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Crossed product algebras associated with topological dynamical systems

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Chapter 6

On the Banach $*$ -algebra crossed product associated with a topological dynamical system

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Abstract. Given an arbitrary topological dynamical system $\Sigma = (X, \sigma)$, where X is a compact Hausdorff space and σ a homeomorphism of X , we introduce and analyze the associated Banach $*$ -algebra crossed product $\ell^1(\Sigma)$. The C^* -envelope of this algebra is the usual C^* -crossed product of $C(X)$ by the integers under the automorphism of $C(X)$ induced by σ . While the connections between the structure of this C^* -algebra and the properties of Σ are well-studied, such considerations concerning $\ell^1(\Sigma)$ are new. We derive equivalences between topological dynamical properties of Σ and structural properties of $\ell^1(\Sigma)$ that have well-known analogues in the C^* -algebra context, but also obtain a result on this so-called interplay whose counterpart in the case of $C^*(\Sigma)$ is false.

6.1. Introduction

The interplay between topological dynamical systems and C^* -algebras has been intensively studied, e.g. in [13] and [14] where for an arbitrary topological dynamical system $\Sigma = (X, \sigma)$ one associates a crossed product C^* -algebra $C^*(\Sigma)$ with it. This is the C^* -crossed product of $C(X)$ by the integers under the automorphism of $C(X)$ induced by σ . It is shown in [8], [9] and [10] that a number of connections between topological dynamics and C^* -algebras as appearing in [13] and [14] have an analogue for a certain dense $*$ -subalgebra $k(\Sigma)$ of $C^*(\Sigma)$. Conversely, analogues of results obtained in the setup in [8], [9] and [10] have later been proven in the context of the interplay between Σ and $C^*(\Sigma)$. Namely, the

result [11, Corollary 4.4], which says that the commutant of $C(X)$ in $C^*(\Sigma)$ always has the intersection property for ideals, is an analogue of [9, Theorem 6.1] and [10, Theorem 3.1].

One way of obtaining $C^*(\Sigma)$ is as the enveloping C^* -algebra of a certain Banach $*$ -algebra, $\ell^1(\Sigma)$, which we define in Section 6.2. Because $\ell^1(\Sigma)$ has sufficiently many Hilbert space representations, $C^*(\Sigma)$ contains a $*$ -isomorphic copy of $\ell^1(\Sigma)$ as a dense $*$ -subalgebra. Furthermore, $\ell^1(\Sigma)$ contains a $*$ -isomorphic copy of $k(\Sigma)$ as a dense $*$ -subalgebra. The inclusions are easily seen to be strict and we may write, up to $*$ -isomorphisms, $k(\Sigma) \subsetneq \ell^1(\Sigma) \subsetneq C^*(\Sigma)$. Inspired by the fact that several theorems on the interplay between Σ and $C^*(\Sigma)$ have analogues if the latter is replaced by $k(\Sigma)$, we investigate $\ell^1(\Sigma)$ and in particular the connection between its structural properties and the topological dynamical properties of Σ . Although $\ell^1(\Sigma)$ is the Banach $*$ -algebra of crossed product type which is most naturally associated with Σ , it seems that this connection has only briefly been studied so far (cf. [10]).

In Section 6.2 we define the properties of topological dynamical systems that we shall investigate, recall the simple key result Lemma 6.2.1 and go through the construction of $\ell^1(\Sigma)$ in detail. Furthermore, we introduce some basic definitions, and an extension theorem (Theorem 6.2.5), from the theory of ordered linear spaces which we use, together with elementary theorems on Banach $*$ -algebras, to deduce a certain extension result for states on Banach $*$ -algebras (Proposition 6.2.10) that will be useful to us. We also recall the explicit description of a certain collection of pure states of $C^*(\Sigma)$, as appearing in [13] and [14], that we shall exploit in our setup. In Section 6.3 we describe the commutant of $C(X)$ in $\ell^1(\Sigma)$, which we denote by $C(X)'$, explicitly and conclude that $C(X)' = C(X)$ precisely when Σ is topologically free (Theorem 6.3.2). We show that $C(X)'$ is commutative, hence the unique maximal abelian Banach $*$ -subalgebra of $\ell^1(\Sigma)$ that contains $C(X)$ (Proposition 6.3.3), and describe its character space (Theorem 6.3.4). We prove that $C(X)'$ has non-zero intersection with every non-zero closed (not necessarily self-adjoint) ideal of $\ell^1(\Sigma)$, regardless of Σ (Theorem 6.3.7). In Section 6.4 we use Theorem 6.3.7 to conclude a number of analogues of results on the interplay between Σ and $C^*(\Sigma)$. Theorem 6.4.1 says that $C(X)$ has the intersection property for closed ideals in $\ell^1(\Sigma)$ if and only if it is a maximal abelian Banach $*$ -subalgebra of $\ell^1(\Sigma)$, which is in turn equivalent to topological freeness of Σ . Theorem 6.4.2 states that $\ell^1(\Sigma)$ is simple if and only if Σ is minimal, under the condition that X is infinite. Theorem 6.4.5 says that if X is infinite, $\ell^1(\Sigma)$ is prime if and only if Σ is topologically transitive. We also give simple counter examples of Theorem 6.4.2 and Theorem 6.4.5 when X is finite. In Section 6.5 we use a deep result from abstract harmonic analysis (Theorem 6.5.1) to prove that every closed ideal of $\ell^1(\Sigma)$ is self-adjoint if and only if Σ is free (Theorem 6.5.2). This is a result lacking an obvious analogon in the context of the interplay between Σ and $C^*(\Sigma)$ as closed ideals of C^* -algebras are always self-adjoint.

