilon, -\omega, -q)} = \\
&= \lim_{\varepsilon\to 0}\frac{\delta^2 S_{on-shell}}{\delta E(z=\varepsilon, \omega, q)\delta E(z=\varepsilon, -\omega, -q)} \\
&\times \frac{\delta E(\omega, q, z)}{\delta a_2(\omega, q, z)}\frac{\delta E(-\omega, -q, z)}{\delta a_0(-\omega, -q, z)} = \\
&= -\frac{\omega q}{q^2-\omega^2}\frac{\mathcal{B}_E(\omega, q)}{\mathcal{A}_E(\omega, q)} \,.
\end{aligned} \tag{2.112}$$

Note that these expression agree with the Ward identity for the $U(1)$ conserved current J^μ,

$$\omega\langle J^0(\omega, q)J^0(-\omega, -q)\rangle - q\langle J^0(\omega, q)J^2(-\omega, -q)\rangle = 0 \,. \tag{2.113}$$

We evaluate numerically the ratio $\mathcal{B}_E/\mathcal{A}_E$ on the solution of equation (2.47) with incoming-wave near horizon behavior (2.52). In Fig. 2.3 we present numerical results for the real and imaginary parts of the $\mathcal{B}_E/\mathcal{A}_E$ for different values of ω, q in the case of $b = 0$. The holographic zero sound corresponds to the peak in the spectral density.

2.5.2 Fluctuations of the transverse component of the gauge field

In this subsection we will compute numerically the holographic two-point function for the transverse current $\langle J^1(x)J^1(y)\rangle$. Let us put $b = 0$.

In the near horizon regime $z \to \infty$ the bulk solution, corresponding to the retarded current-current propagator in the dual field theory, takes the incoming-wave form

$$a_1 = C\frac{e^{i\omega z}}{z}\,, \tag{2.114}$$

and in the vicinity of the boundary, the equation of motion becomes

$$a_1'' - (q^2 - \omega^2)a_1 = 0\,, \tag{2.115}$$

with a general solution being a combination of F_I and F_{II} (2.107), (2.108),

$$a_1 = \mathcal{A}_a F_I + \mathcal{B}_a F_{II}\,. \tag{2.116}$$

The results of numerical evaluations of the holographic two-point function $\langle J^1(q)J^1(-q)\rangle = \frac{\mathcal{B}_a}{\mathcal{A}_a}$ are presented in Fig. 2.4. We see that it does not reveal any structure.

2.5.3 Non-vanishing magnetic field

In the case of $b \neq 0$ fluctuations of the longitudinal $E(x^0, x^2, z)$ and transverse $a_1(x^0, x^2, z)$ components of the gauge field are no longer decoupled[8]. They are described by the action (2.68), which can be written as

$$\begin{aligned} S = \int dz\, \Big(\Big(&-(\mathcal{G}_E E')' + \mathcal{U}_E E - \frac{1}{2}(\mathcal{C}^{(1)})' a_1 \Big) E + \\ &+ \Big(-(\mathcal{G}_a a_1')' + \mathcal{U}_a a_1 - \frac{1}{2}(\mathcal{C}^{(1)})' E \Big) a_1 \Big) + \\ &+ [\mathcal{G}_E E E' + \mathcal{G}_a a_1 a_1' + \mathcal{C}^{(1)} E a_1]_{z=0}^{z=\infty}\,. \end{aligned} \tag{2.117}$$

[8]We thank R. Davison for pointing this out to us.

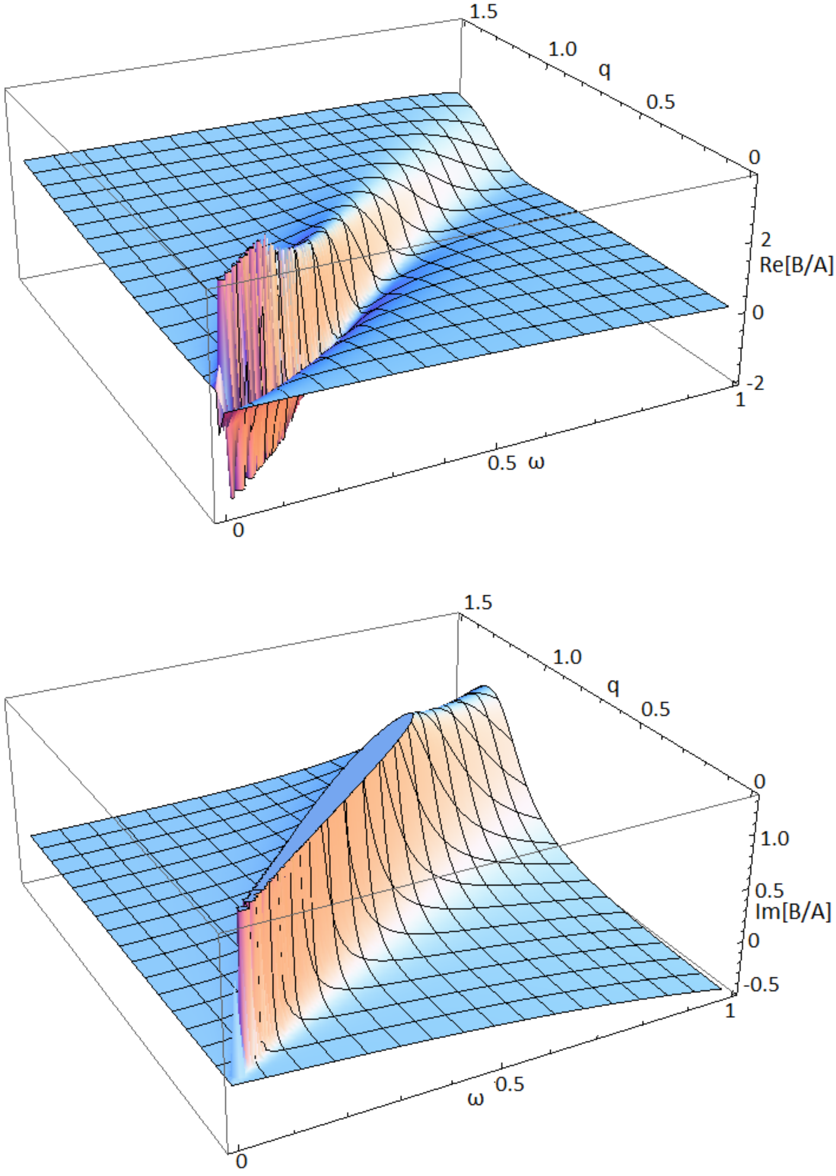

Figure 2.3. Real and imaginary parts of the $\mathcal{B}_E/\mathcal{A}_E$ in the $D3/Dp$ system with $d = 2+1$ dimensional intersection. The spectrum of excitations is exhausted by the holographic zero sound mode with the speed of sound $u_0 = \frac{1}{\sqrt{2}}$, and the attenuation $\Gamma_q \simeq q^2$.

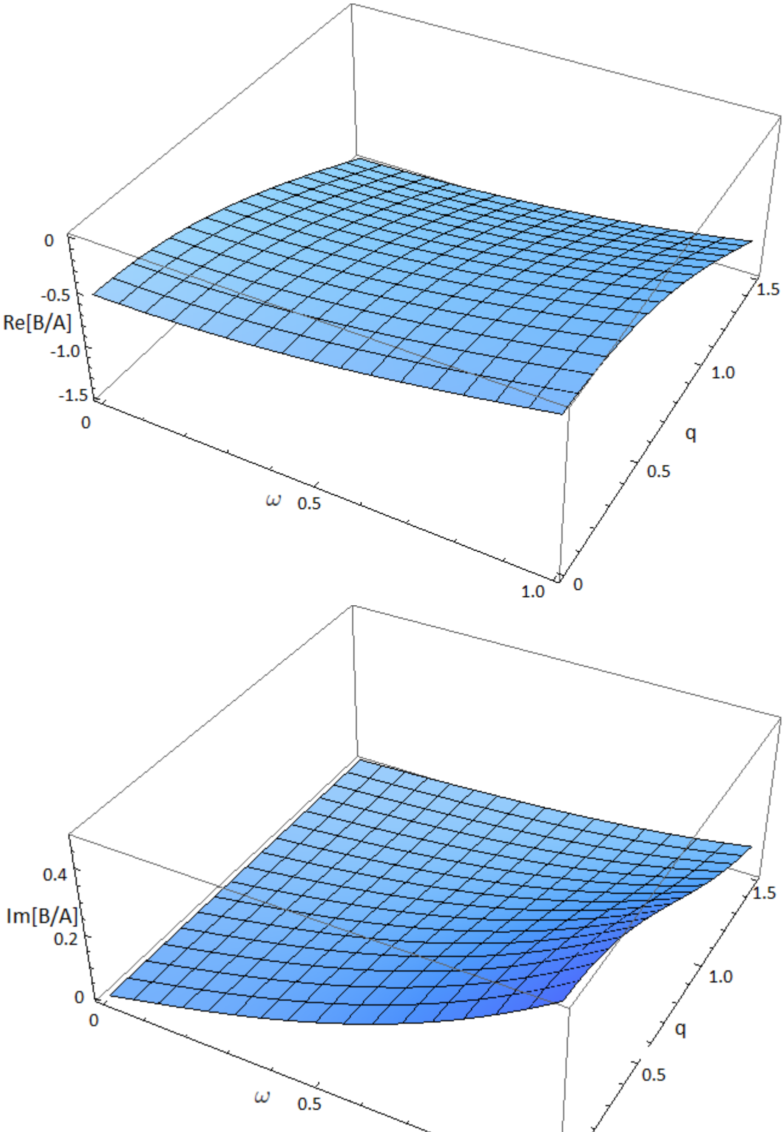

Figure 2.4. Real and imaginary parts of the correlation function $\langle J^1(-q)J^1(q)\rangle$ in the $D3/D7$ system with $d = 2+1$ dimensional intersection. No non-trivial collective excitation modes are observed. For small frequencies and momenta $\omega, q \ll 1$, the imaginary part of the correlation function behaves as $\mathrm{Im}[\langle J^1(-q)J^1(q)\rangle] \sim \omega$, independently of a particular value of q.

The first two lines vanish on shell. In the last line the cross term does not contribute to the variation of the on-shell action by the boundary $z = 0$ values of the fields E and a_1, because

$$\mathcal{C}^{(1)}|_{z=0} = 0\,. \tag{2.118}$$

The on-shell action is then given by the boundary term

$$S_{on-shell} \simeq \lim_{\varepsilon\to 0} \int d\omega dq \left(\frac{1}{q^2-\omega^2} E E' + a_1 a_1' \right)_{z=\varepsilon} . \tag{2.119}$$

Near the boundary the solutions to equations of motion are given by

$$E = \mathcal{A}_E F_I + \mathcal{B}_E F_{II}\,, \qquad a_1 = \mathcal{A}_a F_I + \mathcal{B}_a F_{II}\,, \tag{2.120}$$

where $F_{I,\,II}$ are defined by (2.107), (2.108).

To compute current-current two-point function numerically, we follow [46], where general system of coupled equations in the bulk is studied. For arbitrary two independent solutions $\Phi_{(1)},\ \Phi_{(2)}$ of the coupled system of fluctuation equations (2.47), (2.48), we define the matrix $H = \left(\Phi_{(1)},\ \Phi_{(2)}\right)$. Near the boundary it is expanded as

$$H = \mathcal{A} F_I + \mathcal{B} F_{II}\,. \tag{2.121}$$

On-shell action (2.119) may be rewritten as

$$S_{on-shell} \simeq \int d\omega dq\, \Phi^T\, M\, \Phi'\,, \tag{2.122}$$

where

$$M = \begin{pmatrix} \frac{1}{q^2-\omega^2} & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.123}$$

The matrix of correlation functions is then given by (see eq. (2.34) in [46])

$$G \simeq M \mathcal{B} \mathcal{A}^{-1}\,. \tag{2.124}$$

In such a form the current-current correlation matrix G is explicitly independent of a linear change of fields

$$\Phi_{(1)} \to r_1\Phi_{(1)} + r_2\Phi_{(2)}\,,\ \Phi_{(2)} \to r_3\Phi_{(1)} + r_4\Phi_{(2)}\ \Rightarrow\ H \to HR\,, \tag{2.125}$$

where $R = \begin{pmatrix} r_1 & r_3 \\ r_2 & r_4 \end{pmatrix}$ is some arbitrary non-degenerate matrix. If $\Phi_{(1),(2)} = \begin{pmatrix} E^{(1),(2)} \\ a_1^{(1),(2)} \end{pmatrix}$ are some arbitrary independent solutions, then due to (2.121) we get

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_E^{(1)} & \mathcal{A}_E^{(2)} \\ \mathcal{A}_a^{(1)} & \mathcal{A}_a^{(2)} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} \mathcal{B}_E^{(1)} & \mathcal{B}_E^{(2)} \\ \mathcal{B}_a^{(1)} & \mathcal{B}_a^{(2)} \end{pmatrix}, \tag{2.126}$$

and therefore using (2.124) we obtain

$$G \simeq \frac{1}{\mathcal{A}_E^{(1)}\mathcal{A}_a^{(2)} - \mathcal{A}_a^{(1)}\mathcal{A}_E^{(2)}} \begin{pmatrix} \frac{\mathcal{B}_E^{(1)}\mathcal{A}_a^{(2)} - \mathcal{B}_E^{(2)}\mathcal{A}_a^{(1)}}{q^2-\omega^2} & \frac{\mathcal{B}_E^{(2)}\mathcal{A}_E^{(1)} - \mathcal{B}_E^{(1)}\mathcal{A}_E^{(2)}}{q^2-\omega^2} \\ \mathcal{B}_a^{(1)}\mathcal{A}_a^{(2)} - \mathcal{B}_a^{(2)}\mathcal{A}_a^{(1)} & \mathcal{A}_E^{(1)}\mathcal{B}_a^{(2)} - \mathcal{B}_a^{(1)}\mathcal{A}_E^{(2)} \end{pmatrix}, \tag{2.127}$$

Near-horizon solutions are given by (2.76), (2.77), which we can write as a linear combination of two independent solutions

$$\tilde{\Phi}_{(1)} = \begin{pmatrix} (1 - i\omega z)e^{i\omega z} \\ (1 - i\sqrt{\omega^2 - b^2q^2}z)e^{i\sqrt{\omega^2-b^2q^2}z} \end{pmatrix}, \tag{2.128}$$

$$\tilde{\Phi}_{(2)} = \begin{pmatrix} (1 - i\omega z)e^{i\omega z} \\ -(1 - i\sqrt{\omega^2 - b^2q^2}z)e^{i\sqrt{\omega^2-b^2q^2}z} \end{pmatrix} \tag{2.129}$$

Arbitrary near-horizon behavior, with the most general form (up to simultaneous rescaling of all fields by the same factor) may therefore be written as a linear combination of these two solutions,

$$\Phi = \begin{pmatrix} (1 - i\omega z)e^{i\omega z} \\ c(1 - i\sqrt{\omega^2 - b^2q^2}z)e^{i\sqrt{\omega^2-b^2q^2}z} \end{pmatrix} = \frac{1+c}{2}\tilde{\Phi}_{(1)} + \frac{1-c}{2}\tilde{\Phi}_{(2)}\,. \tag{2.130}$$

On the other hand, fluctuation equations may be rewritten as

$$\Omega_1\Phi'' + \Omega_2\Phi' + \Omega\Phi = 0\,, \tag{2.131}$$

with matrices $\Omega_{1,2,3}$, being determined from (2.47), (2.48). Therefore linear combination of near-horizon solutions (2.129) results in the same linear combination of the solutions near the boundary. Recall that the matrix correlation function (2.127) is the same for any such a non-degenerate linear combination.

We therefore fix two arbitrary near-horizon conditions, say (2.129), determine corresponding coefficients $\mathcal{A}_E^{(1),(2)}$, $\mathcal{A}_a^{(1),(2)}$ and $\mathcal{B}_E^{(1),(2)}$, $\mathcal{B}_a^{(1),(2)}$

by integrating numerically fluctuation equations (2.47), (2.48) up to the boundary and matching corresponding solutions with (2.120), and compute the correlation matrix (2.127). Each of the four components of the correlation matrix shows a gapped zero sound mode.

In figure 2.5 we plot the real and imaginary parts of the G_{22} component, for $b = 0.001$. We see the gapped zero sound mode, with the gap which scales as $\omega_c \sim b$, in agreement with analytic result $\omega_c = \frac{b}{K(1/2)}$ of the previous section.

2.6 Discussion

In this chapter we have studied current-current two-point functions at strong coupling. We have considered the current-current correlators at finite momenta, but did not observe any nontrivial structure in the spectral function, other than the zero sound[9].

It is instructive to compare the holographic density-density correlator with the form expected from the random phase approximation and reviewed in Section II. Within RPA the zero sound mode presents itself as a smeared delta-function like peak in Fig. 1, the Lindhard particle-hole continuum starts at $\mathbf{q} \simeq \mathbf{w}/v_F$ and sharply ends at $\mathbf{q} \simeq 2\mathbf{q}_F$. The absence of the Lindhard continuum in the holographic computations can be explained by parametrically large values of the Landau parameters. The key point is eq. (2.11) which implies that since the zero sound velocity that we observe is $\mathcal{O}(1)$, the value of Fermi velocity scales like $v_F \sim 1/\sqrt{F_0 F_1}$. The regime of validity of our calculations is limited to $\mathbf{w} \sim \mathbf{q}$, and therefore the Lindhard continuum cannot be observed for parametrically large values of the Landau parameters. In the following we offer some speculations on how such a scenario can play out.

We can argue that the Fermi velocity is parametrically small. Recall that $q \simeq \mathbf{q}\sqrt{\lambda}/\mu$. Hence, eq. (2.60) implies that the zero sound attenuation is $\alpha \sim \mathbf{w}^2\sqrt{\lambda}/\mu$. According to [47] this can be expressed in terms of the quasi-particle lifetime as

$$\alpha \simeq \frac{1}{\tau}\frac{m^*}{\mu}v_F^2 F_2^2 \sim \frac{\mathbf{w}^2}{\mu}F_0^2 F_2^2 \tag{2.132}$$

[9]Note that our models are different from those studied in [7, 8], where poles at finite momenta were observed in the holographic two-point functions of operators with nonvanishing charge under global $U(1)$.

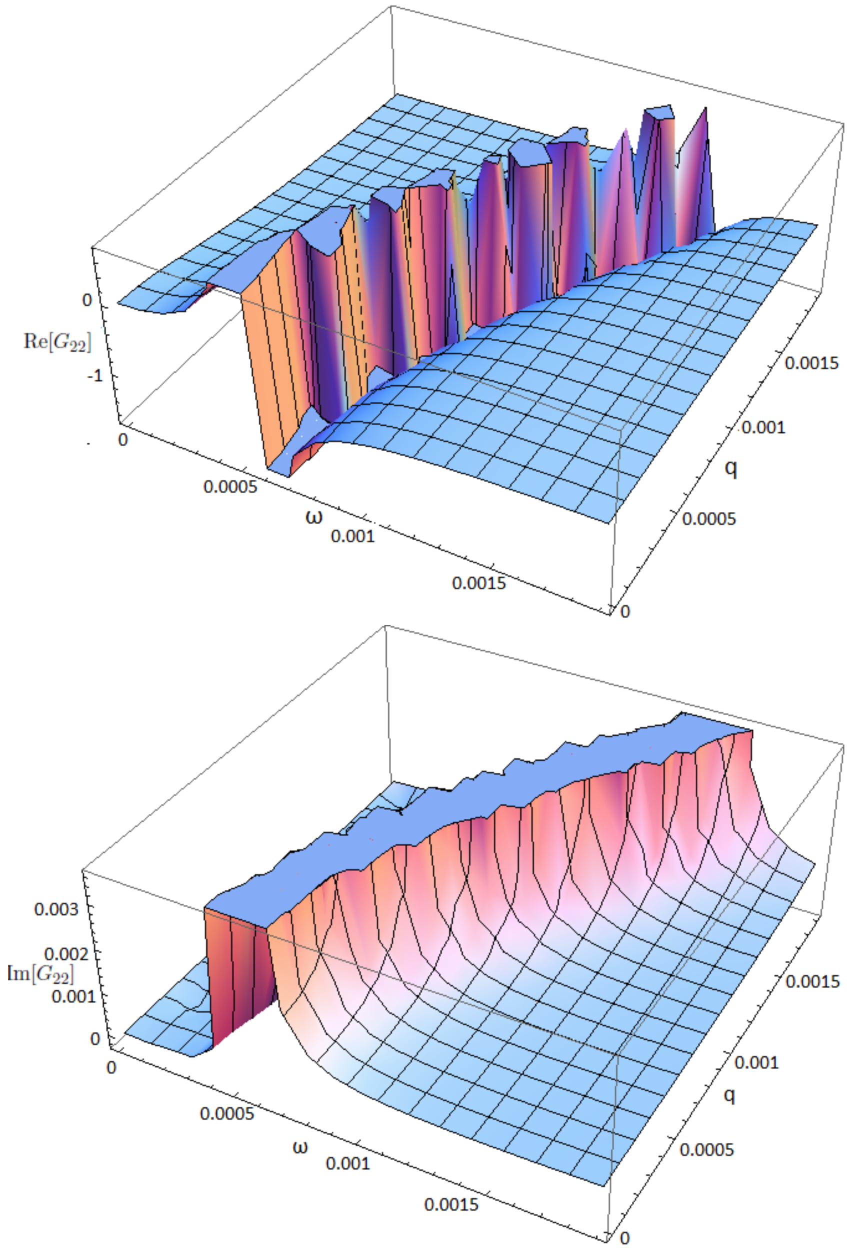

Figure 2.5. Real and imaginary parts of the component G_{22} of the correlation matrix (2.124) in the $D3/D7$ system with $d = 2 + 1$ dimensional intersection, for the magnetic field $b = 0.001$. The spectrum of excitations is exhausted by a gapped zero sound mode, with the value of the gap $\omega_c \sim b$.

To derive the second approximate equality we used the Fermi liquid estimate $1/\tau \sim \mathbf{w}^2 m^* {F_0}^2/\mathbf{q}_F^2 \sim \mathbf{w}^2 F_0^2/E_F^*$. Eq. (2.132) implies that the Landau parameters are indeed parametrically large, $F_0^2 F_2^2 \sim \sqrt{\lambda}$.

We have analyzed the system in the presence of magnetic field and observed a gap in the excitation spectrum $\mathbf{w}_c$. We derived a scaling relation $\mathbf{w}_c \simeq B/\mu$. Note that the gap in the spectrum of non relativistic fermions scales linearly with B, while the relativistic fermions obey $\sqrt{B}$ scaling; Kohn's theorem implies that the gap in the spectrum of excitations is not changed when the pairwise interaction is turned on. In our setup charged fermions interact and can exchange momentum with $\mathcal{N} = 4$ SYM degrees of freedom; the linear scaling of the gap with the magnetic field is consistent with the assumption that the effective degrees of freedom have an effective mass $m^* \simeq \mu$. [According to eq. (2.10) this implies that $F_1 = \mathcal{O}(1)$; a scenario consistent with the discussion above may involve a parametrically large $F_0 \sim \lambda^{1/4}$ but finite F_n, $n > 0$.] We do not quite understand the mechanism of dynamical mass generation at finite density – it is clearly very different from dynamical mass generation in a strongly interacting fermion system at zero density[10].

We already emphasized that a priori the very existence of zero sound is nontrivial, given the interaction of the charged matter with the uncharged superconformal degrees of freedom. It would be interesting to make this picture more precise and to see whether there is any relation to the recent studies of fermions in magnetic fields in the context of holography [48–50, 52, 51]. It would also be interesting to compare our results with the correlators computed in the charged magnetic brane background [53].

At this point it is worth recalling the relation between the charge density and the value of the chemical potential, given by (2.35). As usual, the value of the charge density is proportional to $\hat{d}$, $\rho \simeq N_c \lambda^{(p-5)/4} \hat{d}$ and the proportionality coefficient strongly depends on the dimensionality of the probe brane. The incompressibility $\partial \hat{d}/\partial \mu$ is a smooth non-vanishing function of μ, b. This implies that we cannot rule out the existence of gapless modes in our system[11]. Indeed, the analysis that led to the existence of the zero sound implicitly assumed $\omega \sim q \sim b$, and can be shown to break down for $|\omega| < bq$. We leave the search for gapless quasi normal modes for future work. Let us also note that a smooth compressibility is not

[10] The holographic dual of the latter involves repulsion of the probe brane from the bulk of of the AdS space; see e.g. [41] for a recent discussion.

[11] We thank D. Son for pointing this out to us.

compatible with the existence of Landau levels for the effective fermions.

Appendix is devoted to the subject of higher derivative corrections to the DBI action for the probe Dp brane. The possibility of breakdown of the DBI description in the extreme infrared (very close to the horizon) was pointed out in [54]. Two possible causes were identified in the presence of the electric flux on the brane: strong back-reaction and vanishing of the effective string tension. The strength of back-reaction from the flavor branes is governed by the ratio N_f/N_c. We did not investigate $1/N_c$ corrections in this chapter, although it is a very interesting problem. Instead, we explored the effects of the breakdown of the DBI description due to the vanishing of the effective string tension near the horizon. The strength of this effect is controlled by an inverse power of 't Hooft coupling. In principle, such effects can be described by going to higher orders in the α' expansion of the effective action for open strings, which corresponds to adding higher derivative terms to the DBI Lagrangian. Unfortunately we are not aware of the precise structure of higher derivative corrections to DBI in the presence of the worldvolume electric field. However we were able to model this situation by writing generic higher derivative terms which become important near the horizon and completely change the effective metric for fluctuations there.

The effect of such terms is confined to a very small region (which scales as an inverse power of 't Hooft coupling in suitable units); outside of this region the second order differential equations derived from the DBI are applicable. In principle, one can solve the higher order fluctuation equation outwards from the horizon, and then feed the resulting solution into the second order equation. From the point of view of the latter, this amounts to modifying the boundary conditions: an outgoing wave (with a small coefficient) is added to the incoming wave near the horizon. We verify that this does not introduce any qualitative new features in the two-point functions.

2.7 Appendix: Higher-derivative corrections to $L_{DBI}(a_1)$

The DBI description might break down in the near-horizon region [54], and therefore higher derivative corrections become essential in that region. Consider higher derivative correction to the DBI Lagrangian of the form

[55, 56]

$$\frac{\epsilon}{2}\sqrt{-g}g^{\mu\lambda}g^{\nu\rho}g^{\alpha\beta}(\nabla_\alpha F_{\lambda\rho})(\nabla_\beta F_{\mu\nu})\,, \tag{2.133}$$

Alternatively, using the Bianchi identity, one may rewrite it as

$$\tilde{\epsilon}\sqrt{-g}g_{\mu\nu}\nabla_\lambda F^{\lambda\mu}\nabla_\sigma F^{\sigma\nu}\,. \tag{2.134}$$

Here $\epsilon \sim \ell_s^2 \sim \frac{1}{\sqrt{\lambda}}$.

