



Universiteit
Leiden
The Netherlands

Quantum computation with Majorana zero modes in superconducting circuits

Heck, B. van

Citation

Heck, B. van. (2015, May 6). *Quantum computation with Majorana zero modes in superconducting circuits*. *Casimir PhD Series*. Retrieved from <https://hdl.handle.net/1887/32939>

Version: Not Applicable (or Unknown)

License: [Leiden University Non-exclusive license](#)

Downloaded from: <https://hdl.handle.net/1887/32939>

Note: To cite this publication please use the final published version (if applicable).

Cover Page



Universiteit Leiden



The handle <http://hdl.handle.net/1887/32939> holds various files of this Leiden University dissertation.

Author: Heck, Bernard van

Title: Quantum computation with Majorana modes in superconducting circuits

Issue Date: 2015-05-06

Chapter 3

Coulomb-assisted braiding of Majorana modes in a Josephson junction array

Non-Abelian anyons have a topological charge that provides a nonlocal encoding of quantum information [6]. In superconducting implementations [45, 46] the topological charge equals the electrical charge modulo $2e$, shared non-locally by a pair of mid-gap states called Majorana modes [26]. This mundane identification of topological and electrical charge by no means diminishes the relevance for quantum computation. To the contrary, it provides a powerful way to manipulate the topological charge through the well-established sub- e charge sensitivity of superconducting electronics [28, 59].

Following this line of thought, a hybrid device called a *top-transmon* was recently proposed, which combines the adjustable charge sensitivity of a superconducting charge qubit (the *transmon* [37, 60]) to read out and rotate a topological (*top*) qubit [61]. A universal quantum computer with highly favorable error threshold can be constructed [62] if these operations are supplemented by the braiding of Majorana modes, which is a non-Abelian operation on the degenerate ground state [14, 27].

Here we show how Majorana modes can be braided by means of charge-sensitive superconducting electronics. (Braiding was not implemented in Ref. [61] nor in other studies of hybrid topological/non-topological superconducting qubits [35, 63–66].) We exploit the fact that the charge-sensitivity can be switched on and off *with exponential accuracy* by varying the magnetic flux through a split Josephson junction [60]. This provides a macroscopic handle on the Coulomb interaction of pairs of Majorana modes, which makes it possible to transport and exchange them in a Josephson junction array.

We compare and contrast our approach with that of Sau, Clarke, and Tewari, who showed (building on the work of Alicea et al. [21]) how non-Abelian braiding statistics

could be generated by switching on and off the tunnel coupling of adjacent pairs of Majorana modes [67]. The tunnel coupling is controlled by a gate voltage, while we rely on Coulomb interaction controlled by a magnetic flux. This becomes an essential difference when electric fields are screened too strongly by the superconductor to be effective. (For an alternative non-electrical approach to braiding, see Ref. [68].)

The basic procedure can be explained quite simply, see Sec. 3.2, after the mechanism of the Coulomb coupling is presented in Sec. 3.1. We make use of two more involved pieces of theoretical analysis, one is the derivation of the low-energy Hamiltonian of the Coulomb coupled Majorana modes (using results from Refs. [36, 69]), and the other is the calculation of the non-Abelian Berry phase [20] of the exchange operation. To streamline the chapter the details of these two calculations are given in Appendices.

3.1 Majorana-Coulomb Hamiltonian

3.1.1 Single island

The basic building block of the Josephson junction array is the Cooper pair box [70], see Fig. 3.1, consisting of a superconducting island (capacitance C) connected to a bulk (grounded) superconductor by a split Josephson junction enclosing a magnetic flux Φ . The Josephson energy E_J is a periodic function of Φ with period $\Phi_0 = h/2e$. If the two arms of the split junction are balanced, each with the same coupling energy E_0 , the Josephson energy

$$E_J = 2E_0 \cos(\pi\Phi/\Phi_0) \quad (3.1)$$

varies between 0 and $2E_0 > 0$ as a function of $|\Phi| < \Phi_0/2$.

When the island contains no Majorana modes, its Hamiltonian has the usual form [40]

$$H = \frac{1}{2C}(Q + q_{\text{ind}})^2 - E_J \cos \phi, \quad (3.2)$$

in terms of the canonically conjugate phase ϕ and charge $Q = -2ei d/d\phi$ of the island. The offset q_{ind} accounts for charges on nearby gate electrodes. We have chosen a gauge such that the phase of the pair potential is zero on the bulk superconductor.