6.2. Definitions and preliminaries

Throughout this paper we consider topological dynamical systems $\Sigma = (X, \sigma)$, where X is a compact Hausdorff space and $\sigma : X \rightarrow X$ is a homeomorphism. Here $\mathbb{Z}$ acts on X via iterations of σ , namely $x \mapsto \sigma^n(x)$ for $n \in \mathbb{Z}$ and $x \in X$. We denote by $\text{Per}^\infty(\sigma)$ and

$\text{Per}(\sigma)$ the sets of aperiodic points and periodic points, respectively. If $\text{Per}^\infty(\sigma) = X$, Σ is called free and if $\text{Per}^\infty(\sigma)$ is dense in X , Σ is called topologically free. Moreover, for an integer n we write $\text{Per}^n(\sigma) = \text{Per}^{-n}(\sigma) = \{x \in X : \sigma^n(x) = x\}$, and $\text{Per}_n(\sigma)$ for the set of all points belonging to $\text{Per}^n(\sigma)$ but to no $\text{Per}^k(\sigma)$ with $|k|$ non-zero and strictly less than $|n|$. When $n = 0$ we have $\text{Per}^0(\sigma) = X$. We write $\text{Per}(x) = k$ if $x \in \text{Per}_k(\sigma)$, with $k > 0$. Note that if $\text{Per}(y) = k$, with $k > 0$, and $y \in \text{Per}^n(\sigma)$, then $k|n$. For a subset $S \subseteq X$ we denote its interior by S^0 and its closure by $\bar{S}$. When a periodic point y belongs to the interior of $\text{Per}_k(\sigma)$ for some positive integer k we call y a periodic interior point. We denote the set of all such points by $\text{PIP}(\sigma)$. Note that $\text{PIP}(\sigma)$ does not coincide with $\text{Per}(\sigma)^0$ in general, as the following example shows. Let $X = [0, 1] \times [-1, 1]$ be endowed with the standard subspace topology from $\mathbb{R}^2$ and let σ be the homeomorphism of X defined as reflection in the x -axis. Then clearly $\text{Per}(\sigma) = X$. Furthermore, $\text{Per}_1(\sigma) = [0, 1] \times \{0\}$, hence $\text{Per}_1(\sigma)^0 = \emptyset$, and $\text{Per}_2(\sigma) = X \setminus \text{Per}_1(\sigma)$, so that $\text{Per}_2(\sigma)^0 = \text{Per}_2(\sigma)$. We conclude that $\text{PIP}(\sigma) = \text{Per}_2(\sigma) \subsetneq X = \text{Per}(\sigma)^0$. Incidentally, $\text{Per}_2(\sigma)$ in this example also shows that the sets $\text{Per}_n(\sigma)$ are in general not closed, as opposed to the sets $\text{Per}^n(\sigma)$ which are easily seen to always be closed.

The following topological lemma will be important to us throughout this paper. For a proof we refer to [11, Lemma 2.1].

Lemma 6.2.1. *The union of $\text{Per}^\infty(\sigma)$ and $\text{PIP}(\sigma)$ is dense in X .*

Given a dynamical system $\Sigma = (X, \sigma)$ and a point $x \in X$, we denote by $\mathcal{O}_\sigma(x) = \{\sigma^n(x) : n \in \mathbb{Z}\}$ the orbit of x in the system. Recall that a dynamical system $\Sigma = (X, \sigma)$ is called minimal if every orbit of Σ is dense in X . It is called topologically transitive if for any pair of non-empty open sets U, V of X , there exists an integer n such that $\sigma^n(U) \cap V \neq \emptyset$.

