Holography, fermi surfaces and criticality

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Chapter 2

The holographic dictionary

2.1 The basic entries

We are now ready to consider the theoretical background of our work and to work out in some detail the results we will use. This essentially corresponds to constructing the detailed dictionary entries and formulate rules for the boundary terms in the action. We start with pure AdS space (we will need more later, to introduce temperature). We will work on the Poincaré patch of AdS space rather than global AdS. For most of the calculations it is much more appropriate to use the dimensionless inverse of the r coordinate:

$$z \equiv \frac{L}{r} \tag{2.1}$$

While the radial distance goes from $r = 0$ in the interior to infinity, now $z = 0$ corresponds to the boundary while $z = \infty$ is the deep interior. We might have a situation where there is a lowest bound on r, e.g. the position of a black hole horizon r_h (and there will be, if the temperature is finite). Then the deep IR is at $z_h = L/r_h$ instead of infinity. The AdS$_{D+1}$ metric in z coordinate is

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + \sum_{i=1}^{D-1} dx_i^2 + dz^2\right). \tag{2.2}$$

Deformations away from AdS space are allowed as long as the small z asymptotics (AdS boundary) is unchanged. We will only consider equilibrium physics in this thesis, which corresponds to stationary and homogenous geometries. We will also only consider isotropic systems, i.e.

isotropic geometries in the bulk. This makes all components of the metric depend only on z and allows at most two free functions parametrizing deformations from AdS. We can therefore write the most general metric as

$$ds^2 = \frac{1}{z^2}\left(-f(z)h(z)dt^2 + \sum_{i=1}^{D-1} dx_i^2 + \frac{dz^2}{f(z)}\right) \tag{2.3}$$

where we recognize $f(z)$ as the red shift factor (warp function). For AdS asymptotics we must have $f(z) = 1 + O(z)$ and $h(z) = 1 + O(z)$ for $z \to 0$. General stability conditions also make both f and h everywhere non-negative. Finally, $f(z_h) = 0$ indicates the existence of a horizon at $z = z_h$.

2.1.1 Thermodynamics

Finite temperature

The basis of the dictionary is given by the identification of the partition functions given in (1.6). The first new dictionary entry we introduce is temperature, originally proposed by Witten in [115]. It is a direct consequence of the basic fact that temperature enters kinematics of a field theory by imposing periodicity of Euclidean time. Consider first an AdS space in imaginary time. A well-known (but not unique) solution with periodic Euclidean time $\tau \equiv it$ is the Schwarzschild black hole. This solution corresponds to metric (2.3) with $h = 0$ and

$$f(z) = 1 - \frac{4\pi M}{D\pi^{D/2}\Gamma(D/2+1)} z^D, \tag{2.4}$$

where M is the black hole mass. This solution is only defined up to the horizon at z_h, the outermost (smallest z) radial slice where the red shift function vanishes: $f(z_h) = 0$. It is only smooth if the time is periodic with the period

$$\frac{1}{T_{BH}} \equiv \beta = \frac{z_h}{2\pi} \tag{2.5}$$

where T_{BH} is the Hawking temperature of the black hole. Since the space-time coordinates (t, x) are directly identified in the dictionary, the compactification of imaginary time retains the same meaning in the boundary theory: $T_{BH} = T_{bnd}$. Notice that the temperature in field theory equals the temperature of the black hole and *not the temperature of the bulk*, as the latter is always zero. This is of more than academic interest as it means that the bulk fields live at $T = 0$ and should be treated by the usual field theory and not thermal field theory.

Free energy

The next dictionary entry, especially important when dealing with exotic systems where very few principles are known to hold, is that of free energy of the field theory, as the laws of thermodynamics are general enough that they can always be used as the starting point. This directly follows from the relation of free energy $\mathcal{F}_{bnd}$ to partition function Z_{bnd} as the defining equality:

$$e^{-\beta\mathcal{F}_{CFT}} = \langle Z_{CFT}\rangle_{\text{CFT}}. \tag{2.6}$$

According to GKPW formula, the right-hand side equals the bulk on-shell action with appropriate boundary conditions. We thus find:

$$e^{-\beta\mathcal{F}_{CFT}} = \langle e^{-\int d\tau\mathcal{L}_{bulk}+S_{bnd}}\rangle_{\text{AdS}} \tag{2.7}$$

where we have included the possibility of boundary interactions on the gravity side. In classical gravity, i.e. for large N and large gN the bulk expectation value is obtained simply by plugging in the on-shell solutions into $S_{bulk} + S_{bnd}$. Taking into account (2.5) we get the factor of β in the exponent of Z_{bulk} too, so

$$\mathcal{F}_{CFT} = S_{bulk}(\Phi_{\text{on-shell}}) + S_{bnd}(\Phi_{\text{on-shell}}). \tag{2.8}$$

This simple but very important rule was given in [115]. Then we can follow all the usual thermodynamic identities to find other thermodynamic potentials, as well as their derivatives. Notice again that we cannot equate $\mathcal{F}_{CFT}$ to any *thermodynamic* quantity in the bulk, as the latter is at zero temperature.