In this section we put $L = 1$. The induced $AdS_4 \times S^4$ metric on the trivially embedded Dp brane world-volume then takes the form

$$ds^2 = \rho^2(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2) + \frac{d\rho^2}{\rho^2} + d\Omega_4^2\,, \tag{2.135}$$

This corresponds to non-vanishing Christoffel symbols in the AdS subspace,

$$\Gamma^\rho_{\rho\rho} = -\frac{1}{\rho}\,, \quad \Gamma^\rho_{ij} = -\rho^3\eta_{ij}\,, \quad \Gamma^i_{\rho j} = \frac{1}{\rho}\delta^i_j\,, \tag{2.136}$$

where $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = 1$. We fix the background value of $A_0'(\rho)$ (2.34) (with $\bar{B} = 0$) and study the dynamics of the fluctuation field $a_1(\rho, x^0, x^2)$. Consequently, the non-vanishing components of the field strength tensor covariant derivatives

$$\nabla_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu} - \Gamma^\tau_{\alpha\mu}F_{\tau\nu} - \Gamma^\tau_{\alpha\nu}F_{\mu\tau} \tag{2.137}$$

are given by

$$\nabla_1 F_{0\rho} = -\frac{1}{\rho}F_{01}\,, \quad \nabla_\rho F_{0\rho} = \partial_\rho F_{0\rho}\,, \quad \nabla_\rho F_{01} = \partial_\rho F_{01} - \frac{2}{\rho}F_{01}\,, \tag{2.138}$$

$$\nabla_1 F_{01} = \rho^3 F_{0\rho}\,, \quad \nabla_2 F_{01} = \partial_2 F_{01}\,, \quad \nabla_0 F_{01} = \partial_0 F_{01} - \rho^3 F_{\rho 1}\,, \tag{2.139}$$

$$\nabla_2 F_{12} = \partial_2 F_{12} + \rho^3 F_{1\rho}\,, \quad \nabla_0 F_{12} = \partial_0 F_{12}\,, \quad \nabla_\rho F_{12} = \partial_\rho F_{12} - \frac{2}{\rho}F_{12}\,, \tag{2.140}$$

$$\nabla_2 F_{\rho 1} = \partial_2 F_{\rho 1} - \frac{1}{\rho}F_{21}\,, \quad \nabla_0 F_{\rho 1} = \partial_0 F_{\rho 1} - \frac{1}{\rho}F_{01}\,, \quad \nabla_\rho F_{\rho 1} = \partial_\rho F_{\rho 1}\,. \tag{2.141}$$

Let us now substitute the quantities (2.138)-(2.141) into the Lagrangian (2.133), which becomes in momentum representation,

$$\Delta L = \epsilon[-\rho^2(\partial_\rho A_0)^2 - \rho^4(\partial_\rho^2 A_0)^2 + \frac{1}{\rho^2}\left(\frac{(q^2-\omega^2)^2}{\rho^2} + 5q^2 - 6\omega^2\right)a_1^2$$
$$+ 2(\rho^2 + q^2 - \omega^2)(\partial_\rho a_1)^2 + \rho^4(\partial_\rho^2 a_1)^2 + \frac{4(\omega^2 - q^2)}{\rho}a_1\,\partial_\rho a_1]\,. \tag{2.142}$$

To obtain the corrected equation of motion of the background field $\partial_\rho A_0$, we put the a_1 fluctuations to zero and write the total, DBI + corrections, Lagrangian as

$$L = \rho^2\sqrt{1-(\partial_\rho A_0)^2} - \epsilon[\rho^2(\partial_\rho A_0)^2 + \rho^2(\partial_\rho^2 A_0)^2] \,. \tag{2.143}$$

The corresponding equation of motion

$$\frac{\rho^2\partial_\rho A_0}{\sqrt{1-(\partial_\rho A_0)^2}} + 2\epsilon[\rho^2\,\partial_\rho A_0 - \partial_\rho\left(\rho^4\,\partial_\rho^2 A_0\right)] = \hat{d}^2 \tag{2.144}$$

is solved to first order in ϵ by

$$\partial_\rho A_0 = \frac{\hat{d}^2}{\sqrt{\rho^4+\hat{d}^4}} + \delta\,\partial_\rho A_0\,, \tag{2.145}$$

where we have denoted the correction to the background as

$$\delta\,\partial_\rho A_0 = -\frac{2\hat{d}^2\epsilon\rho^6(\hat{d}^8+16\hat{d}^4\rho^4+3\rho^8)}{(\rho^4+\hat{d}^4)^4}\,. \tag{2.146}$$

Note that (2.146) approaches zero as $\mathcal{O}(\rho^6)$, near the horizon $\rho=0$. Therefore the correction to the behavior of the background potential $\partial_\rho A_0$ does not substantially affect the near-horizon physics. Using the $z=1/\rho$ radial coordinate, and considering the near horizon limit $\omega z \gg 1$, we obtain from (2.142) the correction to the near-horizon DBI Lagrangian

$$\Delta L = \epsilon\left((q^2-\omega^2)z^2(2a_1'^2+(q^2-\omega^2)a_1^2) + (2a_1'+za_1'')^2\right)\,. \tag{2.147}$$

This is to be added to the quadratic DBI near-horizon Lagrangian,

$$L_{DBI} = z^2(a_1'^2-\omega^2 a_1^2)\,. \tag{2.148}$$

As a result we obtain the following near-horizon Lagrangian:

$$\begin{aligned} L = &\left(\left(1+2\epsilon(q^2-\omega^2)\right)z^2+2\epsilon\right)a_1'^2 + \left(-\omega^2+\epsilon(q^2-\omega^2)^2\right)z^2 a_1^2 \\ &+ 2\epsilon(za_1'^2)' + \epsilon z^2(a_1'')^2\,. \end{aligned} \tag{2.149}$$

Up to a total derivative term[12] and $\mathcal{O}(\epsilon)$ modification of the DBI behavior, this Lagrangian therefore may be rewritten as

$$L = z^2(a_1'^2-\omega^2 a_1^2) + \epsilon z^2(a_1'')^2\,, \tag{2.150}$$

[12] Corresponding boundary terms $2\epsilon z a_1'^2$, evaluated on non-perturbed solution $e^{i\omega z}/z$, vanish when $z\gg 1$.

with associated equation of motion

$$a_1'' + \frac{2}{z}a_1' + \omega^2 a - \epsilon\left(a_1'''' + \frac{4}{z}a_1'''\right) = 0\,. \tag{2.151}$$

To estimate the relative significance of the correction and DBI terms, let us compare terms a_1'' and $\epsilon\left(a_1'''' + \frac{4}{z}a_1'''\right)$, when evaluated on the non-perturbed near-horizon solution $e^{i\omega z}/z$:

$$a_1'' \simeq \frac{1+(\omega z)^2}{z^3}\,, \qquad \epsilon\left(a_1'''' + \frac{4}{z}a_1'''\right) \simeq \epsilon\frac{\omega^4}{z}\,. \tag{2.152}$$

We observe that this correction is negligible.

Unfortunately we are not aware of the exact form of the higher derivative corrections to the DBI action in the presence of the electric field on the world-volume of the probe Dp brane. In the following we will simply assume a particular expression for the higher derivative corrections to the Lagrangian for the transverse fluctuations:

$$L = z^2(a_1'^2 - \omega^2 a_1^2) + \epsilon z^{2+\nu}(a_1'')^2\,, \tag{2.153}$$

with $\nu > 0$. To estimate the significance of the correction term we need to compare contributions to the equation of motion from the terms $\mathcal{O}(1)$

$$a_1'' \simeq \frac{1+(\omega z)^2}{z^3} \tag{2.154}$$

and $\mathcal{O}(\epsilon)$

$$\epsilon\left(z^\nu a_1'''' + 2(\nu+2)z^{\nu-1}a_1''' + (\nu+1)(\nu+2)z^{\nu-2}a_1''\right) \simeq \epsilon z^{\nu-5}(1+(\omega z)^4)\,. \tag{2.155}$$

Therefore, if $0 < \nu \leq 2$, the correction becomes significant when $z \gg \frac{1}{(\epsilon\omega^2)^{1/\nu}}$ (see the hierarchy of scales in Fig. 2.6). If $\nu > 2$, considering modes with sufficiently low frequency $\omega < \epsilon^{1/(\nu-2)}$ the correction becomes significant when $z \gg \epsilon^{1/(2-\nu)}$ (see Fig. 2.7). Finally, if $\nu > 2$ and $\omega > \epsilon^{1/(\nu-2)}$, the Fig. 2.6 is applicable, and the correction is significant when $z \gg \frac{1}{(\epsilon\omega^2)^{1/\nu}}$. Hence, in the region $z \ll \frac{1}{(\epsilon\omega^2)^{1/\nu}}$ the DBI description is valid, provided that $0 < \nu \leq 2$ or $\nu > 2$, $\omega > \epsilon^{1/(\nu-2)}$. The DBI description is valid in the region $z \ll \epsilon^{1/(2-\nu)}$ for $\nu > 2$, $\omega < \epsilon^{1/(\nu-2)}$.

The behavior of a_1 in the limit $z \gg 1$ where the DBI description is valid, is different from the incoming-wave (2.114): it has a qualitative form

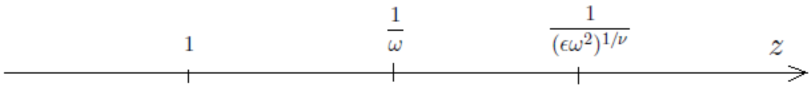

Figure 2.6. Hierarchy of scales in the near-horizon region for $0 < \nu \leq 2$.

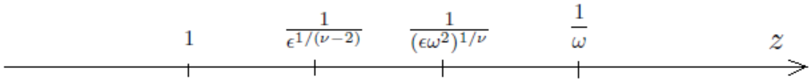

Figure 2.7. Hierarchy of scales in the near-horizon region for $\nu > 2$ and $\omega < \epsilon^{1/(\nu-2)}$.

of "incoming wave" + $\mathcal{O}(\epsilon)$ "outgoing wave". It is worth noting that the effect of higher derivative corrections on the current-current correlation function is essentially the same as an effect of non-zero b-field. We verified that such a modification does not lead to any nontrivial structure in the spectral density.

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Chapter 3

S-parameter, Technimesons, and Phase Transitions in Holographic Tachyon DBI Models

3.1 Introduction and summary

Systems of strongly interacting fermions have applications in many realms, including condensed matter (e.g. graphene) and particle physics (e.g. technicolor models). A simple way to introduce interaction between fermions involves adding a quartic term to the lagrangian of N free fermions, resulting in the Nambu-Jona-Lasinio model (see e.g. [1] for a review). In three space-time dimensions the model is renormalizable to all orders in the $1/N$ expansion: one can take a double scaling limit where the coupling is tuned to the critical value, while the UV cutoff is sent to infinity, keeping the physical mass fixed. Dynamical mass generation at sufficiently large values of the coupling is an important feature which is believed to happen in other strongly interacting fermion systems.

Unfortunately one often has to resort to approximate methods to describe the physics in the vicinity of the phase transition from the massless phase to the one with a gap. This is because the transition happens at the intermediate values of the coupling where both the weak coupling and and strong coupling expansions break down. Nevertherless such description is often very useful for phenomenological reasons: for example, the walking technicolor models are precisely of this type, since they stay very close to the putative conformal fixed point for the long RG time. In [2] a tachyon dynamics in the AdS space-time was shown to holographically model this type of physics; this has been further studied in [3] in the context of a particular holographic model based on the Tachyon DBI action in AdS. The mass of the tachyon is tuned to the critical value (the BF bound) and at the same time the UV cutoff is sent to infinity, so that the physical scale measured, for example, by the meson masses, stays fixed.

In this chapter we study some phenomenological applications of the model proposed in [3] which, in turn, was motivated by the holographic description of the dynamics of the D3 and D7 branes intersecting along 2+1 dimensions [4]. We restrict our attention to four space-time dimensions. In the next Section we investigate the phase diagram of the holographic model at finite temperature and charge density. We show that the phase transition at finite temperature between the symmetric and the massive phase is generalized into the phase transition line in the temperature-charge density plane. Furthermore, depending on the value of the quartic coefficient in the tachyon potential, the phase transition line can either stay first order, or possess a critical point where the order of the phase transition changes from second to first. This is somewhat similar to the situation with the (conjectural) phase diagram of QCD with massless quarks and constitutes an interesting prediction for the phase diagram of strongly interacting fermions.

In Section 3 we explore a possibility of using the holographic TDBI model in the context of holographic walking technicolor. We couple the tachyon bilinear to the gauge fields in the adjoint representations of $SU(N_f)_L$ and $SU(N_f)_R$ which contain electroweak gauge group (setting $N_f = 2$ and embedding the electroweak group as $SU(2) \times U(1) \subset SU(2)_L \times SU(2)_R$ constitutes the simplest setup). Tachyon condensate breaks electroweak symmetry and generates masses for the W and Z bosons giving rise to a model of holographic walking technicolor. We compute the Peskin-Takeuchi S-parameter for a variety of tachyon potentials and observe that it is positive and does not go to zero. In Section 4 we compute the masses of the lightest scalar technimesons for a certain family of the tachyon potentials and observe that even though there is no parametrically light "technidilaton", the lowest lying meson can be an order of magnitude lighter then the next one.

We conclude in Section 5. Appendix contains application of the holographic RG to the holographic tachyon TDBI model, where a picture for the running of the double trace coupling, expected from field theoretic considerations, is reproduced.

3.2 Holographic TDBI at finite temperature and chemical potential

In this Section we consider holographic tachyon DBI model at finite temperature and chemical potential. We consider AdS-Schwarzshild black hole to account for a non-vanishing temperature, and we turn on a background flux of the $U(1)$ gauge potential which corresponds to to the finite density in the dual field theory. We describe the phase with broken conformal symmetry by the dual picture with non-vanishing tachyon field in the bulk, while conformally symmetric field theory state corresponds to the identically vanishing tachyon in the bulk. We compute holographically free energies of both phases and determine the resulting phase diagram.

Perhaps the future development of the results of this Section will mostly lie in the realm of condensed matter physics. However, let us make a slight detour and remind the reader a closely related problem, a phase diagram of QCD at finite temperature and chemical potential. (See e.g. [5, 6] for recent reviews). The phase structure of QCD is roughly the following. If the temperature is low and we increase the density, then at some value of the density the system is expected to undergo a first order phase transition to the states where the hadrons dissociate. At sufficiently large density, the system gets into the color superconducting phase. In this phase confined bound state of two quarks goes to Coulomb bound state, in a process similar to Cooper pairing in the microscopic description of a superconductor. Increasing the temperature destroys Cooper pairing mechanism for the quarks, eventually giving rise to a quark-gluon plasma. This is believed to be a preferred high temperature state for any values of the chemical potential, however the phase transition from the hadronic state is first order for larger densities, but second order for smaller densities (for massless quarks). As we will see below, we can observe somewhat similar phase structure for certain TDBI models, though either the orders of first and second order phase transitions are interchanged or we have two critical points at which the order of phase transition changes.

Phase transitions in the holographic tachyon DBI at finite temperature have been studied in [3], which the reader is encouraged to consult for technical details relevant to the present Section [1]. There it has been established that the order of the phase transition is determined by the behavior of the tachyon potential for very small values of T (the tachyon field). In the BKT limit, where the UV cutoff is taken to infinity, with physical observables held fixed, the solution must have a fixed ratio between the two asymptotics near the boundary of AdS. The value of the coefficient in front of the T^4 term in the tachyon potential determines whether increasing the value of T at the black hole horizon corresponds to the smaller or larger temperatures. In the former case, the transition is second order, while in the latter case it is first order. In the following we repeat this analysis in the presence of finite density.

Consider finite temperature AdS_{d+1}-Schwarzshild metric

$$ds^2 = r^2\left(-F(r)dt^2 + (dx^1)^2 + ... + (dx^{d-1})^2\right) + \frac{dr^2}{r^2F(r)}\,, \tag{3.1}$$

where $F(r) = 1 - \left(\frac{r_h}{r}\right)^d$, and turn on non-vanishing flux $\dot{A}_0$. Tachyon-DBI action then takes the form

$$S_{TDBI} = -\int_{r_h}^{\infty} dr \int d^dx r^{d-1} V(T)\sqrt{1 + r^2\dot{T}^2F - \dot{A}_0^2}\,. \tag{3.2}$$

From equation of motion for gauge flux we obtain

$$\dot{A}_0^2 = \frac{\hat{d}^4(1 + r^2F\dot{T}^2)}{r^{2(d-1)}V^2 + \hat{d}^4}\,, \tag{3.3}$$

where $\hat{d}(\mu_{ch}, r_h)$ is a constant of integration. As usual, up to a normalization constant, $\hat{d}^2$ is proportional to the charge density of the system. Due to (3.3) in the leading order in T we obtain

$$\mu_{ch} = \int_{r_h}^{\infty} \frac{dr\,\hat{d}^2}{\sqrt{\hat{d}^4 + rqn2(d-1)}} = \frac{\hat{d}^2}{(d-2)r_h^{d-2}}\,{}_2F_1\left(\frac{1}{2}, \frac{d-2}{2(d-1)}, \frac{3d-4}{2(d-1)}, -\frac{\hat{d}^4}{r_h^{2(d-1)}}\right)\,. \tag{3.4}$$

[1]In recent work [7] the phase structure of the holographic model of QCD in the Veneziano limit has been analyzed at finite temperature.

Plugging (3.3) in into the action (3.2) we arrive at

$$S_{TDBI}{=}-\int_{r_h}^{\infty}dr\int d^dx r^{2(d-1)}V^2\left((1{+}r^2F\dot{T}^2)(r^{2(d-1)}V^2{+}\hat{d}^4)^{-1}\right)^{1/2}. \tag{3.5}$$

Introduce dimensionless coordinate, $\tilde{r}=r/\hat{d}^{\frac{2}{d-1}}$, and dimensionless temperature, $\tilde{r}_h=r_h/\hat{d}^{\frac{2}{d-1}}$. As a result the action acquires the form

$$S_{TDBI}=-\hat{d}^{\frac{2d}{d-1}}\int_{\tilde{r}_h}^{\infty}d\tilde{r}\int d^dx\frac{\tilde{r}^{2(d-1)}V^2}{\sqrt{1+\tilde{r}^{2(d-1)}V^2}}\sqrt{1+\tilde{r}^2FT'^2}\,, \tag{3.6}$$

where $F=1-(\tilde{r}_h/\tilde{r})^d$ and $T'=\partial T/\partial\tilde{r}$.

Let us define tachyon T value at the horizon, $T_h=T(r_h)$. Equation of motion for the tachyon field, following from the action (3.6), is

$$\frac{\partial}{\partial r}\left[\frac{\tilde{r}^{2d}FV^2T'}{\sqrt{(1+\tilde{r}^2FT'^2)(1+\tilde{r}^{2(d-1)}V^2)}}\right] - \tilde{r}^{2(d-1)}V^2\frac{2+\tilde{r}^{2(d-1)}V^2}{(1+\tilde{r}^{2(d-1)}V^2)^{3/2}}\sqrt{1+\tilde{r}^2FT'^2}\,\partial_T\log V=0\,. \tag{3.7}$$

Using (3.7) and imposing the boundary condition $T(\tilde{r}=\tilde{r}_h)=T_h$, we find

$$T'(\tilde{r}=\tilde{r}_h)=\frac{2+\tilde{r}_h^{2(d-1)}V^2(T_h)}{d\tilde{r}_h(1+\tilde{r}_h^{2(d-1)}V^2(T_h))}\partial_T\log V(T_h)\,. \tag{3.8}$$

When $T\sim T_h\ll 1$ and $m\simeq m_{BF}^2=-d^2/4$ we obtain linearized equation of motion

$$\left(\frac{\tilde{r}^{2d}FT'}{\sqrt{1+\tilde{r}^{2(d-1)}}}\right)'+\frac{d^2\tilde{r}^{2(d-1)}}{4}\frac{2+\tilde{r}^{2(d-1)}}{(1+\tilde{r}^{2(d-1)})^{3/2}}T=0 \tag{3.9}$$

and boundary conditions

$$T(\tilde{r}=\tilde{r}_h)=T_h\,,\qquad T'(\tilde{r}=\tilde{r}_h)=-\frac{dT_h}{4\tilde{r}_h}\frac{2+\tilde{r}_h^{2(d-1)}}{1+\tilde{r}_h^{2(d-1)}}\,. \tag{3.10}$$

Near the boundary $\tilde{r}\to\infty$ behavior of $T(\tilde{r})$ is given by equation

$$T''+\frac{d+1}{\tilde{r}}T'+\frac{d^2}{4\tilde{r}^2}T=0\,. \tag{3.11}$$

Let us now specialize to $d = 4$ case. Near-boundary behavior is then described by equation

$$T'' + \frac{5}{\tilde{r}}T' + \frac{4}{\tilde{r}^2}T = 0\,. \tag{3.12}$$

which is solved by

$$T(\tilde{r}) \simeq \frac{1}{\tilde{r}^2}\left(c_1 \log \tilde{r} + c_2\right) \quad \Rightarrow \quad T(r) = \frac{1}{r^2}\left(c_1 \log \frac{r}{\hat{d}^{2/3}} + c_2\right)\,. \tag{3.13}$$

Let us denote

$$g = \hat{d}^{2/3} \tag{3.14}$$

The constants c_1 and c_2 can be determined by solving equation of motion (3.7) numerically. If we consider instead linearized equation (3.9) in BKT limit with boundary conditions (3.10), we obtain c_2/c_1, which is a function of $\tilde{r}_h = r_h/g$. In the case of vanishing temperature and vanishing chemical potential the near-boundary behavior of tachyon field is given by

$$T(r) = \frac{1}{r^2}\left(C_1 \log \frac{r}{\mu} + C_2\right) \tag{3.15}$$

Clearly it must be the same as (3.13) . Matching these equations, we obtain

$$\frac{g}{\mu} = C_0 \exp\left(\Xi\left(\frac{r_h/\mu}{g/\mu}\right)\right)\,, \tag{3.16}$$

where we have denoted $C_0 = e^{-C_2/C_1}$ and $\Xi = c_2/c_1$. Equation (3.16) can be solved numerically, which gives critical values of temperature r_h and g measured in units of μ. The result appears in figure 1. We have checked that when $\hat{d} = 0$ the critical temperature is equal to $2\,C_0$, which is a correct limiting value [3].

To determine which state in the canonical ensemble is preferred, we need to compare the free energies. Similarly to [3], we focus on the near-critical region, where tachyon field is either vanishing or small. The difference in free energies between non-vanishing tachyon and vanishing tachyon phases is given by

$$\mathcal{F}(r_h, \hat{d}) = S_{TDBI}(T \equiv 0) - S_{TDBI}(T)\,, \tag{3.17}$$

where the last term in the r.h.s. is evaluated on the solution, satisfying $T(r = r_h) = T_h$ boundary condition. Due to $V(0) = 1$ one obtains, using

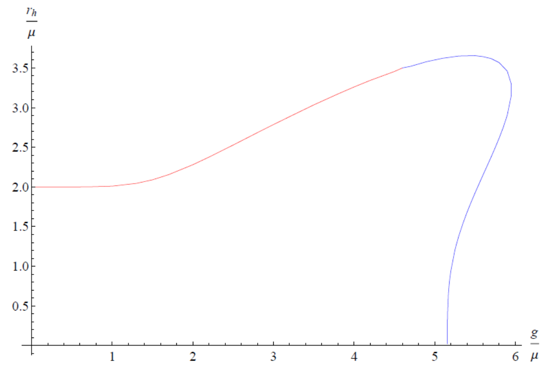

Figure 3.1. Phase diagram for conformal phase transition in $(g/\mu,\ r_h/\mu)$ plane. Order of phase transition changes at the point $\tilde{r}_h \equiv r_h/g = 0.75$. Blue part of the curve describes second order phase transition and red part of the curve describes first order phase transition.

(3.6)

$$\mathcal{F}(r_h, \hat{d}) = \hat{d}^{8/3} \int_{\tilde{r}_h}^{\infty} d\tilde{r} \int d^4x \tilde{r}^6 \left(\frac{V^2}{\sqrt{1+\tilde{r}^6 V^2}} \sqrt{1+\tilde{r}^2 F T'^2} - \frac{1}{\sqrt{1+\tilde{r}^6}} \right) . \tag{3.18}$$

We will need the form of the tachyon potential near $T = 0$:

$$V(T) = 1 + \frac{1}{2} m^2 T^2 + \frac{a}{4} T^4 + \cdots , \tag{3.19}$$

where $m^2 \simeq m_{BF}^2 = -4$ and a is the coefficient of the quartic term which, as explained in [3], determines the order of the phase transition in the case of vanishing density. Below we will see that at finite density the situation is more subtle, and the first order phase transition line can join the second order phase transition line at a critical point, provided the value of a is chosen accordingly.

In the BKT limit we have $T \sim T_h \ll 1$ and $m^2 = m_{BF}^2 = -4$. We compute (3.18) up to the fourth order in T_h,

$$\mathcal{F}(r_h, \hat{d}) = \mathcal{F}_2(r_h, \hat{d}) + \mathcal{F}_4(r_h, \hat{d}) + \cdots , \tag{3.20}$$

where

$$\mathcal{F}_2 = \hat{d}^{8/3} \int_{\tilde{r}_h}^{\infty} d\tilde{r} \int d^4x \frac{1}{2\sqrt{1+\tilde{r}^6}} \left(\tilde{r}^8 F T'^2 - 4\tilde{r}^6 \frac{2+\tilde{r}^6}{1+\tilde{r}^6} T^2 \right)$$

$$\mathcal{F}_4 = -\hat{d}^{8/3} \int_{\tilde{r}_h}^{\infty} d\tilde{r} \int d^4x \frac{\tilde{r}^6}{8(1+\tilde{r}^6)^{5/2}} \Big(F^2 T'^4 \tilde{r}^4 (1+\tilde{r}^6)^2 + \tag{3.21}$$

$$+ 8F\tilde{r}^2 T^2 T'^2 (1+\tilde{r}^6)(2+\tilde{r}^6) + 2T^4 (8(\tilde{r}^6 - 2)$$

$$- a(1+\tilde{r}^6)(2+\tilde{r}^6)) \Big) \tag{3.22}$$

The quadratic terms vanish on shell, up to the boundary term, which also vanishes, because

$$F(\tilde{r} = \tilde{r}_h) = 0\,, \qquad T(\tilde{r} = \infty) = 0\,. \tag{3.23}$$

We solve numerically equation (3.9) with boundary conditions (3.10), for each particular $\tilde{r}_h = r_h/g$. This gives us $T = T_1$, which is a solution of the first order in T_h. The first correction to this solution is obtained when we take into account quartic in T_h terms in the action for T, and therefore the corrected solution is $T = T_1 + T_3$, where T_3 is of the third order in T_h. Therefore we need to compute in the leading order

$$\mathcal{F}(T_1 + T_3) = \mathcal{F}_2(T_1 + T_3) + \mathcal{F}_4(T_1 + T_3)\,. \tag{3.24}$$

For brevity let us rewrite (3.70) as

$$\mathcal{F}_2 = \int dr [\alpha(r) T^2 + \beta(r) T'^2]$$

$$\mathcal{F}_4 = \int dr [a(r) T^4 + b(r) T^2 T'^2 + c(r) T'^4]\,. \tag{3.25}$$

Let us use integration by parts to bring $\mathcal{F}_{2,4}$ to the form

$$\mathcal{F}_2 = \int dr\, T[\alpha T - (\beta T')'] \equiv \int dr\, T P_1$$

$$\mathcal{F}_4 = \int dr\, T \left[a T^3 + \frac{b}{2} T T'^2 - \left(\frac{b}{2} T' T^2 \right)' - (c T'^3)' \right] \equiv \int dr\, T P_3 \tag{3.26}$$

where $P_{1,3}$ are polynomials of the T, T', T'' of the degree specified by the subscript. From the variation

$$\delta\mathcal{F} = 2 \int dr\, \delta T [\alpha T - (\beta T')'] + 4 \int dr\, \delta T \left[a T^3 + \frac{b}{2} T T'^2 - \left(\frac{b}{2} T' T^2 \right)' - (c T'^3)' \right] \tag{3.27}$$

we obtain equation of motion

$$2P_1(T_1+T_3)+4P_3(T_1+T_3)=0\,, \tag{3.28}$$

which we can solve perturbatively as

$$P_1(T_1)=0\,,\quad P_1(T_3)+2P_3(T_1)=0\,. \tag{3.29}$$

Using these equations in the expansion of (3.24)

$$\begin{aligned}\mathcal{F} &= \int dr (T_1+T_3)P_1(T_1+T_3)+(T_1+T_3)P_3(T_1+T_3)\\ &= \int dr T_1P_1(T_3)+T_3P_1(T_1)+T_1P_3(T_1)+\cdots\end{aligned} \tag{3.30}$$

we obtain

$$\mathcal{F}(T)\simeq -\int dr T_1 P_3(T_1) = -\mathcal{F}_4(T_1)\,. \tag{3.31}$$

We then evaluate quartic terms, $\mathcal{F}_4(\hat{d},a)$, on the numerically found solution T_1. Equation $\mathcal{F}_4(\tilde{r}_h,\,a)=0$ gives values of ratio $r_h/g=\tilde{r}_h$ for each particular a at which order of phase transition changes. This equation is valid only for those values of r_h and $\hat{d}$ which are close to critical ones. We solve this equation numerically for each particular value of the parameter a, that is we find $\tilde{r}_h^{(c)}(a)$. The result is plotted in figure 2. Notice that when $\hat{d}$ is sent to zero, $\tilde{r}_h$ goes to infinity, and the special value $a\simeq 6.47$ becomes the same as in the case of vanishing chemical potential [3]. Also notice that when $6.47\leq a\leq 7.03$ there are two points $\tilde{r}_h$ at which the order of phase transition changes.