We denote by α the automorphism of $C(X)$ induced by σ via $\alpha(f) = f \circ \sigma^{-1}$ for $f \in C(X)$. Via $n \mapsto \alpha^n$, the integers act on $C(X)$ by iterations. Given a topological dynamical system $\Sigma = (X, \sigma)$, we shall endow the set

$$\ell^1(\Sigma) = \{a : \mathbb{Z} \rightarrow C(X) : \sum_{k \in \mathbb{Z}} \|a(k)\|_\infty < \infty\},$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $C(X)$, with the structure of a Banach $*$ -algebra. As in [12], we understand a Banach $*$ -algebra (or involutive Banach algebra) to be a complex Banach algebra with an *isometric* involution. We define scalar multiplication and addition on $\ell^1(\Sigma)$ as the natural pointwise operations. Multiplication is defined by convolution twisted by α as follows:

$$(ab)(n) = \sum_{k \in \mathbb{Z}} a(k) \cdot \alpha^k(b(n-k)),$$

for $a, b \in \ell^1(\Sigma)$. We define the involution, $*$, as

$$a^*(n) = \overline{\alpha^n(a(-n))},$$

for $a \in \ell^1(\Sigma)$. The bar denotes the usual pointwise complex conjugation. Finally, we define a norm on $\ell^1(\Sigma)$ by

$$\|a\| = \sum_{k \in \mathbb{Z}} \|a(k)\|_\infty,$$

for $a \in \ell^1(\Sigma)$. It is not difficult to check that when endowed with these operations, $\ell^1(\Sigma)$ is indeed a Banach *-algebra. A useful way of working with $\ell^1(\Sigma)$ is to write an element $a \in \ell^1(\Sigma)$ in the form $a = \sum_{k \in \mathbb{Z}} a_k \delta^k$, for $a_k = a(k)$ and $\delta = \chi_{\{1\}}$ where, for $n, m \in \mathbb{Z}$,

$$\chi_{\{n\}}(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

It is then clear that $\delta^* = \delta^{-1}$ and that $\delta^n = \chi_{\{n\}}$, where $n \in \mathbb{Z}$. In the rest of this paper we shall use the notation a_k rather than $a(k)$, for $a \in \ell^1(\Sigma)$ and $k \in \mathbb{Z}$. Clearly one may canonically view $C(X)$ as a closed abelian *-subalgebra of $\ell^1(\Sigma)$, namely as $\{a\delta^0 : a \in C(X)\}$. Thus $\ell^1(\Sigma)$ is generated as a Banach *-algebra by an isometrically isomorphic copy of $C(X)$ and the unitary element δ , subject to the relation $\delta f \delta^* = \alpha(f)$ for $f \in C(X)$. We let $k(\Sigma) = \{\sum_k f_k \delta^k : \text{only finitely many } f_k \text{ are non-zero}\}$ and note that $k(\Sigma)$ is a dense *-subalgebra of $\ell^1(\Sigma)$. We write the canonical projection of norm one from $\ell^1(\Sigma)$ to $C(X)$ as E , where $E(\sum_k a_k \delta^k) = a_0$. Note that if $a = \sum_k a_k \delta^k$ then $E(a\delta^{*j}) = a_j$ for every integer j , and that $E(fag) = fE(a)g$ for $f, g \in C(X)$. Furthermore it is easy to show that for a finite collection of scalars $\lambda_i \geq 0$ and $a_i \in \ell^1(\Sigma)$, $E(\sum_i \lambda_i a_i^* a_i) \geq 0$ and that $E(\sum_i \lambda_i a_i^* a_i) = 0$ implies $\sum_i \lambda_i a_i^* a_i = 0$.

In our analysis of $\ell^1(\Sigma)$ we shall make use of the theory of ordered linear spaces in the sense of [3], whence we recall a number of basic definitions appearing there. Although all linear spaces in [3] have the reals as scalar field, we restate the definitions here for linear spaces over a field $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 6.2.2. Let V be a linear space. A linear ordering of V is a binary relation $\leq$, not necessarily anti-symmetric, on V such that

- (i) $v \leq w$ and $w \leq u$ implies $v \leq u$ for all $v, w, u \in V$;
- (ii) $v \leq w$ implies $v + u \leq w + u$ for all $v, w, u \in V$;
- (iii) $v \leq w$ and $\lambda \geq 0$ implies $\lambda v \leq \lambda w$ for all $v, w \in V$ and $\lambda \in \mathbb{R}$.

An ordered linear space is a linear space over $\mathbb{F}$ with a linear ordering.

Definition 6.2.3. Let V be an ordered linear space with linear ordering $\leq$. A linear subspace $W \subseteq V$ is said to be cofinal if given any $v \in V$ with $0 \leq v$, there exists a $w \in W$ such that $v \leq w$.

Definition 6.2.4. Let V be an ordered linear space with linear ordering $\leq$. A linear functional $f : V \rightarrow \mathbb{F}$, where $\mathbb{F}$ is the scalar field of V , is said to be positive if $0 \leq v$ implies $0 \leq f(v)$ for all $v \in V$.

We are now ready to state the following extension theorem, which will be a key result for us.

Theorem 6.2.5. [3, Theorem 1.6.1.] Suppose W is a cofinal linear subspace of an ordered real linear space V . Then a positive linear functional on W can be extended to a positive linear functional on V .

To apply this theorem in our setup, we need to recall two results from the theory of Banach $*$ -algebras. First, however, we recall the standard linear ordering on a Banach $*$ -algebra.