2.1.2 Sources and expectation values

Scalar field

The observables of a CFT have correlation functions of their operators $\mathcal{O}$, carrying certain quantum numbers. These correlation functions are formally generated in the standard way by taking functional derivatives of

$$\langle\mathcal{O}\mathcal{O}\ldots\mathcal{O}\rangle = \frac{\delta^n}{\delta^n\Phi_0}\langle e^{\int\Phi_0\mathcal{O}}\rangle_{\text{CFT}}. \tag{2.9}$$

Recalling our duality discussion, we should identify the source Φ_0 with a field in AdS $\Phi(x)$ restricted to the boundary where conformal symmetries are realized, relating $\lim_{z\to 0}\Phi(z)$ to Φ_0. The boundary conditions should ensure that the source is the leading (non-normalizable) component of the solution at the boundary. Let us see how such a procedure works for a scalar field and for a gauge field. The results to follow are mostly from [114, 115] with some slight refinements summarized in [2, 25]. In this case the bulk action and the equations of motion are trivially

$$S_{bulk} = -\int d^D x \left(D^\dagger_\mu \Phi D^\mu \Phi + m^2 \Phi^2 \right) \tag{2.10}$$

$$\left(z^{D-1}\partial_z z^{1-D}\partial_z + k^2 - \frac{m^2}{z^2} \right) \Phi = 0 \tag{2.11}$$

We are looking for a solution which remains finite at the boundary $z \to 0$. Making a power-law ansatz $\Phi \sim z^\alpha$, we find that exponents of the near-boundary asymptotic of the field Φ are $\Delta_\pm = D/2 \mp \sqrt{(D/2)^2 + m^2}$. Here, Δ_- corresponds to the leading and Δ_+ to the subleading branch. One can actually find the exact solution in the whole AdS space in terms of modified Bessel functions, giving general solution of the form

$$\Phi(z) = \Phi_S z^{D/2} K_{\Delta - D/2}(kz) + \Phi_R z^{D/2} I_{\Delta - D/2}(kz) \tag{2.12}$$

where K and I are modified Bessel functions of first and second kind, respectively and

$$\Delta = \Delta_+ = D/2 + \sqrt{\left(\frac{D}{2}\right)^2 + m^2}. \tag{2.13}$$

The normalizable solution is proportional to Φ_R while the non-normalizable one is the Φ_S branch. Therefore, according to the dictionary, Φ_R is the response (expectation value) and Φ_S the source. Consider now the one-point function $\langle O \rangle$. The variation of the bulk action for such a configuration is found by substituting the solution into S_{bulk}:

$$\delta S_{bulk} = \int_0^\infty dz \int d^D x \sqrt{-g}\, 2\delta\Phi (D^\dagger_\mu D^\mu - m^2)\Phi - 2\int d^D x \sqrt{-h}\, \delta\Phi \partial_z \Phi|_{z=0} \tag{2.14}$$

where h is the induced metric on the boundary. The first term vanishes for the solution of (2.11). For the second term the characteristic AdS/CFT steps come. First, we see that the bulk action in general diverges at the UV boundary $z \to 0$ and needs to be regularized. The last, divergent part of (2.14) can be removed by the boundary counterterm

$$S_{bnd} = \int d^D x \sqrt{-h} \Phi^2 \tag{2.15}$$

This is exactly the Dirichlet term familiar from elementary analysis: its meaning is to fix the boundary data Φ_0. So consistency if the bulk theory *requires* it to be reconstructible from the boundary.

At second order we find the two-point correlator for the boundary field $\mathcal{O}$

$$\langle \mathcal{O}(\mathbf{x}_1) \mathcal{O}(\mathbf{x}_2) \rangle = \frac{\partial^2 S}{\partial \Phi(x_1) \partial \Phi(x_2)} \sim \frac{\text{const.}}{|x_1 - x_2|^{2\Delta}} \tag{2.16}$$

with Δ defined in (2.13). Therefore, the seemingly arbitrary definition of Δ in (2.12) is chosen to match the conformal dimension of the boundary field. We see that the operator $\mathcal{O}$ scales in accordance with the predictions of CFT with conformal dimension Δ. Also if additional terms asuch as interactions are added in the bulk, it is clear that the UV asympotics will still be determined by m, or else (if the additional terms are irrelevant at the boundary) the asymptotic AdS geometry will be unstable. So another dictionary entry is that *conformal dimension in field theory is determined by the bulk mass of the field.*

Gauge fields, field strengths and densities

The procedure above is readily generalized to gauge fields. In this thesis we will need only the Abelian $U(1)$ field so we focus on that. Let us start from the well known Maxwell action. By partial integration, bulk action evaluates to

$$S = -\frac{1}{4} \int_0^\infty dz \int d^D x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \lim_{z_0 \to 0} \int d^D x \sqrt{-g} F_{\mu\nu} A^\mu n^\nu |_{z_0} + \\ + \int_0^\infty dz \int d^D x \sqrt{-g} A_\nu \partial_\mu F^{\mu\nu} \tag{2.17}$$

where n^ν is the unit normal vector to the boundary. To cancel the boundary contribution we precisely need the von Neumann term $S_{ct} = \int d^D x \sqrt{-h} F_{\mu\nu} A^\mu$ that fixes the field strength at the boundary. Now that we have the boundary action, we can proceed to find the dictionary entries. The solution to the Maxwell equations near the AdS boundary is a linear function in z. For the component A_0 we can write

$$A_0 = A_0^{(0)} + A_0^{(1)} z + O(z^2), \tag{2.18}$$

so the leading term, $A_0^{(0)}$, is the source and $A_0^{(1)}$ is the response. The boundary action is $S_{bnd} = -A_0^{(0)} A_0^{(1)} + \ldots$: the leading and subleading term are linearly coupled to each other. It becomes clear that $A_0^{(1)}$ can be identified with negative charge density ρ while its source $A_0^{(0)}$ has the meaning of chemical potential μ (i.e. background scalar potential). For a spatial component of the gauge field, we can write