A segment of a semiconductor nanowire (typically InAs) on the superconducting island can have Majorana mid-gap states bound to the end points [45, 46]. For N segments there can be $2N$ Majorana modes on the island. They have identical creation and annihilation operators $\gamma_n = \gamma_n^\dagger$ satisfying

$$\gamma_n \gamma_m + \gamma_m \gamma_n = 2\delta_{nm}. \quad (3.3)$$

The topological charge of the island equals the fermion parity

$$\mathcal{P} = i^N \prod_{n=1}^{2N} \gamma_n. \quad (3.4)$$

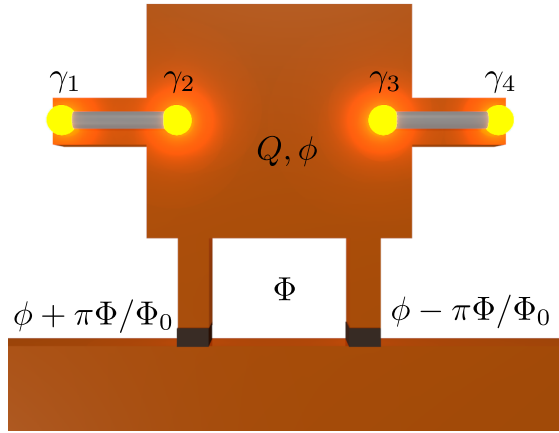


Figure 3.1: Cooper pair box, consisting of a superconducting island (brown) connected to a bulk superconductor by a split Josephson junction (black, with the gauge-variant phase differences indicated). The island contains Majorana modes (yellow) at the end points of a nanowire (grey). These are coupled by the Coulomb charging energy, tunable via the flux Φ through the Josephson junction.

The eigenvalues of \mathcal{P} are ± 1 , depending on whether there is an even or an odd number of electrons on the island.

The Majorana operators do not enter explicitly in H , but affect the spectrum through a constraint on the eigenstates [36],

$$\Psi(\phi + 2\pi) = (-1)^{(1-\mathcal{P})/2} \Psi(\phi). \quad (3.5)$$

This ensures that the eigenvalues of Q are even multiples of e for $\mathcal{P} = 1$ and odd multiples for $\mathcal{P} = -1$. Since \mathcal{P} contains the product of all the Majorana operators on the island, the constraint (3.5) effectively couples distant Majorana modes — without requiring any overlap of wave functions.

We operate the Cooper pair box in the regime that the Josephson energy E_J is large compared to the single-electron charging energy $E_C = e^2/2C$. The phase ϕ (modulo 2π) then has small zero-point fluctuations around the value $\phi_{\min} = 0$ which minimizes the energy of the Josephson junction, with occasional 2π quantum phase slips.

In Appendix 3.A we derive the effective low-energy Hamiltonian for $E_J \gg E_C$,

$$H_{\text{eff}} = -E_J + \sqrt{2E_C E_J} - U \mathcal{P}, \quad (3.6)$$

$$U = 16(E_C E_J^3 / 2\pi^2)^{1/4} e^{-\sqrt{8E_J/E_C}} \cos(\pi q_{\text{ind}}/e). \quad (3.7)$$

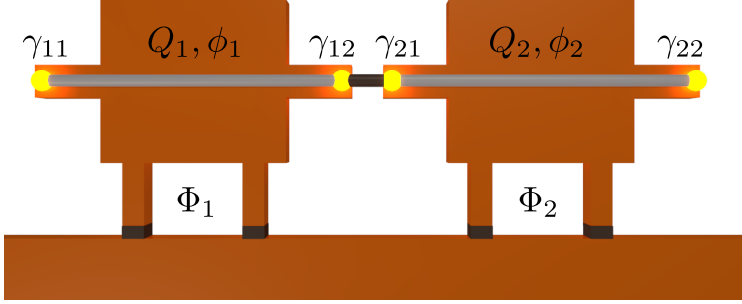


Figure 3.2: Two Cooper pair boxes, each containing a pair of Majorana modes. Single electrons can tunnel between the superconducting islands via the overlapping Majorana's γ_{12} and γ_{21} . This tunnel coupling has a slow (cosine) dependence on the enclosed fluxes, while the Coulomb coupling between the Majorana's on the same island varies rapidly (exponentially).

The energy minimum $-2E_0$ at ϕ_{\min} is increased by $\sqrt{2E_C E_J}$ due to zero-point fluctuations of the phase. This offset does not contain the Majorana operators, so it can be ignored. The term $-U\mathcal{P}$ due to quantum phase slips depends on the Majorana operators through the fermion parity. This term acquires a dynamics for multiple coupled islands, because then the fermion parity of each individual island is no longer conserved.

3.1.2 Multiple islands

We generalize the description to multiple superconducting islands, labeled $k = 1, 2, \dots$, each connected to a bulk superconductor by a split Josephson junction enclosing a flux Φ_k . (See Fig. 3.2.) The Josephson junctions contribute an energy

$$H_J = -\sum_k E_{J,k} \cos \phi_k, \quad E_{J,k} = 2E_0 \cos(\pi\Phi_k/\Phi_0). \quad (3.8)$$

We assume that the charging energy is dominated by the self-capacitance C of each island, so that it has the additive form

$$H_C = \sum_k \frac{1}{2C} (Q_k + q_{\text{ind},k})^2. \quad (3.9)$$

While both E_0 and C may be different for different islands, we omit a possible k -dependence for ease of notation. There may be additional fluxes enclosed by the regions between the islands, but we do not include them to simplify the expressions. None of these simplifications is essential for the operation of the device.