Definition 6.2.6. Let A be a Banach $*$ -algebra. We introduce a linear ordering on A by defining, for $a, b \in A$, $a \leq b$ if and only if $b - a = \sum_{i=1}^n \lambda_i c_i^* c_i$ for some $c_i \in A$ and λ_i non-negative real numbers. An element $a \in A$ is called non-negative if $0 \leq a$.

We can now state the results we need.

Lemma 6.2.7. [12, Lemma I.9.8] Let A be a unital Banach algebra. If a is an element of A such that the spectral radius of $1 - a$ is strictly less than one, then there exists $b \in A$ with $b^2 = a$. Furthermore, if A is a Banach $*$ -algebra and if a is self-adjoint, then a self-adjoint element can be chosen as the above b .

Lemma 6.2.8. [12, Lemma I.9.9] If A is a unital Banach $*$ -algebra, then every positive linear functional ω of A is continuous and $\|\omega\| = \omega(1)$.

Recall the following definition.

Definition 6.2.9. Let A be a unital Banach $*$ -algebra. A positive linear functional f on A such that $f(1) = 1$ is called a state (we also use this terminology for positive linear functionals on arbitrary $*$ -subalgebras of A). We denote by $S(A)$ the set of states of A endowed with the weak $*$ topology. The space $S(A)$ is then a convex and compact subset of the unit ball in the dual of A by Lemma 6.2.8 together with the Banach-Alaoglu theorem. We call the extreme points of $S(A)$ pure states. If A is commutative, its pure states are precisely its characters whence we shall refer to the pure states of A as the characters of A in this situation.

The following extension result for Banach $*$ -algebras will be a cornerstone in what follows.

Proposition 6.2.10. Let A be a unital Banach $*$ -algebra and let B be a Banach $*$ -subalgebra containing the unit element of A . Then every state of B extends to a state of A , and every pure state of B extends to a pure state of A .

Proof. Viewing the set of self-adjoint elements of A as a real linear space, it follows from Lemma 6.2.7 that the set of self-adjoint elements of B constitutes a real linear cofinal subspace of it. To see this, suppose a is a self-adjoint element of A . Then $b = 1 - \frac{a}{\|a\| + \epsilon}$, where ϵ is some positive real number, is a self-adjoint element such that $1 - b$ has norm, and hence spectral radius, strictly less than one. Hence by Lemma 6.2.7 there exists $c \in \ell^1(\Sigma)$ such that $b = c^*c$, and thus $(\|a\| + \epsilon) - a = (\|a\| + \epsilon)c^*c$. Hence $a \leq (\|a\| + \epsilon)$ and clearly $(\|a\| + \epsilon) \in B$. By Theorem 6.2.5 this implies that, when we view the set of self-adjoint elements of A as a real linear space and the set of self-adjoint elements of B as a real linear

subspace of it, positive linear functionals on the latter extend to positive linear functionals on the former. Since every element in a Banach *-algebra can be written uniquely as $a = a_1 + ia_2$ with a_1, a_2 self-adjoint, it follows easily that a positive complex linear functional on B has a positive complex linear extension to A . Invoking Theorem 6.2.8 we thus see that states of B have state extensions to A . Now suppose that φ is a pure state of B . The set of state extensions of φ to A is a weak*-compact convex set by the Banach-Alaoglu theorem, whence by the Krein-Milman theorem there exists an extension $\tilde{\varphi}$ that is extreme amongst all state extensions of φ . To see that $\tilde{\varphi}$ is a pure state, suppose there exist states $\tilde{\psi}_1, \tilde{\psi}_2$ and λ with $0 < \lambda < 1$ such that $\tilde{\varphi} = \lambda\tilde{\psi}_1 + (1 - \lambda)\tilde{\psi}_2$. Denoting by ψ_i , for $i = 1, 2$, the restriction of $\tilde{\psi}_i$ to B we then have $\varphi = \lambda\psi_1 + (1 - \lambda)\psi_2$. Since φ is pure, we see that $\psi_1 = \psi_2 = \varphi$. Thus $\tilde{\psi}_1$ and $\tilde{\psi}_2$ also yield φ when restricted to B . Since $\tilde{\varphi}$ is extreme amongst all state extensions of φ , we conclude that $\tilde{\psi}_1 = \tilde{\psi}_2 = \tilde{\varphi}$. $\square$

Although $\ell^1(\Sigma)$ is our main object of study, we shall sometimes refer to its enveloping C^* -algebra, which we denote by $C^*(\Sigma)$. For a detailed account of the general theory of the interplay between Σ and $C^*(\Sigma)$ we refer to [13] and [14]. For particular aspects of it akin to the work in this paper, see [11]. A few known results on $C^*(\Sigma)$ appearing in [13] and [14] need to be mentioned here, however. As is explained in [13, Section 3.2], $\ell^1(\Sigma)$ has sufficiently many Hilbert space representations, whence its enveloping C^* -algebra, $C^*(\Sigma)$, contains a dense *-isomorphic copy of it. It follows from [2, Theorem 2.7.5 (i)] that restriction gives a bijection between the states of $C^*(\Sigma)$ and those of $\ell^1(\Sigma)$, and between the pure states of $C^*(\Sigma)$ and those of $\ell^1(\Sigma)$.