In figure 1 we have taken $a=6.41$ for which phase transition is the second order one for $\frac{r_h}{g}<0.75$ and the first order one for $\frac{r_h}{g}>0.75$. This corresponds respectively to the blue and red parts of the phase transition curve in figure 1. [2]

The other option is to take the value of a at which we have two critical points where the order of phase transition changes. Then for the temperature bellow some critical value, $\tilde{r}_h^{(c)}<r_h^{(c,1)}$ we have second order phase

[2]One may use the top-down approach based on the $D3-D7$ system to derive the phase diagram of $N=4$ super Yang-Mills coupled to $N=2$ matter at finite temperature and chemical potential. It also exhibits the phase transition of the second order at small temperatures. See [8] for a recent discussion.

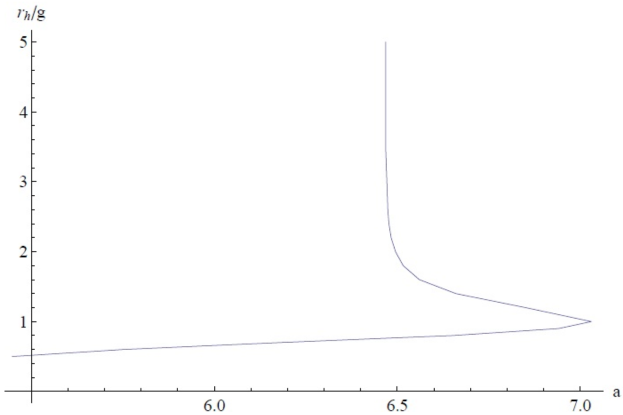

Figure 3.2. The ratio $\tilde{r}_h = r_h/g$ at which the order of phase transition changes, as a function of UV parameter a. It is determined by the sign of $\mathcal{F}_4$ in the conformal symmetry broken phase. On the left side of the curve $\mathcal{F}_4 < 0$ and phase transition is of the first order, on the right side $\mathcal{F}_4 > 0$ and phase transition is of the second order.

transition, for $\tilde{r}_h^{(c,1)} < \tilde{r}_h^{(c)} < \tilde{r}_h^{(c,2)}$ we have first order order phase transition, and finally for $\tilde{r}_h^{(c)} > \tilde{r}_h^{(c,2)}$ we have second order phase transition. The critical point $\tilde{r}_h^{(c,2)}$ therefore resembles the one in the QCD phase diagram.

As emphasized in [3], the behavior of the free energy for small values of the tachyon condensate determines the order of the phase transition, provided the phase diagram has a simple form. This was the case in all examples studied in [3]. We believe this remains true once the finite density is turned on, but to show this some further numerical work is necessary.

3.3 S parameter

3.3.1 Review of technicolor and S, T, U parameters

Consider the system of 2 techniquark matter fields $(\tilde{u}, \tilde{d})$ with color charges, transforming in fundamental representation of the gauge group $SU(N_c)$. Quark fields are coupled to gauge field in the adjoint representation of the gauge group. In the ultraviolet regime these quarks are massless, and therefore the system possesses the $SU(2)_L \times SU(2)_R$ chiral symmetry. Therefore we can couple the doublet of left quarks $Q = (\tilde{u}_L\,,\ \tilde{d}_L)$ to bosons of weak gauge group $SU(2)_L$, leaving 2 right quarks $\tilde{u}_R$ and $\tilde{d}_R$ in the singlet representation sector of the weak gauge transformations. We also give each quark field the hypercharge Y, characterizing its representation under the action of the $U(1)_Y$ gauge group.

We look at introduced 2 quarks as a set of strongly-interacting fermionic fields of the physics beyond the Standard Model. At some energy scale due to the strong interaction these quarks may form a chiral condensate, breaking the chiral symmetry down to $SU(2)_{diag}$. [3] In the vacuum with spontaneously broken chiral symmetry the 2 quarks acquire a mass. In the technicolor models the phenomenon of the chiral symmetry breaking via techniquark condensation is used to explain the spontaneous electroweak symmetry breaking, realized therefore as a dynamical symmetry breaking. Furthermore, the extended technicolor models combine these 2 techniquarks with SM matter fields in some specific multiplets in such a way that condensation of techniquarks gives masses to SM matter fields. In the simplest technicolor models such quartic fermionic terms, generated at high scale Λ_{ETC}, lead to flavor changing neutral currents matrix elements that are way above experimental bounds. A walking technicolor models, where the system spends a long RG time in a vicinity of a putative RG fixed point and the anomalous dimension of the technimeson condensate $\gamma \simeq 1$, has been proposed to alleviate this problem. (See [10] for a recent review of walking technicolor and references therein). The theory is necessarily strongly coupled, and it is natural to use holography in this context.

To create a possibility for experimental tests of theories describing physics beyond the Standard Model, Pesking and Takeuchi [11, 12] introduced dimensionless parameters $S\,,\ T\,,\ U$, measuring an impact of a hidden sector of heavy beyond-SM fundamental matter fields coupled to electro-weak gauge bosons. They argued (following [13]) that the most important impact arises from oblique corrections: vacuum polarization diagrams, which renormalize gauge boson propagators. Peskin-Takeuchi parameters are expressed via these vacuum polarization amplitudes, and we will review their argument in a greater detail bellow. For each beyond-SM theory we therefore may compute $S\,,\ T\,,\ U$ parameters and see whether the results lie within the boundaries set by the deviation of experimental

[3] It was shown in [9] that under general assumptions in large-N_c chromodynamics the chiral symmetry breaks spontaneously.

data from Standard Model predictions.

The quantum corrections of matter fields to the propagators of the SM gauge fields come from the vacuum polarization amplitudes

$$\int d^4x e^{iq\cdot x}\langle J_a^\mu(x) J_b^\nu(0)\rangle = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi_{ab}(q^2)\,, \tag{3.32}$$

where $a, b = 1, 2, 3, Q$ and we are assuming mostly plus signature of the metric. The expression (3.32) should be computed for the matter fields of the SM and for the hidden matter sector of the beyond-SM physics.

For weak currents $J_i\,, i = 1, 2$, weak isospin current J_3 and electromagnetic current J_Q we have the vacuum polarization amplitudes

$$\Pi_{11}\,,\ \Pi_{22}\,,\ \Pi_{33}\,,\ \Pi_{3Q}\,,\ \Pi_{QQ}\,. \tag{3.33}$$

If we know these amplitudes, then using expression for the electroweak interaction Lagrangian

$$L = \frac{e}{\sqrt{2}s}\left(W_\mu^+ J_+^\mu + W_\mu^- J_-^\mu\right) + \frac{e}{sc}Z_\mu\left(J_3^\mu - s^2 J_Q^\mu\right) + eA_\mu J_Q^\mu \tag{3.34}$$

we can obtain 1PI self-energies for the electroweak gauge bosons, and 1PI mixing for Z-boson and photon,

$$\Pi_{AA} = e^2\Pi_{QQ}\,, \quad \Pi_{ZA} = \frac{e^2}{sc}(\Pi_{3Q} - s^2\Pi_{QQ})\,, \ldots\,. \tag{3.35}$$

Then with the help of Schwinger-Dyson equations we can derive full quantum propagators for the electroweak gauge fields.

Now, in the interaction Lagrangian (3.34) we have parameters e and

$$s^2 \equiv \sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2}\,. \tag{3.36}$$

Quantum corrections due to the vacuum polarization amplitudes boil down to the renormalization of these parameters,

$$e_\star^2(q^2) \equiv \frac{e^2}{1 - e^2\Pi_{QQ}(q^2)}\,, \tag{3.37}$$

$$s_\star^2(q^2) \equiv s^2 - sc\frac{\Pi_{ZA}(q^2)}{q^2 - \Pi_{AA}(q^2)}\,. \tag{3.38}$$

Then, the renormalized parameter $s_\star$ enters the measured left-right Z-decay asymmetry,

$$A_{LR}(q^2) = \frac{2(1-4s_\star^2)}{1+(1-4s_\star^2)^2}\,, \tag{3.39}$$

and therefore the renormalization of the gauge fields propagators (coming mainly as the oblique corrections due to loops of heavy fermions) can be measured experimentally.

Let us also define θ_0 as

$$\sin(2\theta_0) = \sqrt{\frac{4\pi\alpha_{\star,0}(m_Z^2)}{\sqrt{2}G_F m_Z^2}}\,. \tag{3.40}$$

Here m_Z and G_F are experimentally measured. And $\alpha_{\star,0}(m_Z^2)$ is a running electromagnetic coupling, which is computed due to known physics up to $q^2 = m_Z^2$ scale. The running starts from the measured $\alpha(q^2=0) = e^2/(4\pi)$.

The renormalization comes from SM and from physics beyond the SM. In the SM the most important contribution comes from t-quark loops (see e.g. [14] Chapter 21),

$$s^2 - s_\star^2 = -\frac{3\alpha c^2}{16\pi s^2}\frac{m_t^2}{m_Z^2}\,, \tag{3.41}$$

$$s_\star^2 - s_0^2 = -\frac{3\alpha}{16\pi(c^2-s^2)}\frac{m_t^2}{m_Z^2}\,. \tag{3.42}$$

Let us now describe quantum corrections due to vacuum polarization diagrams of beyond-SM physics. First of all for heavy fermion we can expand vacuum polarization amplitudes around $q^2=0$,

$$\Pi_{QQ}(q^2) = q^2\Pi'_{QQ}(0)\,, \quad \Pi_{3Q}(q^2) = q^2\Pi'_{3Q}(q^2)\,, \tag{3.43}$$

$$\Pi_{33}(q^2) = \Pi_{33}(0) + q^2\Pi'_{33}(0)\,, \tag{3.44}$$

$$\Pi_{11}(q^2) = \Pi_{11}(0) + q^2\Pi'_{11}(0)\,, \tag{3.45}$$

where prime denotes differentiation w.r.t. q^2 and we have made use of the fact that Ward identity for electromagnetic field ensures $\Pi_{QQ}(0) = 0$ and $\Pi_{3Q}(0) = 0$. Also we have $\Pi_{11} = \Pi_{22}$. We have therefore six parameters defining vacuum polarization amplitudes of heavy fermions. We make a renormalization, fixing values of three well-measured parameters, which are α, G_F and m_Z. Three parameters which are left are free of UV divergencies, and we combine these into

$$\alpha S = 4e^2 \left(\Pi'_{33}(0) - \Pi'_{3Q}(0) \right) , \tag{3.46}$$

$$\alpha T = \frac{e^2}{s^2 c^2 m_Z^2} \left(\Pi_{11}(0) - \Pi_{33}(0) \right) , \tag{3.47}$$

$$\alpha U = 4e^2 \left(\Pi'_{11}(0) - \Pi'_{33}(0) \right) . \tag{3.48}$$

In addition to SM corrections (3.41) and (3.42) we can write down contribution of beyond-SM physics via these parameters:

$$\frac{m_W^2}{m_Z^2} - c_0^2 = \frac{\alpha c^2}{c^2 - s^2} \left(-\frac{1}{2} S + c^2 T + \frac{c^2 - s^2}{4s^2} U \right) , \tag{3.49}$$

$$s_\star^2 - s_0^2 = \frac{\alpha}{c^2 - s^2} \left(\frac{1}{4} S - s^2 c^2 T \right) . \tag{3.50}$$

Thus we explicitly constructed a set of experimentally measured quantities, quantum corrections to which may be separately computed from the SM, (3.41), (3.42), and from a hidden sector, (3.49), (3.50), with the latter being expressed via Peskin-Takeuchi parameters. Let us use vector and axial-vector isospin currents

$$J_V^\mu = \bar\psi \gamma^\mu \tau_3 \psi \,, \qquad J_A^\mu = \bar\psi \gamma^\mu \gamma^5 \tau_3 \psi \,, \tag{3.51}$$

to express left isospin current as

$$J_3^\mu = \frac{1}{2} (J_V^\mu - J_A^\mu) \,. \tag{3.52}$$

Consider also electromagnetic current, expressed via isospin and hypercharge currents in a usual way,

$$J_Q^\mu = J_V^\mu + \frac{1}{2} J_Y^\mu \,. \tag{3.53}$$

Assuming the conservation of parity by technicolor interactions we can express isospin current correlator via vector and axial vector isospin correlators, $\Pi_{33} = \frac{1}{4}(\Pi_{VV} + \Pi_{AA})$. We also note that due to isospin conservation $\langle J_3 J_Y \rangle = 0$ (otherwise in technicolor models there would have been a preferred isospin direction), we obtain $\Pi_{3Q} = \frac{1}{2}\Pi_{VV}$. Therefore

$$S = -4\pi (\Pi'_{VV}(q^2) - \Pi'_{AA}(q^2))|_{q^2=0} \,. \tag{3.54}$$

The holographic tachyon DBI was introduced in [3] to describe a system of strongly interacting fermions, which can be made into the walking technicolor theory. However for the purpose of computing the S-parameter and technimeson masses, we do not even need to specify that the holographic TDBI model describes strongly interacting fermions. Instead, it is sufficient to treat the holographic model as a black box, which produces two-point functions for the vector and axial currents and features spontaneous breaking of the axial symmetry. Then, these currents are coupled to the SM gauge fields to produce spontaneous symmetry breaking of the electroweak gauge group. The resulting contribution to the S-parameter is given by (3.54) and is computed below.

3.3.2 Computation of the S parameter from the tachyon DBI action

As pointed out above, the holographic tachyon DBI theory provides a natural model of the walking technicolor scenario. The important feature of the holographic approach is that we can isolate the impact of beyond-SM sector of the theory. For this purpose we just have to consider a corresponding set of fields in the bulk, and study its classical dynamics[4].

We need to construct a dual to a strongly interacting theory with the $SU(2)_L \times SU(2)_R$ global symmetry in the UV. The global currents $j_\mu^{(L)}$ and $j_\mu^{(R)}$ give rise to the bulk fields $A_M^{(L,R)}$, living in adjoint of $SU(2)_{L,R}$, with gauge transformations

$$A_M^{(L)} \to U_L A_M^{(L)} U_L^\dagger + i\partial_M U_L U_L^\dagger \,, \; A_M^{(R)} \to U_R A_M^{(R)} U_R^\dagger + i\partial_M U_R U_R^\dagger \,. \quad (3.55)$$

Tachyon field $T(r, x)$ lives in bi-fundamental of $SU(2)_L \times SU(2)_R$, i.e., its gauge transformations are given by

$$T \to U_L T U_R^\dagger \,. \quad (3.56)$$

Tachyon action with $SU(2)_L \times SU(2)_R$ local symmetry in the bulk is then

$$S = -\int d^4x dr \, \mathrm{Tr}\, V(|T|) \left(\sqrt{-G^{(L)}} + \sqrt{-G^{(R)}} \right) \,, \quad (3.57)$$

[4] Previous work dedicated to holographic technicolor and S parameter includes [15–46]; see also [47–53] for recent related work.

where

$$G^{(R)}_{MN} = G_{MN} + F^{(R)}_{MN}\,, \quad G_{MN} = g_{MN} + (D_{(M}T)^{\dagger} D_{N)} T\,,$$
$$G^{(R)} = \det\, G^{(R)}_{MN}\,, \tag{3.58}$$

and similar for the left; and covariant derivative of tachyon field is given by

$$D_M T = \partial_M T + i A^{(L)}_M T - i T A^{(R)}_M\,. \tag{3.59}$$

Similar actions for the tachyon have been introduced in [54].

We have $A^{(L,R)}_M = (A^{(L,R)}_r, A^{(L,R)}_\mu)$ and we partly fix the gauge symmetry putting

$$A^{(L)}_r = 0\,, \qquad A^{(R)}_r = 0\,. \tag{3.60}$$

Let us introduce gauge fields in the bulk, dual to vector and axial currents on the boundary:

$$A^{(L)}_M = \frac{1}{2}(V_M - A_M)\,, \qquad A^{(R)}_M = \frac{1}{2}(V_M + A_M) \tag{3.61}$$

Suppose we have background tachyon field $T(r) = \langle T(r)\rangle I$, with real-valued vacuum average $\langle T(r)\rangle = T_0(r)$, satisfying equation of motion at vanishing gauge fields,

$$\frac{d}{dr}\left(\frac{r^5 \dot{T}_0}{\sqrt{1 + r^2 \dot{T}_0^2}}\right) = \frac{r^3 \partial_T \log V(T_0)}{\sqrt{1 + r^2 \dot{T}_0^2}} \tag{3.62}$$

Such a background tachyon field breaks the symmetry down to $SU(2)_{diag}$, which means $U_L = U_R$. Its non-zero covariant derivative components, due to the gauge choice (3.60) and definition (3.61) are (the fact that T couples only to axial field A means that axial symmetry is broken)

$$D_r T = \dot{T}_0\, I\,, \qquad D_\mu T = -i A_\mu T_0\,. \tag{3.63}$$

In what follows we consider the case of just one flavor of quark fields. The results can be generalized to arbitrary number of flavors, because for the holographic computation of two-point functions higher order non-abelian terms in gauge field Lagrangian do not play any role. We therefore have

$$G_{MN} = g_{MN} + \partial_M T_0 \partial_N T_0 + A_M A_N T_0^2\,. \tag{3.64}$$

Let us denote for brevity

$$\mathcal{G}_{MN} = g_{MN} + \partial_M T_0 \partial_N T_0 = \text{diag}\left(-r^2,\ r^2,\ r^2,\ r^2,\ \frac{1 + r^2\dot{T}_0^2}{r^2}\right), \quad (3.65)$$

and let us write down an inverse matrix to (3.65)

$$\mathcal{M}^{MN} \equiv (\mathcal{G}^{-1})^{MN} = \text{diag}\left(-\frac{1}{r^2},\ \frac{1}{r^2},\ \frac{1}{r^2},\ \frac{1}{r^2},\ \frac{r^2}{1 + r^2\dot{T}_0^2}\right). \quad (3.66)$$

We also denote

$$\sqrt{-G} = \sqrt{-\det \|\, G_{MN} \,\|}, \quad (3.67)$$

$$\sqrt{-G_0} = \sqrt{-\det \|\, \mathcal{G}_{MN} \,\|} = r^3 \sqrt{1 + r^2\dot{T}_0^2}\,,$$

$$K_{MN} = G_{MN} - \mathcal{G}_{MN} = A_M A_N T_0^2\,. \quad (3.68)$$

Up to second order in A we expand

$$\sqrt{-G} = \sqrt{-G_0}\, \exp\left(\frac{1}{2}\text{tr}\log(1 + \mathcal{M}K)\right)$$

$$= r^3\sqrt{1 + r^2\dot{T}_0^2}\left(1 + \frac{T_0^2}{2r^2}\eta^{\mu\nu}A_\mu A_\nu\right). \quad (3.69)$$

Expanding action (3.57) up to second power of gauge fields and replacing left and right gauge fields with vectors and axials we get

$$S = -\int d^4x dr V(T_0)\sqrt{-G}\left(2 + \frac{1}{4}(G^{-1})^{M_1M_2}(G^{-1})^{N_1N_2}\left(F^{(L)}_{M_1N_1}F^{(L)}_{M_2N_2}\right.\right.$$
$$\left.\left. + \ F^{(R)}_{M_1N_1}F^{(R)}_{M_2N_2}\right)\right)$$
$$= -\int d^4x dr V(T_0) r^3 \sqrt{1 + r^2\dot{T}_0^2}\,[2 + \frac{T_0^2}{r^2}\eta^{\mu\nu}A_\mu A_\nu \quad (3.70)$$
$$+ \frac{1}{8}\mathcal{M}^{M_1M_2}\mathcal{M}^{N_1N_2}(F^{(V)}_{M_1N_1}F^{(V)}_{M_2N_2} + F^{(A)}_{M_1N_1}F^{(A)}_{M_2N_2})]\,.$$

Using expression (3.66) for $\mathcal{M}$ and throwing away what is independent of gauge fields we proceed to

$$S = -\int d^4x dr V(T_0) r^3 \sqrt{1 + r^2\dot{T}_0^2}\,[\frac{1}{4(1 + r^2\dot{T}_0^2)}\eta^{\mu\nu}(\dot{V}_\mu\dot{V}_\nu + \dot{A}_\mu\dot{A}_\nu)$$
$$+ \frac{1}{8r^4}\eta^{\mu\nu}\eta^{\lambda\rho}(F^{(V)}_{\mu\lambda}F^{(V)}_{\nu\rho} + F^{(A)}_{\mu\lambda}F^{(A)}_{\nu\rho}) + \frac{T_0^2}{r^2}\eta^{\mu\nu}A_\mu A_\nu]\,. \quad (3.71)$$

We now go to momentum representation,

$$V_\mu(x,r) = \int \frac{d^4q}{(2\pi)^2} V_\mu(q,r) e^{-iq_\lambda x^\lambda}\,,$$
$$A_\mu(x,r) = \int \frac{d^4q}{(2\pi)^2} A_\mu(q,r) e^{-iq_\lambda x^\lambda}\,, \tag{3.72}$$

which results in

$$S{=}-\int d^4q dr V(T_0) r^3 \sqrt{1{+}r^2\dot{T}_0^2}\Big[\frac{1}{4(1{+}r^2\dot{T}_0^2)}\eta^{\mu\nu}(\dot{V}_\mu \dot{V}_\nu{+}\dot{A}_\mu\dot{A}_\nu) \tag{3.73}$$
$$+\frac{q^2}{4r^4}\left(V_\mu V_\nu\left(\eta^{\mu\nu}{-}\frac{q^\mu q^\nu}{q^2}\right){+}A_\mu A_\nu\left(\eta^{\mu\nu}\left(1{+}\frac{4T_0^2 r^2}{q^2}\right)-\frac{q^\mu q^\nu}{q^2}\right)\right)\Big]\,,$$

where all squared gauge fields are just a short notation for q-mode and $-q$-mode product.

Let us split radial and momentum dependence as follows:

$$V_\mu(q,r) = v_\mu(q) v(q,r)\,, \qquad A_\mu(q,r) = a_\mu(q) a(q,r)\,. \tag{3.74}$$

(We can use residual gauge symmetry to gauge-fix $q^\mu V_\mu(q,\Lambda) = q^\mu A_\mu(q,\Lambda) = 0$.) Let us also split the action (3.73) to axial and vector parts:

$$S = S_V + S_A\,, \tag{3.75}$$

where

$$S_V = -\frac{1}{4}\int d^4q dr \frac{r^3}{\sqrt{1+r^2\dot{T}_0^2}} V(T_0) v_\mu(q) v_\nu(-q) \tag{3.76}$$
$$\times\left(\dot{v}_q(r)\dot{v}_{-q}(r)\eta^{\mu\nu}{+}\frac{q^2(1+r^2\dot{T}_0^2)}{r^4}\left(\eta^{\mu\nu}{-}\frac{q^\mu q^\nu}{q^2}\right) v_q(r) v_{-q}(r)\right)\,,$$

$$S_A{=}-\frac{1}{4}\int d^4q dr \frac{r^3}{\sqrt{1{+}r^2\dot{T}_0^2}} V(T_0) a_\mu(q) a_\nu(-q)\left(\dot{a}_q(r)\dot{a}_{-q}(r)\eta^{\mu\nu}\right.$$
$$\left.+\frac{q^2(1+r^2\dot{T}_0^2)}{r^4}\left(\eta^{\mu\nu}\left(1+\frac{4T_0^2 r^2}{q^2}\right)-\frac{q^\mu q^\nu}{q^2}\right) a_q(r) a_{-q}(r)\right)\,. \tag{3.77}$$

We are interested in transverse components of gauge fields:

$$v_\mu^T(q) = P_{\mu\lambda}\eta^{\lambda\nu} v_\nu(q)\,,\ a_\mu^T(q) = P_{\mu\lambda}\eta^{\lambda\nu} a_\nu(q)\,,\ P_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\,, \tag{3.78}$$

which are described by

$$S_V^T = -\frac{1}{4}\int d^4q dr \frac{r^3}{\sqrt{1+r^2\dot{T}_0^2}} V(T_0) v_\mu^T(q) v_\nu^T(-q)\eta^{\mu\nu} \times \left(\dot{v}_q(r)\dot{v}_{-q}(r) + \frac{q^2(1+r^2\dot{T}_0^2)}{r^4} v_q(r)v_{-q}(r)\right), \tag{3.79}$$

$$S_A^T = -\frac{1}{4}\int d^4q dr \frac{r^3}{\sqrt{1+r^2\dot{T}_0^2}} V(T_0) a_\mu^T(q) a_\nu^T(-q)\eta^{\mu\nu} \times \left(\dot{a}_q(r)\dot{a}_{-q}(r) + \frac{q^2(1+r^2\dot{T}_0^2)}{r^4}\left(1+\frac{4T_0^2r^2}{q^2}\right) a_q(r)a_{-q}(r)\right), \tag{3.80}$$

Corresponding equations of motion are

$$\ddot{v}_q(r) + \frac{\sqrt{1+r^2\dot{T}_0^2}}{r^3V(T_0)}\frac{d}{dr}\left(\frac{r^3V(T_0)}{\sqrt{1+r^2\dot{T}_0^2}}\right)\dot{v}_q(r) - \frac{q^2(1+r^2\dot{T}_0^2)}{r^4}v_q(r) = 0, \tag{3.81}$$

$$\ddot{a}_q(r) + \frac{\sqrt{1+r^2\dot{T}_0^2}}{r^3V(T_0)}\frac{d}{dr}\left(\frac{r^3V(T_0)}{\sqrt{1+r^2\dot{T}_0^2}}\right)\dot{a}_q(r) - \frac{q^2(1+r^2\dot{T}_0^2)}{r^4}\left(1+\frac{4T_0^2r^2}{q^2}\right)a_q(r) = 0. \tag{3.82}$$

We see that if there is no tachyon background, then equations of motion for vector and axial vector fields become the same.

We must ensure that near-horizon behavior of vector and axial vector fields is regular. The precise boundary conditions in the bulk depend strongly on the tachyon background. Below we consider concrete tachyon potentials and determine the corresponding boundary conditions. We also require

$$v(q, r=\infty) = 1, \qquad a(q, r=\infty) = 1. \tag{3.83}$$

We solve equations of motion for $v(q,r)$ and $a(q,r)$ with these boundary conditions and plug the solutions into (3.79) and (3.80). As a result we obtain (recall that at the boundary tachyon field vanishes)

$$S_V^{on-shell} = -\frac{1}{4}\int d^4q\Lambda^3\eta^{\mu\nu}v_\mu^T(q)v_\nu^T(-q)\dot{v}(q,\Lambda)\,, \tag{3.84}$$

$$S_A^{on-shell} = -\frac{1}{4}\int d^4q\Lambda^3\eta^{\mu\nu}a_\mu^T(q)a_\nu^T(-q)\dot{a}(q,\Lambda)\,. \tag{3.85}$$

Due to AdS/CFT correspondence

$$i\int d^4xe^{iqx}\langle j_V^\mu(x)j_V^\nu(0)\rangle = \frac{\delta^2 S_V^{on-shell}}{\delta v_\mu^T(q)\delta v_\nu^T(-q)}|_{v=0}\,, \tag{3.86}$$

and similarly for the axial current. Consequently using (3.32) we get

$$\Pi_{\mu\nu}^V = P_{\mu\nu}\Pi_V(q^2) = \frac{\delta^2 S_V^{on-shell}}{\delta v_\mu^T(q)\delta v_\nu^T(-q)}\,. \tag{3.87}$$

Therefore correlation functions for vector and axial currents are given by

$$\Pi_V(q^2) = -\frac{1}{2}\Lambda^3\dot{v}(q,\Lambda)\,, \tag{3.88}$$

$$\Pi_A(q^2) = -\frac{1}{2}\Lambda^3\dot{a}(q,\Lambda)\,. \tag{3.89}$$

Propagators for vector and axial-vector currents in the field theory become the same if tachyon background vanishes. Non-vanishing tachyon background breaks chiral symmetry, and therefore generally speaking we have non-vanishing S parameter, defined as

$$S = -4\pi\frac{d}{dq^2}\left[\Pi_V(q^2) - \Pi_A(q^2)\right]_{q^2=0}\,. \tag{3.90}$$

With the help of holographic expressions (3.88) and (3.89) we obtain

$$S = 2\pi\Lambda^3\frac{d}{dq^2}(\dot{v}(q^2,\Lambda) - \dot{a}(q^2,\Lambda))\,. \tag{3.91}$$

The infrared behavior is specific for each particular tachyon potential and we discuss it bellow. Now let us consider near-boundary region. In the near-boundary region $r \gg 1$ we can totally neglect tachyon field, which makes equations of motion for vector and axial vector fields the same:

$$\ddot{v} + \frac{3}{r}\dot{v} - \frac{q^2}{r^4}v = 0\,, \tag{3.92}$$

$$\ddot{a} + \frac{3}{r}\dot{a} - \frac{q^2}{r^4}a = 0\,. \tag{3.93}$$

In practical computations one has to make sure that the last term in (3.82) is small, $T_0^2 r^2/q^2 \sim (q^2r^2)^{-1} \ll 1$ in near-boundary region. This is important, because momentum q competes in smallness with $1/r$ when one is computing S parameter. Cutoff is supposed to be sent to infinity first, for each value of momentum q. The solutions to these equations, normalized by near-boundary condition (3.83), are

$$v = 1 - \frac{q^2}{2r^2}\log r + C_v\frac{1}{r^2}\,, \tag{3.94}$$

$$a = 1 - \frac{q^2}{2r^2}\log r + C_a\frac{1}{r^2}\,, \tag{3.95}$$

where $C_v(q^2)$ and $C_a(q^2)$ define asymptotic near-boundary behavior of the vector fields, have dimension two and go to finite constants when $q^2 = 0$. Therefore substituting (3.94) and (3.95) into (3.91) we find

$$S = 4\pi\frac{d}{dq^2}(C_a - C_v)|_{q^2=0}\,. \tag{3.96}$$

Notice that the S parameter is expressed only via the coefficients $C_{v,a}$, describing near-boundary behavior of vector and axial-vector gauge fields, and does not depend on the cutoff Λ.