The set of Majorana's on the k -th island is indicated by γ_{kn} with $n = 1, 2, \dots, 2N_k$. The fermion parities $\mathcal{P}_k = i^{N_k} \prod_n \gamma_{kn}$ of neighboring islands k and k' are coupled with strength E_M by the overlapping Majorana's γ_{kn} and $\gamma_{k'm}$. We denote the gauge-invariant phase difference [40] by $\theta_{kk'} = \phi_k - \phi_{k'} + (2\pi/\Phi_0) \int_{k \rightarrow k'} \mathbf{A} \cdot d\mathbf{l}$. The corresponding tunnel Hamiltonian [26]

$$H_{kk'} = \Gamma_{kk'} \cos(\theta_{kk'}/2), \quad \Gamma_{kk'} = iE_M \gamma_{kn} \gamma_{k'm}, \quad (3.10)$$

is 4π -periodic in the gauge-invariant phase difference, as an expression of the fact that single electrons (rather than Cooper pairs) tunnel through the mid-gap state. For example, in the two-island geometry of Fig. 3.2 one has

$$H_{12} = iE_M \gamma_{12} \gamma_{21} \cos(\theta_{12}/2), \quad (3.11a)$$

$$\theta_{12} = \phi_1 - \phi_2 - \pi(\Phi_1 + \Phi_2)/\Phi_0. \quad (3.11b)$$

In Appendix 3.A we derive the effective low-energy Hamiltonian in the regime $E_J \gg E_C, E_M$,

$$H_{\text{eff}} = \text{const} - \sum_k U_k \mathcal{P}_k + \sum_{k,k'} \Gamma_{kk'} \cos \alpha_{kk'}, \quad (3.12)$$

$$\alpha_{kk'} = \lim_{\phi_k, \phi_{k'} \rightarrow 0} \frac{1}{2} \theta_{kk'}. \quad (3.13)$$

The single sum couples Majorana's within an island, through an effective Coulomb energy U_k . The double sum couples Majorana's in neighboring islands by tunneling. Both the Coulomb and tunnel couplings depend on the fluxes through the Josephson junctions, but in an entirely different way: the tunnel coupling varies slowly $\propto \cos(\pi\Phi/\Phi_0)$ with the flux, while the Coulomb coupling varies rapidly $\propto \exp[-4\sqrt{(E_0/E_C) \cos(\pi\Phi/\Phi_0)}]$.

3.1.3 T-junction

Since \mathcal{P}_k and $\Gamma_{kk'}$ in the Majorana-Coulomb Hamiltonian (3.12) do not commute, the evolution of the eigenstates upon variation of the fluxes is nontrivial. As we will demonstrate, it can provide the non-Abelian braiding statistic that we are seeking.

Similarly to earlier braiding proposals [21, 67], the minimal setup consists of three superconductors in a T-junction. (See Fig. 3.3.) Each superconductor contains a pair of Majorana modes γ_k, γ'_k , with a tunnel coupling between γ'_1, γ'_2 , and γ'_3 . The Majorana-Coulomb Hamiltonian (3.12) takes the form

$$H_{\text{eff}} = iE_M (\gamma'_1 \gamma'_2 \cos \alpha_{12} + \gamma'_2 \gamma'_3 \cos \alpha_{23} + \gamma'_3 \gamma'_1 \cos \alpha_{31}) - \sum_{k=1}^3 U_k i \gamma_k \gamma'_k, \quad (3.14)$$

with gauge-invariant phase differences

$$\alpha_{12} = -(\pi/2\Phi_0)(\Phi_1 + \Phi_2 + 2\Phi_3), \quad (3.15a)$$

$$\alpha_{23} = (\pi/2\Phi_0)(\Phi_2 + \Phi_3), \quad (3.15b)$$

$$\alpha_{31} = (\pi/2\Phi_0)(\Phi_1 + \Phi_3). \quad (3.15c)$$

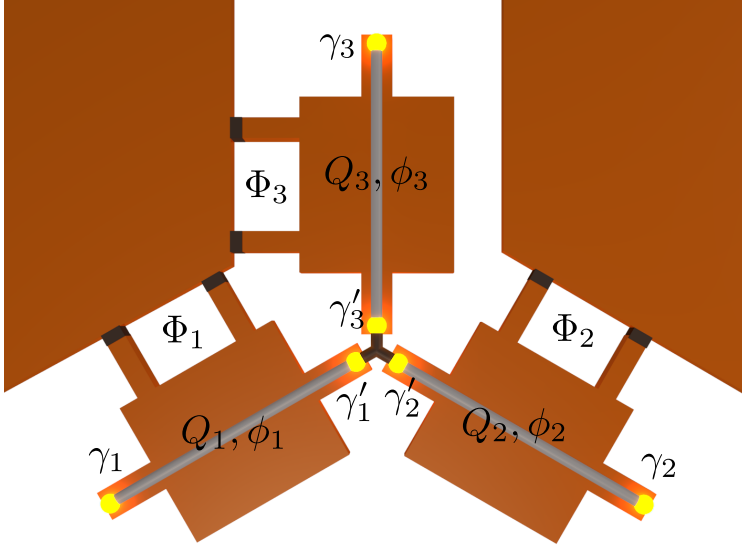


Figure 3.3: Three Cooper pair boxes connected at a T-junction via three overlapping Majorana modes (which effectively produce a single zero-mode γ_0 at the center). This is the minimal setup required for the braiding of a pair of Majorana's, controlled by the fluxes through the three Josephson junctions to a bulk superconductor.