For $x \in X$ we denote by μ_x the functional on $C(X)$ that acts as point evaluation in x . The pure state extensions of such point evaluations to $\ell^1(\Sigma)$ will play a prominent role in this paper. We shall exploit the fact that the pure state extensions to $C^*(\Sigma)$ of point evaluations on $C(X)$ have been described explicitly. We now recall some basic facts about them, without proofs. For further details and proofs, we refer to [14, §4]. For $x \in \text{Per}^\infty(\sigma)$ there is a unique pure state extension of μ_x , denoted by φ_x , given by $\varphi_x = \mu_x \circ E$ (here E denotes the continuous extension to $C^*(\Sigma)$ of the projection $E : \ell^1(\Sigma) \rightarrow C(X)$ as defined above). The set of pure state extensions of μ_y for $y \in \text{Per}(\sigma)$ is parametrized by the unit circle as $\{\varphi_{y,t} : t \in \mathbb{T}\}$. We denote the GNS-representations of $C^*(\Sigma)$ associated with the pure state extensions above by $\tilde{\pi}_x$ and $\tilde{\pi}_{y,t}$, respectively. For $x \in \text{Per}^\infty(\sigma)$, $\tilde{\pi}_x$ is the representation of $C^*(\Sigma)$ on ℓ^2 , whose standard basis we denote by $\{e_i\}_{i \in \mathbb{Z}}$, defined on the generators as follows. For $f \in C(X)$ and $i \in \mathbb{Z}$ we have $\tilde{\pi}_x(f)e_i = f \circ \sigma^i(x) \cdot e_i$, and $\tilde{\pi}_x(\delta)e_i = e_{i+1}$. For $y \in \text{Per}(\sigma)$ with $\text{Per}(y) = p > 0$ and $t \in \mathbb{T}$, $\tilde{\pi}_{y,t}$ is the representation on $\mathbb{C}^p$, whose standard basis we denote by $\{e_i\}_{i=0}^{p-1}$, defined as follows. For $f \in C(X)$ and $i \in \{0, 1, \dots, p-1\}$ we set $\tilde{\pi}_{y,t}(f)e_i = f \circ \sigma^i(y) \cdot e_i$. For $j \in \{0, 1, \dots, p-2\}$, $\tilde{\pi}_{y,t}(\delta)e_j = e_{j+1}$ and $\tilde{\pi}_{y,t}(\delta)e_{p-1} = t \cdot e_0$. We also mention that the unitary equivalence class of $\tilde{\pi}_x$ is determined by the orbit of x , and that of $\tilde{\pi}_{y,t}$ by the orbit of y and the parameter t . We shall abuse notation slightly and use the same symbol for a pure state of $C^*(\Sigma)$ as for its restriction to $\ell^1(\Sigma)$, and similarly for the associated GNS-representation. Hence, denoting by Φ the set of all pure state extensions of point evaluations on $C(X)$ to $\ell^1(\Sigma)$, we have $\Phi = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,t} : y \in \text{Per}(\sigma), t \in \mathbb{T}\}$. We shall also use the subset Φ' of Φ defined by $\Phi' = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,t} : y \in \text{PIP}(\sigma), t \in \mathbb{T}\}$.

When speaking of closed ideals in $\ell^1(\Sigma)$, we shall always mean closed two-sided ideals

that are not necessarily self-adjoint. We say that $\ell^1(\Sigma)$ is simple if it lacks proper non-zero closed ideals, and we call $\ell^1(\Sigma)$ prime if the intersection of any pair of non-zero closed ideals is non-zero.

6.3. The commutant of $C(X)$

We will now analyze the commutant of $C(X)$ in $\ell^1(\Sigma)$, denoted by $C(X)'$ and defined as

$$C(X)' = \{a \in \ell^1(\Sigma) : af = fa \text{ for all } f \in C(X)\}.$$

One easily concludes that it is a Banach $*$ -subalgebra of $\ell^1(\Sigma)$. We need the following topological lemma, for a proof of which we refer to [11, Lemma 3.1].

Lemma 6.3.1. *The system $\Sigma = (X, \sigma)$ is topologically free if and only if $\text{Per}^n(\sigma)$ has empty interior for all positive integers n .*

We give the following concrete description of $C(X)'$.