$$A_i = A_i^{(0)} + A_i^{(1)} z + O(z^2) \tag{2.19}$$

and equate the subleading term $A_i^{(1)}$ to the current J_i while $A_x^{(0)}$ is its source. Therefore, we arrive at the conclusion that the subleading and leading term of the bulk gauge field encode the current density and its source, i.e. background $U(1)$ field. We can rephrase this conclusion in terms of electric and magnetic field strengths in the bulk if we assume spacetime homogeneity. In this case transverse electric field is simply $E_i = -i\omega A_i$ and the radial magnetic field is $B_i = i\epsilon_{ijk} k_j A_k$. We can now say that the bulk radial electric field stands for the charge density while the radial magnetic field in the bulk is the magnetic field at the boundary. For the transverse fields, we get that transverse bulk electric field encodes for the electric field at the boundary, while transverse bulk magnetic field stands for spatial current on field theory side.[1] Notice that the fields at the boundary obey *global* rather than gauge currents. This is an important property of the dictionary: gauge symmetry in the bulk becomes a local symmetry at the boundary. Another manifestation of this principle is the $SO(D-1)$ rotational invariance in field theory. In AdS, $SO(D-1)$ is a gauge symmetry, a consequence of diffeomorphism

[1] This fails for the component A_z. Obviously, since the radial coordinate does not exist on field theory side, A_z cannot be dual to any component of the current. In fact, it has no physical sense at all and one should put $A_z = 0$ in holographic setups. To see this, remember that nonzero radial gauge field implies a nonzero radial flux through the boundary. This would violate the RG flow interpretation of the radial direction – we do not know how to interpret radial flow of *matter* along z. For that reason we always put $A_z = 0$.

invariance, in the sense that an $SO(D-1)$ rotation transforms AdS space into itself but in different coordinates.

There is a way to use AdS/CFT in the canonical ensemble using the method of alternative quantization for the gauge field. From (2.19), we see that the leading term has the same asymptotics as the derivative of the subleading term. By a Legendre transform we can thus swap the roles of $F_{\mu\nu}$ and A_μ in the boundary term and regard J_μ as fixed instead of the source E_μ. For example, suppose the gauge field has the form $A = A_0 dt$. Then the boundary action is $S_{ct} = \mu\rho + \ldots$: the two coefficients are linearly coupled to each other, and we can identify $a_0 \mapsto \mu, b_0 \mapsto \rho$: *leading and subleading term in the gauge field component* A_0 *correspond to chemical potential and charge density in field theory.*

2.2 Holographic superconductors: a tutorial

In this subsection we will present a worked-out example where the general formalism of holography is applied on perhaps the simplest possible nontrivial system: a charged scalar boson coupled to the $U(1)$ Maxwell field and gravity. This is the famous holographic superconductor model, proposed in 2008 by Hartnoll, Horowitz and Herzog [47, 46], and Gubser [40]. It is immediately clear that the term superconductor is not quite satisfying: not only are there no fermionic degrees of freedom but the $U(1)$ symmetry is global and not gauged, thus more akin to the situation in a superfluid. Nevertheless, it is the most famous application of AdS/CFT on complex systems, encapsulating all important elements.

Let us first recall the effective Landau-Ginzburg theory of superconductivity. There, one replaces the microscopic treatment of Cooper pairs by an effective theory for the charged bosonic order parameter Φ. One then constructs the free energy in the vicinity of the transition point in accordance with general symmetry requirements. The result is a phenomenological action which can describe the dependence of the pair density on temperature near the critical point, as well as the Higgsing phenomenon, i.e. breaking of the gauge $U(1)$ symmetry by the condensate. Since the holographic description will take $U(1)$ to be a global rather a local symmetry, This last ingredient is missing in the holographic version. The holographic superconductivity an important breakthrough. Not only does it give an example on how to treat in principle the condensation of any order parameter holographically, but it does so in a novel way: directly from a critical system and this is reflected in non-standard transport properties which reproduce the experimental results for superconducting materials.

Following the original papers, we specify to the case of $D = 3$ in this section. The bulk action is easy to write from the symmetry requirements:

$$S_{bulk} = \int dz \int d^3x \left[R + 6 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - D_\mu^\dagger \Phi D^\mu \Phi - m^2 \Phi^2 - V_{int}(|\Phi|) \right] \tag{2.20}$$

where the covariant derivative is

$$D_\mu = \partial_\mu - iqA_\mu \tag{2.21}$$

and the potential V_{int} can be an arbitrary function in the bottom-up setup. We will opt for the simplest case and set it to zero. At finite temperature nothing changes dramatically upon introducing a finite potential. The ansatz (2.3) can be used for the metric. For simplicity, let us assume spherical symmetry, isotropy and an electric-only configuration of the Maxwell field for now, writing

$$A = A_0(z) dt \tag{2.22}$$

The 00 and zz components of the Einstein equations read:

$$3f - z\partial_z f - 3 = \frac{1}{2} \left((\partial_z \Phi)^2 - V + (\partial_z A_0)^2 + q^2 \Phi^2 A_0^2 \right) \tag{2.23}$$

$$3f - z\partial_z f - 3zf \frac{\partial_z h}{h} - 3 = \frac{1}{2} \left((\partial_z \Phi)^2 + V + (\partial_z A_0)^2 + q^2 \Phi^2 A_0^2 \right) \tag{2.24}$$

while the Maxwell equation for F_{0z} reads

$$\partial_z \left(\frac{1}{\sqrt{h}} \partial_z A_0 \right) = 2q^2 \frac{\Phi^2}{z^3 \sqrt{fh}}. \tag{2.25}$$