As we vary $|\Phi_k|$ between 0 and $\Phi_{\max} < \Phi_0/2$, the Coulomb coupling U_k varies between two (possibly k -dependent) values U_{\min} and U_{\max} . We require $U_{\max} \gg U_{\min}$, which is readily achievable because of the exponential flux sensitivity of the Coulomb coupling expressed by Eqs. (3.1) and (3.7). We call the Coulomb couplings U_{\max} and U_{\min} *on* and *off*, respectively. We also take $U_{\max} \ll E_M$, meaning that the Coulomb coupling is weaker than the tunnel coupling. This is not an essential assumption, but it allows us to reduce the 6-Majorana problem to a 4-Majorana problem, as we will now show.

Consider first the case that $U_k = 0$ for all k . Then the Hamiltonian (3.14) has four eigenvalues equal to zero: three of these represent the Majorana's γ_k far away from the junction, while the fourth Majorana,

$$\gamma_0 = \frac{1}{\sqrt{3}}(\gamma'_1 + \gamma'_2 + \gamma'_3) \quad (3.16)$$

is situated at the T-junction. The T-junction contributes also two nonzero eigenvalues $\pm \frac{1}{2}E_{\text{gap}}$, separated by the gap

$$E_{\text{gap}} = E_M \sqrt{\cos^2 \alpha_{12} + \cos^2 \alpha_{23} + \cos^2 \alpha_{31}}. \quad (3.17)$$

For Φ_{\max} well below Φ_0 and $U_{\max} \ll E_M$ these two gapped modes can be ignored, and only the four Majorana's $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ need to be retained.

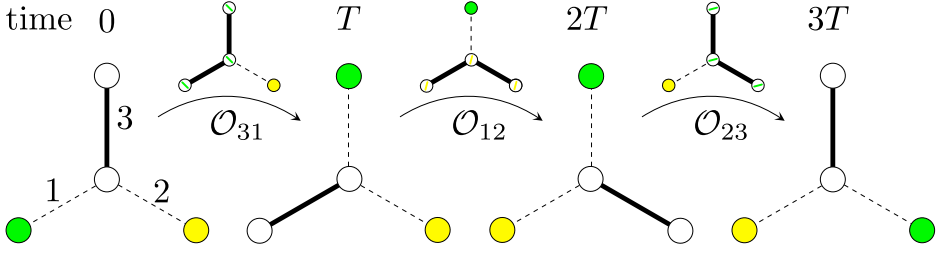


Figure 3.4: Schematic of the three steps of the braiding operation. The four Majorana's of the T-junction in Fig. 3.3 (the three outer Majorana's $\gamma_1, \gamma_2, \gamma_3$ and the effective central Majorana γ_0) are represented by circles and the Coulomb coupling is represented by lines (solid in the *on* state, dashed in the *off* state). White circles indicate Majorana's with a large Coulomb splitting, colored circles those with a vanishingly small Coulomb splitting. The small diagram above each arrow shows an intermediate stage, with one Majorana delocalized over three coupled sites. The three steps together exchange the Majorana's 1 and 2, which is a non-Abelian braiding operation.

The Hamiltonian H_{int} that describes the Coulomb interaction of these four Majorana's for nonzero U_k is given, to first order in U_k/E_M , by

$$H_{\text{int}} = \sum_{k=1}^3 \Delta_k i\gamma_0 \gamma_k, \quad \Delta_k = -(2E_M/E_{\text{gap}})\beta_k U_k, \quad (3.18)$$

$$\beta_1 = \cos \alpha_{23}, \quad \beta_2 = \cos \alpha_{31}, \quad \beta_3 = \cos \alpha_{12}. \quad (3.19)$$

3.2 Majorana braiding

The Hamiltonian (3.18) describes four flux-tunable Coulomb-coupled Majorana modes. Although the coupling studied by Sau, Clarke, and Tewari [67] has an entirely different origin (gate-tunable tunnel coupling), their Hamiltonian has the same form. We can therefore directly adapt their braiding protocol to our control parameters.