Proposition 6.3.2. *$C(X)' = \{\sum_k a_k \delta^k \in \ell^1(\Sigma) : \text{supp}(a_k) \subseteq \text{Per}^k(\sigma) \text{ for all } k\}$. Consequently, $C(X)' = C(X)$ if and only if the dynamical system is topologically free.*

Proof. The assertion is an adaption of [8, Corollary 3.4] to our context, but we include a proof here for the reader's convenience. Suppose $a = \sum_k a_k \delta^k \in C(X)'$. For any f in $C(X)$ we then have

$$\begin{aligned} f a_k &= E(f a \delta^{*k}) = E(a f \delta^{*k}) = E(a \delta^{*k} \alpha^k(f)) = \\ &= E(a \delta^{*k}) \cdot \alpha^k(f) = a_k \cdot \alpha^k(f). \end{aligned}$$

Hence for all $x \in X$

$$f(x) a_k(x) = a_k(x) f \circ \sigma^{-k}(x).$$

Therefore, if $a_k(x)$ is not zero we have that $f(x) = f \circ \sigma^{-k}(x)$ for all $f \in C(X)$. It follows that $\sigma^{-k}(x) = x$, i.e. x belongs to $\text{Per}^k(\sigma)$. Since $\text{Per}^k(\sigma)$ is closed, $\text{supp}(a_k) \subseteq \text{Per}^k(\sigma)$. Conversely, if $\text{supp}(a_k) \subseteq \text{Per}^k(\sigma)$ for all k , then $f(x) a_k(x) = a_k(x) f \circ \sigma^{-k}(x)$ for every k and x , so that $(fa)_k = (af)_k$ for all k , i.e. a commutes with f . This establishes the description of $C(X)'$.

Moreover, by Lemma 6.3.1, Σ is topologically free if and only if for every nonzero integer k the set $\text{Per}^k(\sigma)$ has empty interior. So when the system is topologically free, we see from the above description of $C(X)'$ that an element a in $C(X)'$ necessarily belongs to $C(X)$. If Σ is not topologically free, however, $\text{Per}^k(\sigma)$ has non-empty interior for some non-zero k and hence there is a non-zero function $f \in C(X)$ such that $\text{supp}(f) \subseteq \text{Per}^k(\sigma)$. Then $f \delta^k \in C(X)' \setminus C(X)$ by the above. $\square$

The following elementary result is an adaption of [8, Proposition 2.1] to our setup.

Proposition 6.3.3. *The commutant $C(X)'$ of $C(X)$ is abelian, and thus it is the unique maximal abelian Banach $*$ -subalgebra of $\ell^1(\Sigma)$ containing $C(X)$.*

Proof. Suppose $a, b \in C(X)'$. By definition of the multiplication in $\ell^1(\Sigma)$ it follows that for an integer n , $(ab)_n = \sum_k a_k \cdot \alpha^k(b_{n-k})$. As $a \in C(X)'$, it follows from Proposition 6.3.2 that $(ab)_n = \sum_k a_k \cdot b_{n-k}$. Similarly, $(ba)_n = \sum_k b_k \cdot a_{n-k}$. Thus $(ab)_n = (ba)_n$ for all integers n and hence $ab = ba$. $\square$

We now determine the characters of $C(X)'$ in terms of the sets Φ and Φ' as introduced in Section 6.2.

Theorem 6.3.4. *The characters of $C(X)'$ are precisely the restrictions of elements in Φ to $C(X)'$. Furthermore, the restriction map is injective on Φ' .*

Proof. By Proposition 6.2.10, characters of $C(X)'$ have pure state extensions to $\ell^1(\Sigma)$. Of course, the restriction of a character of $C(X)'$ to $C(X)$ is a character of the latter, and since we have concluded that characters of $C(X)'$ have pure state extensions to $\ell^1(\Sigma)$, it follows that every character of $C(X)'$ is the restriction of some element of Φ . So to prove the first assertion it suffices to show that every element of Φ is multiplicative on $C(X)'$. By Proposition 6.3.2, $C(X)'$ is the closed linear span of its monomials, by which we mean elements of the form $f_n \delta^n$. This implies that it suffices to check multiplicativity on the monomials of $C(X)'$. Let $f_n \delta^n, f_m \delta^m \in C(X)'$ and note that by Proposition 6.3.2 this implies that $\text{supp}(f_n) \subseteq \text{Per}^n(\sigma)$ and $\text{supp}(f_m) \subseteq \text{Per}^m(\sigma)$. Suppose first that $x \in \text{Per}^\infty(\sigma)$ and consider the pure state $\varphi_x = \mu_x \circ E$ on $\ell^1(\Sigma)$. By definition of φ_x , it follows that

$$\varphi_x(f_n \delta^n) = \begin{cases} 0, & n \neq 0; \\ f_n(x), & n = 0, \end{cases}$$

and similarly for $f_m \delta^m$. Furthermore $f_n \delta^n f_m \delta^m = f_n \cdot f_m \circ \sigma^{-n} \delta^{n+m}$ and thus

$$\varphi_x(f_n \delta^n f_m \delta^m) = \begin{cases} 0, & n \neq -m; \\ f_n(x) \cdot f_m \circ \sigma^{-n}(x), & n = -m. \end{cases}$$

As $x \in \text{Per}^\infty(\sigma)$ the support conditions on f_n and f_m imply that

$$\varphi_x(f_n \delta^n f_m \delta^m) = \begin{cases} 0, & n, m \text{ not both zero}; \\ f_n(x) f_m(x), & n = m = 0. \end{cases}$$

So clearly $\varphi(f_n \delta^n f_m \delta^m) = \varphi(f_n \delta^n) \varphi(f_m \delta^m)$.