Tachyon field describes chiral symmetry breaking at energy scale given holographically by $r \ll \mu$. In that region we have essentially different dynamics of axial vector and vector gauge fields. In what follows we measure all dimensionfull quantities in units of dynamically generated scale μ.

3.3.3 Soft Wall

Consider tachyon potential

$$V(T) = (1 + (A-2)T^2)e^{-AT^2}\,, \tag{3.97}$$

with $A > 2$. Near the horizon in this potential tachyon field behaves as $T_0(r) = 1/r^{A/2}$. Correspondingly Lagrangian for vector field fluctuation is

$$L_v = r^{3-\frac{A}{2}}e^{-\frac{A}{r^A}}\left(\dot{v}^2 + \frac{q^2A^2}{4}r^{-A-4}v^2\right)\,. \tag{3.98}$$

It is useful to redefine

$$v = r^{\frac{A+2}{2}} e^{\frac{A}{2r^A}} \psi_v \tag{3.99}$$

and consider Lagrangian for ψ_v

$$L_v = r^{\frac{10+A}{2}} \dot{\psi}_v^2 + \frac{A^4}{4} r^{\frac{3(2-A)}{2}} \psi_v^2 \,. \tag{3.100}$$

Solution of the corresponding equation of motion is a linear combination of Bessel functions $I_{\pm\alpha}\left(\frac{A}{2r^A}\right)$ times a power of r, of which the regular combination behaves as

$$\psi_v = r^{\frac{A}{4}-2} e^{-\frac{A}{2r^A}} \,. \tag{3.101}$$

Correspondingly

$$v = r^{\frac{3A}{4}-1} \,. \tag{3.102}$$

Near horizon Lagrangian for axial field is

$$L_a = r^{3-\frac{A}{2}} e^{-\frac{A}{r^A}} \left(\dot{a}^2 + \frac{A^2}{r^{2(A+1)}} a^2 \right) \,. \tag{3.103}$$

It is convenient to make a redefinition

$$a = r^{\frac{A+2}{2}} e^{\frac{A}{2r^A}} \psi_a \,. \tag{3.104}$$

The near-horizon Lagrangian for axial field is now

$$L_a = r^{\frac{10+A}{2}} \dot{\psi}_a^2 + \frac{A^2(A^2+4)}{4} r^{\frac{3(2-A)}{2}} \psi_a^2 \,. \tag{3.105}$$

Similarly to the case with vector field we choose the regular solution, which is

$$\psi_a = r^{\frac{A}{4}-2} \exp\left(-\frac{\sqrt{A^2+4}}{2r^A} \right) \,. \tag{3.106}$$

Correspondingly near-horizon behavior of axial field is given by

$$a = r^{\frac{3A}{4}-1} \exp\left(-\frac{\sqrt{A^2+4}-A}{2r^A} \right) \,. \tag{3.107}$$

To summarize: we have the following near-horizon boundary conditions:

$$T_0(r) = \frac{1}{r^{A/2}} \,,\ v(r) = r^{\frac{3A}{4}-1} \,,\ a(r) = r^{\frac{3A}{4}-1} \exp\left(-\frac{\sqrt{A^2+4}-A}{2r^A} \right) . \tag{3.108}$$

We present results of numeric evaluations of the S parameter for different values of A in figure 3.

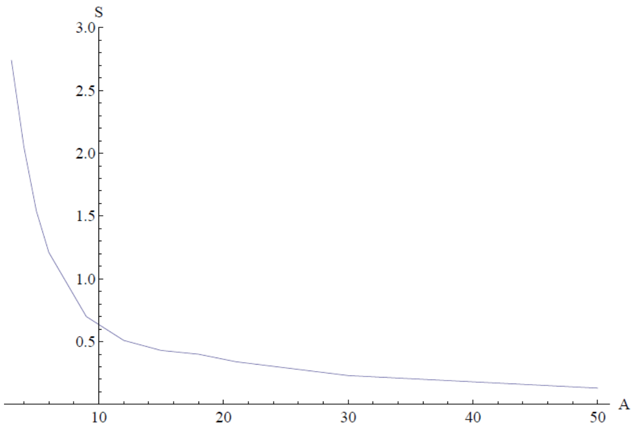

Figure 3.3. S parameter in the soft wall potential depending on the value of the parameter A.

3.3.4 Hard wall

Consider hard wall tachyon potential

$$V(T) = (\cos T)^4 \,. \tag{3.109}$$

The IR regime of the field theory corresponds to the near hard wall region of AdS space, $r \simeq \mu$, where μ is the dynamically generated scale. Let us measure all dimensional quantities in units of μ. Then the hard wall is located at $r = 1$. When $r \simeq 1$, the tachyon field behaves as

$$T(r) \simeq \frac{\pi}{2} - c\sqrt{r-1}\,, \qquad c = \sqrt{\frac{5}{2}}\,. \tag{3.110}$$

Plugging (3.110) into equations of motion for vector and axial-vector gauge fields, (3.81) and (3.82), and considering the region near $r = 1$, we obtain

$$\ddot{v} + \frac{5}{2(r-1)}\dot{v} - \frac{5q^2}{8(r-1)}v = 0 \tag{3.111}$$

$$\ddot{a} + \frac{5}{2(r-1)}\dot{a} - \frac{5(q^2+\pi^2)}{8(r-1)}a = 0\,. \tag{3.112}$$

The solutions are given by

$$\begin{aligned} v &= \frac{c_1^v}{(r-1)^{1/2}}\left(1+\frac{d_1}{r-1}\right)+c_2^v+\mathcal{O}(\sqrt{r-1})\,, \\ a &= \frac{c_1^a}{(r-1)^{1/2}}\left(1+\frac{d_2}{r-1}\right)+c_2^a+\mathcal{O}(\sqrt{r-1})\,, \end{aligned} \tag{3.113}$$

where d_1 and d_2 stand for known functions of q^2. We require momentum density T^{0r} to vanish at $r = 1$. The momentum density is given by (to compute it perturb the background metric by small g_{0r} and keep only terms of the action which are linear in g_{0r})

$$\begin{aligned} T^{0r} &\simeq \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta g_{0r}} \simeq \frac{V(T)}{\sqrt{1+r^2\dot{T}^2}}\eta^{ij}\left[\dot{V}_i F_{0j}^{(V)} + \dot{A}_i F_{0j}^{(A)}\right] \\ &\simeq (r-1)^{5/2}(v\dot{v}+a\dot{a})\,. \end{aligned} \tag{3.114}$$

We therefore choose the boundary conditions near the wall

$$v = 1\,, \qquad a = 1\,. \tag{3.115}$$

Similarly as we have done in the soft wall case, we can now compute the S parameter. Numerics give $S \simeq 2.6$.

3.4 Lightest mesons

In this Section we compute the sigma-mesons spectrum in soft wall potential. Consider fluctuation of the tachyon field $\tau(r,t)$ around the vacuum configuration $T_0(r)$. Expanding the TDBI action

$$S = -\int d^4x dr V(T) r^3 \left(1 + r^2(\dot{T}_0+\dot{\tau})^2 - \frac{1}{r^2}(\partial_t\tau)^2\right)^{1/2}\,. \tag{3.116}$$

we arrive at the action for fluctuation field

$$S = -\int d^3x d\omega dr \left(G(r)\dot{\tau}^2 + U(\tau)\tau^2\right)\,. \tag{3.117}$$

Perform a Fourier transform

$$\tau(r,t) = \int \frac{d\omega}{2\pi}\tau_\omega(r)e^{i\omega\tau}\,, \tag{3.118}$$

where $\omega^2 = m^2$ is the squared mass of the tachyon excitation mode. For the soft wall potential (we consider $A > 2$ to get a discrete spectrum of sigma-mesons, see [3] for details)

$$V(T) = (1 + (A-2)T^2)e^{-AT^2} \tag{3.119}$$

we obtain

$$G(r) = \frac{e^{-AT_0^2} r^5 (1 + (A-2)T_0^2)}{2(1 + r^2 \dot{T}_0^2)^{3/2}} \tag{3.120}$$

$$\begin{aligned} U(r) = & \frac{\partial}{\partial r}\left(\frac{e^{-AT_0^2} r^5 T_0 \dot{T}_0 (2 + A(A-2)T_0^2)}{\sqrt{1 + r^2 \dot{T}_0^2}}\right) \\ & + e^{-AT_0^2} r^3 (-2 + AT_0^2 (10 - 3A + 2(A-2)AT_0^2)) \sqrt{1 + r^2 \dot{T}_0^2} \\ & - m^2 \frac{e^{-AT_0^2} r (1 + (A-2)T_0^2)}{2\sqrt{1 + r^2 \dot{T}_0^2}} \end{aligned} \tag{3.121}$$

Near the horizon $r = 0$ the background tachyon field behaves as $T_0 = \frac{1}{r^{A/2}}$. Therefore the Lagrangian for fluctuating field is

$$L = \frac{4}{A^2} r^{\frac{10+A}{2}} e^{-A/r^A} \dot{\tau}^2 - m^2 r^{\frac{2-A}{2}} e^{-A/r^A} \tau^2 \,. \tag{3.122}$$

It is convenient to make a redefinition $\tau = e^{A/(2r^A)} \psi$ and consider the field ψ with the Lagrangian

$$L_\psi = r^{\frac{10+A}{2}} \dot{\psi}^2 + \frac{A^4}{4} r^{\frac{3(2-A)}{2}} \psi^2 \tag{3.123}$$

The solution of equation of motion for the field ψ is a linear combination of Bessel functions $I_{\pm\alpha}(A/(2r^A))$, times a power of r. We choose the regular combination of Bessel functions, which is

$$I_\alpha(A/(2r^A)) - I_{-\alpha}(A/(2r^A)) \simeq r^{A/2} e^{-A/(2r^A)} \,. \tag{3.124}$$

Corresponding near-horizon behavior of fluctuation tachyon field is

$$\tau(r) = r^{\frac{A}{4}-2} \,. \tag{3.125}$$

We therefore impose the near-horizon conditions

$$\tau(\epsilon) = 1 \,, \quad \tau'(\epsilon) = \left(\frac{A}{4} - 2\right)\frac{1}{\epsilon} \,. \tag{3.126}$$

We then integrate equation of motion for τ with these boundary conditions up to the near-boundary region. We fit the result with the expression

$$\tau(r) = \frac{1}{r^2}(c_1 \log r + c_2)\,. \tag{3.127}$$

The ratio c_1/c_2 must be equal to this ratio for the background field T_0. This determines the discrete mass spectrum of tachyon excitations.

We compute numerically the values m_1^2 and m_2^2 of the masses of the first two excitations as a function of the parameter A of the tachyon potential (3.119). We plot the result of numerics in figure 4.

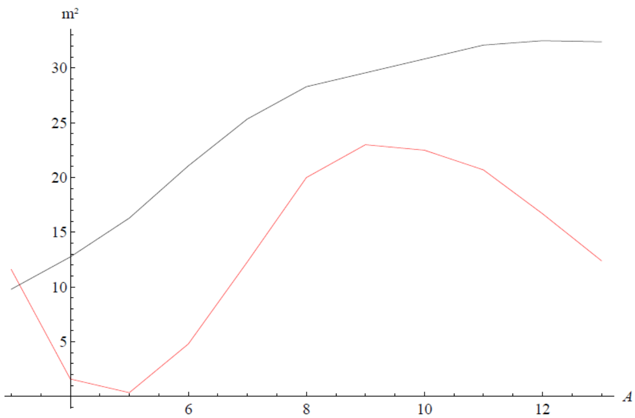

Figure 3.4. The values of of the masses of the first two excitations of the tachyon as a function of the parameter A of the tachyon potential (3.119). The m_1^2 (rescaled by a factor of ten) is plotted in red and the m_2^2 is plotted in black.

3.5 Conclusions

In this chapter we have considered strongly coupled systems which are described holographically by the tachyon DBI action in the AdS space-time. These models are renormalizable: the UV cutoff can be taken to infinity while the dynamically generated mass scale stays fixed. We investigated the phase diagram of these models at finite temperature and charge density. For smaller values of temperature and chemical potential the system resides in the phase with broken conformal symmetry. This phase is separated by a phase transition line from the phase with restored symmetry.

We observe that depending on the form of the tachyon potential, the order of the phase transition may change, and hence one or more critical point appears in the diagram. We have also used the TDBI action to describe holographically dynamical electroweak symmetry breaking. We have computed the S parameter using our holographic TDBI model for generic soft-wall tachyon potential, and for hard wall tachyon potential. The S-parameter takes generic positive values and does not appear to vanish in the parameter space that we investigated. We have also computed the masses of the lowest lying scalar mesons and observed that even though there is no parametrically light scalar, the lightest meson can be made at least an order of magnitude lighter than the next one.

3.6 Appendix: Conformal phase transition and double-trace coupling running

Consider a gauge field theory, coupled to matter fields with a single-trace UV Lagrangian. When we go to lower energies, integrating out higher momentum modes, we generally notice [55–57] that effective Wilsonian Lagrangian contains double-trace operators. We have to study the RG running of coupling constants for double-trace operators if we want to study the fate of the theory at low energies. Depending on the parameters defining the theory the beta-functions for double-trace operators can exhibit essentially different behavior; varying these parameters can lead to phase transitions between different IR phases of the theory. Here our focus will be on the particular type of these phase transitions, called conformal phase transitions in [58]. In this Section we review the field theory expectations for the physics associated with conformal phase transitions (CPT). We then use the technology of holographic Wilsonian RG to see how these expectations are reproduced in a particular holographic model based on the Tachyon DBI action in AdS space.

3.6.1 Conformal phase transitions and Wilsonian RG

Consider a gauge theory with $SU(N_c)$ gauge group, coupled to N_f massless Dirac fermions in the Veneziano limit, where both N_c and N_f are taken to infinity, with the ratio $x = N_f/N_c$ held fixed. It has a qualitatively different RG behavior depending on the value of x. Let us look at the IR effective field theory; three possible regimes can be identified.

When $x > 11/2$ the theory loses asymptotic freedom and is free in the IR; when $x_c < x < 11/2$, where $x_c \simeq 4$ (see e.g. [47]) is not known precisely, the IR theory is in the interacting Coulomb phase. This interval in x , where the theory flows to a conformal fixed point in the IR, is called "conformal window". However for x smaller than x_c the IR theory acquires a mass gap and chiral symmetry is broken, due to the presence of chiral condensate.

The model studied in [57] is slightly different from the example above, but exhibits similar behavior. The advantage is that the beta function for the double-trace operator can be computed exactly [57]. Suppose that we have some strongly interacting theory, for which all single-trace operators have vanishing beta-functions, e.g., orbifold theories [59, 60] or non-supersymmetric deformations of $\mathcal{N} = 4$ super Yang-Mills theory [61]. To see whether the theory has conformal fixed points we therefore have to study double-trace couplings [55–57]. Denote by $\mathcal{O}$ a single-trace operator, and consider a double trace term in the Lagrangian, $L_{dt} = f\mathcal{O}^2$, where f is a double-trace coupling constant. In [57] (and, earlier, to the one-loop level in [55, 56]) it has been shown that depending on the parameters of the theory the beta function for f either has a real zero (and then the theory flows to a conformal fixed point) or it does not (and then the theory generates a mass gap).

We will observe a similar behavior in the holographic model based on the tachyon DBI action in AdS space-time. First we introduce a bulk scalar field, dual to the field theory operator $\mathcal{O}$. We choose it to be the tachyon field T, described by the tachyon DBI action. Now, we want to study renormalization of the corresponding double-trace coupling f. We will use the holographic Wilsonian renormalization as described in [62]. [5] The full AdS action is written as a sum of the bulk action (in our case it is the tachyon DBI action), defined up to the cutoff Λ, and the boundary action at $r = \Lambda$,

$$S[T] = \int_0^\Lambda dr d^d x L_0[T] + \int d^d x L_B[T]_{r=\Lambda} \,. \tag{3.128}$$

To obtain holographically correlation functions that are invariant under the RG flow, one has to require invariance of the action S under the change of Λ: this is a holographic implementation of the Callan-Symanzik equation. The boundary term S_B encodes all degrees of freedom from

[5] See also [63].

the integrated out region $r > \Lambda$ of the AdS space, and is written down as a sum of multi-trace operators with corresponding coupling constants multiplying these operators. Solving for S_B the holographic RG (HRG) equation we determine running of the dual field theory coupling constants.

Below we apply this method to the tachyon-DBI action in the AdS space and find the RG behavior of the double-trace coupling f, depending on the mass m of the tachyon field. The non-vanishing tachyon field in the bulk is a preferred state when $m^2 < m_{BF}^2 = -\frac{d^2}{4}$ [3]. We conclude that f exhibits a walking behavior between the IR scale Λ_{IR} and the UV cutoff scale Λ_{UV} which are related as $\Lambda_{IR} = \Lambda_{UV} \exp\left(-\frac{\pi}{\sqrt{m_{BF}^2 - m^2}}\right)$. Such a relation confirms that our holographic model exhibits a conformal phase transition. This was also observed in [3], where a similar relation between the UV cutoff and the physical observables of the theory, such as e.g. meson masses, was established.

Finally we remind the reader what happens as the tachyon mass squared is lowered below the BF bound. According to the AdS/CFT dictionary, the dimension of the operator $\mathcal{O}$, dual to the tachyon field T, is given by

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2} \,. \tag{3.129}$$

The two possible scaling dimensions in (3.129), Δ_- and Δ_+ of the operator $\mathcal{O}$ are realized in the two conformal fixed points: the UV and IR respectively. When we turn on a double-trace deformation $f\mathcal{O}^2$ in the UV theory, the theory flows to the IR conformal fixed point, where dimension of $\mathcal{O}$ becomes equal to Δ_+ [64]. When the value of m^2 is lowered below $-d^2/4$, the two fixed points merge and then disappear, and the Miranski scaling emerges [2].

3.6.2 Double trace running from tachyon DBI

Consider the tachyon-DBI bulk action for the tachyon field $T(r)$ of the mass m, defined up to UV cutoff scale $r = \Lambda$ in AdS_{d+1}:

$$S_0 = -\int_0^\Lambda dr \int d^d x r^{d-1} V \sqrt{1 + r^2 \dot{T}^2} \,, \tag{3.130}$$

where tachyon potential is expanded around $T = 0$ as

$$V(T) = 1 + \frac{m^2 T^2}{2} + \cdots \,, \tag{3.131}$$

and we denote differentiation w.r.t. r by dot. Suppose we integrate out all degrees of freedom in the bulk which correspond to $r > \Lambda$. Then we generate holographic Wilsonian effective action

$$S = S_0 + S_B[T, \Lambda]\,, \tag{3.132}$$

where S_B is boundary term, which encodes integrated out degrees of freedom.

Boundary condition at $r = \Lambda$ is given by

$$\Pi = \frac{\partial S_B}{\partial T}\,, \tag{3.133}$$

where we have introduced momentum Π, canonically conjugate to the tachyon field T:

$$\Pi = -\frac{\delta S_0}{\delta T(r=\Lambda)} = \frac{\Lambda^{d+1} V \dot{T}}{\sqrt{1+\Lambda^2 \dot{T}^2}}\,. \tag{3.134}$$

Using boundary condition (3.133) one may then express

$$\dot{T} = \frac{\partial S_B/\partial T}{\Lambda\sqrt{\Lambda^{2d}V^2 - (\partial S_B/\partial T)^2}}\,. \tag{3.135}$$

If we denote $S_0 = \int_{r_h}^{\Lambda} dr \int d^d x L_0$, then holographic RG equation is

$$\frac{\partial S_B}{\partial \Lambda} + L_0(r=\Lambda) + \frac{\partial S_B}{\partial T}\dot{T}(\Lambda) = 0\,. \tag{3.136}$$

With the help of (3.130) and (3.135) this eventually acquires the form

$$\frac{\partial S_B}{\partial \Lambda} = \Lambda^{d-1} V \sqrt{1 - \frac{1}{\Lambda^{2d}V^2}\left(\frac{\partial S_B}{\partial T}\right)^2}\,. \tag{3.137}$$

Action S_B implicitly contains boundary metric factor $\sqrt{-\det g_b} = \Lambda^d$. Let us make this factor explicit, defining dimensionless boundary action $\mathcal{S}$ as

$$S_B = \Lambda^d \mathcal{S}\,. \tag{3.138}$$

Let us also define new cutoff coordinate,

$$\epsilon = \log\frac{\Lambda}{\mu}\,, \tag{3.139}$$

where μ is some constant, introduced for dimensional reasons. HRG equation (3.137) therefore gets rewritten as

$$\partial_\epsilon \mathcal{S} + d\mathcal{S} = V\sqrt{1 - \left(\frac{\partial_T \mathcal{S}}{V}\right)^2}\,. \tag{3.140}$$

Let us expand the boundary action as

$$\mathcal{S} = C(\epsilon) + J(\epsilon)T + \frac{1}{2}f(\epsilon)T^2\,. \tag{3.141}$$

Plugging it into (3.140) and matching terms of the same order in T, we obtain

$$\partial_\epsilon C = \sqrt{1 - J^2} - dC\,, \tag{3.142}$$

$$\partial_\epsilon J = -\frac{fJ}{\sqrt{1 - J^2}} - dJ\,, \tag{3.143}$$

$$\partial_\epsilon f = \frac{m^2}{\sqrt{1 - J^2}} - \frac{f^2}{(1 - J^2)^{3/2}} - df\,. \tag{3.144}$$

We can solve these equations by putting $J \equiv 0$ and making $\bar{f} = -(f + d/2)$ satisfy equation

$$\partial_\epsilon \bar{f} = \bar{f}^2 - \frac{d^2}{4} - m^2\,. \tag{3.145}$$

Let us denote $\kappa^2 = -\frac{d^2}{4} - m^2 \equiv m_{BF}^2 - m^2$, then solution to (3.145) may be written as

$$\bar{f} = \kappa \tan(\kappa\epsilon)\,. \tag{3.146}$$

We conclude that double-trace coupling $\bar{f}$ exhibits a walking behavior between UV scale

$$\Lambda_{UV} = \mu \exp\left(\frac{\pi}{2\kappa}\right) \tag{3.147}$$

and IR scale

$$\Lambda_{IR} = \mu \exp\left(-\frac{\pi}{2\kappa}\right)\,. \tag{3.148}$$

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Chapter 4

Stringy holography at finite density

4.1 Introduction

In the usual AdS/CFT setting gauge theory on the boundary has a dual description in terms of closed string theory in the bulk. Most often, a limit of small curvature is taken to yield a low energy theory of strings, supergravity. In the $\mathcal{N} = 4$ supersymmetric Yang-Mills case this limit implies strong 't Hooft coupling of field theory. A distinct example of non-gravitational theory with a holographically dual description is the Little String Theory [1, 2]. It can be viewed as the theory of N coincident NS5-branes, taken at vanishing string coupling, $g_s = 0$, where the bulk degrees of freedom decouple. The coupling constant of the low-energy $U(N)$ gauge degrees of freedom, living on the NS5-branes world-volume, stays unaffected by taking this limit, and is equal to $g_5 = \ell_s$, where ℓ_s is string length in type-IIB string theory (see [3] for a review).

The holographic dual of the Little String Theory [2, 4, 5] is the theory of closed strings in the background of NS5-branes, with the geometry $R^{5,1} \times R^{\phi} \times SU(2)_N$, the two-form field and the linear dilaton. The CFT on $SU(2)$ is described by WZW action at level N. The bulk physics (in the double scaling limit) can be reformulated as the string theory on $R^5 \times \frac{SL(2,R)_N}{U(1)} \times SU(2)_N$ space-time. This is due to the fact that the gauged WZW model on $SL(2,R)_N/U(1)$ gives rise to the classical "cigar" geometry of the two-dimensional black hole with the asymptotically linear dilaton [6, 7]. In the large N limit the bulk theory reduces to supergravity[1].

Generally one expects that a lot of nontrivial physics drastically simplifies in the limit of infinitely many degrees of freedom (large N limit), both in the boundary field theory and from the dual bulk perspective. For example, one expects the large N physics of a field theory at finite temperature and density to have "classical" nature, resulting, in particular, in the mean field critical exponents. (Another recent example of this is given by the stringy nature of finite-momentum and zero-frequency singularity of the current-current two-point functions, observed in [8], where the results of [9–15] were extensively used.)

The low energy excitations in Little String Theory at finite temperature have been considered in [16] [2]. The closed string description involves the gauged WZW (gWZW) action with the $SL(2,R)/U(1)$ target space-time and $\mathcal{N}=2$ world-sheet supersymmetry. In [16] the two-point functions of the stress-energy tensor and the $U(1)$ current have been computed holographically; their pole structure indicates the presence of hydrodynamic modes. This has been also verified by solving fluctuation equations in supergravity approximation in the background of a large number of NS5-branes.

In this chapter we study string theory in the background of a direct product of the two-dimensional charged black hole [23] and flat space. The string theory in the two-dimensional charged black hole background is described by the gWZW action with the $\frac{SL(2,R)\times U(1)_x}{U(1)}$ target space-time [24–26]. Here $U(1)_x$ is a compact circle, which is Kaluza-Klein reduced, and $U(1)$ subgroup of $SL(2,R)\times U(1)_x$ is gauged asymetrically. The left-moving sector of the gauged $U(1)$ is a linear combination of the left-moving sector of the $U(1)_x$ and the left-moving sector of the $U(1)$ subgroup of the $SL(2,R)$. The coefficient of this linear combination determines the charge to mass ratio of the resulting black hole. The right-moving sector of the gauged $U(1)_x$ is the right-moving sector of the $U(1)$ subgroup of the $SL(2,R)$.

This bulk system is holographically dual to the boundary quantum field theory at finite temperature and charge density. (One can think of the resulting system as little string theory at finite density, but we do

[1] The radius of the $SU(2)$ sphere is $R_{sph}=\sqrt{N}\ell_s$. Therefore the large N limit is equivalent to the limit of small ℓ_s/R_{sph}.

[2] See also e.g. [17–22] for some preceding holographic study of the Little String Theory.

not study the field theoretic interpretation here in detail.) The inverse temperature is equal to $\beta = 2\pi\sqrt{k}\frac{\cos^2(\psi/2)}{\cos\psi}$, where $\psi \in [0, \pi/2]$ is the parameter of asymmetric gauging. The finite charge density in the field theory is described holographically by the background $U(1)$ potential in the bulk, $A_t(u) \simeq -\frac{q}{u}$, where $q = M \sin\psi$ is the charge and M is the mass of the black hole; u is the radial coordinate in the bulk.

The vertex operator of the string ground state in this model was constructed in [25]. In this chapter we construct the vertex operators which describe massless closed string excitations in this model, which constitute the NS-NS sector of type-II supergravity. We also construct the gauge field vertex operators, which are obtained by Kaluza-Klein reduction on $U(1)_x$ from graviton and antisymmetric tensor field vertex operators. The graviton in the bulk is dual to the stress-energy tensor on the boundary; the gauge field in the bulk is dual to the charge current on the boundary. We study the low energy excitations of the system by computing holographically the two-point functions for the charge current and the stress-energy tensor and reading off the dispersion relation from their poles. We find two distinct gapless modes in the shear channel; the dispersion relation of one of them is independent of the charge to mass ratio of the black hole. The two modes merge in the limit of vanishing charge, producing the shear mode which was observed in [16]. We confirm these results by solving fluctuation equations of the type-II supergravity. The situation in the sound channel is similar.