We have three fluxes Φ_1, Φ_2, Φ_3 to control the couplings. The braiding operation consists of three steps, see Table 3.1 and Fig. 3.4. (Ref. [67] had more steps, involving 6 rather than 4 Majorana's.) At the beginning and at the end of each step two of the couplings are *off* ($\Phi_k = 0$) and one coupling is *on* ($|\Phi_k| = \Phi_{\text{max}}$). We denote by $\mathcal{O}_{kk'}$ the step of the operation that switches the coupling that is *on* from k to k' . This is done by first increasing $|\Phi_{k'}|$ from 0 to Φ_{max} and then decreasing $|\Phi_k|$ from Φ_{max} to 0, keeping the third flux fixed at 0.

During this entire process the degeneracy of the ground state remains unchanged (twofold degenerate), which is a necessary condition for an adiabatic operation. If,

time	Φ_1	Φ_2	Φ_3
0	0	0	$-\Phi_{\max}$
	Φ_{\max}	0	$-\Phi_{\max}$
T	Φ_{\max}	0	0
	Φ_{\max}	Φ_{\max}	0
$2T$	0	Φ_{\max}	0
	0	Φ_{\max}	$-\Phi_{\max}$
$3T$	0	0	$-\Phi_{\max}$

Table 3.1: Variation of the flux through the three Josephson junctions during the braiding operation, at time steps corresponding to the diagrams in Fig. 3.4. The flux Φ_3 is varied in the opposite direction as Φ_1, Φ_2 , to ensure that the coupling parameters $\Delta_k \propto \beta_k$ do not change sign during the operation.

instead, we would first have first decreased $|\Phi_k|$ and then increased $|\Phi_{k'}|$, the ground state degeneracy would have switched from two to four at some point during the process, precluding adiabaticity.

We start from coupling 3 *on* and couplings 1,2 *off*. The braiding operation then consists, in sequence, of the three steps \mathcal{O}_{31} , \mathcal{O}_{12} , and \mathcal{O}_{23} . Note that each coupling Δ_k appears twice in the *on* state during the entire operation, both times with the same sign s_k .

The step $\mathcal{O}_{kk'}$ transfers the uncoupled Majorana at site k' to site k in a time T . The transfer is described in the Heisenberg representation by $\gamma_k(T) = \mathcal{U}^\dagger(T)\gamma_k\mathcal{U}(T)$. We calculate the unitary evolution operator $\mathcal{U}(T)$ in the adiabatic $T \rightarrow \infty$ limit in Appendix 3.B, by integrating over the Berry connection. In the limit $U_{\min} \rightarrow 0$ we recover the result of Ref. [67],

$$\gamma_k(T) = -s_k s_{k'} \gamma_{k'}(0). \quad (3.20)$$

The result after the three steps is that the Majorana's at sites 1 and 2 are switched, with a difference in sign,

$$\gamma_1(3T) = -s_1 s_2 \gamma_2(0), \quad \gamma_2(3T) = s_1 s_2 \gamma_1(0). \quad (3.21)$$

The corresponding unitary time evolution operator,

$$\mathcal{U}(3T) = \frac{1}{\sqrt{2}}(1 + s_1 s_2 \gamma_1 \gamma_2) = \exp\left(\frac{\pi}{4} s_1 s_2 \gamma_1 \gamma_2\right), \quad (3.22)$$

has the usual form of an adiabatic braiding operation [27]. For a nonzero U_{\min} the coefficient $\pi/4$ in the exponent acquires corrections of order U_{\min}/U_{\max} , see Appendix 3.B.

If one repeats the entire braiding operation, the Majorana's 1 and 2 have returned to their original positions but the final state differs from the initial state by a unitary

operator $\mathcal{U}(3T)^2 = s_1 s_2 \gamma_1 \gamma_2$ and not just by a phase factor. That is the hallmark of non-Abelian statistics [14].

3.3 Discussion

In summary, we have proposed a way to perform non-Abelian braiding operations on Majorana modes, by controlling their Coulomb coupling via the magnetic flux through a Josephson junction. Majorana modes are themselves charge-neutral particles (because they are their own antiparticle), so one may ask how there can be any Coulomb coupling at all. The answer is that the state of a pair of Majorana modes in a superconducting island depends on the parity of the number of electrons on that island, and it is this dependence on the electrical charge modulo $2e$ which provides an electromagnetic handle on the Majorana's.

The Coulomb coupling can be made exponentially small by passing Cooper pairs through a Josephson junction between the island and a bulk (grounded) superconductor. The control parameter is the flux Φ through the junction, so it is purely magnetic. This is a key difference with braiding by electrostatically controlled tunnel couplings of Majorana modes [67]. Gate voltages tend to be screened quite efficiently by the superconductor, so magnetic control is advantageous. Another advantage is that the dependence of the Coulomb coupling on the flux is governed by macroscopic electrical properties (capacitance of the island, resistance of the Josephson junction). Tunnel couplings, in contrast, require microscopic input (separation of the Majorana modes on the scale of the Fermi wave length), so they tend to be more difficult to control.