Now suppose that $y \in \text{Per}_k(\sigma)$ for some integer $k > 0$. Let $t \in \mathbb{T}$ and consider $\varphi_{y,t}$. Then

$$\varphi_{y,t}(f_n \delta^n) = (\tilde{\pi}_{y,t}(f_n \delta^n) e_0, e_0) = \begin{cases} t^{\frac{n}{k}} f_n(y), & \text{if } k|n; \\ 0, & \text{otherwise.} \end{cases}$$

and similarly for $f_m \delta^m$. Furthermore

$$\varphi_{y,t}(f_n \delta^n f_m \delta^m) = \begin{cases} t^{\frac{n+m}{k}} f_n(y) f_m \circ \sigma^{-n}(y), & \text{if } k|(n+m); \\ 0, & \text{otherwise.} \end{cases}$$

As $y \in \text{Per}_k(\sigma)$, however, the support conditions on f_n and f_m imply that

$$\varphi_{y,t}(f_n \delta^n f_m \delta^m) = \begin{cases} t^{\frac{n+m}{k}} f_n(y) f_m(y), & \text{if } k|n \text{ and } k|m; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\varphi_{y,t}(f_n \delta^n f_m \delta^m) = \varphi_{y,t}(f_n \delta^n) \varphi_{y,t}(f_m \delta^m)$.

To prove the second assertion, note firstly that by the discussion in the end of Section 6.2 we know that for $x \in \text{Per}^\infty(\sigma)$, φ_x is the unique pure state extension of μ_x from $C(X)$ to $\ell^1(\Sigma)$. Hence no other element in Φ than φ_x itself coincides with φ_x on $C(X)'$. Now suppose $y \in \text{PIP}(\sigma)$ with $y \in \text{Per}_k(\sigma)^0$ for some positive integer k , and let $t \in \mathbb{T}$. Suppose, by contradiction, that there are two pure state extensions $\varphi_{y,s}$ and $\varphi_{y,t}$ of μ_y , with $s \neq t$, that coincide on $C(X)'$. Let $f \in C(X)$ be a function such that $f(y) = 1$ and $\text{supp}(f) \subseteq \text{Per}^k(\sigma)$. Then by Theorem 6.3.2 it follows that $f \delta^k \in C(X)'$. Furthermore

$$\varphi_{y,s}(f \delta^k) = f(y)s = s \neq t = f(y)t = \varphi_{y,t}(f \delta^k),$$

which is a contradiction. $\square$

Before continuing our investigation of $C(X)'$, we state a result on ideals of $\ell^1(\Sigma)$.

Proposition 6.3.5. *Let I be a proper (not necessarily closed or self-adjoint) ideal in $\ell^1(\Sigma)$. Viewing I as a subset of $C^*(\Sigma)$ under the canonical embedding, the closure of I in $C^*(\Sigma)$ is proper as well.*

Proof. Since $\ell^1(\Sigma)$ is unital and hence has no proper dense ideals, the closed ideal J generated by the set $\{a^*a : a \in I\}$ is self-adjoint and proper in $\ell^1(\Sigma)$. We prove first that there is a state of $\ell^1(\Sigma)$ that vanishes on J . To see this, consider the state, of the Banach $*$ -subalgebra $\mathbb{C} + J \subseteq \ell^1(\Sigma)/J$, defined by $\lambda + J \mapsto \lambda$ for $\lambda \in \mathbb{C}$. By Proposition 6.2.10 it has a state extension, f say, to $\ell^1(\Sigma)/J$. Denoting by $\pi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma)/J$ the natural quotient map it is clear that $f \circ \pi$ is a state of $\ell^1(\Sigma)$ that vanishes on J . Suppose now that the closure of I inside $C^*(\Sigma)$ coincides with $C^*(\Sigma)$. Then there is a sequence $(a_n) \in I$ that converges to 1 in the norm of $C^*(\Sigma)$. The sequence $(a_n^*a_n)$ then converges to 1 as well. The elements of this sequence, however, are in J whence the state extension of $f \circ \pi$ to $C^*(\Sigma)$ vanishes on 1 by continuity, a contradiction. $\square$

We make the following definition.

Definition 6.3.6. A Banach $*$ -subalgebra B of $\ell^1(\Sigma)$ is said to have the intersection property for closed ideals if for every non-zero closed ideal $I \subseteq \ell^1(\Sigma)$ we have $I \cap B \neq \{0\}$.

Finally we are ready to prove the main result of this section.