Finally we study fluctuation equations in the low-energy limit in heterotic gravity [23]. We find one gapless mode in the shear channel. Comparing this result with the thermodynamics of the charged black hole [27] we find that the ratio of shear viscosity to entropy density is equal to $\eta/s = 1/(4\pi)$, independently of the charge to mass ratio of the black hole.

The rest of this chapter is organized as follows. In section 2 we review the thermodynamics of the two-dimensional charged black hole and derive the dispersion relation of the shear hydrodynamic mode. In section 3 we apply the BRST quantization method of the coset models, and the covariant quantization of the string to construct the holomorphic and anti-holomorphic physical vertex operators of the massless states on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset. In section 4 we use these vertex operators and write down the vertex operators of graviton, antisymmetric tensor field and gauge fields. In that section we also compute the two-point functions of these vertex operators and discuss the low-energy excitation modes. We also briefly discuss finite-momentum and zero-frequency singularity of the correlation functions. In section 5 we solve fluctuation equations in type-II supergravity to verify the dispersion relations, derived in section 4. We discuss our results in section 6. Appendix A is devoted to a review of some rudimentary conformal field theory and derivation of the gWZW action on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset. In Appendix B we solve fluctuation equations in heterotic gravity. We find one mode in the shear channel. Matching its dispersion relation to the one, written in section 2, we obtain that $\eta/s = 1/(4\pi)$ for any charge to mass ratio.

4.2 Thermodynamics of the charged black hole

The metric of the two-dimensional charged black hole [23] with mass M and charge q in suitable coordinates can be written as [26, 27]

$$
\begin{aligned}
ds^2 &= -f(u)dt^2 + \frac{\hat{k}}{4}\frac{du^2}{u^2 f(u)}\,, \\
f(u) &= \frac{(u-u_+)(u-u_-)}{u^2}\,, \quad u_\pm = M \pm \sqrt{M^2-q^2}
\end{aligned}
\tag{4.1}
$$

with the background $U(1)$ gauge field and the dilaton field being equal to

$$
\begin{aligned}
A_t(u) &= q\left(\frac{1}{u_+} - \frac{1}{u}\right)\,, \\
\Phi &= \Phi_0 - \frac{1}{2}\log\left(\frac{u\sqrt{\hat{k}}}{2}\right)\,, \quad \Phi_0 = -\frac{1}{2}\log\left(\frac{M u_+\sqrt{\hat{k}}}{u_+ + u_-}\right)\,.
\end{aligned}
\tag{4.2}
$$

The gauge potential vanishes at the outer horizon, $A_t(u_+) = 0$. Define parameter ψ by the equation

$$
\frac{u_-}{u_+} = \tan^2\frac{\psi}{2}\,. \tag{4.3}
$$

For the full description of thermodynamics of two-dimensional charged black hole the reader is referred to [27], we just review their results which are useful for us. It is convenient for further purposes to denote the background dilaton slope (see eq. (4.2)) as $Q = 2/\sqrt{\hat{k}}$. Requiring the metric (4.1) near external horizon $u = u_+$ to be regular, we find the temperature of the charged black hole

$$\beta = \frac{4\pi}{Q}\frac{u_+}{u_+ - u_-} = \frac{4\pi}{Q}\frac{\cos^2\frac{\psi}{2}}{\cos\psi}\,. \tag{4.4}$$

Asymptotical $u \gg 1$ value of the gauge potential (see eq. (4.2)) is equal to the chemical potential:

$$\mu = \frac{q}{u_+} = \sqrt{\frac{u_-}{u_+}} = \tan\frac{\psi}{2}\,. \tag{4.5}$$

The entropy of the two-dimensional charged black hole is given by [27]

$$S_{bh}(M,q) = \frac{2\pi}{Q}(M + \sqrt{M^2 - q^2})\,. \tag{4.6}$$

Using (4.6) and evaluating the grand canonical partition sum $\mathcal{Z}$ one observes [27] that the grand canonical potential $\Omega \sim -\log\mathcal{Z}$ vanishes, and therefore the pressure vanishes.

Consider black brane background space-time $CBH_2 \times R^{d-1}$, which is a direct product of the two-dimensional charged black hole and flat $d-1$-dimensional space. Denote by X the direction of R^{d-1} of propagation of all the excitation, and denote by Y some transverse direction of R^{d-1}. In the shear channel excitation modes appear as poles of the two-point function $\langle T_{XY}T_{XY}\rangle$ of the stress-energy tensor T_{MN}, with the dispersion relation of the low-energy mode given by

$$\omega = -\frac{i\eta}{(M+P)/V}p^2\,, \tag{4.7}$$

where p is the momentum and ω is the frequency of the mode; η is the shear viscosity, M/V and P/V are energy and pressure densities.

Because for the two-dimensional charged black hole the pressure vanishes, we obtain

$$\omega = -\frac{i\eta}{M/V}p^2\,. \tag{4.8}$$

Using (4.6) one can express,

$$M = \frac{QS_{bh}}{2\pi(1+\cos\psi)}\,, \tag{4.9}$$

and therefore the hydrodynamics predicts a shear pole with the dispersion relation

$$\omega = -4\pi i \frac{\eta}{s} \frac{\sqrt{\hat{k}} \cos^2(\psi/2)}{2} p^2 \,, \tag{4.10}$$

where $s = S/V$ is the entropy density. Bellow we are going to compare this result with the computation in heterotic gravity and derive the value of η/s.

4.3 The physical state conditions and vertex operators

4.3.1 BRST quantization

The gauging of the $U(1)$ subgroup from the $SL(2,R) \times U(1)$ group in the gWZW model on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset is realized by adding the $U(1)$ non-dynamical gauge field to the system, and adding corresponding action terms to the $SL(2,R) \times U(1)$ WZW action. The $U(1)$ subgroup is gauged left-right asymetrically, and anomaly-free condition must be satisfied. The details of the construction are reviewed in Appendix A. The end product is the gWZW action

$$\begin{aligned} S_g = S[g] &+ \frac{1}{2\pi} \int d^2 z \partial x \bar{\partial} x \\ &+ \frac{1}{2\pi} \int d^2 z \left[A\, \tilde{k} + \bar{A}\, k + A\bar{A} \left(2 + \mathrm{Tr}(g^{-1}\sigma^3 g \sigma^3) \cos\psi \right) \right] . \end{aligned} \tag{4.11}$$

Here we have denoted the currents of the gauged $U(1)$ subgroup as

$$k = \sqrt{\hat{k}} \mathrm{Tr}(\partial g g^{-1} \sigma^3) \cos\psi + 2 \sin\psi \partial x = \frac{2}{\sqrt{\hat{k}}} j^3 \cos\psi + 2 \sin\psi \partial x \,, \tag{4.12}$$

$$\tilde{k} = \sqrt{\hat{k}} \mathrm{Tr}\left(g^{-1} \bar{\partial} g \sigma^3 \right) = -\frac{2}{\sqrt{\hat{k}}} \tilde{j}^3 \,. \tag{4.13}$$

To determine physical spectrum of the quantum model on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset, we are going to use BRST quantization method [28] (see [7] where this method was applied to build the $SL(2,R)/U(1)$ model). The path integral for the theory is

$$Z = \int [dG][dA][d\bar{A}] \exp\left(-S_g[G, A, \bar{A}]\right)$$
$$= \int [dG][du][dv] \det \partial \det \bar{\partial} \exp\left(-S[G] + S[w]\right) . \qquad (4.14)$$

Represent functional determinants in terms of the gauge ghost fields

$$\det \partial \det \bar{\partial} = \int [db][dc][d\tilde{b}][d\tilde{c}] \exp\left(-\frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \tilde{b}\partial\tilde{c})\right) . \qquad (4.15)$$

Ghosts satisfy OPEs

$$c(z)b(w) \sim \frac{1}{z-w} + \cdots , \qquad \tilde{c}(\bar{z})\tilde{b}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}} + \cdots . \qquad (4.16)$$

Fix the gauge symmetry, for concreteness fix $v = 1$, therefore $u = w$. Consequently the path integral is given by

$$Z = \int [dG][dw][db][dc][d\tilde{b}][d\tilde{c}] \exp\left(-S[G] + S[w] - \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \tilde{b}\partial\tilde{c})\right) \qquad (4.17)$$

and the total action is given by

$$S_q = S[G] - S[w] + \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \tilde{b}\partial\tilde{c}) \qquad (4.18)$$

Notice from this action that the correlation function for w has the wrong sign:

$$\langle \partial w(z_1)\partial w(z_2)\rangle = \frac{1}{2(z_1 - z_2)^2} . \qquad (4.19)$$

Perform variations $(\delta G, \delta w, \delta b)$ in the action (4.18),

$$\delta S_q = -\frac{1}{2\pi}\int d^2z \left[\hat{k}\mathrm{Tr}(\partial G G^{-1}\bar{\partial}(G^{-1}\delta G)) + 2\partial w\bar{\partial}\delta w - \delta b\bar{\partial}c\right] . \qquad (4.20)$$

For the transformations with a local Grassmann parameter η:

$$\delta G = \eta c G T_L , \quad \delta w = \eta c , \quad \delta b = \eta(k + 2\partial w) \qquad (4.21)$$

we therefore obtain

$$\delta S_q = \frac{1}{2\pi}\int d^2z (\bar{\partial}\eta)\, c(k + 2\partial w) . \qquad (4.22)$$

If η is a global parameter, then S_q is invariant. Corresponding transformation is BRST symmetry transformations. Then from (4.22) for anti-holomorphic $\eta(\bar{z})$ we can read off holomorphic component of the corresponding conserved Noether current:

$$j_{BRST} = c(k + 2\partial w)\,. \tag{4.23}$$

Notice that

$$\langle (k(z_1) + 2\partial w(z_1))(k(z_2) + 2\partial w(z_2))\rangle = 0\,. \tag{4.24}$$

Therefore corresponding BRST charge

$$Q_{BRST} = \frac{1}{2\pi i}\oint dz j_{BRST} \tag{4.25}$$

is nilpotent. Similarly one finds the anti-holomorphic component of the BRST current

$$\tilde{j}_{BRST} = \tilde{c}(\tilde{k} + 2\bar{\partial}\tilde{w})\,. \tag{4.26}$$

Physical states of the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset model are the BRST-closed states of the $SL(2,R)\times U(1)$ model:

$$Q_{BRST}|\text{phys}\rangle = 0\,, \quad \tilde{Q}_{BRST}|\text{phys}\rangle = 0\,, \tag{4.27}$$

and are defined up to BRST-exact states.

Denote null bosonic currents as

$$J = k + 2\partial w\,, \qquad \tilde{J} = \tilde{k} + 2\bar{\partial}\tilde{w}\,. \tag{4.28}$$

BRST physical state conditions (4.27) therefore become

$$J_n|\text{phys}\rangle = 0\,,\ \ n \geq 0\,, \quad \tilde{J}_n|\text{phys}\rangle = 0\,,\ \ n \geq 0\,. \tag{4.29}$$

The BRST-exact massless state is obtained by acting with J_{-1} and $\tilde{J}_{-1}$ on the BRST-closed ground state.

4.3.2 Ground state vertex operator

The ground state vertex operator V_t of the $\frac{SL(2,R)\times U(1)}{U(1)}$ model was constructed in [25] as a ground state vertex operator of the $SL(2,R)\times U(1)$ model invariant under gauge $U(1)$ transformations. This vertex operator describes a tachyon, which due to GSO projection is projected out of the NS-NS sector. The lowest NS states are massless, and they are described by the vertex operator $V^M = j^M_{-1} V_t$ (and similarly for anti-holomorphic vertex operator). We derive these massless vertex operators bellow in this section. In this subsection we find it useful to reproduce the result of citeGiveon:2003ge4 using BRST quantization, developed in the previous subsection.

Suppose $\Phi(z,\bar{z})$ is a vertex operator on $SL(2,R) \times U(1)$. Then

$$V_t(z,\bar{z}) = \Phi(z,\bar{z}) \exp\left(im_L w_L + im_R w_R\right) \tag{4.30}$$

where

$$w(z,\bar{z}) = w_L(z) + w_R(\bar{z}) \tag{4.31}$$

is a vertex operator on the coset $\frac{SL(2,R)\times U(1)}{U(1)}$ if the physical state condition (4.27) are satisfied. We obtain

$$k_0 \cdot V_t(z,\bar{z}) = -im_L V_t(z,\bar{z})\,, \tag{4.32}$$

$$\tilde{k}_0 \cdot V_t(z,\bar{z}) = -im_R V_t(z,\bar{z})\,. \tag{4.33}$$

The non-compact w-circle contains only momentum modes and does not contain any winding modes, therefore

$$m_L = \frac{M}{R} - WR\,, \quad m_R = \frac{M}{R} + WR \quad \Rightarrow \quad m_L = m_R\,. \tag{4.34}$$

Let us denote $m_L = m_R = N$. The ground state on $SL(2,R) \times U(1)$ is described by the vertex operator [3]

$$\Phi(z,\bar{z}) = V_{jm\bar{m}} e^{2in_L x_L + 2in_R x_R}\,, \tag{4.35}$$

therefore

$$k_0 \cdot \Phi(z,\bar{z}) = 2\left(\frac{m\cos\psi}{\sqrt{\hat{k}}} - in_L \sin\psi\right)\Phi(z,\bar{z})\,, \tag{4.36}$$

$$\tilde{k}_0 \cdot \Phi(z,\bar{z}) = -\frac{2\bar{m}}{\sqrt{\hat{k}}}\Phi(z,\bar{z})\,. \tag{4.37}$$

[3] Define $x \sim x + \pi$, so that $n_{L,R}$ are integers. The $V_{jm\bar{m}}$ is the $SL(2,R)$ ground state primary field, see details in Appendix A.

The BRST physical state conditions (4.32), (4.33) due to (4.34) therefore imply

$$2\left(\frac{m\cos\psi}{\sqrt{\hat{k}}} - in_L\sin\psi\right) = -iN \tag{4.38}$$

$$\frac{2\bar{m}}{\sqrt{\hat{k}}} = iN \tag{4.39}$$

and consequently

$$m\cos\psi + \bar{m} - i\sqrt{\hat{k}}n_L\sin\psi = 0\,, \tag{4.40}$$

as was derived in [25].

4.3.3 Vertex operators of massless states in type-II superstring theory

The stress-energy tensor, which follows from the action (4.18), has the following holomorphic (left-moving) component [4]

$$T(z) = \frac{1}{\hat{k}}\eta_{AB}j^A j^B - \partial x\partial x + \partial w\partial w \tag{4.41}$$

and similarly for anti-holomorphic component. Therefore $\mathcal{O}((z-w)^{-2})$ terms of the OPEs of the stress-energy tensor and the ground state primary V_t are given by

$$T(z)V_t(w,\bar{w}) = \frac{1}{(z-w)^2}\left(-\frac{j(j+1)}{\hat{k}} + \frac{n_L^2 - m_L^2}{2}\right)V_t(w,\bar{w}) + \cdots \tag{4.42}$$

$$\tilde{T}(\bar{z})V_t(w,\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^2}\left(-\frac{j(j+1)}{\hat{k}} + \frac{n_R^2 - m_R^2}{2}\right)V_t(w,\bar{w}) + \cdots \tag{4.43}$$

In what follows we are going to perform Kaluza-Klein reduction of the $U(1)_x$ circle, therefore $n_L = n_R = 0$.

In this chapter we are interested in the NS-NS vertex operators of the massless closed string excitations in type-II superstring theory. These

[4] The term with w corresponds to the coset Kazama-Suzuki construction [29, 30], where $T_{G/H} = T_G - T_H$, in the following way. From BRST condition due to (4.23) one obtains, schematically, $\partial w = -\frac{1}{2}k$. Therefore contribution of w to the stress-energy tensor $T(z)$ is $T_w(z) = \partial w\partial w = \frac{1}{4}kk$. Then we expect $T_H = -T_w$, which is indeed the case: $T_H(z)k(0) = k(z)/z^2$.

operators in the $(-1,-1)$ picture are constructed as (anti)symmetrized direct products of massless holomorphic $\mathbf{V}^\mu(z) = e^{-\varphi}\psi^\mu_{-1/2}\cdot V_t$ and anti-holomorphic $\tilde{\mathbf{V}}^\mu(\bar{z}) = e^{-\tilde{\varphi}}\tilde{\psi}^\mu_{-1/2}\cdot V_t$ vertex operators. Here $\psi^\mu(z)$ and $\tilde{\psi}^\mu(\bar{z})$ are world-sheet fermions, and $\varphi\,,\tilde{\varphi}$ are bosonized superconformal ghosts. The only non-trivial super-Virasoro physical state condition, which one needs to impose on the massless states, is $G_{1/2}\cdot\mathbf{V}^\mu(z) = 0$, and similarly for anti-holomorphic vertex operator, where $G(z) = \sum_r G_r/z^{r+3/2}$ is the supercurrent.

The other option, which is what we are going to use in this chapter, is to consider vertex operators V^μ and $\tilde{V}^\mu$ in zero-ghost picture. They are obtained from $(-1,-1)$ picture vertex operators $\mathbf{V}^\mu$ and $\tilde{\mathbf{V}}^\mu$ by acting with the picture changing operators $e^\varphi G$ and $e^{\tilde{\varphi}}\tilde{G}$. As a result one obtains vertex operator in zero-ghost picture

$$V^\mu = G_{-1/2}\cdot\psi_{-1/2}\cdot V_t = (j^\mu_{-1} + p\cdot\psi_{-1/2}\psi^\mu_{-1/2})\cdot V_t\,, \tag{4.44}$$

where j^μ is the current, supersymmetric to the fermion ψ^μ, and p^μ is the momentum of the state. Similar expression is true for anti-holomorphic vertex operator. The only non-trivial super-Virasoro constraint which one should impose in zero-ghost picture is $L_1\cdot V^\mu = 0$. Moreover, the L_1 here is actually the amplitude of the stress-energy tensor $L_1^{(b)}$ for only bosonic modes: in the r.h.s. of (4.44) contribution of fermions is automatically annihilated by the fermionic stress-energy tensor amplitude $L_1^{(f)} = \psi^\nu_{1/2}\psi_{\nu 1/2}$. Therefore instead of studying massless NS-NS states in type-II superstring theory we can study gravity multiplet in bosonic string theory.

4.3.4 (Anti)holomorphic vertex operators of massless modes in the $R\times\frac{SL(2,R)}{U(1)}$ coset model

In this subsection we review the construction of (anti)holomorphic vertex operators [16], describing massless (right-)left-moving excitations in the gWZW model on $R\times\frac{SL(2,R)}{U(1)}$ [6, 7]. The classical background is the two-dimensional black hole with the linear dilaton in a direct product with a real line. The real line is parametrized by the flat coordinate X, which we choose as a direction of propagation of all the excitations. The momentum is equal to p.

The authors of [16] considered graviton vertex operator in $(-1,-1)$ picture on the world-sheet with $\mathcal{N}=2$ supersymmetry. We perform a picture changing and consider vertex operators in zero-ghost picture. Due to noted in the previous subsection, we can actually study bosonic string and then make contact with the results of [16].

Without loss of generality let us focus on holomorphic vertex operators. The ground state vertex operator of the $R \times \frac{SL(2,R)}{U(1)}$ coset theory is

$$V_t = e^{ipX} e^{iNw} V_{jm} \,. \tag{4.45}$$

This state must be closed under the action of the null $U(1)$ BRST current,

$$J = j^3 - \sqrt{\hat{k}} \partial w \,, \tag{4.46}$$

which imposes the condition

$$iN = \frac{2m}{\sqrt{\hat{k}}} \,. \tag{4.47}$$

The most general holomorphic vertex operator of the massless state (which is a gauge field from the space-time point of view) on $R \times \frac{SL(2,R)}{U(1)}$ is

$$V^\chi = (a_+ j^+_{-1} j^-_0 + a_- j^-_{-1} j^+_0 + a_3 j^3 + a_w \partial w + a_X \partial X) V_t \,. \tag{4.48}$$

Mass-shell Virasoro constraint (see (4.42)) gives

$$L_0 V^\chi = V^\chi \quad \Rightarrow \quad -\frac{j(j+1)}{\hat{k}-2} + \frac{p^2 - N^2}{4} = 0 \tag{4.49}$$

Closeness of (4.48) w.r.t. J_1 (see (4.46)) reduces the number of parameters by one, giving the most general BRST-closed state

$$\begin{aligned} V^\chi = (a_+ j^+_{-1} j^-_0 + a_- j^-_{-1} j^+_0 + \frac{2}{\sqrt{\hat{k}}} (a_+ (m+j)(m-1-j) \\ - a_- (m-j)(m+1+j)) \partial w + AJ + a_X \partial X) V_t \,. \end{aligned} \tag{4.50}$$

Also V^χ is defined up to BRST-exact state JV_t, which makes one more parameter unphysical, leaving us with a gauge field in three dimensional $R \times \frac{SL(2,R)}{U(1)}$ with three polarization parameters.

Gauge field in three dimensions has one transverse physical d.o.f. Two of the three d.o.f. are eliminated in the following way. First, we impose

Virasoro constraint $L_1 V^\chi = 0$. Second, the state V^χ, which satisfies this constraint, is defined up to the null state $L_{-1}V_t$

$$L_{-1}V_t = \left(\frac{1}{\hat{k}-2}(j^+_{-1}j^-_0 + j^-_{-1}j^+_0 - 2mj^3) + iN\partial w + ip\partial X \right) V_t\,. \quad (4.51)$$

As a result we are left with one transverse d.o.f.

The $L_1 V^\chi = 0$ constraint gives

$$\begin{aligned} &a_+(m+j)(m-1-j) + a_-(m-j)(m+1+j) \\ &+ \frac{iN}{\sqrt{\hat{k}}}(a_+(m+j)(m-1-j) - a_-(m-j)(m+1+j)) = a_X \frac{ip}{2}\,. \end{aligned} \quad (4.52)$$

Let us parametrize the solution to this equation by two independent parameters $a_X,\ a$:

$$a_+ = \frac{a_X \frac{ip}{4} + a\left(1 - \frac{2m}{\hat{k}}\right)}{(m+j)(m-1-j)}\,, \quad a_- = \frac{a_X \frac{ip}{4} - a\left(1 + \frac{2m}{\hat{k}}\right)}{(m-j)(m+1+j)}\,. \quad (4.53)$$

Therefore the most general massless left-moving state, satisfying all the Virasoro and $U(1)$ gauge BRST constraints (and defined up to BRST-exact state $J_{-1}V_t$ and null Virasoro state $L_{-1}V_t$) is described by holomorphic vertex operator

$$\begin{aligned} V^\chi = &\left(\frac{a_X \frac{ip}{4} + a\left(1 - \frac{2m}{\hat{k}}\right)}{(m+j)(m-1-j)} j^+_{-1}j^-_0 + \frac{a_X \frac{ip}{4} - a\left(1 + \frac{2m}{\hat{k}}\right)}{(m-j)(m+1+j)} j^-_{-1}j^+_0 \right. \\ &\left. + \frac{4a}{\sqrt{\hat{k}}}\partial w + a_X \partial X \right) V_t\,. \end{aligned} \quad (4.54)$$

Now notice that for

$$a = ma_1\,, \qquad a_X = -i(\hat{k}-2)pa_1 \quad (4.55)$$

we obtain that the state (4.54) is

$$V^\chi_0 = a_1(-2mJ_{-1} - (\hat{k}-2)L_{-1})V_t\,. \quad (4.56)$$

Such a state is a pure gauge (BRST-exact).

Therefore the most general physical state, which satisfies all the constraints and which is not a pure gauge, is a state for which

$$\frac{a}{a_X} \neq \frac{im}{(\hat{k}-2)p}\,. \quad (4.57)$$

Any such state is orthogonal to the V_0^χ state (4.56), due to Virasoro and BRST physical state conditions.

The two-point function of the most general physical state (4.54) is

$$\langle V^\chi(p,j,m)V^\chi(-p,j,-m)\rangle = \frac{(\hat{k}^2-\hat{k}p^2-4m^2)(ma_X+i(\hat{k}-2)pa)^2}{2\hat{k}^2(m^2-j^2)(m^2-(j+1)^2))} \langle V_t(p,j,m)V_t(-p,j,-m)\rangle\,. \tag{4.58}$$

When (4.57) is not satisfied, we are dealing with the null pure gauge state, which is a linear combination of timelike and longitudinal polarizations, that is for such a state

$$ma_X+i(\hat{k}-2)pa=0\,. \tag{4.59}$$

Finally let us make contact with the result of [16]. The two holomorphic supercurrents of $\mathcal{N}=2$ supersymmetric $SL(2,R)/U(1)$ gWZW theory are

$$G^+=\psi^+j^-\,,\quad G^-=\psi^-j^+\,. \tag{4.60}$$

Applying the picture-changing operator $G^+_{-1/2}+G^-_{-1/2}$ to the physical holomorphic vertex operator of [16] we obtain the vertex operator of the form

$$V^\chi=(\partial X+a_+j^+_{-1}j^-_0+a_-j^-_{-1}j^+_0+\text{fermions})V_t\,. \tag{4.61}$$

Here due to (4.54) we have

$$a_+=\frac{ip/4}{(m+j)(m-1-j)}\,,\quad a_-=\frac{ip/4}{(m-j)(m+1+j)}\,. \tag{4.62}$$

Due to (4.58) we obtain that the two-point function of this vertex operator has poles at $m=\pm j$. Bellow we discuss these poles in detail and show that actually only $m=-j$ pole is present, which after taking into account the mass-shell condition precisely reproduces the dispersion relation of the gapless low-energy mode, found in [16].

4.3.5 (Anti)holomorphic vertex operators of massless modes in the $R\times\frac{SL(2,R)\times U(1)}{U(1)}$ coset model

In this subsection we are going to construct (anti)holomorphic vertex operators, describing massless (right-)left-moving string excitations in the

$R \times \frac{SL(2,R)\times U(1)}{U(1)}$ model. The classical geometry of this model is a geometry of the 2d charged black hole in a direct product with a real line. We choose this real line as a direction of propagation of all the excitations, and parametrize it by the coordinate X. The momentum of propagation is p.

The vertex operator operators must satisfy BRST and Virasoro physical state conditions. Recall the null BRST currents (4.28):

$$J = \frac{2}{\sqrt{\hat{k}}} j^3 \cos\psi + 2\sin\psi \partial x + 2\partial w \tag{4.63}$$

$$\tilde{J} = -\frac{2}{\sqrt{\hat{k}}} \tilde{j}^3 + 2\bar{\partial}\tilde{w} \,. \tag{4.64}$$

Notice that the anti-holomorphic sector is the same as for the model of the previous subsection: anti-holomorphic (right-moving) sector of the circle, $U(1)_{\tilde{x}}$, is disconnected from the rest of the geometry.

Consider holomorphic sector. Ground state vertex operator is

$$V_t = e^{ipX} e^{iNw} V_{jm} \,. \tag{4.65}$$

This state must be closed w.r.t. BRST current (4.63), which imposes the constraint

$$iN = -\frac{2m\cos\psi}{\sqrt{\hat{k}}} \,. \tag{4.66}$$

The most general massless holomorphic vertex operator on the $R \times \frac{SL(2,R)\times U(1)}{U(1)}$ is given by

$$V^{\chi} = (a_+ j^+_{-1} j^-_0 + a_- j^-_{-1} j^+_0 + a_w \partial w + a_X \partial X + b_x \partial x + AJ) V_t \,. \tag{4.67}$$

It must be closed w.r.t. BRST current (4.63), which requires

$$a_w = b_x \sin\psi - \frac{2}{\sqrt{\hat{k}}} \cos\psi (a'_+ - a'_-) \,. \tag{4.68}$$

where we have denoted for brevity

$$a'_+ = a_+(m+j)(m-1-j)\,, \qquad a'_- = a_-(m-j)(m+1+j)\,. \tag{4.69}$$

The most general massless state, closed w.r.t. (4.63), is therefore described by the vertex operator

$$\begin{aligned} V^{\chi} = \Big(& a_+ j^+_{-1} j^-_0 + a_- j^-_{-1} j^+_0 + a_X \partial X + b_x \partial x \\ & + \left(b_x \sin\psi - \frac{2}{\sqrt{\hat{k}}} \cos\psi (a'_+ - a'_-) \right) \partial w + AJ \Big) V_t \,. \end{aligned} \tag{4.70}$$

One d.o.f. in (4.70) is unphysical due to the fact that each state V^χ is defined up to BRST-exact state $J_{-1}V_t$. Therefore there remain four d.o.f. of the gauge field V^χ in four dimensional target space. Two of them are unphysical, and are eliminated due to Virasoro constraints, as we show bellow.