Both Ref. [67] and the present proposal share the feature that the gap of the topological superconductor is not closed during the braiding operation. (The measurement-based approach to braiding also falls in this category [71].) Two other proposals [21, 68] braid the Majorana's by inducing a topological phase transition (either by electrical or by magnetic means) in parts of the system. Since the excitation gap closes at the phase transition, this may be problematic for the required adiabaticity of the operation.

The braiding operation is called topologically protected, because it depends on the *off/on* sequence of the Coulomb couplings, and not on details of the timing of the sequence. As in any physical realization of a mathematical concept, there are sources of error. Non-adiabaticity of the operation is one source of error, studied in Ref. [72]. Low-lying sub-gap excitations in the superconducting island break the adiabatic evolution by transitions which change the fermion parity of the Majorana's.

Another source of error, studied in Appendix 3.B, is governed by the *off/on* ratio U_{\min}/U_{\max} of the Coulomb coupling. This ratio depends exponentially on the ratio of the charging energy E_C and the Josephson energy E_J of the junction to the bulk superconductor. A value $E_J/E_C \simeq 50$ is not unrealistic [60], corresponding to $U_{\min}/U_{\max} \simeq 10^{-5}$.

The sign of the Coulomb coupling in the *on* state can be arbitrary, as long as it does not change during the braiding operation. Since $U_{\max} \propto \cos(\pi q_{\text{ind}}/e)$, any

change in the induced charge by $\pm e$ will spoil the operation. The time scale for this quasiparticle poisoning can be milliseconds [57], so this does not seem to present a serious obstacle.

A universal quantum computation using Majorana modes requires, in addition to braiding, the capabilities for single-qubit rotation and read-out of up to four Majorana's [6]. The combination of Ref. [61] with the present proposal provides a scheme for all three operations, based on the interface of a topological qubit and a superconducting charge qubit. This is not a topological quantum computer, since single-qubit rotations of Majorana modes lack topological protection. But by including the topologically protected braiding operations one can improve the tolerance for errors of the entire computation by orders of magnitude (error rates as large as 10% are permitted [62]).

3.A Derivation of the Majorana-Coulomb Hamiltonian

3.A.1 Single island

Considering first a single island, we start from the Cooper pair box Hamiltonian (3.2) with the parity constraint (3.5) on the eigenstates. Following Ref. [69], it is convenient to remove the constraint by the unitary transformation

$$\tilde{H} = \Omega^\dagger H \Omega, \quad \Omega = \exp[i(1 - \mathcal{P})\phi/4]. \quad (3.23)$$

The transformed wave function $\tilde{\Psi}(\phi) = \Omega^\dagger \Psi(\phi)$ is then 2π -periodic, without any constraint. The parity operator \mathcal{P} appears in the transformed Hamiltonian,

$$\tilde{H} = \frac{1}{2C} \left(Q + \frac{1}{2}e(1 - \mathcal{P}) + q_{\text{ind}} \right)^2 - E_J \cos \phi. \quad (3.24)$$

For a single junction the parity is conserved, so eigenstates of H are also eigenstates of \mathcal{P} and we may treat the operator \mathcal{P} as a number. Eq. (3.24) is therefore the Hamiltonian of a Cooper pair box with effective induced charge $q_{\text{eff}} = q_{\text{ind}} + e(1 - \mathcal{P})/2$. The expression for the ground state energy in the Josephson regime $E_J \gg E_C$ is in the literature [37, 73],

$$E_{\text{ground}} = -E_J + \sqrt{2E_C E_J} - 16(E_C E_J^3 / 2\pi^2)^{1/4} e^{-\sqrt{8E_J/E_C}} \cos(\pi q_{\text{eff}}/e). \quad (3.25)$$

The first term $-E_J$ is the minimal Josephson energy at $\phi_{\text{min}} = 0$. Zero-point motion, with Josephson plasma frequency $\omega_p = \sqrt{8E_C E_J}/\hbar$, adds the second term $\sqrt{2E_C C_J} = \frac{1}{2}\hbar\omega_p$. The third term is due to quantum phase slips with transition amplitudes $\tau_{\pm} \simeq \exp(\pm i\pi q_{\text{eff}}/e) \sqrt{\hbar\omega_p E_J} \exp(-\hbar\omega_p/E_J)$ by which ϕ increments by $\pm 2\pi$.

Using $\mathcal{P}^2 = 1$, the ground state energy (3.25) may be written in the form

$$E_{\text{ground}} = -E_J + \sqrt{2E_C E_J} - U\mathcal{P}, \quad (3.26)$$

with U defined in Eq. (3.7). Higher levels are separated by an energy $\hbar\omega_p$, which is large compared to U for $E_J \gg E_C$. We may therefore identify $E_{\text{ground}} = H_{\text{eff}}$ with the effective low-energy Hamiltonian of a single island in the large- E_J limit.