The ii component of Einstein equations can be shown to be a linear combination of the remaining two and can be left out. The equations for this simple system are clearly quite involved. This is typical for the bulk physics of holographic systems: the full solution has to be obtained numerically, while analytical estimates can be made in the near-horizon and near-boundary limit. The former is of importance for the phase diagram and analysis of the condensate formation. We will discuss it after we solve a more basic question: how to impose the boundary conditions and calculate the quantities on the field theory side? To that end, we can use the results obtained earlier for the near-boundary asymptotics of the scalar field – it turns out that coupling to the gauge field is always a subleading term for $z \to 0$ and does not change the asymptotics. Schematically, the near-boundary solution is therefore

$$\Phi(z \to 0) = \Phi^{(1)} z^{3-\Delta} + \Phi^{(2)} z^{\Delta}. \tag{2.26}$$

The boundary action is important for the calculation of free energy at the boundary. The scaling dimension is set by the bulk mass; as before, we have $\Delta = D/2 + \sqrt{D^2/4 + m^2}$. According to the dictionary, $\Phi^{(2)}$ sources the boundary field while $\Phi^{(1)}$ is its VEV. For a solution that holographically encodes spontaneous symmetry breaking, we must seek for a spontaneously generated VEV without a source for the scalar and gauge field:[2]

$$S^{(1)}_{bnd-\Phi} = \oint d^3x \sqrt{-h}\Phi^2|_{z\to 0} \tag{2.27}$$

For completeness we give also the boundary action for the metric and the gauge field. This is the Hawking-Gibbons term for the metric and imposing the chemical potential $\mu = A_0(z_0)$ through a Dirichlet boundary condition for A_0. This gives altogether:

$$S_{bnd} = \oint d^3x \sqrt{-h}\left(-2K + 4 + A_0 \partial_z A_0 + \Phi^2\right). \tag{2.28}$$

2.2.1 Scalar condensate and phase transitions

In the presence of a nonzero electrostatic potential the scalar has an effective negative mass: $-m^2_{eff}\Phi^2 \sim -q^2 f h A_0^2 \Phi^2/z^2$. For a large enough charge q, it is reasonable to expect the scalar order parameter to condense. This is precisely what happens. Note that this means that the spontaneous breaking of the global $U(1)$ invariance in field theory is described by the spontaneous breaking of a local symmetry in the bulk, i.e. Higgsing in the bulk. Upon solving the equations of motion (2.23-2.25) with appropriate boundary conditions, one is able to find a solution with non-vanishing scalar field. On the field theory side, the operator dual to Φ will condense,

[2] We can also employ the alternative quantization, where the subleading term becomes the source. Fixing the subleading term however is not enough to cancel the divergence, and we need to add an explicit counterterm so the boundary action becomes $S^{(2)}_{bnd-\Phi} = \oint d^3x\sqrt{-h}(\Phi^2 + 2\Phi n_z \partial^z \Phi)$.

breaking now the global $U(1)$ symmetry.[3] Solving the system (2.23-2.25) numerically, one obtains the dependence of the condensate value $\langle\Phi\rangle$ on temperature. The result is a textbook order-disorder transition with the *mean field* scaling of the condensate with temperature:

$$\langle\Phi_{1,2}\rangle \propto \left(1 - \frac{T}{T_c}\right)^{\beta_{1,2}}. \tag{2.29}$$

One can then proceed to calculate the free energy which indeed reveals the existence of a second order phase transition, and with mean field exponents, thus reproducing the predictions of the Landau-Ginzburg theory. This finding encapsulates the essential features of holographic superconductivity – a scalar with arbitrary mass Higsses in the bulk leading to a global order-disorder transition on the field theory side.

Hartnoll et al have proceeded to compute conductivities [47] and found excellent qualitative agreement with experiment. In the standard quantization, Φ_1 condenses and backreacts on the gauge field. We can then compute the conductivity of the system as the ration of the current and the external field – the corresponding bulk quantities are the subleading and the leading term of a spatial component of the gauge field. The resulting curve looks like that of conventional BCS superconductors. Doing the same in alternative quantization, for Φ_2 (see the footnote on this page), one finds that conductivity mimics the one seen in unconventional superconductors. This was the first triumph of AdS/CMT in approaching the experiment [45].