Imposing Virasoro constraint $L_1 V^\chi = 0$, with account to (4.66), we obtain condition

$$a'_+ + a'_- + \frac{2m\cos^2\psi}{\hat{k}}(a'_+ - a'_-) = a_X\frac{ip}{2} + b_x\frac{m\sin 2\psi}{2\sqrt{\hat{k}}}\,. \tag{4.71}$$

We parametrize the solution to this equation as

$$a_+ = \frac{a_X\frac{ip}{4} + a\left(1 - \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m+j)(m-1-j)} \tag{4.72}$$

$$a_- = \frac{a_X\frac{ip}{4} - a\left(1 + \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m-j)(m+1+j)}\,. \tag{4.73}$$

Using the mass-shell Virasoro condition (see (4.42))

$$L_0 V^\chi = V^\chi \quad \Rightarrow \quad -\frac{j(j+1)}{\hat{k}-2} + \frac{p^2 - N^2}{4} = 0\,. \tag{4.74}$$

we can re-write

$$(m+j)(m-1-j) = -\frac{\hat{k}-2}{4}p^2 - m\left(1 - \frac{2m\cos^2\psi}{\hat{k}}\right) + m^2\sin^2\psi\,, \tag{4.75}$$

$$(m-j)(m+1+j) = -\frac{\hat{k}-2}{4}p^2 + m\left(1 + \frac{2m\cos^2\psi}{\hat{k}}\right) + m^2\sin^2\psi\,. \tag{4.76}$$

These expressions are useful for computations, described bellow.

To summarize, the most general massless physical state V^χ on the $R\times\frac{SL(2,R)\times U(1)}{U(1)}$, satisfying all the physical constraints, is

$$\begin{aligned} V^\chi = \Bigg(& \frac{a_X\frac{ip}{4} + a\left(1 - \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m+j)(m-1-j)} j^+_{-1} j^-_0 \\ & + \frac{a_X\frac{ip}{4} - a\left(1 + \frac{2m\cos^2\psi}{\hat{k}}\right) + \frac{b_x m\sin 2\psi}{4\sqrt{\hat{k}}}}{(m-j)(m+1+j)} j^-_{-1} j^+_0 \\ & + a_X\partial X + b_x\partial x + \left(b_x\sin\psi - \frac{4a\cos\psi}{\sqrt{\hat{k}}}\right)\partial w\Bigg) V_t\,. \end{aligned} \tag{4.77}$$

This state is defined up to the null Virasoro state (recall $n_{L,R} = 0$ due to Kaluza-Klein reduction of the x-circle)

$$L_{-1}V'_{jm} = \left(\frac{1}{\hat{k}-2}(j^+_{-1}j^-_0 + j^-_{-1}j^+_0 - 2mj^3) + iN\partial w + ip\partial X\right) V_t\,. \quad (4.78)$$

Using (4.75) and (4.76) one then demonstrates that for

$$a_X = -i(\hat{k}-2)pa_1\,, \quad a = ma_1\,, \quad b_x = -2m\sqrt{\hat{k}}\tan\psi a_1 \qquad (4.79)$$

the state (4.77) is non-physical (it is the sum of the BRST-exact and the null Virasoro states)

$$V_0^\chi = -a_1\left((\hat{k}-2)L_{-1} + \frac{m\sqrt{\hat{k}}}{\cos\psi}j\right) V_t\,. \qquad (4.80)$$

The two-point function of the vertex operator (4.77) is given by

$$\begin{aligned}\langle V^\chi V^\chi\rangle =& (m^2-j^2)^{-1}(m-(j+1)^2)^{-1}(c_1(i(\hat{k}-2)pa+ma_X)^2 \qquad (4.81)\\ &+c_2(i(\hat{k}-2)pb_x-2m\sqrt{\hat{k}}\tan\psi a_X)^2+c_3(b_x+2a\sqrt{\hat{k}}\tan\psi)^2)\langle V_tV_t\rangle\,,\end{aligned}$$

where

$$\begin{aligned}c_1 =& \frac{1}{4\hat{k}^2(\hat{k}-2)}(\hat{k}(\hat{k}-2)^2\cos^2\psi p^2 - 8(\hat{k}-2)\cos^4\psi m^2\\ &-\hat{k}^2(\hat{k}-2)(p^2-2) + 2\hat{k}(\sin^2 2\psi + 2\hat{k}\sin^4\psi)m^2)\,, \qquad (4.82)\end{aligned}$$

$$c_2 = \frac{\cos^2\psi}{32\hat{k}^2(\hat{k}-2)}(2((\hat{k}-2)^2\cos 2\psi + 4 - 4\hat{k} - \hat{k}^2)m^2 + (\hat{k}-2)^2\hat{k}p^2)\,, \quad (4.83)$$

$$\begin{aligned}c_3 =& \frac{\cot\psi}{32\hat{k}^2}((8m^2(\hat{k}^2-2m^2) + 2(\hat{k}^2-4)m^2p^2 - (\hat{k}-2)^2\hat{k}p^4)\sin 2\psi\\ &-m^2(8m^2+(\hat{k}-2)^2p^2)\sin 4\psi)\,. \qquad (4.84)\end{aligned}$$

When the (4.79) is satisfied, we are dealing with the null state V_0^χ with zero norm.

Like in the previous section, where we derived the vertex operator (4.61), we now proceed to writing down the vertex operators $V^x = (\partial x +$

...)V_t and $V^X = (\partial X + ...)V_t$, where dots denote contribution from $j^{\pm}_{-1}$ currents:

$$V^X = \left(\partial X + \frac{ip/4}{(m+j)(m-1-j)} j^+_{-1} j^-_0 + \frac{ip/4}{(m-j)(m+1+j)} j^-_{-1} j^+_0\right) V_t \quad (4.85)$$

$$V^x = \left(\partial x + \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m+j)(m-1-j)} j^+_{-1} j^-_0 - \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m-j)(m+1+j)} j^-_{-1} j^+_0\right) V_t \,. \quad (4.86)$$

4.4 Vertex operators of massless NS-NS states, and correlation functions

In the previous section we constructed holomorphic and anti-holomorphic vertex operators, describing respectively left-moving and right-moving massless excitations of the string on the $\frac{SL(2,R)\times U(1)_x}{U(1)}$ coset. The state of the closed string is described by the vertex operator which is a direct product of holomorphic and anti-holomorphic vertex operators. In this section we will construct the vertex operators for graviton and antisymmetric tensor field, which are massless NS-NS states of type-II gravity. Kaluza-Klein reduction on $U(1)_x$, applied to graviton and antisymmetric tensor field vertex operators, gives vertex operators for gauge fields. We will split the vertex operators into two decoupled from each other groups, and find correlation functions for vertex operators within each group.

Denote $M = a, X, x$, and $\mu = a, X$, where a labels non-compactified directions, transverse to the direction X of propagation of all the excitations, and x is a coordinate of the compactified circle. Then, $V^M = j^M V_t$ are holomorphic physical vertex operators and $\tilde{V}^M = \tilde{j}^M V_t$ are anti-holomorphic physical vertex operators of the massless left-moving and right-moving states.

Here $j^a = \partial x^a$ and $\tilde{j}^a = \bar{\partial} x^a$. Due to (4.85) and (4.86) the j^x and j^X are elements of two different BRST and Virasoro cohomology classes, and are defined by

$$j^X = \partial X + \frac{ip/4}{(m+j)(m-1-j)} j^+_{-1} j^-_0 + \frac{ip/4}{(m-j)(m+1+j)} j^-_{-1} j^+_0 \quad (4.87)$$

$$j^x = \partial x + \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m+j)(m-1-j)} j^+_{-1} j^-_0 - \frac{(\sqrt{\hat{k}}/4)\tan\psi}{(m-j)(m+1+j)} j^-_{-1} j^+_0 \,. \quad (4.88)$$

Due to anti-holomorphic version of (4.61),

$$\tilde{j}^X = \bar{\partial}X + \frac{ip/4}{(\bar{m}+j)(\bar{m}-1-j)} \tilde{j}^+_{-1}\tilde{j}^-_0 + \frac{ip/4}{(\bar{m}-j)(\bar{m}+1+j)} \tilde{j}^-_{-1}\tilde{j}^+_0 \,. \tag{4.89}$$

Finally, $\tilde{j}^x = \bar{\partial}x$.

Notice that due to (4.81) the normalized two-point functions $\langle j^x j^x \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$, $\langle j^x j^X \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$, $\langle j^X j^X \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$ have simple poles at $m = \pm j$, and due to (4.58) (the anti-holomorphic version of it) the two-point function $\langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} / \langle V_t V_t \rangle$ has simple poles at $\bar{m} = \pm j$.

The two-point function for ground state of the $SL(2, R)$ model is given by (see e.g. [14, 15] for a recent discussion)

$$\langle V_{j,m,\bar{m}} V_{j,-m,-\bar{m}} \rangle = \nu \frac{\Gamma\left(1 - \frac{2j+1}{\hat{k}-2}\right)\Gamma(-2j-1)\Gamma(1+j+m)\Gamma(1+j-\bar{m})}{\Gamma\left(1 + \frac{2j+1}{\hat{k}-2}\right)\Gamma(2j+1)\Gamma(-j+m)\Gamma(-\bar{m}-j)} \tag{4.90}$$

where ν is some number. Notice that due to factors of $\Gamma(-j+m)$ and $\Gamma(-\bar{m}-j)$ in the denominator, the (4.90) has simple zeroth are $j = m$ and $j = -\bar{m}$. Therefore the two-point functions $\langle j^x j^x \rangle_{jm\bar{m}}$, $\langle j^x j^X \rangle_{jm\bar{m}}$, $\langle j^X j^X \rangle_{jm\bar{m}}$ have simple pole at $m = -j$, while the simple pole at $m = j$ is canceled, and the two-point function $\langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}}$ has simple pole at $\bar{m} = j$, while the pole at $\bar{m} = -j$ is canceled.

4.4.1 Vertex operators and their correlation functions

Graviton vertex operator is

$$G^{MN} = (j^M \tilde{j}^N + j^N \tilde{j}^M) V_t \,. \tag{4.91}$$

Antisymmetric tensor field vertex operator is

$$B^{MN} = (j^M \tilde{j}^N - j^N \tilde{j}^M) V_t \,. \tag{4.92}$$

Gauge field vertex operators are:

$$A^\mu = G^{x\mu} = (j^x \tilde{j}^\mu + \bar{\partial}x j^\mu) V_t \tag{4.93}$$

$$B^\mu = B^{x\mu} = (j^x \tilde{j}^\mu - \bar{\partial}x j^\mu) V_t \,. \tag{4.94}$$

We have the following groups of vertex operators defined by the spin w.r.t. to the rotations in the transverse non-compactified space (for which the coordinates are labeled by small Latin indices).[5] In the sound channel the spin is zero, and one considers the fields G^{XX}, A^X B^X. In the shear channel the spin is one, and one considers the fields G^{Xa}, B^{Xa}, A^a, B^a. In the scalar channel the spin is two, and one considers the fields G^{ab} and B^{ab}. Due to the rotational symmetry in the transverse space, vertex operators from different groups are decoupled from each other.

Shear channel

In the shear channel we have vertex operators

$$G^{Xa} = (j^X \bar{\partial} x^a + \tilde{j}^X \partial x^a) V_t \tag{4.95}$$

$$B^{Xa} = (j^X \bar{\partial} x^a - \tilde{j}^X \partial x^a) V_t \tag{4.96}$$

$$A^a = G^{xa} = (j^x \bar{\partial} x^a + \bar{\partial} x \partial x^a) V_t \tag{4.97}$$

$$B^a = B^{xa} = (j^x \bar{\partial} x^a - \bar{\partial} x \partial x^a) V_t \tag{4.98}$$

Notice that all these vertex operators are coupled to each other. We can consider instead two groups of operators:

the first group is

$$S^{Xa} = \frac{1}{2}(G^{Xa} + B^{Xa}) = j^X \bar{\partial} x^a V_t \tag{4.99}$$

$$W^a = \frac{1}{2}(A^a + B^a) = j^x \bar{\partial} x^a V_t \tag{4.100}$$

and the second group is

$$R^{Xa} = \frac{1}{2}(G^{Xa} - B^{Xa}) = \tilde{j}^X \partial x^a V_t \tag{4.101}$$

$$U^a = \frac{1}{2}(A^a - B^a) = \bar{\partial} x \partial x^a V_t \, . \tag{4.102}$$

We call the operators from the first group S-system and the operators from the second group R-system. The S-system is decoupled from the R-system. For the vertex operators of the S-system the two-point functions are

$$\langle S^{Xa} S^{Xb} \rangle = -\frac{1}{2} \delta^{ab} \langle j^X j^X \rangle_{jm\bar{m}} \tag{4.103}$$

[5] See e.g. [31] for a recent discussion in the holographic context.

$$\langle W^a W^b \rangle = -\frac{1}{2}\delta^{ab}\langle j^x j^x \rangle_{jm\bar{m}} \tag{4.104}$$

$$\langle S^{Xa} W^b \rangle = -\frac{1}{2}\delta^{ab}\langle j^x j^X \rangle_{jm\bar{m}} \tag{4.105}$$

These correlation functions have a simple pole at $j = -m$.

For the vertex operators of the R-system the two-point functions are

$$\langle R^{Xa} R^{Xb} \rangle = -\frac{1}{2}\delta^{ab}\langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}} \tag{4.106}$$

$$\langle U^a U^b \rangle = \frac{1}{4}\delta^{ab} \tag{4.107}$$

$$\langle R^{Xa} U^b \rangle = 0\,. \tag{4.108}$$

These correlation functions have a simple pole at $j = \bar{m}$.

Due to holographic correspondence we obtain correlation functions of the shear components of the stress-energy tensor of the dual field theory:[6]

$$\langle G^{Xa} G^{Xb} \rangle = \langle T^{Xa} T^{Xb} \rangle = -\frac{1}{2}\delta^{ab}\left(\langle j^X j^X \rangle_{jm\bar{m}} + \langle \tilde{j}^X \tilde{j}^X \rangle_{jm\bar{m}}\right) \tag{4.109}$$

The correlation functions for the transverse components of the charge current are

$$\langle J^a J^b \rangle = \langle A^a A^b \rangle = -\frac{1}{2}\delta^{ab}\left(\langle j^x j^x \rangle_{jm\bar{m}} - \frac{1}{2}\right) \tag{4.110}$$

Finally,

$$\langle J^a T^{Xb} \rangle = \langle A^a G^{Xb} \rangle = -\frac{1}{2}\delta^{ab}\langle j^x j^X \rangle_{jm\bar{m}}\,. \tag{4.111}$$

We conclude that in the shear/transverse diffusion channel we have modes with the dispersion relations $m = -j$ and $\bar{m} = j$.

Sound channel

In the sound channel we have vertex operators

$$G^{XX} = j^X \tilde{j}^X V_t \tag{4.112}$$

$$A^X = G^{xX} = (j^x \tilde{j}^X + \bar{\partial} x j^X) V_t \tag{4.113}$$

[6] One also may be interested in computing correlation functions of the operator, dual to B_{MN}-field. See [32, 33], where the primary operator in $\mathcal{N} = 4$ SYM, holographically dual to the B-field in $AdS_5 \times S^5$, was found.

$$B^X = B^{xX} = (j^x\tilde{j}^X - \bar{\partial}xj^X)V_t\,. \tag{4.114}$$

Notice that A^X and B^X are coupled. Consider instead decoupled gauge fields vertex operators

$$W^X = \frac{1}{2}(A^X + B^X) = j^x\tilde{j}^X V_t \tag{4.115}$$

$$U^X = \frac{1}{2}(A^X - B^X) = \bar{\partial}xj^X V_t\,. \tag{4.116}$$

Correlation functions are

$$\langle G^{XX}G^{XX}\rangle = \langle j^X j^X\rangle_{jm\bar{m}}\langle\tilde{j}^X\tilde{j}^X\rangle_{jm\bar{m}} \tag{4.117}$$

$$\langle G^{XX}W^X\rangle = \langle j^x j^X\rangle_{jm\bar{m}}\langle\tilde{j}^X\tilde{j}^X\rangle_{jm\bar{m}} \tag{4.118}$$

$$\langle G^{XX}U^X\rangle = 0 \tag{4.119}$$

$$\langle W^X W^X\rangle = \langle j^x j^x\rangle_{jm\bar{m}}\langle\tilde{j}^X\tilde{j}^X\rangle_{jm\bar{m}} \tag{4.120}$$

$$\langle U^X U^X\rangle = -\frac{1}{2}\langle j^X j^X\rangle_{jm\bar{m}}\,. \tag{4.121}$$

The correlation functions (4.117), (4.118) and (4.120) have simple poles at $j = -m$ and $j = \bar{m}$ and the correlation function (4.121) has simple pole at $j = -m$.

Due to holographic correspondence we obtain correlation functions of the longitudinal component of the stress-energy tensor of the dual field theory:

$$\langle T^{XX}T^{XX}\rangle = \langle G^{XX}G^{XX}\rangle = \langle j^X j^X\rangle_{jm\bar{m}}\langle\tilde{j}^X\tilde{j}^X\rangle_{jm\bar{m}}\,. \tag{4.122}$$

The correlation function of the longitudinal component of the charge current is

$$\langle J^X J^X\rangle = \langle A^X A^X\rangle = \langle j^x j^x\rangle_{jm\bar{m}}\langle\tilde{j}^X\tilde{j}^X\rangle_{jm\bar{m}} - \frac{1}{2}\langle j^X j^X\rangle_{jm\bar{m}}\,. \tag{4.123}$$

Finally,

$$\langle J^X T^{XX}\rangle = \langle A^X G^{XX}\rangle = \langle j^x j^X\rangle_{jm\bar{m}}\langle\tilde{j}^X\tilde{j}^X\rangle_{jm\bar{m}}\,. \tag{4.124}$$

Therefore in the sound channel we have modes with the dispersion relations $m = -j$ and $\bar{m} = j$.

Scalar channel

In the scalar channel one considers G^{ab} and B^{ab}, correlation functions for which do not have poles at $j = -m$ and $j = \bar{m}$.

4.4.2 Low-energy modes

In the previous subsection we concluded that there are modes with the dispersion relations $m = -j$ and $\bar{m} = j$ in the shear and sound channels of the quantum field theory holographically dual to the charged black brane. Now we are going to show, considering small ω and q, that these modes are actually gapless modes.

The frequency is determined by the asymptotic behavior of the tachyon vertex operator $V_t \sim e^{i\omega t}$ (see [25]), and is given by (for $\psi \neq \pi/2$, that is in non-extremal case)

$$\omega = \frac{(1 - \tan^2(\psi/2))(m - \bar{m})}{\sqrt{\hat{k}}} \tag{4.125}$$

Due to the gauge physical state condition (4.40) we have $\bar{m} = -m\cos\psi$, and therefore (also after Wick rotation $m \to im$)

$$\omega = \frac{2im\cos\psi}{\sqrt{\hat{k}}} \,. \tag{4.126}$$

Due to the mass-shell condition

$$-\frac{j(j+1)}{\hat{k}} + \frac{p^2 - N^2}{4} = 0 \tag{4.127}$$

where $N \sim m \sim \omega$, for $\omega \sim p^2$ and $p \ll 1$ we obtain

$$j = \frac{\hat{k}}{4} p^2 \,. \tag{4.128}$$

Therefore the S-system possesses the low-energy excitation mode with the dispersion relation

$$\omega = -i\frac{\sqrt{\hat{k}}}{2}\cos\psi\, p^2 \tag{4.129}$$

while for the R-system we obtain the mode with the dispersion relation

$$\omega = -i\frac{\sqrt{\hat{k}}}{2}\, p^2 \,. \tag{4.130}$$

In the extremal case $\psi = \pi/2$ the two-point functions in the S-system, due to (4.129), behave as $\langle SS \rangle \sim 1/\omega$, indicating local criticality, while the dispersion relation (4.130) of the R-system stays unaffected.

At zero density $q = 0$ we have $\psi = 0$, and the mode (4.129) coincides with the mode (4.130). In this case $U(1)_x$ completely decouples from $SL(2,R)/U(1)$, and we recover the results of [16] for the model on the $SL(2,R)/U(1)$. Due to (4.4) we obtain

$$\omega = -i\frac{1}{4\pi T}p^2 \tag{4.131}$$

Comparing it with the shear mode dispersion relation at zero density

$$\omega = -i\frac{\eta}{sT}p^2 \tag{4.132}$$

we recover the result of [16]

$$\frac{\eta}{s} = \frac{1}{4\pi}\,. \tag{4.133}$$

4.4.3 "$2k_F$" singularity

Expression (4.90) for the groundstate two-point function contains the factor of $\Gamma\left(1 - \frac{2j+1}{\hat{k}-2}\right)$. Due to the physical state mass-shell condition (4.127) we obtain

$$j = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + (\hat{k}-2)(p^2 - N^2)}\,. \tag{4.134}$$

At zero frequency $N = 0$. Therefore equation $2j + 1 = \hat{k} - 2$, which defines singularity of $\Gamma\left(1 - \frac{2j+1}{\hat{k}-2}\right)$, has a zero frequency and finite momentum solution. The value of the momentum is given by

$$p_\star^2 = \frac{1}{\ell_s^2}\left(\hat{k} - 2 - \frac{1}{\hat{k}-2}\right)\,. \tag{4.135}$$

Note that p_* is independent of the chemical potential $\mu = \tan\frac{\psi}{2}$. The singular behavior of $\langle V_{jm\bar{m}} V_{jm\bar{m}} \rangle$ at $\omega = 0$ and $p = p_*$ was compared by Polchinski and Silverstein [8] with "$2k_F$" singularities in current correlation functions of condensed matter systems (see e.g. [34]).

4.5 Type-II gravity approximation

In this section we will compute the two-point functions for graviton, antisymmetric tensor field and gauge fields in the background of the 2d charged black hole (in a direct product with a flat space). Our purpose is to verify the dispersion relations (4.129-(4.130).

One-loop beta-functions for the NS-NS fields of type-II gravity are given by (see e.g. [35],Polchinski:1998rr4)

$$\beta^G_{MN} = R_{MN} + 2\nabla_M \partial_N \Phi - \frac{1}{4} H_M{}^{LS} H_{NLS}\,, \tag{4.136}$$

$$\beta^\Phi = c + \frac{1}{16\pi^2}\left(4(\partial\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2\right)\,, \tag{4.137}$$

$$\beta^B_{MN} = \nabla_L H^L{}_{MN} - 2(\partial_L \Phi) H^L{}_{MN}\,. \tag{4.138}$$

Corresponding equations of motion are $\beta^{G,\Phi,B} = 0$.

Here the field strength of antisymmetric tensor B_{MN} is given by

$$H_{MNL} = \partial_M B_{NL} + \partial_N B_{LM} + \partial_L B_{MN}\,. \tag{4.139}$$

The beta-functions (4.216)-(4.218) are invariant w.r.t. the gauge symmetry

$$\delta B_{MN} = \partial_M \Lambda_N - \partial_N \Lambda_M\,. \tag{4.140}$$

Requirement of the world-sheet conformal invariance gives the equations of motion $\beta^{G,B,\Phi} = 0$. These equations have a black brane solution, which is a direct product of two-dimensional charged black hole (CBH) and flat space, $CBH \times R^{d-1}$:

$$\begin{aligned} g_{MN} &= \mathrm{diag}\{-f(r), 1/f(r), 1, ..., 1\}\,, \\ f(r) &= 1 - 2Me^{-Qr} + q^2 e^{-2Qr}\,, \end{aligned} \tag{4.141}$$

$$\Phi = \Phi_0 - \frac{Qr}{2}\,, \qquad F_{tr} = F(r) = Qqe^{-Qr}\,. \tag{4.142}$$

where[7]

$$g_{tx} = B_{xt} = -B_{tx} = A_t\,. \tag{4.143}$$

The string theory solution, described in the previous section, implies $Q = 2/\sqrt{k}$.

[7] We thank A. Giveon for pointing out to us the role of this equation in the 2d charged black hole solution of type-II superstring theory.

Consider fluctuations h_{MN}, b_{MN} and φ around this solution. Use the diffeomorphism invariance to fix $h_{Mr} = 0$. Use gauge invariance (4.140) to fix $b_{Mr} = 0$. Among $d+1$ space-time coordinates, denoted by capital Latin indices, we have t, r coordinates of the charged black hole and $d-1$ flat coordinates. Let us consider $CBH \times R^3$. Choose X to be the R^3 direction of propagation of excitations (with momentum p) and choose Y to be the R^3 direction, transverse to propagation of excitations. Finally x is the direction of R^3 which we are going to Kaluza-Klein reduce. Fluctuations depend on t, r, X. The dependence on t and X in momentum representation boils down to the factor $e^{-i\omega t + ipX}$.

Perform Kaluza-Klein reduction of the x coordinate. Small Greek indices are used for non-reduced coordinates, $M = \mu, x$. It is convenient, as we did in the world-sheet consideration, to consider fluctuations of the fields

$$S_{MN} = \frac{1}{2}(h_{MN} + b_{MN}) \tag{4.144}$$

$$R_{MN} = \frac{1}{2}(h_{MN} - b_{MN})\,. \tag{4.145}$$

The fields (4.144) belong to the S-system and the fields (4.145) belong to the R-system, we are using the same terminology as in the previous section.

Let us consider shear fluctuations in the reduced space $CBH \times R^2$: R_{tY}, R_{XY}, S_{tY}, S_{XY} and transverse components of gauge fields w_Y and u_Y (see bellow). The ansatz for graviton and two-form field in the non-reduced space $CBH \times R^3$ in terms of the fields on the reduced space $CBH \times R^2$ is

$$G = \begin{pmatrix} A_t^2 - f & 0 & 0 & R_{tY}+S_{tY}+A_t(u_Y+w_Y) & A_t \\ 0 & 1/f & 0 & 0 & 0 \\ 0 & 0 & 1 & R_{XY}+S_{XY} & 0 \\ R_{tY}+S_{tY}+A_t(u_Y+w_Y) & 0 & R_{XY}+S_{XY} & 1 & u_Y+w_Y \\ A_t & 0 & 0 & u_Y+w_Y & 1 \end{pmatrix} \tag{4.146}$$

$$B = \tag{4.147}$$

$$\begin{pmatrix} 0 & 0 & 0 & S_{tY}-R_{tY}+A_t(w_Y-u_Y) & -A_t \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{XY}-R_{XY} & 0 \\ -(S_{tY}-R_{tY})-A_t(w_Y-u_Y) & 0 & -(S_{XY}-R_{XY}) & 0 & -(w_Y-u_Y) \\ A_t & 0 & 0 & w_Y-u_Y & 0 \end{pmatrix}.$$

Before proceeding, rescale

$$\mathbf{r} = rQ\,, \quad \mathbf{w} = \omega/Q\,, \quad \mathbf{p} = p/Q\,, \tag{4.148}$$

which eliminates Q dependence from the equations of motion.

Due to string theory result we know that the R-system fields R and u are decoupled from the S-system fields S and w. We will find out that this decoupling is true in gravity computations as well. To find equations of motion for the fields $R_{\mu\nu}$, $S_{\mu\nu}$, w_μ and u_μ we compute the beta functions $\beta^R_{MN} = \beta^G_{MN} - \beta^B_{MN}$ and $\beta^S_{MN} = \beta^G_{MN} + \beta^B_{MN}$ for $(MN) = (tY),\ (rY),\ (XY),\ (xY)$.

Consider first equations of motion in the R-system.

β^R_{tY}:

$$pf(pR_{tY} + \omega R_{XY}) + f^2(2\Phi' R'_{tY} - R''_{tY}) - A_t(f^2 u''_Y \\ + f(f' - 2f\Phi')u'_Y + (\omega^2 - p^2 f + (f^2/A_t)(A''_t - 2A'_t\Phi')u_Y) = 0\,. \tag{4.149}$$

β^R_{rY}:

$$\omega R'_{tY} + pfR'_{XY} = 0\,. \tag{4.150}$$

β^R_{XY}:

$$\omega(pR_{tY} + \omega R_{XY}) + f(fR''_{XY} + (f' - 2f\Phi')R'_{XY}) = 0\,. \tag{4.151}$$

β^R_{xY}:

$$f^2 u''_Y + f(f' - 2f\Phi')u'_Y + (\omega^2 - p^2 f)u_Y = 0\,. \tag{4.152}$$

Notice that for the CBH background $\Phi' = -1/2$ and $A''_t = -A'_t$. Therefore one sees that u_Y contribution to R_{tY} equation (4.149) vanishes due to u_Y equation (4.152). Therefore we see that $R_{\mu Y}$ and u_Y fluctuations decouple, as expected from the string theory computations (4.108).