3.A.2 Multiple islands

We now turn to the case of multiple islands with tunnel coupling. To be definite we take the geometry of two islands shown in Fig. 3.2. The full Hamiltonian is $H = H_1 + H_2 + H_{12}$, where H_1 and H_2 are two copies of the Cooper box Hamiltonian (3.2) and H_{12} is the tunnel coupling from Eq. (3.11).

To obtain 2π -periodicity in both phases ϕ_1 and ϕ_2 , we make the unitary transformation $\tilde{H} = \Omega^\dagger H \Omega$ with

$$\Omega = e^{i(1-\mathcal{P}_1)\phi_1/4} e^{i(1-\mathcal{P}_2)\phi_2/4}. \quad (3.27)$$

The Cooper pair box Hamiltonians are transformed into

$$\tilde{H}_k = \frac{1}{2C} (Q_k + eq_k + q_{\text{ind},k})^2 - E_{J,k} \cos \phi_k, \quad (3.28)$$

with $q_k = \frac{1}{2}(1 - \mathcal{P}_k)$. The tunnel coupling transforms into

$$\tilde{H}_{12} = \frac{1}{2} e^{-iq_1\phi_1} \Gamma_{12} e^{iq_2\phi_2} e^{i\pi(\Phi_1 + \Phi_2)/2\Phi_0} + \text{H.c.}, \quad (3.29)$$

where $\Gamma_{12} = iE_M \gamma_{12} \gamma_{21}$ and H.c. stands for Hermitian conjugate. Since $e^{iq\phi} = \cos \phi + iq \sin \phi$, the transformed tunnel coupling \tilde{H}_{12} is 2π -periodic in ϕ_1 and ϕ_2 .

For $E_J \gg E_C$ the phases remain close to the value which minimizes the sum of the Josephson energies to the bulk superconductor and between the islands. To leading order in $E_M/E_J \ll 1$ this minimal energy is given by

$$\begin{aligned} \mathcal{E}_{\min} &= -E_{J,1} - E_{J,2} + \Gamma_{12} \cos[\pi(\Phi_1 + \Phi_2)/2\Phi_0] \\ &\quad + \mathcal{O}(E_M^2/E_J). \end{aligned} \quad (3.30)$$

The Josephson coupling of the islands changes the plasma frequency $\omega_{p,k}$ for phase ϕ_k by a factor $1 + \mathcal{O}(E_M/E_J)$, so the zero-point motion energy is

$$\frac{1}{2} \hbar \omega_{p,k} = \sqrt{2E_C E_{J,k}} + E_M \times \mathcal{O}(E_C/E_J)^{1/2}. \quad (3.31)$$

The transition amplitudes τ_{\pm} for quantum phase slips of phase ϕ_k are similarly affected,

$$\tau_{\pm,k} = -U_k \mathcal{P}_k + E_M e^{-\hbar \omega_{p,k}/E_{J,k}} \times \mathcal{O}(E_C/E_J)^{1/4}. \quad (3.32)$$

These are the contributions to the effective Hamiltonian $H_{\text{eff}} = \mathcal{E}_{\min} + \sum_k (\frac{1}{2} \hbar \omega_{p,k} + \tau_{+,k} + \tau_{-,k})$ for $E_J \gg E_C, E_M$,

$$\begin{aligned} H_{\text{eff}} &= \left(-U_1 \mathcal{P}_1 - U_2 \mathcal{P}_2 + \Gamma_{12} \cos[\pi(\Phi_1 + \Phi_2)/2\Phi_0] \right) \\ &\quad \times [1 + \mathcal{O}(E_M/E_J)] + \text{const.} \end{aligned} \quad (3.33)$$

Eq. (3.12) in the main text generalizes this expression for two islands to an arbitrary number of coupled islands.

3.B Calculation of the Berry phase of the braiding operation

We evaluate the unitary evolution operator \mathcal{U} of the braiding operation in the adiabatic limit. This amounts to a calculation of the non-Abelian Berry phase (integral of Berry connection) of the cyclic variation of the interaction Hamiltonian $H_{\text{int}}(\Delta_1, \Delta_2, \Delta_3)$.

In the Fock basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ the interaction Hamiltonian (3.18) of 4 Majorana modes is given by the occupation number of the two fermionic operators $c_1 = (\gamma_1 - i\gamma_2)/2$ and $c_2 = (\gamma_0 - i\gamma_3)/2$. It takes the form

$$H_{\text{int}} = \begin{pmatrix} -\Delta_3 & 0 & 0 & -i\Delta_1 - \Delta_2 \\ 0 & \Delta_3 & -i\Delta_1 - \Delta_2 & 0 \\ 0 & i\Delta_1 - \Delta_2 & -\Delta_3 & 0 \\ i\Delta_1 - \Delta_2 & 0 & 0 & \Delta_3 \end{pmatrix}. \quad (3.34)$$

The eigenvalues are doubly degenerate at energy $\pm\varepsilon = \pm\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$ (up to a flux-dependent offset, which only contributes an overall phase factor to the evolution operator). The two degenerate ground states at $-\varepsilon$ are distinguished by an even (e) or odd (o) quasiparticle number,

$$|e\rangle = \sqrt{\frac{\varepsilon - \Delta_3}{2\varepsilon}} \begin{pmatrix} i \frac{\varepsilon + \Delta_3}{\Delta_1 + i\Delta_2} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (3.35a)$$

$$|o\rangle = \sqrt{\frac{\varepsilon + \Delta_3}{2\varepsilon}} \begin{pmatrix} 0 \\ i \frac{\varepsilon - \Delta_3}{\Delta_1 + i\Delta_2} \\ 1 \\ 0 \end{pmatrix}. \quad (3.35b)$$

This parameterization is smooth and continuous except along the line $\Delta_1 = \Delta_2 = 0$.