Remarkably, a neutral scalar can also condense. The above mechanism clearly cannot be the cause of the formation of neutral hair. What is the mechanism here? The explanation lies in the generalization of the tachyonic instability to AdS known as the Breitenlohner-Freedman (BF) bound [25, 47, 46] and the geometry of the charged black hole. The BF bound is the value for which the square root in Δ becomes imaginary. In $D+1$ dimension it reads:

$$m^2 < m_{BF}^2 = -\frac{D^2}{4L^2} \tag{2.30}$$

[3] For low masses, the scalar field has two quantizations with the non-standard alternative quantization similar to the Legendre transform to the canonical ensemble as described earlier. The two possible choices for the boundary conditions – fixing the VEV versus fixing the source – lead to two different field theories, with different properties.

where L is the radius of the space. In AdS_4 the BF value is thus $-9/4L^2$. In the presence of non-zero chemical potential, this system has a different geometry in deep interior dual to the IR of a CFT. The near-horizon region of the charged black hole has the geometry $\mathrm{AdS}_2 \otimes \mathbb{R}^2$: it is a direct product of the $x-y$ plane and a *two-dimensional* AdS space, distinct from the AdS_4 where the system as a whole lives. AdS_2 has the BF bound $m^2 < m^2_{BF}$. Dimension is reduced from $D+1=4$ to $D+1=2$ but the radius of the AdS_2 is smaller than the radius of AdS_4: $L_2 = L/\sqrt{6}$. Therefore, the BF bound in the interior is $m^2 < -6/4L^2$. This means that there is a window of the values of m where $m^2_{\mathrm{AdS4}} < m^2 < m^2_{\mathrm{AdS2}}$, so a scalar which is stable in AdS_4 will still condense in AdS_2 [47]. The field theory meaning of this effect is the breaking of the discrete (Ising) $\mathbb{Z}_2$ gauge symmetry. This is a truly novel result of the holographic theory. The fact that the physics on field theory side can be explained by analyzing near-horizon geometry is an important lesson we will take from this review section.

2.3 Holographic dictionary for fermions

We now proceed to the object of this thesis: fermions. The essential problem for fermions is the well-known fact the Dirac fermion is a constrained system: the equations of motion are of first order, only half of the components of Dirac field are independent degrees of freedom while the rest are uniquely determined by them. The sign problem does *not* plague holography at least at the leading (tree) level. This is because the quasiparticle picture is preserved in the bulk, in the sense that we will consider weakly interacting fermions coupled to external fields only. Besides, we know that two-point correlation functions and expectation values (densities) are dual to tree-level objects in the bulk, thus one does not need to face the loop effects where the fermionicity strikes harder.[4]

[4]Occasionally, it is laconically claimed that the fermion sign problem is eliminated by holography as in the limit of classical gravity/large N strongly coupled field theory the bulk physics is classical. This is not entirely true: while *gravity* is treated classically in this limit as the gravitational constant $\kappa_{D+1} \to 0$, this does not tell us anything about the matter fields. Indeed, these in general require the same QFT treatment no matter if we take classical gravity limit, SUGRA limit or neither.

2.3.1 Equations of motion

While already the original AdS/CFT works include fermions as the field theory side is supersymmetric, it was not *a priori* clear how to construct dictionary entries for a fermionic observable in field theory. This problem was addressed in [84, 7, 56]. A more systematic rephrasing of the solution, which takes the viewpoint of holographic regularization, was given in [16]. We will mainly follow the reasoning of the latter reference as it is the most logically coherent exposition of the problem. Whereas the boundary action S_{bnd} needed to be picked by hand in earlier formulations, [16] shows that it follows logically from the requirement that the theory should be regular in the UV.

Kinematics and holography

Let us first discuss the kinematics of Dirac fermion; we have already announced that this will be the main source of trouble. The Dirac algebra in full AdS space ($D+1$-dimensional) is represented by gamma matrices Γ_μ, $\mu = 0, \ldots D$, and $\Gamma_D \equiv \Gamma_z$. The restriction of this representation to D dimensions, i.e. on the boundary, we will denote by γ_μ ($\mu = 0, \ldots D-1$). Recalling the table of the representations of Dirac algebra in various dimensions, we find that in odd number of dimensions $D+1$, i.e. for D even, there is a single spinor representation, whereas for D odd there are two irreducible representations of the Dirac algebra. We will mainly deal with this case in the thesis. In this case, Ψ is a bispinor and we can decompose it into two spinors $\Psi_\pm$. The choice of projection operator $\Pi_\pm$ is non-unique. In holography there is a natural choice which preserves all symmetries in the boundary theory: projection on the radial direction. Thus the projectors are $\Pi_\pm = (1 \pm \Gamma_z)/2$.

Dynamics

We are now ready to write the Dirac equation. We can always write it as a pair of coupled equations for $\Psi_\pm$. As we know, the Dirac equation reads

$$(\not{D} - m)\Psi = 0. \tag{2.31}$$

The covariant derivative includes the coupling to any gauge fields present and to the metric through the spin connection:

$$\not{D} = e^\mu_a \left(\partial_\mu + \frac{1}{8}\omega^{bc}_\mu \left[\Gamma_b, \Gamma_c\right] - iqe^\mu_a A_a \right). \tag{2.32}$$

From now on, we will denote the local tangential coordinates by Latin indices and the metric coordinates by Greek indices. The inverse vielbein is e_a^μ. From now on we will study a fermion in the homogenous background coupled to isotropic A_0 gauge field, describing a field theory at finite density. Taking into account homogeneity and isotropy of the system in transverse direction, we can partially Fourier-transform so that the derivative becomes $\partial_\mu \mapsto (-i\omega, ik, \partial_z)$. The spin connection, given in general by $\omega_\mu^{bc} = e_\nu^b \partial_\mu e^{\nu c} + e_\nu^b e^{\sigma c} \Gamma^\nu_{\sigma\mu}$, has only two nonzero components, ω_0^{0z} and ω_i^{iz}:

$$\begin{aligned} \omega_0^{0z} &= e_0^0 e^{zz} \Gamma^0_{z0} = \frac{1}{2} e_0^0 e^{zz} g^{00} \partial_z g_{00} = e^{zz} \partial_z e_0^0 \\ \omega_i^{iz} &= e_i^i e^{zz} \Gamma^i_{zi} = \frac{1}{2} e_i^i e^{zz} g^{ii} \partial_z g_{ii} = e^{zz} \partial_z e_i^i, \end{aligned} \tag{2.33}$$

