Crossed product algebras associated with topological dynamical systems

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Stellingen

Propositions belonging to the thesis

Crossed product algebras associated with topological dynamical systems

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The following definitions concern Stellingen 1 and 2.

For $i = 1, 2$, let X_i be a compact Hausdorff space, σ_i a homeomorphism of X_i and Σ_i the corresponding topological dynamical system. Denote by $C^*(\Sigma_i)$ the associated C^*-crossed product. The analytic crossed product $A^+(\Sigma_i)$ associated with Σ_i is the (nonselfadjoint) closed subalgebra of $C^*(\Sigma_i)$ generated by $C(X) \cup \{\delta\}$. It follows from [2, Theorem 1] that Σ_1 and Σ_2 are topologically conjugate if and only if $A^+(\Sigma_1)$ and $A^+(\Sigma_2)$ are isomorphic as algebras. Denote by $\ell^1_+(\Sigma_i)$ the (nonselfadjoint) closed subalgebra of $\ell^1(\Sigma_i)$ generated by $C(X) \cup \{\delta\}$. It follows from the more general work in [1] that Σ_1 and Σ_2 are topologically conjugate if and only if $\ell^1_+(\Sigma_1)$ and $\ell^1_+(\Sigma_2)$ are isomorphic as algebras.

1. For $i = 1, 2$, let $\Sigma_i = (X_i, \sigma_i)$ be a topological dynamical system as above. Denote by $k^+(\Sigma_i)$ the (nonselfadjoint) subalgebra of $k(\Sigma_i)$ generated by $C(X) \cup \{\delta\}$. Then Σ_1 and Σ_2 are topologically conjugate if and only if $k^+(\Sigma_1)$ and $k^+(\Sigma_2)$ are isomorphic as algebras.

2. As in the cases of $A^+(\Sigma_i)$ and $\ell^1_+(\Sigma_i)$, the above result holds true if the X_i are merely locally compact and one constructs the analogues of $k^+(\Sigma_i)$ using $C_0(X_i)$ as "coefficient algebras" instead of $C(X_i)$.

3. Let Σ be a topological dynamical system consisting of r disjoint finite orbits of orders $n_1, n_2, \ldots, n_r$. Then

$$C^*(\Sigma) \cong M_{n_1}(C(\mathbb{T})) \oplus M_{n_2}(C(\mathbb{T})) \oplus \ldots \oplus M_{n_r}(C(\mathbb{T}))$$

as C^*-algebras. Consequently, the C^*-crossed product of a topological dynamical system on a finite space is never simple. It also follows that systems consisting of one single finite orbit constitute the only examples of minimal systems that have non-simple C^*-algebras.

4. Let $\Sigma = (X, \sigma)$ be such that $X = \{x, \sigma(x)\}$ and $\sigma(x) \neq x$. Then $C^*(\Sigma) \cong M_2(C(\mathbb{T}))$. Denote

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id}^* & -\mathrm{id}^* \end{pmatrix}.$$

Then the map $\mathrm{Ad}\, U$ defined, for $a \in M_2(C(\mathbb{T}))$, by $a \mapsto UaU^*$ is a nice concrete example of an automorphism of $C^*(\Sigma)$ which does not preserve the "coefficient algebra" $C(X)$.

5. Recall from Chapter 5 that the commutant, $C(X)'$, of $C(X)$ in $C^*(\Sigma)$ is commutative. Denoting by X' the spectrum of $C(X)'$ we may then write $C(X)' = C(X')$. Consider the topological dynamical system $\Sigma' = (X', \sigma')$ where σ' is the homeomorphism induced by the automorphism $\mathrm{Ad}\,\delta$ of $C(X)'$. If $C(X) \subsetneq C(X)'$ or, equivalently, if the system Σ is not topologically free, the algebra $C^*(\Sigma)$ is naturally embedded as a proper C^*-subalgebra of $C^*(\Sigma')$. That Σ is not topologically free implies that Σ' is also not topologically free, hence repeating this argument, we get a strictly increasing infinite sequence of C^*-crossed products:

$$C^*(\Sigma) \subsetneq C^*(\Sigma') \subsetneq C^*(\Sigma^{''}) \subsetneq C^*(\Sigma^{'''}) \subsetneq \ldots$$

6. By making use of the formula in [3, Proposition 1] for approximating elements in $C^*(\Sigma)$ with elements in $k(\Sigma)$ by generalized Cèsaro sums, it is possible to give more direct proofs of some of the existing theorems regarding the interplay between a system Σ and its C^*-algebra $C^*(\Sigma)$.

7. Denote, for $a \in C^*(\Sigma)$, its generalized Fourier coefficents by $(a(n))_{n=-\infty}^{\infty}$, where $a(n) \in C(X)$ for all n. If $aa^* = 1$, then, for all $x \in X$,

$$\sum_{n=-\infty}^{\infty} |a(n)(x)|^2 = 1.$$

8. Throughout the history of the world championships in football, Sweden has won a larger number of medals than the Netherlands.

9. A mathematician is a person finding abstract nonsense less cumbersome than concrete nonsense.

10. The fact that professional mathematicians can often see the material taught in undergraduate courses from perspectives not yet available to the students can sometimes make it hard for them to recall which concepts are particularly difficult to grasp for someone being introduced to them for the first time.

References

[1] Davidson, K.R., Katsoulis, E.G., *Isomorphisms between topological conjugacy algebras*, J. Reine Angew. Math. **621** (2008), 29-51.

[2] Power, S.C., *Classification of analytic crossed product algebras*, Bull. London Math. Soc. **24** (1992), 368-372.

[3] Tomiyama, J., *Structure of Ideals and Isomorphisms of C^*-crossed Products by Single Homeomorphisms*, Tokyo J. Math. **23** (2000), 1-13.