Introduce diff-invariant quantity

$$Z = pR_{tY} + \omega R_{XY}\,. \tag{4.153}$$

Solving following from this definition equation

$$Z' = pR'_{tY} + \omega R'_{XY} \tag{4.154}$$

together with R_{rY} equation (4.150) one obtains

$$R'_{tY} = -\frac{pfZ'}{\omega^2 - p^2 f}\,, \qquad R'_{XY} = \frac{\omega Z'}{\omega^2 - p^2 f}\,. \tag{4.155}$$

Plugging expressions (4.155) into R_{tY} equation (4.149) one obtains equation (the same equation is obtained if one plugs (4.155) into R_{XY} equation (4.151))

$$Z'' + \left(\frac{\omega^2 f'}{f(\omega^2 - p^2 f)} - 2\Phi'\right) Z' + \frac{\omega^2 - p^2 f}{f^2} Z = 0\,. \tag{4.156}$$

Together with decoupled from it transverse gauge field equation (4.152) for u_Y these are fluctuation equations for shear components of R-system.

Consider now S and w fluctuation equations of the S-system.

β^S_{tY}:

$$\begin{aligned}&-\omega^2 A_t w_Y + f(p(pS_{tY} + \omega S_{XY}) + A_t(w_Y(p^2 - 2A_t'^2) - 2A_t' S_{tY}' \\ &- f' w_Y')) - f^2(-2\Phi'(S_{tY}' + A_t w_Y') + 2A_t'(w_Y' - \Phi' w_Y) + A_t'' w_Y \\ &+ S_{tY}'' + A_t w_Y'') = 0\,.\end{aligned} \tag{4.157}$$

β^S_{rY}:

$$2\omega A_t' w_Y + \omega S_{tY}' + pf S_{XY}' = 0\,. \tag{4.158}$$

β^S_{XY}:

$$\omega(pS_{tY} + \omega S_{XY}) + f(S_{XY}'(f' - 2\Phi' f) + f S_{XY}'') = 0\,. \tag{4.159}$$

β^S_{xY}:

$$(\omega^2 - f(p^2 - 2A_t'^2))w_Y + f(2A_t' S_{tY}' + (f' - 2\Phi' f)w_Y' + f w_Y'') = 0\,. \tag{4.160}$$

Introduce diff-invariant quantity

$$V = pS_{tY} + \omega S_{XY}\,. \tag{4.161}$$

Then solving equation

$$V' = pS_{tY}' + \omega S_{XY}'\,. \tag{4.162}$$

together with β^S_{rY} equation (4.158) we obtain

$$S_{XY}' = \frac{\omega(2pA_t' w_Y + V')}{\omega^2 - p^2 f}\,, \quad S_{tY}' = -\frac{2\omega^2 A_t' w_Y + pfV'}{\omega^2 - p^2 f}\,. \tag{4.163}$$

Plugging into β^S_{tY} equation (4.157) the expressions (4.163) together with w''_Y, expressed from w_Y equation (4.160), we arrive at (the same result is obtained by plugging (4.163) into β^S_{XY} equation (4.159)) [8]

$$V'' + \left(\frac{\omega^2 f'}{f(\omega^2 - p^2 f)} - 2\Phi'\right) V' + \frac{\omega^2 - p^2 f}{f^2} V \qquad (4.164)$$
$$+ \frac{2pA'_t}{f}\left(f w'_Y + \frac{\omega^2 f'}{\omega^2 - p^2 f} w_Y\right) = 0\,.$$

Finally, using S'_{tY}, expressed in (4.163), in w_Y equation (4.160) we obtain

$$w''_Y + \frac{f' - 2f\Phi'}{f} w'_Y + \left(\frac{\omega^2 - p^2 f}{f^2} + \frac{2A'^2_t}{f}\left(1 - \frac{2\omega^2}{\omega^2 - p^2 f}\right)\right) w_Y \qquad (4.165)$$
$$- \frac{2pA'_t}{\omega^2 - p^2 f} V' = 0\,.$$

We see that in the S-system tensor field shear components are coupled to gauge field transverse component, which agrees with string computation (4.105).

Let us look for poles of the correlation functions in R-system and in S-system. Notice that R-system is just S-system at vanishing background flux $A'_t = 0$; compare equation (4.164) with equation (4.229) and equation (4.165) with equation (4.152) to see that. Therefore it is sufficient to study S-system.

Introduce new radial coordinate $u = e^{\mathbf{r}}$. Then inner and outer horizons are located at

$$u_\pm = M \pm \sqrt{M^2 - q^2}\,. \qquad (4.166)$$

The equations of motion (4.164) and (4.165) become (also take into account $q = \sqrt{u_+ u_-}$)

$$\frac{d^2 w_Y}{du^2} + \left(\frac{1}{u - u_-} + \frac{1}{u - u_+}\right)\frac{dw_Y}{du} + \frac{1}{(u - u_-)(u - u_+)} \qquad (4.167)$$
$$\times \left(\frac{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)}{(u - u_-)(u - u_+)} - \frac{2u_+ u_- \mathbf{w}^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)}\right.$$
$$\left. + \frac{2u_+ u_-}{u^2}\right) w_Y + \frac{2\mathbf{p}\sqrt{u_+ u_-}}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u - u_-)(u - u_+)}\frac{d\mathbf{V}}{du} = 0\,.$$

[8] Also take into account $\Phi' = -1/2$ and $A''_t = -A'_t$.

$$\frac{d^2\mathbf{V}}{du^2}+\frac{1}{u}\left(2+\frac{\mathbf{w}^2u^2}{\mathbf{w}^2u^2-\mathbf{p}^2(u-u_-)(u-u_+)}\left(\frac{u_+}{u-u_+}+\frac{u_-}{u-u_-}\right)\right)\frac{d\mathbf{V}}{du}$$
$$-\frac{2\mathbf{p}\sqrt{u_+u_-}}{u^2}\frac{dw_Y}{du}+\frac{\mathbf{w}^2u^2-\mathbf{p}^2(u-u_-)(u-u_+)}{(u-u_-)^2(u-u_+)^2}\mathbf{V} \tag{4.168}$$
$$-\frac{2\mathbf{p}\sqrt{u_+u_-}}{u}\frac{\mathbf{w}^2}{\mathbf{w}^2u^2-\mathbf{p}^2(u-u_-)(u-u_+)}\left(\frac{u_+}{u-u_+}+\frac{u_-}{u-u_-}\right)w_Y=0\,.$$

In the near horizon limit $v = u - u_+ \ll 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2w_Y}{dv^2}+\frac{1}{v}\frac{dw_Y}{dv}+\frac{\mathbf{w}^2u_+^2}{(u_+-u_-)^2v^2}w_Y=0 \tag{4.169}$$

$$\frac{d^2\mathbf{V}}{dv^2}+\frac{1}{v}\frac{d\mathbf{V}}{dv}+\frac{\mathbf{w}^2u_+^2}{(u_+-u_-)^2v^2}\mathbf{V}=0\,. \tag{4.170}$$

The incoming-wave solutions are

$$w_Y(u)=C_1(u-u_+)^{\frac{-i\mathbf{w}u_+}{u_+-u_-}}\,,\quad \mathbf{V}(u)=C_2(u-u_+)^{\frac{-i\mathbf{w}u_+}{u_+-u_-}}\,. \tag{4.171}$$

In the asymptotic region $u \gg 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2w_Y}{du^2}+\frac{2}{u}\frac{dw_Y}{du}+\frac{\mathbf{w}^2-\mathbf{p}^2}{u^2}w_Y=0 \tag{4.172}$$

$$\frac{d^2\mathbf{V}}{du^2}+\frac{2}{u}\frac{d\mathbf{V}}{du}+\frac{\mathbf{w}^2-\mathbf{p}^2}{u^2}\mathbf{V}=0 \tag{4.173}$$

with the solution

$$w_Y=\mathcal{A}_w u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}+\mathcal{B}_w u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})} \tag{4.174}$$

$$\mathbf{V}=\mathcal{A}_V u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}+\mathcal{B}_V u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}\,. \tag{4.175}$$

We solve numerically the equations (4.235), (4.236) with boundary conditions (4.239) and find two linearly-independent solutions $(w_Y^{(1)},\mathbf{V}^{(1)})$ and $(w_Y^{(2)},\mathbf{V}^{(2)})$ (for two independent choices of $C_{1,2}$). The correlation matrix is given by [37] $\mathcal{G}\simeq\mathcal{B}\mathcal{A}^{-1}$, where the matrices of leading and subleading coefficients are determined by (4.242) and (4.243):

$$\mathcal{A}=\begin{pmatrix}\mathcal{A}_V^{(1)} & \mathcal{A}\mathcal{A}_V^{(2)}\\ \mathcal{A}_w^{(1)} & \mathcal{A}_w^{(2)}\end{pmatrix}\,,\quad \mathcal{B}=\begin{pmatrix}\mathcal{B}_V^{(1)} & \mathcal{B}_V^{(2)}\\ \mathcal{B}_w^{(1)} & \mathcal{B}_w^{(2)}\end{pmatrix}\,. \tag{4.176}$$

Zeroes of the determinant of the matrix of the leading behavior coefficients [9]

$$\mathcal{A}_V^{(1)}\mathcal{A}_w^{(2)} - \mathcal{A}_V^{(2)}\mathcal{A}_w^{(1)} \tag{4.177}$$

define the dispersion relation of low-energy mode, which is given by

$$\mathbf{w} = -i\mathbf{p}^2 \cos\psi \,. \tag{4.178}$$

Due to (4.232) and $Q = 2/\sqrt{\hat{k}}$ the dispersion relation (4.245) coincides with the dispersion relation (4.129), obtained for the S-system by the world-sheet computation.

From the S-system result (4.245) we conclude that in the R-system $\langle ZZ \rangle$ correlation function has pole at

$$\mathbf{w} = -i\mathbf{p}^2 \,, \tag{4.179}$$

which coincides with the pole (4.130), obtained by the world-sheet computation. Notice that as in [16] the supergravity result does not receive stringy corrections.

4.6 Discussion

In this chapter we have used the holographically dual string theory to study quantum field theory at finite temperature and chemical potential. The string theory was defined by the gWZW model on the $\frac{SL(2,R)\times U(1)_x}{U(1)} \times R^{d-1}$, with the $U(1)$ gauged asymmetrically [25] coset and the covariant quantization of the string, we have constructed vertex operators, representing massless NS-NS states of the string. The gauge fields vertex operators were obtained by the Kaluza-Klein reduction of the graviton and the two-form field vertex operators on the $U(1)_x$.

We have found that these vertex operators split into two decoupled systems. This implies that the boundary low energy theory splits into two decoupled models, as far as the two-point functions are concerned. At low energies the Green's functions of stress energy tensor and global $U(1)$ current exhibit two gapless poles. Corresponding dispersion relations are (4.129) and (4.130) in the shear and sound channels. The dispersion relation (4.130) does not depend on the charge to mass ratio of the charged

[9] See e.g. [38] where computation of correlation matrix in the different system of two coupled differential equations is explained in detail.

black hole background. When the charge density is zero, the dispersion relation (4.129) coincides with the dispersion relation (4.130). We have verified these results by computations in type-II supergravity; the supergravity results exactly coincide with superstring results. We speculate that the system is described at low energies by a decoupled sum of two non-interacting fluids. It would be interesting to make this picture more precise.

The current and stress-energy tensor two-point correlation functions, which we have computed, also possess finite-momentum zero-frequency singularity. As in [8] it originates from the two-point function of the vertex operator of the ground state of the WZW model on $SL(2, R)$. This "$2k_F$" singularity is a purely stringy effect [8], absent in the supergravity approximation: from (4.135) it follows, that the momentum p_*, measured in units of inverse curvature radius, scales as $(Rp_*)^2 \simeq \hat{k}\ell_s^2 p_*^2 \sim \hat{k}^2$ when $\hat{k}$ is large. Therefore in supergravtiy approximation p_* is parametrically large.

We have also studied the shear channel in heterotic gravity (see Appendix B), and found one low-energy mode. Matching its dispersion relation to the one obtained from the thermodynamics of the 2d charged black hole, we have derived $\eta/s = 1/(4\pi)$ for any charge to mass ratio. It would be interesting to obtain this result from heterotic string theory as well. However naive construction of the heterotic string theory, based on the $\frac{SL(2,R)\times U(1)_x}{U(1)}$ coset model (where $U(1)_x$ is holomorphic, that is a part of internal space from purely bosonic left-moving sector of heterotic string theory), appears to contain $U(1)$ chiral anomaly. Indeed, naively, to construct heterotic string, based on the coset model used in this chapter, one takes the gWZW action (4.11) and adds to it the Dirac term $S_f \simeq \int d^2z \, \mathrm{Tr}\, \tilde{\Psi}(\partial + A)\tilde{\Psi}$, where anti-holomorphic (right-moving) fermions $\tilde{\Psi} \in sl(2, R) \ominus u(1)$ are superpartners of the anti-holomorphic bosonic currents on $SL(2, R)/U(1)$, and A is the $U(1)$ gauge field. Due to such chiral interaction, on the quantum level the anomaly appears, and the theory becomes inconsistent.

This issue was actually resolved in a different heterotic coset stringy realization of the 2d charged black hole [24]. As it was observed there, the chiral anomaly due to fermions should be compensated by the classical anomaly of gWZW action for bosons [39]. In fact, bosonization of the fermions results in the chiral anomaly due to fermions appearing on the classical level, just as in the anomalous gWZW action [40]. Therefore separately the bosonic and fermionic parts of the action are not invariant under $U(1)$ gauge transformation, while their sum is invariant. It is not clear however how these ideas can be directly applied to the model, based on the bosonic action (4.11), which was constructed [25] to be anomaly-free on its own.

4.7 Appendix A: Conventions and review of the gWZW model on the $\frac{SL(2,R)\times U(1)}{U(1)}$

4.7.1 Conventions

Put the string length equal to one, $\alpha' \equiv \frac{\ell_s^2}{2} = \frac{1}{2}$. The contribution to the world-sheet stress-energy tensor, coming from the coordinates $X^\mu(z,\bar{z})$ of the flat subspace of the target space-time, is given by

$$T_{flat}(z) = -\partial X^\mu(z)\partial X_\mu(z)\,, \tag{4.180}$$

and similarly for the anti-holomorphic part $\tilde{T}(\bar{z})$. The Polyakov action is

$$S_P = \frac{1}{2\pi}\int d^2z\,\partial X\bar{\partial}X\,. \tag{4.181}$$

The two-point function is

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\frac{1}{2}\eta^{\mu\nu}\left(\log(z-w)+\log(\bar{z}-\bar{w})\right)\,. \tag{4.182}$$

The Kac-Moody holomorphic (left-moving) and anti-holomorphic (right-moving) currents of the WZW model at level $\hat{k}$ are given by

$$j(z) = j_A t^A = -\frac{\hat{k}}{2}\partial g g^{-1}\,, \qquad \tilde{j}(\bar{z}) = \tilde{j}_A t^A = \frac{\hat{k}}{2}g^{-1}\bar{\partial}g\,. \tag{4.183}$$

Here hermitean generators of a gauge algebra are

$$t^A = j_0^A\,, \qquad [t^A,t^B] = if^{ABC}t^C\,. \tag{4.184}$$

For $SL(2,R)$, which is the group we are interested in, the following expressions in terms of Pauli matrices take place:

$$j_0^A = \frac{1}{2}\sigma^A\,, \qquad f^{ABC} = \epsilon^{ABC}\,, \tag{4.185}$$

and indices are raised and lowered with the help of $\eta_{AB} = \text{diag}\{1, 1, -1\}$. In Euclidean realization of $SL(2,R)$ we put $\eta^{AB} = \delta^{AB}$.

The holomorphic currents of Kac-Moody algebra satisfy the following OPE

$$j^A(z) j^B(w) = \frac{\frac{\hat{k}}{2}\eta^{AB}}{(z-w)^2} + \frac{i f^{ABC}}{z-w} j_C(w)\,, \tag{4.186}$$

and similarly for the anti-holomorphic currents.

The holomorphic component of the stress-energy is given by the Sugawara expression

$$T(z) = \frac{1}{\kappa}\eta_{AB} j^A(z) j^B(z)\,, \tag{4.187}$$

similar expression is true for the anti-holomorphic component. Here

$$\kappa = \hat{k} + c_V\,. \tag{4.188}$$

For $SU(2)$ (and for Euclidean $SL(2,R)$) the index of the adjoint representation is $c_V = 2$ and for $SL(2,R)$ it is $c_V = -2$.

The groundstate representation space of the $SL(2,R)$ currents is formed by the primary fields $V_j(x, \bar{x}; w, \bar{w})$, characterized by the index j. This index determines the value of Casimir operator of $SL(2,R)$. The $(x, \bar{x})$ coordinates can be regarded as the boundary coordinates of the $SL(2,R)$ target space-time, and $(w, \bar{w})$ are world-sheet coordinates. One can replace the boundary coordinates with the numbers $(m, \bar{m})$, defined via transformation

$$V_{j;m,\bar{m}}(w, \bar{w}) = \int d^2x\, x^{j+m} \bar{x}^{j+\bar{m}} V_j(x, \bar{x}; w, \bar{w})\,. \tag{4.189}$$

OPE of $SL(2,R)$ currents and $SL(2,R)$ primaries are [10]

$$J^3(z) V_{j;m,\bar{m}}(w,\bar{w}) = \frac{m}{z-w} V_{j;m,\bar{m}}(w,\bar{w}) + \cdots\,, \tag{4.192}$$

$$J^\pm(z) V_{j;m,\bar{m}}(w,\bar{w}) = \frac{m \mp j}{z-w} V_{j;m\pm 1,\bar{m}}(w,\bar{w}) + \cdots\,.$$

[10] For Euclidean $SL(2,R)$,

$$J^3(z) V_{j;m,\bar{m}}(w,\bar{w}) = \frac{im}{z-w} V_{j;m,\bar{m}}(w,\bar{w}) + \cdots\,. \tag{4.190}$$

Therefore

$$\eta_{33} (J^3)^2(z) V_{j;m,\bar{m}}(w,\bar{w}) = -\frac{m^2}{z-w} V_{j;m,\bar{m}}(w,\bar{w}) \tag{4.191}$$

is true for both Euclidean and Minkowski signatures.

From this one finds how the $SL(2,R)$ currents act on the primaries:

$$J_0^3 \cdot V_{j;m,\bar{m}}(w,\bar{w}) = mV_{j;m,\bar{m}}(w,\bar{w}) \,, \tag{4.193}$$
$$J_0^\pm \cdot V_{j;m,\bar{m}}(w,\bar{w}) = (m \mp j)V_{j;m\pm1,\bar{m}}(w,\bar{w}) \,, \tag{4.194}$$

with all other $J_n^A \cdot V_{j;m,\bar{m}}(w,\bar{w}) = 0$, $n \geq 1$.

Second order $SL(2,R)$ Casimir operator is given by

$$C_2 = \eta_{AB} J_0^A J_0^B \equiv -(J_0^3)^2 + \frac{1}{2}\{J_0^+, J_0^-\} \,. \tag{4.195}$$

Here

$$J_0^1 = \frac{1}{2}(J_0^+ + J_0^-) \,, \qquad J_0^2 = \frac{i}{2}(J_0^- - J_0^+) \,. \tag{4.196}$$

It takes place

$$C_2 \cdot V_j(w,\bar{w}) = -j(j+1)V_j(w,\bar{w}) \,. \tag{4.197}$$

This expression is also true for Euclidean $SL(2,R)$, due to (4.190). Then clearly for $SL(2,R)$ algebra with currents of weight $\hat{k}$,

$$L_0 \cdot V_j(w,\bar{w}) = -\frac{j(j+1)}{\hat{k}-2} V_j(w,\bar{w}) \,, \tag{4.198}$$

which gives the conformal dimension of V_j

$$\Delta_j = -\frac{j(j+1)}{\hat{k}-2} \,. \tag{4.199}$$

In the superstring theory one considers the total bosonic currents J^a, which include contributions from world-sheet fermions, dual to $SL(2,R)$ currents. The level of total $SL(2,R)$ currents is equal to $\hat{k}+2$, if $\hat{k}$ denotes the level of purely bosonic currentt, and therefore the conformal dimension of the $V_{jm\bar{m}}$ is equal to $\Delta_j = -\frac{j(j+1)}{\hat{k}}$.

4.7.2 Gauged WZW model on the $\frac{SL(2,R)\times U(1)}{U(1)}$

Let us review the derivation [25] of the gWZW action on the $\frac{SL(2,R)\times U(1)}{U(1)}$ coset.

Perform the following asymmetric gauging of the $U(1)$ subgroup of $SL(2,R)\times U(1)$ group with the parameter τ:

$$(g,\ x_L,\ x_R) \sim \left(e^{\tau \cos\psi \sigma_3/\sqrt{\hat{k}}} g e^{\tau\sigma_3/\sqrt{\hat{k}}},\ x_L + \tau \sin\psi,\ x_R \right) \,. \tag{4.200}$$

The condition that the gauge transformation leaves the action invariant is

$$\mathrm{Tr}\left(T_L^2 - T_R^2\right) = 0\,, \tag{4.201}$$

where T_L and T_R are the generators of left-moving and right-moving sectors of the gauged $U(1)$ group.

Let us write the element of the $SL(2,R) \times U(1)$ group as

$$G = \begin{pmatrix} g & 0 \\ 0 & \exp\left(\sqrt{\frac{2}{\hat{k}}}\, x\right) \end{pmatrix}\,. \tag{4.202}$$

Then G is a field of the $SL(2,R) \times U(1)$ WZW model at level $\hat{k}$:

$$\begin{aligned} S[G] &= \frac{\hat{k}}{4\pi}[\int d^2z \mathrm{Tr}(G^{-1}\partial G G^{-1}\bar{\partial}G) - \frac{1}{3}\int_B \mathrm{Tr}(G^{-1}dG)^3] \\ &= \frac{\hat{k}}{4\pi}[\int d^2z \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial}g) - \frac{1}{3}\int_B \mathrm{Tr}(g^{-1}dg)^3] \\ &+ \frac{1}{2\pi}\int d^2z \partial x \bar{\partial} x\,. \end{aligned} \tag{4.203}$$

The gauge transformation (4.200) acts on G-field as

$$G \to e^{T_L \tau} G e^{T_R \tau}\,, \tag{4.204}$$

where the generators of left and right sectors of the $u(1)$ algebra are

$$T_L = \begin{pmatrix} \frac{1}{\sqrt{\hat{k}}}\cos\psi\sigma^3 & 0 \\ 0 & \sqrt{\frac{2}{\hat{k}}}\sin\psi \end{pmatrix}\,, \quad T_R = \begin{pmatrix} \frac{1}{\sqrt{\hat{k}}}\sigma^3 & 0 \\ 0 & 0 \end{pmatrix}\,. \tag{4.205}$$

These generators satisfy anomaly-free condition (4.201. Because of this condition is satisfied we can make gauge fields non-dynamical, as it is shown bellow.

Consider compensator fields (gauge field ‘prepotentials’):

$$U = \exp\left(-uT_L\right)\,, \qquad V = \exp\left(-vT_R\right)\,. \tag{4.206}$$

Define gauge transformation of compensator fields as

$$u \to u + \tau\,, \qquad v \to v + \tau\,. \tag{4.207}$$

The combination UGV is clearly invariant under gauge transformations (4.200), and therefore the WZW-action $S[UGV]$ is gauge-invariant.

But it contains terms which are quadratic in derivatives of compensator field u and quadratic in derivatives of compensator field v. Such terms make the compensator (gauge) d.o.f. dynamical, and therefore the theory with the action $S[UGV]$ instead of gauging some degrees of freedom away adds more degrees of freedom.

Therefore let us consider instead the gWZW action

$$S_g = S[UGV] - \frac{1}{2\pi}\int d^2z\partial w\bar{\partial}w\,, \tag{4.208}$$

where we have introduced gauge-invariant field

$$w = u - v\,. \tag{4.209}$$

Due to the Polyakov-Wiegmann indentity

$$S[UGV] = S[G] + S[U] + S[V] \tag{4.210}$$
$$+\frac{\hat{k}}{2\pi}\int d^2z\text{Tr}\left[G^{-1}\bar{\partial}G\partial VV^{-1}+U^{-1}\bar{\partial}U\partial GG^{-1}+U^{-1}\bar{\partial}UG\partial VV^{-1}G^{-1}\right].$$

Here

$$S[U] = \frac{1}{2\pi}\int d^2z\partial u\bar{\partial}u\,,\quad S[V] = \frac{1}{2\pi}\int d^2z\partial v\bar{\partial}v\,, \tag{4.211}$$

and therefore

$$S[U] + S[V] - \frac{1}{2\pi}\int d^2z\partial w\bar{\partial}w = \frac{1}{2\pi}\int d^2z(\partial v\bar{\partial}u + \partial u\bar{\partial}v) \tag{4.212}$$
$$= \frac{1}{\pi}\int d^2zA\bar{A}\,, \tag{4.213}$$

where

$$A = -\partial v\,,\qquad \bar{A} = -\bar{\partial}u\,. \tag{4.214}$$

The action term (4.212) is non-dynamical, as it is expected in gWZW model with asymmetric gauging, satisfying anomaly-free condition (4.201). As a result, the gWZW action on the $\frac{SL(2,R)\times U(1)}{U(1)}$ is given by

$$S_g = S[g] + \frac{1}{2\pi}\int d^2z\partial x\bar{\partial}x \tag{4.215}$$
$$+\frac{1}{2\pi}\int d^2z\left[A\sqrt{\hat{k}}\text{Tr}\,(g^{-1}\bar{\partial}g\sigma^3)+\bar{A}\left(\sqrt{\hat{k}}\text{Tr}(\partial gg^{-1}\sigma^3)\cos\psi+2\sin\psi\partial x\right)+\right.$$
$$\left.+\,A\bar{A}\left(2+\text{Tr}(g^{-1}\sigma^3g\sigma^3)\cos\psi\right)\right].$$

4.8 Appendix B: Heterotic gravity approximation

In the type-II supergravity, considered in the section 5, two gauge fields appear as $G_{x\mu}$ and $B_{x\mu}$ components after Kaluza-Klein reduction of the compact x coordinate. The two-dimensional charged black hole is also a solution [23] of heterotic supergravity equations of motion. In this section we compute graviton and gauge field two-point functions in the two-dimensional charged black hole background in heterotic supergravity. In this case there is just one background gauge field. We solve fluctuation equations of motion for the shear components of graviton and the transverse component of the gauge potential and find one hydrodynamic mode. Matching the obtained dispersion relation with the result obtained in the study of thermodynamics of the 2d charged black hole we derive shear viscosity to entropy ratio for any value of ψ.

The two-loop beta-functions of bosonic fields in heterotic string theory are [41, 23]

$$\beta^G_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu\partial_\nu\Phi - \frac{1}{2}g^{\lambda\rho}F_{\mu\rho}F_{\nu\lambda}\,, \tag{4.216}$$

$$\beta^\Phi = \frac{1}{4}F^2 - R + c + 4(\partial\Phi)^2 - 4\nabla^2\Phi\,, \tag{4.217}$$

$$\beta^A_\nu = g^{\mu\lambda}(\nabla_\mu F_{\nu\lambda} - 2F_{\nu\lambda}\partial_\mu\Phi)\,. \tag{4.218}$$

Corresponding equations of motion, $\beta^{G,B,\Phi} = 0$, have the $CBH \times R^{d-1}$ solution,

$$g_{\mu\nu} = \text{diag}\{-f(r), 1/f(r), 1, ..., 1\}\,, \tag{4.219}$$
$$f(r) = 1 - 2Me^{-Qr} + q^2e^{-2Qr}\,,$$

$$\Phi = \Phi_0 - \frac{Qr}{2}\,, \qquad F_{tr} = F(r) = \sqrt{2}Qqe^{-Qr}\,. \tag{4.220}$$

Here $Q = 2/\sqrt{k}$.

Consider fluctuations $h_{\mu\nu}$, a_μ and φ around this solution. Use the diffeomorphism invariance to fix $h_{\mu r} = 0$. Among $d+1$ space-time coordinates we have t, r coordinates of CBH and $d-1$ flat coordinates. Let us consider $CBH \times R^2$. Choose X to be the R^2 direction of propagation of excitations (with momentum p) and choose Y to be the R^2 direction, transverse to the direction of propagation of excitations. Fluctuations depend on t, r, X. The dependence on t and X in momentum representation boils down to the factor $e^{-i\omega t+ipX}$.