Note that they can be formally written as total derivatives and as a consequence they can be absorbed in the redefinition of the fermion field in the following way. The equation of the form

$$\Gamma^z e_z^z \left[\partial_z + \partial_z \left(e_0^0 + (D-1)\, e_i^i\right)\right] \Psi + (\ldots)\Psi = 0, \tag{2.34}$$

where $(\ldots)$ denotes all terms containing no radial derivatives, can be rewritten as $\Gamma^z e_z^z \partial_z \psi + (\ldots)\psi = 0$ upon rescaling the Dirac field as

$$\Psi \mapsto \psi \equiv \Psi \sqrt{g_{00}\,(g_{ii})^{D-1}} = \Psi \sqrt{-g g^{zz}}. \tag{2.35}$$

This rescaling works generally for single parameter metrics. From now on throughout this chapter we will use the rescaling (2.35) and work with ψ and $\psi_\pm$ instead of Ψ and $\Psi_\pm$.

With the rescaling for the Dirac field, we can write the Dirac equation for ψ

$$\left[e_z^z \partial_z - \Gamma^z (iqe_0^\mu A^0 + m)\right] \psi = 0. \tag{2.36}$$

Next we decompose the equation into the equations for $\psi_\pm$. The result can be written as:

$$(\partial_z + e^{zz} m)\psi_\pm \pm \mathcal{T} \psi_\mp = 0 \tag{2.37}$$

where $\mathcal{T}$ is the transverse covariant derivative rescaled by the vielbein e_z^z:

$$-i\mathcal{T} = e_z^z e^{00} \gamma_0 (-\omega + qA_0) + e_z^z e^{ii} \gamma_i k_i. \tag{2.38}$$

Starting from the Dirac equation (2.38), we can eliminate either ψ_+ or ψ_- and readily derive a second order equation of motion for $\psi_\pm$. Using that $\not{\mathcal{T}}\not{\mathcal{T}} = -T_0 T^0 + T_i T^i \equiv \mathcal{T}^2$, we can invert $\not{\mathcal{T}}$ to rewrite

$$\frac{\not{\mathcal{T}}}{\mathcal{T}^2}\left(\partial_z + e_z^z m\right)\psi_+ = -\psi_- \tag{2.39}$$

and use the ψ_- equation to obtain

$$\left(\partial_z - m e_z^z\right)\frac{\not{\mathcal{T}}}{\mathcal{T}^2}\left(\partial_z + m e_z^z\right)\psi_+ = -\not{\mathcal{T}}\psi_+. \tag{2.40}$$

This finally brings us to the second-order form of the Dirac equation, for the spinor ψ_+. Denoting it as

$$\left(\partial_{zz} + \mathcal{P}\partial_z + \mathcal{Q}_+\right)\psi_+ = 0 \tag{2.41}$$

we have for the coefficients

$$\begin{aligned} \mathcal{P}(z) &= -[\partial_z, \not{\mathcal{T}}]\frac{\not{\mathcal{T}}}{\mathcal{T}^2} \\ \mathcal{Q}_+(q, m, \omega, k; z) &= -2m e_z^z + (\partial_z m e_z^z) - [\partial_z, \not{\mathcal{T}}]\frac{\not{\mathcal{T}}}{\mathcal{T}^2} m e_z^z + \mathcal{T}^2 \end{aligned} \tag{2.42}$$

For the second component ψ_- we get the same equation but with $\mathcal{Q}_- = \mathcal{Q}_+(-q, -m, -\omega, -k)$.

Of course, the second order equation implies the Dirac equation but is not equivalent to it. The necessary and sufficient condition for ψ_+, the solution of (2.41), to be also the solution to (2.37), reads

$$\psi_- = \frac{1}{\not{\mathcal{T}}}\left(\partial_z + m e_z^z\right)\psi_+. \tag{2.43}$$

It is instructive to solve the simplest case: that of pure AdS with no gauge fields. The field is rescaled as $\psi = \Psi / z^{(D+3)/2}$, and the second order equation for ψ_+ becomes

$$\left(\partial_{zz} - \frac{2m}{z} - \frac{m}{z^2}\right)\Psi_+ = 0 \tag{2.44}$$

which we readily recognize as the Bessel equation. It yields the following general solution:

$$\psi_+(z) = \frac{1}{z}\left(\psi_0^{(1)} K_{m+1/2}(kz) + \psi_0^{(2)} K_{m-1/2}(kz)\right), \tag{2.45}$$

where $K_{m\pm 1/2}$ are modified Bessel functions of the second kind. The near boundary asymptotics of the non-rescaled field Ψ_+ behaves as $\Psi_+ = \Psi_+^{(1)} z^{D/2-m} + \Psi_+^{(2)} z^{D/2+m}$. Clearly, $\Psi_+^{(1)}$ is always the leading, source term. But what is the response? Naively, it can be $\Psi_+^{(2)}$ as the subleading term. In the boundary action (2.48) we have however Ψ_- coupled linearly to the source Ψ_+ (which, with appropriate boundary conditions, becomes $\Psi_+^{(1)}$). Therefore, the response is Ψ_- with appropriate boundary asymptotics. Dirac equation tells that $\Psi_-^{(1)} \propto \Psi_+^{(2)}$ so we conclude that the response is $\Psi_-^{(1)}$.