If we avoid this line the Berry connection can be readily evaluated. It consists of three anti-Hermitian 2×2 matrices \mathcal{A}_k ,

$$\mathcal{A}_k = \begin{pmatrix} \langle e | \frac{d}{d\Delta_k} | e \rangle & 0 \\ 0 & \langle o | \frac{d}{d\Delta_k} | o \rangle \end{pmatrix}. \quad (3.36)$$

Off-diagonal terms in \mathcal{A}_k are zero because of global parity conservation. Explicitly,

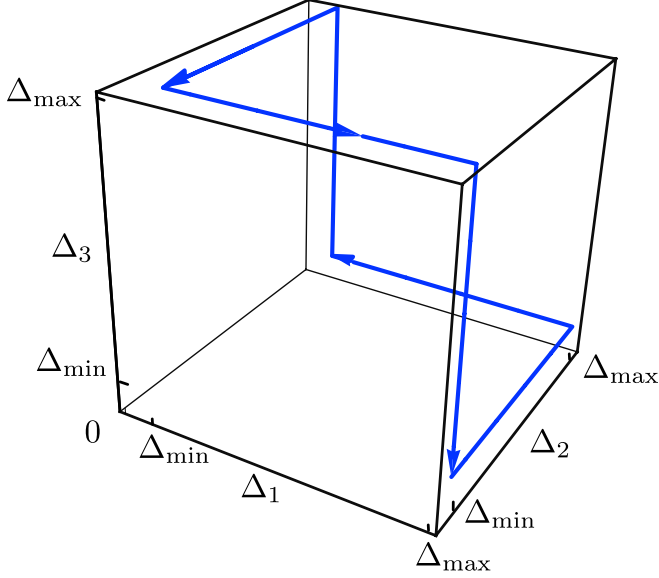


Figure 3.5: The braiding path in three-dimensional parameter space along which the Berry phase is evaluated. This path corresponds to the flux values in Table 3.1, with couplings $\Delta_k = \Delta_{\min}$ for $\Phi_k = 0$ and $\Delta_k = \Delta_{\max}$ for $|\Phi_k| = \Phi_{\max}$. The ratio $\Delta_{\min}/\Delta_{\max}$ in the figure is exaggerated for clarity.

we have

$$\mathcal{A}_1 = \frac{\Delta_2}{\Delta_1^2 + \Delta_2^2} \begin{pmatrix} i \frac{\varepsilon + \Delta_3}{2\varepsilon} & 0 \\ 0 & i \frac{\varepsilon - \Delta_3}{2\varepsilon} \end{pmatrix}, \quad (3.37)$$

$$\mathcal{A}_2 = \frac{-\Delta_1}{\Delta_1^2 + \Delta_2^2} \begin{pmatrix} i \frac{\varepsilon + \Delta_3}{2\varepsilon} & 0 \\ 0 & i \frac{\varepsilon - \Delta_3}{2\varepsilon} \end{pmatrix}, \quad (3.38)$$

$$\mathcal{A}_3 = 0. \quad (3.39)$$

A closed path \mathcal{C} in parameter space has Berry phase [20]

$$\mathcal{U} = \exp \left(- \oint_{\mathcal{C}} \sum_k A_k d\Delta_k \right). \quad (3.40)$$

The path \mathcal{C} corresponding to the braiding operation in Fig. 3.4 and Table 3.1 is shown in Fig. 3.5. We take all couplings Δ_k positive, varying between a minimal value Δ_{\min} and maximal value Δ_{\max} . The parametrization (3.35) is well-defined along the entire contour.

The contour integral evaluates to

$$\mathcal{U} = \exp \left[-i \left(\frac{\pi}{4} - \epsilon \right) \sigma_z \right], \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.41)$$

$$\epsilon = \frac{3}{\sqrt{2}} \frac{\Delta_{\min}}{\Delta_{\max}} + \mathcal{O} \left(\frac{\Delta_{\min}}{\Delta_{\max}} \right)^2. \quad (3.42)$$

The limit $\Delta_{\min}/\Delta_{\max} \rightarrow 0$ corresponds to the braiding operator (3.22) in the main text (with $s_1, s_2 > 0$ and $\sigma_z = 1 - 2c_1^\dagger c_1 = i\gamma_1 \gamma_2$).