Plugging $g_{\mu\nu}+h_{\mu\nu}$, $A_\mu+a_\mu$ and $\Phi+\varphi$ with the most general fluctuations, we obtain shear channel expressions (prime denotes differentiation w.r.t. r)

$$\beta^G_{rY}=\frac{ie^{-i\omega t+ipX}}{2f}\left(\omega h'_{tY}+pfh'_{XY}-\omega Fa_Y\right), \tag{4.221}$$

$$\beta^G_{XY}=e^{-i\omega t+ipX}\left(-\frac{1}{2f}(\omega ph_{tY}+\omega^2h_{XY}+ff'h'_{XY}+f^2h''_{XY})\right. \tag{4.222}$$
$$\left.+\,f\Phi'h'_{XY}\right),$$

$$\beta^G_{tY}=\frac{e^{-i\omega t+ipX}}{2}\left(p^2h_{tY}+\omega ph_{XY}-fh''_{tY}+2f\Phi'h'_{tY}+fFa'_Y\right). \tag{4.223}$$

The equations of motion in shear channel are therefore

$$\omega h'_{tY}+pfh'_{XY}-\omega Fa_Y=0\,, \tag{4.224}$$

$$\omega ph_{tY}+\omega^2h_{XY}+ff'h'_{XY}+f^2h''_{XY}-2f^2\Phi'h'_{XY}=0\,, \tag{4.225}$$

$$p^2h_{tY}+\omega ph_{XY}-fh''_{tY}+2f\Phi'h'_{tY}+fFa'_Y=0\,. \tag{4.226}$$

Consider diff-invariant field

$$Z=\omega h_{XY}+ph_{tY}\,. \tag{4.227}$$

Using (4.227) and (4.224) express

$$h'_{tY}=\frac{\omega^2Fa_Y-pfZ'}{\omega^2-p^2f}\,,\quad h'_{XY}=\omega\frac{Z'-pFa_Y}{\omega^2-p^2f}\,. \tag{4.228}$$

The equations (4.225) and (4.226) after one substitutes (4.228) into them, both give rise to the same equation (due to $F'-2F\Phi'=0$)

$$Z''+\left(\frac{\omega^2f'}{f(\omega^2-p^2f)}-2\Phi'\right)Z'+\frac{\omega^2-p^2f}{f^2}Z \tag{4.229}$$
$$-\frac{pF}{f}\left(fa'_Y+\frac{\omega^2f'}{\omega^2-p^2f}a_Y\right)=0\,.$$

Compute the beta-function for gauge field fluctuation a_Y (choose the gauge $a_r = 0$)

$$\beta^A_Y = -\, e^{-i\omega t + ipX} f \left(a''_Y + \frac{f' - 2f\Phi'}{f} a'_Y + \frac{\omega^2 - p^2 f}{f^2} a_Y - \frac{F h'_{tY}}{f} \right) . \quad (4.230)$$

Express h'_{tY} using (4.228). The equation on a_Y is then

$$a''_Y + \frac{f' - 2f\Phi'}{f} a'_Y + \left(\frac{\omega^2 - p^2 f}{f^2} - \frac{\omega^2 F^2}{f(\omega^2 - p^2 f)} \right) a_Y + \frac{pFZ'}{\omega^2 - p^2 f} = 0 . \quad (4.231)$$

Before proceeding, rescale

$$\mathbf{r} = rQ\,, \quad \mathbf{w} = \omega/Q\,, \quad \mathbf{p} = p/Q\,, \quad \mathbf{Z} = Z/Q\,. \quad (4.232)$$

The dependence on Q disappears from both fluctuation equations, and due to (4.220) we obtain

$$f = 1 - 2Me^{-\mathbf{r}} + q^2 e^{-2\mathbf{r}}\,, \quad \Phi = \Phi_0 - \frac{\mathbf{r}}{2}\,. \quad (4.233)$$

Introduce new radial coordinate $u = e^{\mathbf{r}}$. Then inner and outer horizons are located at

$$u_\pm = M \pm \sqrt{M^2 - q^2}\,. \quad (4.234)$$

The equations of motion become (substitute $q = \sqrt{u_+ u_-}$)

$$\frac{d^2 a}{du^2} + \left(\frac{1}{u - u_-} + \frac{1}{u - u_+} \right) \frac{da}{du} + \frac{1}{(u-u_-)(u-u_+)} \times \left(\frac{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u-u_-)(u-u_+)}{(u-u_-)(u-u_+)} - \frac{2u_+u_-\mathbf{w}^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2(u-u_-)(u-u_+)} \right) a + \frac{\mathbf{p}\sqrt{2u_+u_-}}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u-u_-)(u-u_+)} \frac{d\mathbf{Z}}{du} = 0\,, \quad (4.235)$$

$$\frac{d^2 \mathbf{Z}}{du^2} + \frac{1}{u} \left(2 + \frac{\mathbf{w}^2 u^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u-u_-)(u-u_+)} \left(\frac{u_+}{u-u_+} + \frac{u_-}{u-u_-} \right) \right) \frac{d\mathbf{Z}}{du} - \frac{\mathbf{p}\sqrt{2u_+u_-}}{u^2} \frac{da}{du} + \frac{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u-u_-)(u-u_+)}{(u-u_-)^2 (u-u_+)^2} \mathbf{Z} - \frac{\mathbf{p}\sqrt{2u_+u_-}}{u} \frac{\mathbf{w}^2}{\mathbf{w}^2 u^2 - \mathbf{p}^2 (u-u_-)(u-u_+)} \left(\frac{u_+}{u-u_+} + \frac{u_-}{u-u_-} \right) a = 0\,. \quad (4.236)$$

In the near horizon limit $v = u - u_+ \ll 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2a}{dv^2} + \frac{1}{v}\frac{da}{dv} + \frac{\mathbf{w}^2 u_+^2}{(u_+ - u_-)^2 v^2} a = 0\,, \tag{4.237}$$

$$\frac{d^2\mathbf{Z}}{dv^2} + \frac{1}{v}\frac{d\mathbf{Z}}{dv} + \frac{\mathbf{w}^2 u_+^2}{(u_+ - u_-)^2 v^2} \mathbf{Z} = 0\,. \tag{4.238}$$

The incoming-wave solutions are

$$a_Y(u) = C_1(u - u_+)^{\frac{-i\mathbf{w}u_+}{u_+ - u_-}}\,, \quad \mathbf{Z}(u) = C_2(u - u_+)^{\frac{-i\mathbf{w}u_+}{u_+ - u_-}}\,. \tag{4.239}$$

In the asymptotic region $u \gg 1$ equations (4.235) and (4.236) give rise to

$$\frac{d^2a}{du^2} + \frac{2}{u}\frac{da}{du} + \frac{\mathbf{w}^2 - \mathbf{p}^2}{u^2} a = 0\,, \tag{4.240}$$

$$\frac{d^2\mathbf{Z}}{du^2} + \frac{2}{u}\frac{d\mathbf{Z}}{du} + \frac{\mathbf{w}^2 - \mathbf{p}^2}{u^2} \mathbf{Z} = 0\,, \tag{4.241}$$

with the solution

$$a_Y = \mathring{A}_a u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})} + \mathcal{B}_a u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}\,, \tag{4.242}$$

$$\mathbf{Z} = \mathring{A}_Z u^{\frac{1}{2}(-1+\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})} + \mathcal{B}_Z u^{\frac{1}{2}(-1-\sqrt{1+4(\mathbf{p}^2-\mathbf{w}^2)})}\,. \tag{4.243}$$

We solve numerically the equations (4.235), (4.236) with boundary conditions (4.239). Zeroes of determinant of the leading behavior coefficients matrix

$$\mathring{A}_Z^{(1)}\mathring{A}_a^{(2)} - \mathring{A}_Z^{(2)}\mathring{A}_a^{(1)} \tag{4.244}$$

are located at

$$\mathbf{w} = -i\mathbf{p}^2\cos^2(\psi/2)\,. \tag{4.245}$$

Due to (4.232) and $Q = 2/\sqrt{\hat{k}}$ from (4.245) it follows

$$\omega = -i\frac{\sqrt{\hat{k}}\cos^2(\psi/2)}{2}p^2\,. \tag{4.246}$$

Finally, matching the dispersion relation (4.246) to the dispersion relation (4.10), obtained in Section 2 from the consideration of thermodynamics of the 2d charged black hole, we conclude

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{4.247}$$

is valid for any value of ψ, and due to $q = M\sin\psi$ it is valid for any charge density.

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Chapter 5

Discussion

In this concluding chapter we are going to look at the problems solved in this thesis in the general context of the AdS/CFT correspondence and strongly coupled quantum field theories. We are going to discuss strongly coupled IR phases of a non-abelian gauge quantum field theory with the fundamental matter. Consider $SU(N_c)$ gauge theory coupled to N_f flavors of quarks and N_f flavors of anti-quarks. Let us start in the UV where the gauge interaction is weak and one can rely on perturbation theory to compute the beta function for the gauge coupling. If $N_f/N_c < 11/2$ then the gauge coupling grows as one moves towards the IR, *i.e.*, the theory is asymptotically free. If $N_f/N_c = 11/2 - \epsilon$ and $\epsilon \ll 1$, then the theory flows to a perturbative IR fixed point. It is called a Banks-Zaks fixed point.

The question one can ask is whether there is a finite range of values of $x = N_f/N_c$ for which the IR theory is in a non-abelian Coulomb phase. In real-world QCD we have $N_c = 3$ and $N_f = 6$, so that $x = 2$. The IR phase of this QCD is confining. It has been suggested in the literature that for $x_* < x < 11/2$ QCD flows to the IR fixed point, with $x_* \simeq 4$. It is a well-known non-perturbative problem to find the value of x_*, and the exact solution to this problem of a conformal window has not yet been found.

However the solution is known for the supersymmetric QCD. Let us promote the gauge boson to the $\mathcal{N} = 1$ gauge supermultiplet, and promote the quarks to the chiral superfields. The superpartner of the gluon is gluino, which is a Weyl fermion, and the superpartners of the quarks are squarks, each squark is a complex-valued scalar. Therefore we have a vector superfield, N_f chiral superfields and N_f chiral anti-superfields. The resulting theory is asymptotically free when $x = N_f/N_c < 3$. This value differs from the value $11/2$ for a non-supersymmetric QCD because now we have more fields with the gauge group charge.

First observation to make is that in the $\mathcal{N} = 1$ supersymmetric gauge theory the conformal symmetry group $SO(2,4)$ is enhanced to the superconformal symmetry group $SU(2,2|1)$. This allows one to formulate the requirement of unitarity of a conformal field theory in terms of the bound on the R-charge, which is the charge with respect to the $U(1)$ subgroup of the $SU(2,2|1)$ group. The R-charge is expressed in terms of N_f and N_c. Using these facts one derives that a unitary superconformal fixed point in the IR exists when $3/2 < x < 3$, *i.e.*, the lower edge of the conformal window is located at $x_* = 3/2$. In fact in the $\mathcal{N} = 1$ supersymmetric field theory when $N_f < 3N_c/2$ but $N_f > N_c + 1$, a unitary superconformal fixed point exists, but to describe it one needs to switch to a Seiberg-dual magnetic theory, which is IR-free in this region.

Perhaps new methods for solving the problem of a conformal window in a non-supersymmetric QCD are needed. The AdS/CFT correspondence can be suitable for this purpose. In fact in the literature there exist holographic solutions to the problem of a conformal window for QCD in a Veneziano limit, $N_f, N_c \gg 1$, $N_f/N_c =$ finite. The idea of the solution is that if one starts at $x = 11/2$ and decreases the value of x, then at some point $x = x_*$ conformal phase transition takes place, which can be seen in the dual gravity theory in AdS. It has been suggested that as the field theory passes through the point of conformal phase transition, the scalar field in AdS (playing the role of an order parameter for the phase transition) develops a non-trivial profile. So the problem of a conformal window is reformulated holographically as the question of when a non-trivial configuration is a preferred state for this scalar field. Unfortunately the known holographic solutions are far from being rigorous, and are probably good only on a qualitative level. One of the principal obstacles to solving the problem of a conformal window holographically is that one needs to consider string theory in AdS space instead of its supergravity approximation.

Let us fix N_f and N_c such that the IR theory is confining, like real-world QCD. Up till now we have been considering QCD at vanishing temperature T and vanishing chemical potential μ. One can ask what happens as the values of T and μ are increased. This is the problem of the (T, μ) phase diagram of QCD. Although the precise form of the phase diagram of QCD is not known it is known that at large values of temperature or chemical potential conformal symmetry is restored, and the system of quarks goes into either color superconducting phase or quark-gluon plasma phase. We studied a similar problem in chapter 3, describing the quark bi-linear operator by the dual tachyon field in AdS, with the dynamics defined by the tachyon Dirac-Born-Infeld action. When the values of the temperature or the chemical potential are large enough the preferred tachyon state in AdS is trivial: the tachyon identically vanishes. However we have found that as one moves towards the origin of the (T, μ) plane the tachyon prefers to be in the state with a non-trivial profile. For the dual field theory it means that the bi-linear quark operator develops expectation value. It is a second-order phase transition at small values of temperature and a first-order phase transition at small values of chemical potential.

In this thesis we have also studied conformal phase transition which takes place in a strongly coupled quantum field theory conformal in its single-trace sector. This means that the coupling constants of the single-trace gauge-invariant operators (including the gauge coupling) do not run and reside at their fixed points, while the coupling constants of the double-trace, and more generally the multi-trace operators, run along the RG flow. In chapter 3 we studied such a quantum field theory in a Wilsonian holographic renormalization framework. In a Wilsonian holographic renormalization one integrates out a part of the AdS space between the boundary of AdS and the surface located at the fixed radial coordinate $z = b$ of AdS. Integrated out geometry is encoded in the effective boundary action defined at $z = b$. In the model considered in chapter 3 we introduced the tachyon field in AdS, with the dynamics determined by the tachyon-DBI action, dual to a single trace operator in the boundary field theory. We applied holographic renormalization to the tachyon DBI action and derived the effective holographic Wilsonian boundary action. This action satisfies the equation similar to the Callan-Symanzik equation, from which one can determine the running of the multi-trace coupling constants. The question we asked in chapter 3 was whether there exists a fixed point for the beta function of the double-trace coupling constant and when does this fixed point disappear and conformal phase transition takes place. We have found that conformal phase transition occurs when the mass of the tachyon crosses the Breitenlohner-Freedman bound. It is an infinite-order BKT type of phase transition, with an order parameter of the phase transition being a dynamically generated mass scale.

We have reviewed three kinds of phase transitions in a non-abelian gauge theories: the lower edge of a conformal window, the finite temperature/chemical potential and the running of the multi-trace coupling constants; all occurring between symmetric and massive phases. The AdS/CFT correspondence makes it possible to study these phase transitions, with various degrees of rigor and various assumptions and simplifications being made. It would be interesting to construct more rigorous holographic models of conformal phase transition at the lower edge of a conformal window, in particular for the $\mathcal{N} = 1$ supersymmetric gauge theories. Models like that exist in the literature, but the precise theory still remains to be found.

In the holographic examples described above the fermionic bi-linear operator on the boundary field theory was described by the scalar (tachyon) field in the AdS. In chapter 3 we used the tachyon-DBI action to describe dynamics of the tachyon field. Tachyon-DBI action appears in string theory as the low-energy action describing the embedding profile of a probe D-brane. In chapter 3 we considered various potentials for the tachyon field, some of which have been taken from the Dp-brane DBI action ('hard-wall' potential). We have also studied the tachyon DBI action with the generic 'soft-wall' tachyon potential. For a gravitational theory to be physically consistent it must be embeddable into string theory. In chapter 2 we considered the explicit string theory holographic realization of the matter degrees of freedom. Matter fields of a quantum field theory are massless modes of an open strings which stretch between the $D3$-branes, creating the $AdS_5 \times S^5$ background, and a probe branes in this background. Matter is described holographically by the dynamics of the probe branes. Such models, both supersymmetric and non-supersymmetric, have been extensively studied in the literature. The main focus of chapter 2 was the study of the probe brane matter in the background of a constant magnetic field.

Majority of quantum fields theories which have been studied by the methods of AdS/CFT correspondence are taken at an infinite number of degrees of freedom. This is a direct consequence of the fact that $SU(N)$ gauge theory is dual to a classical (supergravity) theory in AdS only when N is large. Any finite N means that quantum corrections in the bulk should be taken into account. At the same time it is known that interaction between a gauge-invariant IR operators is subleading in the $1/N$ expansion, this phenomenon is known as large-N factorization. In chapter 4 we studied holographically a quantum field theory dual to a two-dimensional charged black hole. It can be considered as Little String Theory at a finite density, the latter is defined as the low-energy limit of the $SU(N)$ gauge theory on the world-volume of an $NS5$-branes at a vanishing string coupling. Such a theory can be described holographically. When the N is large, the supergravity approximation of the bulk dynamics is valid (compare with the AdS/CFT correspondence, in which case the supergravity approximation is valid for the large 't Hooft coupling). String theory in the two-dimensional charged black hole background is exactly solvable, and therefore we do not have to resort to the supergravity approximation, which means we can describe holographically a finite-density matter at a finite N.

Results of chapter 4 give rise to the following problem. In chapter 4 we considered type-II superstring theory in a two-dimensional charged black hole background. We have found that the dual field theory behaves as a sum of two non-interacting fluids. Each fluid supports a gapless excitation, and we have verified that at low frequencies and momenta the dispersion relations of these excitations coincide with the dispersion relations obtained in the supergravity approximation. One might wonder whether there exists a theory in a two-dimensional charged black hole background which describes a hydrodynamics of a field theory at a finite density and a finite N. It is known that one can consider classical heterotic string theory in a coset realization of the two-dimensional charged black hole. In chapter 4 we studied heterotic gravity in the two-dimensional charged black hole and found out that the dual field theory is described by hydrodynamics, by deriving a dispersion relation of the diffusion mode in the shear channel. It would be interesting to see whether this dispersion relation is corrected in heterotic string theory, and whether the viscosity over entropy ratio gets stringy corrections. However as one tries to answer these questions the problem comes up. One needs to perform a first quantization and construct a spectrum of heterotic string in a two-dimensional charged black hole. This geometry is represented as a coset over $U(1)$ group. As one tries to construct quantum spectrum of this theory chiral anomaly appears because in heterotic string theory only the right-handed fermions couple to the $U(1)$ gauge field. Usually in string theory on coset space, described by a gauged WZW model, a chiral anomaly of fermions is compensated by a classical anomaly of bosons (which appears due to the asymmetric gauging). It would be interesting to find how this problem is resolved for heterotic string in the two-dimensional charged black hole.

There are numerous applications of the AdS/CFT correspondence to strongly coupled systems which we did not have a chance to discuss in this thesis. Since the AdS/CFT correspondence sometimes turns out to be the only available analytical tool for the study of the low-energy strongly coupled systems we advocate that a lot of effort should be invested into it.

Samenvatting

De AdS/CFT (Anti-de Sitter/conforme veldentheorie) correspondentie is een veelbelovende aanpak om de lage-energie-fasen van sterk gecorreleerde materie te onderzoeken. Eerder is al aangetoond dat deze correspondentie een kwalitatief goede beschrijving kan geven van onder andere confinement en chirale symmetrie-breking in kwantum chromodynamica (QCD) modellen, supergeleiding en de vorming van Fermi oppervlakken in sterk gekoppelde gecondenseerde materie-systemen. In dit proefschrift passen we holografische methodes toe om de eigenschappen van lage-energie fysica te bestuderen.

Allereerst beschouwen we in hoofdstuk 2 een eindige dichtheid van quarks, die holografisch beschreven worden door een als sonde fungerende braan in Anti-de Sitter ruimte met een niet-triviaal ijkveld op de achtergrond van zijn wereldvolume. Daarmee reproduceren we het holografische nulgeluid, dat aanwezig is in het longitudinale kanaal van de stroom-stroom correlatie functie. Dit resultaat hebben we veralgemeniseerd naar gevallen met een magnetisch veld op de achtergrond. Zo'n veld veroorzaakt een kloof in de dispersie van het nulgeluid, met een breedte evenredig met de grootte van het magnetisch veld zolang dit veld klein is. In de afwezigheid van een magnetisch veld vertoont de tweepunts-correlatiefunctie van de transversale stroom een niet-triviale afhankelijkheid van de impuls, wat de aanwezigheid van collectieve excitaties aantoont.

In hoofdstuk 3 bestuderen we de klassieke dynamica van het tachyonveld in een AdS ruimte, beschreven door de tachyon-Dirac-Born-Infeld (DBI) actie. Door de introductie van een zwart gat in de AdS ruimte en een niet-nul achtergronds-ijkveld verkrijgen we een holografisch model dat conforme symmetrie-breking in een sterk gekoppeld systeem met eindige temperatuur en ladingsdichtheid beschrijft. Het fasediagram als functie van temperatuur en chemische potentiaal vertoont gelijkenissen met het fasediagram van QCD. De meeste modellen uit hoofdstuk 3 zijn echter fenomenologisch van aard, aangezien we niet de precieze snaartheoretische vorm van de tachyon potentiaal kennen. Met tachyonen in de AdS-ruimte kunnen we ook dynamische chirale en elektrozwakke symmetrie-breking van lopende technicolor theorieën modelleren. De overeenkomstige S-parameter van de techni-quarks die we afleiden uit het holografische model van de tachyon DBI-actie is positief, en wordt niet nul voor alle tachyon-potentialen die aan bod komen in hoofdstuk 3. Daarnaast bevat hoofdstuk 3 de observatie dat het tachyon-DBI model gebruikt kan worden om een conforme faseovergang naar theorieën met lopende koppelingen, die conform zijn in hun enkelspoorsector, te beschrijven.

In hoofdstuk 4 geven wij de exacte snaartheoretische beschrijving van een kwantumveldentheorie bij eindige temperatuur en ladingsdichtheid. Het voordeel van zo'n beschrijving is dat we daarmee kwantumveldentheorieën kunnen beschrijven met een eindig aantal vrijheidsgraden, zonder de in de holografische beschrijving veelgebruikte grote-N factorisatie in te zetten. De achtegrond binnen in de ruimte bestaat uit een zwarte braan met een niet-triviale ijkveldflux. Deze braan is verkregen als direct product van een tweedimensionaal geladen zwart gat en de vlakke ruimte. Het voordeel van een tweedimensionaal geladen zwart gat is dat er een bekende $SL(2,R)$-coset realisatie van bestaat, en het overeenkomstige Wess-Zumino-Witten model is exact oplosbaar. We hebben de vertex-operatoren van de massaloze Neveu-Schwarz-Neveu-Schwarz toestanden geconstrueerd, die bestaan uit bosonische zwaartekrachtsmultipletten, waarbij we de tweepuntsfuncties van deze vertex-operatoren hebben gevonden. Van de polen van deze tweepuntsfuncties leiden we de dispersie van de lage-energiemodi af. Het blijkt dat de theorie van superzwaartekracht exact hetzelfde resultaat geeft als snaartheorie. We concluderen dat het systeem zich gedraagt als de som van twee niet-wisselwerkende vloeistoffen. In hoofdstuk 4 bestudeerden we ook de heterotische zwaartekracht in de zwarte braanachtergrond met een ijkveldflux. Het lage-energie spectrum van dit model kan beschreven worden met hydrodynamica.

Tenslotte vatten we in hoofdstuk 5 onze resultaten samen, die we dan plaatsen in de algemene context van de AdS/CFT correspondentie.

Summary

The AdS/CFT correspondence is a powerful approach to problems of strongly coupled low-energy phases of matter. It has been proving to be efficient at giving a qualitative description of the phenomena such as confinement and chiral symmetry breaking in QCD-like models, and superconductivity and Fermi surfaces in strongly-coupled condensed matter systems. In this thesis we apply the methods of holography to find out properties of low-energy physics.

We begin in chapter 2 by considering a finite-density system of quarks, realized holographically by a probe brane in Anti-de Sitter (AdS) space, with a non-trivial gauge field background on its world-volume. We reproduce the holographic zero-sound in the longitudinal channel of the current-current correlation function. We generalize this result to the case of a non-vanishing background magnetic field. This field leads to a gap in the zero-sound mode, which scales proportionally to the magnitude of the field when it is small. At vanishing magnetic field the two-point correlation function of the transverse current component exhibits a non-trivial momentum-independent structure, signaling the presence of collective excitations in the system.

In chapter 3 we study the classical dynamics of the tachyon field in an AdS background described by the tachyon-Dirac-Born-Infeld (DBI) action. By considering a black hole in AdS space and switching on a non-vanishing background gauge field we obtained a holographic model of conformal symmetry breaking in a strongly coupled system at finite temperature and charge density. The resulting phase diagram in the temperature-chemical potential plane is reminiscent of the phase diagram of QCD. Most of the models in chapter 3 are phenomenological since we do not know the precise string-theoretic form of the tachyon potential. The tachyon in AdS space also models dynamical chiral and electro-weak symmetry breaking of walking technicolor models. The corresponding S-

parameter of techni-quarks derived holographically from the tachyon-DBI action is positive-valued and does not vanish for the tachyon potentials considered in chapter 3. Another observation made in chapter 3 is that the tachyon-DBI model can be used to describe conformal phase transitions to a walking region in theories conformal in their single-trace sector.

In chapter 4 we provide the exact string theoretic description of a quantum field theory at finite temperature and charge density. The advantage of such a description is that it allows to consider QFT with a finite number of degrees of freedom, therefore avoiding the large-N factorization that is almost always used in systems that are studied holographically. The bulk background is a black brane with a non-trivial gauge field flux. It is obtained as a direct product of the two-dimensional charged black hole and a flat space. The advantage of the two-dimensional charged black hole is that there is a known coset $SL(2,R)$-based realization of it, and the corresponding gauged Wess-Zumino-Witten model is exactly solvable. We construct vertex operators of the massless Neveu-Schwarz-Neveu-Schwarz states, which comprise the bosonic gravity multiplet, and find two-point functions of these vertex operators. From the poles of these two-point functions we infer dispersion relations of the low-energy modes. It turns out that supergravity gives exactly the same result for these dispersion relations as string theory. We conclude that the system behaves as a sum of two non-interacting fluids. In chapter 4 we study heterotic gravity in the black brane background with a gauge field flux. The resulting low-energy spectrum is described by hydrodynamics.

Finally, in chapter 5 we summarize our results and consider them in the general context of the AdS/CFT correspondence.

List of Publications

1. M. Goykhman and E. Ivanov, "Worldsheet Supersymmetry of Pohlmeyer-Reduced $AdS_n x S^n$ Superstrings," JHEP **1109** (2011) 078 [arXiv:1104.0706 [hep-th]].

2. M. Goykhman, A. Parnachev and J. Zaanen, "Fluctuations in finite density holographic quantum liquids," JHEP **1210** (2012) 045 [arXiv:1204.6232 [hep-th]].

3. M. Goykhman, E. Ivanov and S. Sidorov, "Super Landau Models on Odd Cosets," Phys. Rev. D **87** (2013) 025026 [arXiv:1208.3418 [hep-th]].

4. M. Goykhman and A. Parnachev, "S-parameter, Technimesons, and Phase Transitions in Holographic Tachyon DBI Models," Phys. Rev. D **87** (2013) 026007 [arXiv:1211.0482 [hep-th]].

5. M. Goykhman and A. Parnachev, "Stringy holography at finite density," Nucl. Phys. B **874** (2013) 115 [arXiv:1304.4496 [hep-th]].

6. R. A. Davison, M. Goykhman and A. Parnachev, "AdS/CFT and Landau Fermi liquids," *to appear in JHEP* [arXiv:1312.0463 [hep-th]].

Curriculum Vitæ

During the years 2005-2009 I was a bachelor student at the Department of General and Applied Physics of the Moscow Institute of Physics and Technnology (MIPT) in Dolgoprudny, Russia. I continued at the MIPT as a master student from 2009 till 2011. During the last three years of my stay at MIPT I spent a significant period of time at the University Center of the Joint Institute for Nuclear Research in Dubna, the base institute of our high-energy group. My bachelor and master research, dedicated to one- and two-dimensional supersymmetric sigma models, has been done there under the supervision of Prof. Dr. Evgeny Ivanov.

Shortly after my graduation from MIPT I moved to Leiden, The Netherlands to work with Dr. Andrei Parnachev on applications of string theory to strongly coupled systems. I became a PhD student at Lorentz Institute at Leiden University. I benefited a lot from collaboration and discussions with members of the Quantum Matter Theory group.

During my PhD study I attended many schools, including two annual DRSTP schools on high-energy and condensed-matter physics. I gave talks about my work at several schools, seminars and conferences in Russia, Belgium, South Korea, Crete and The Netherlands. After my PhD I will continue to do research, as a postdoc at the University of Chicago.