2.3.2 Boundary action

Let us start again from the minimal bulk action for Dirac fermions coupled to gravity and possibly gauge fields:

$$S_{bulk} = S_{grav} + \int d^{D+1}x\sqrt{-g}\bar{\Psi}(\not{D} - m)\Psi + \ldots, \tag{2.46}$$

where $(\ldots)$ stand for any additional fields in the system. It is assumed that these will not change the UV behavior of fermions nor the AdS asymptotics of the background; they might change the background and thus also the fermionic behavior in IR but we will simply assume a given fixed IR whatever might be the fields which produce it. The issue is how to implement the dictionary. The Dirac action is famously proportional to Dirac equation and thus vanishes on shell. We have seen this also in the scalar sector however. The resolution is the existence of a boundary action, which in fact encodes the full holographic partition function. The objective is to construct it here for fermions. To do so, let us find the variation of the bulk part (disregarding again the parts we know: gravity and bosons). Since we work in a spacetime with a boundary, there will generically be a boundary contribution. Employing partial integration in (2.46) and varying with respect to ψ, we get:

$$\begin{aligned} \delta S_{bulk} &= \delta \int d^{D+1}x\sqrt{-g}\bar{\psi}(\not{D} - m)\psi = \\ &= \int d^D x\sqrt{-h}\bar{\psi}\delta\psi|_{z_0}^{z_h} - \int d^{D+1}x\sqrt{-g}(-\not{D} - m)\bar{\psi}\delta\psi. \end{aligned} \tag{2.47}$$

The second, bulk term vanishes on shell as it is proportional to the equation of motion. The first, boundary term does not vanish however. It is to be evaluated on the boundary of AdS in UV and at z_h in IR.[5] In terms of the radial projections, it reads

$$\delta S = \frac{1}{2}\int d^D x\sqrt{-h}\left(\bar{\psi}_+\delta\psi_- + \bar{\psi}_-\delta\psi_+\right). \tag{2.48}$$

We know from general rules of AdS/CFT that one of the components of ψ will be the source and the other the response, and in the previous subsection we have seen that the leading component of ψ_+ is larger (i.e. decays slower) at the boundary than the leading component of ψ_-. We can therefore pick ψ_+ to be the source. This means that ψ_+ is fixed at the boundary and its variation is zero: $\delta\psi_+ = 0$. The variation of the action now reduces to the first term in (2.48). To cancel ad we can add a counterterm reading

$$S_{ct} = \frac{1}{2}\int d^D x\sqrt{-h}(\bar{\psi}_+\psi_- + \bar{\psi}_-\psi_+) \tag{2.49}$$

and the whole action is given by $S = S_{bulk} + S_{ct}$, so $S_{bnd} \equiv S_{ct}$: the whole boundary contribution can be understood as the counterterm which regularizes the action, eliminating UV divergences and making the on-shell solution satisfy the Dirac equation in the bulk.

For the steps to follow it is convenient to introduce the bulk-to-boundary propagator $\mathcal{G}_\pm(z)$ and to express the solution in terms of $\mathcal{G}_\pm$. The bulk-to-boundary propagator satisfies the equation of motion [114]:

$$(\not{D} - m)\,\mathcal{G}(z) = \delta(z) \tag{2.50}$$

i.e. it is a response to a Dirac delta function source at the boundary. We can now express the solution to Dirac equation in terms of $\mathcal{G}_\pm$ and $\chi_\pm$. The expression for $\psi_\pm$ reads

$$\psi_+ = \mathcal{G}_+^{-1}(z_0)\mathcal{G}_+(z)\chi_+, \quad \psi_- = \mathcal{G}_+^{-1}(z_0)\mathcal{G}_-(z)S\chi_- \tag{2.51}$$

where $S = \lim_{z\to 0}\mathcal{T}/\not{\mathcal{T}}$. Namely, at the boundary the energy-momentum dependence can be shown to drop from the factor $\mathcal{T}/\not{\mathcal{T}}$, leaving only a constant matrix (which of course depends on the representation of gamma matrices, hence we do not specify it here). The convenience of the above representation of ψ is that all z dependence of ψ is encoded in the bulk-to-boundary propagators. Substituting (2.51) into the boundary action, we

[5] The latter is a single point if $z_h \to \infty$ or a slice in the transverse direction if z_h is finite.

obtain an expression for the full on-shell action in terms of the solutions $\mathcal{G}_{\pm}(z)$:

$$S^{on-shell} = \int_{z=z_0} \frac{d\omega d^2k}{(2\pi)^3} \sqrt{-h}\bar{\chi}_+ \mathcal{G}_-(z_0)\mathcal{G}_+^{-1}(z_0)\chi_+. \tag{2.52}$$

The two-point correlator in field theory is therefore

$$G(\omega, k) = \mathcal{G}_-(z_0)\mathcal{G}_+^{-1}(z_0). \tag{2.53}$$

What this illustrates is that the subleading component of Ψ_- is the response to the leading component of Ψ_+. This will be the starting point of the work done in most of Chapter 3 and 4.

2.4 The remainder of the thesis

Having discussed the larger context in the first chapter and the theoretical foundations and previous work on the topic of our research in this, second chapter, we have finished introducing the formal framework of our work. We now outline the work done in this thesis on specific problems with fermion systems. We will use the power of holography to describe strongly coupled systems from a new fundamental perspective, to circumvent the sign problem. We stay exclusively with bottom-up setups. The first reason is their obvious simplicity as compared to top-down constructions which become particularly complicated if fermions are included. A deeper reason is that the conceptual aspects we consider such as the dictionary entry for a Fermi surface, or for a Fermi liquid state, or pathways through which Fermi liquids are destroyed – are not expected to depend much on the exact string action.

Another compromise with consistency that we have to decide about is the choice between self-consistent calculations, with backreaction, versus probe limit calculations. We start from the probe limit and afterwards include backreaction, first on gauge field and then also on geometry. Of course, probe limit suffices at small fermion density, when the backreaction is anyway small, but becomes less and less satisfactory as the density increases. The field theory interpretation is that backreaction probes the stability of the system – unstable quantum critical matter is described by the probe limit calculations, but to arrive at stable phases we need to backreact. In particular, the Fermi-liquid-like phase (which we know empirically to be very stable) requires backreaction.

In Chapter 3 we address the critical theory governing the zero temperature quantum phase transition between strongly renormalized Fermi-liquids as found in heavy fermion intermetallics and possibly high T_c superconductors. From the solutions of Dirac equation in the probe limit in the AdS-RN background, we obtain the spectral functions of fermions in the field theory. By increasing the fermion density away from the relativistic quantum critical point, we observe multiple Fermi surfaces, some of them of distinctly non-Fermi liquid nature while others have some features of the Fermi liquid. Tuning the scaling dimensions of the critical fermion fields we find that the quasiparticle disappears at a quantum phase transition of a purely statistical nature, not involving any symmetry change. The resulting phase has no Fermi surfaces at all.

In Chapter 4 we extend our work by backreacting on gauge field. We provide evidence that the bulk dual to a strongly coupled charged Fermi-liquid-like system has a non-zero fermion density in the bulk. We then calculate density explicitly in the small density approximation, a model we call black hole with Dirac hair. Then we show that the pole strength of the stable quasiparticle characterizing the Fermi surface is encoded in the spatially averaged AdS probability density of a single normalizable fermion wave function in AdS. Recalling Migdal's theorem which relates the pole strength to the Fermi-Dirac characteristic discontinuity in the number density at Fermi energy, we conclude that the AdS dual of a Fermi liquid is described by occupied on-shell fermionic modes in AdS. Encoding the occupied levels in the total spatially averaged probability density of the fermion field directly, we show that an AdS Reissner-Nordström black hole in a theory with charged fermions has a critical temperature, at which the system undergoes a first-order transition to a black hole with a non-vanishing profile for the bulk fermion field. Thermodynamics and spectral analysis support that the solution with non-zero AdS fermion-profile is the preferred ground state at low temperatures.

In Chapter 5 we continue our study of self-consistent (backreacted) models and move toward constructing the full phase diagram of the Dirac-Maxwell-Einstein system and its field theory dual. We compare our Dirac hair model with the electron star model of Hartnoll et all [51], and argue that the electron star and the AdS Dirac hair solution are two limits of the free charged Fermi gas in AdS. Spectral functions of holographic duals to probe fermions in the background of electron stars have a free parameter that quantifies the number of constituent fermions that make up the charge and energy density characterizing the electron star solution. The strict electron star limit takes this number to be infinite. The Dirac hair solution is the limit where this number is unity. This is evident in the behavior of the distribution of holographically dual Fermi surfaces. As we decrease the number of constituents in a fixed electron star background the number of Fermi surfaces also decreases. An improved holographic Fermi ground state should be a configuration that shares the qualitative properties of both limits.

We construct such configuration in Chapter 6. We employ a model which combines the (semiclassical) WKB approximation and its Airy correction with the quantum corrections based on Dirac equation. At high temperatures, the system exhibits a first order thermal phase transition to a charged AdS-RN black hole in the bulk and the emergence of local quantum criticality on the CFT side. This restores the intuition that the transition between the critical AdS-RN liquid and the finite density Fermi system is of van der Waals liquid-gas type. At zero temperature, we find a Berezhinsky-Kosterlitz-Thouless transition from Fermi-liquid-like finite density phase with a sharp Fermi surface to zero density AdS-Reissner-Nordström but in the regime without Fermi surfaces. This suggests that it is indeed the Fermi surface which drives the instability of the AdS-RN quantum critical phase. Based on these findings, we construct the three-dimensional phase diagram, with temperature, conformal dimension and fermion charge.

Even though we have not answered some of the questions we started from, in particular the question of what is the holographic dual to a textbook Landau Fermi liquid and how it is destroyed by strong interactions, we have obtained a qualitative model of how stable Fermi-liquid-*like* quasiparticles become unstable at a quantum critical point and give rise to novel phenomena. These phenomena could not be obtained in a perturbative approach and they illustrate the power of AdS/CFT and its ability to make specific predictions on strongly correlated fermions. These predictions have not been tested experimentally so far. Because of many simplifying assumptions and the lack of ability to construct a microscopic Hamiltonian on the boundary, our results are unlikely to be a good quantitative description of any realistic system. Nevertheless, they make some remarkable qualitative predictions which can be expected to hold also in real-world materials, due to the universality associated to quantum critical behavior. The coming years will surely determine whether the novel physics on display in AdS/CMT is a part of the real world.