Theorem 6.3.7. *$C(X)'$ has the intersection property for closed ideals in $\ell^1(\Sigma)$.*

Proof. Suppose that I is a closed non-zero ideal such that $I \cap C(X)' = \{0\}$. Define J to be the closed ideal generated by the set $\{a^*a : a \in I\}$. Then J is easily seen to be a non-zero closed self-adjoint ideal of $\ell^1(\Sigma)$ contained in I . Hence $\ell^1(\Sigma)/J$ is a unital Banach $*$ -algebra. Clearly $J \cap C(X)' = \{0\}$ and, denoting by $\pi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma)/J$ the natural quotient map, we may algebraically identify $C(X)'$ with its isomorphic embedding $\pi(C(X)')$ in $\ell^1(\Sigma)/J$ and write this embedded algebra as $C(X)'/J$. As in the proof of Proposition 6.2.10, one concludes by Theorem 6.2.5 together with Lemma 6.2.7 that states on $C(X)'/J$ have state extensions to $\ell^1(\Sigma)/J$. We shall use this fact to show that every character of $C(X)'$ has a pure state extension to $\ell^1(\Sigma)$ that vanishes on J . Fix a character ω

of $C(X)'$. Then $\omega \circ (\pi|_{C(X)'})^{-1}$ is clearly a state on $C(X)'/J$. It is easy to see that the set of states of $\ell^1(\Sigma)/J$ that extend $\omega \circ (\pi|_{C(X)'})^{-1}$ constitutes a non-empty weak*-closed subset of the unit ball in the dual of $\ell^1(\Sigma)/J$, whence by the Banach-Alaoglu theorem those states form a weak*-compact subset which is clearly also convex. By the Krein-Milman theorem this set is the closed convex hull of its extreme points, so there is a state extension ω' of $\omega \circ (\pi|_{C(X)'})^{-1}$ to $\ell^1(\Sigma)/J$ that is an extreme point of the set of all those state extensions of $\omega \circ (\pi|_{C(X)'})^{-1}$. Using the same technique as in the proof of Proposition 6.2.10, one concludes that the fact that ω is pure on $C(X)'$ implies that ω' is even an extreme point of the set of all states on $\ell^1(\Sigma)/J$, and hence a pure state of it. For completeness, we give a proof of this fact. Suppose there is a $\lambda \in (0, 1)$ and two states ξ_1, ξ_2 of $\ell^1(\Sigma)/J$ such that $\omega' = \lambda\xi_1 + (1 - \lambda)\xi_2$. Restricting to $C(X)'/J$ we get

$$\omega \circ (\pi|_{C(X)'})^{-1} = \lambda\xi_1|_{C(X)'/J} + (1 - \lambda)\xi_2|_{C(X)'/J}$$

and hence

$$\omega = \lambda\xi_1 \circ \pi|_{C(X)'} + (1 - \lambda)\xi_2 \circ \pi|_{C(X)'}$$

But ω was pure so it follows that $\xi_1 \circ \pi|_{C(X)'} = \xi_2 \circ \pi|_{C(X)'} = \omega$. Finally we conclude that $\xi_1|_{C(X)'/J} = \xi_2|_{C(X)'/J} = \omega \circ (\pi|_{C(X)'})^{-1}$, so ξ_1, ξ_2 were extensions of $\omega \circ (\pi|_{C(X)'})^{-1}$, whence by assumption $\xi_1 = \xi_2 = \omega'$ and thus ω' is a pure state of $\ell^1(\Sigma)/J$. By the GNS-construction, pure states of Banach *-algebras correspond to irreducible Hilbert space representations. This makes it easy to see that $\omega' \circ \pi$ is a pure state of $\ell^1(\Sigma)$ extending ω and vanishing on J . As restrictions to $C(X)'$ of elements in Φ' have unique pure state extensions to $\ell^1(\Sigma)$ by Theorem 6.3.4, it follows that every element in Φ' vanishes on J . This means that all pure state extensions to $\ell^1(\Sigma)$ of point evaluations in $\text{Per}^\infty(\sigma)$ and $\text{PIP}(\sigma)$ vanish on J . Thus clearly $\varphi_x = \mu_x \circ E$ vanishes on J for $x \in \text{Per}^\infty(\sigma)$, and $\mu_y \circ E$, with $y \in \text{PIP}(\sigma)$, vanishes on J as well by the Krein-Milman theorem since it is a state extension of μ_y and we have already concluded that all pure state extensions of μ_y to $\ell^1(\Sigma)$ vanish on J . Thus the states $\{\mu_x \circ E : x \in \text{Per}^\infty(\sigma) \cup \text{PIP}(\sigma)\}$ all vanish on J . Now suppose $a \in J$ and fix an arbitrary $k \in \mathbb{Z}$. By the above $\mu_x \circ E(a\delta^{*k}) = a_k(x) = 0$ for all $x \in \text{Per}^\infty(\sigma) \cup \text{PIP}(\sigma)$. By Lemma 6.2.1 we conclude that $a_k \equiv 0$. As k was arbitrary, it follows that $a = 0$ and we can finally conclude that $J = \{0\}$ and hence also that $I = \{0\}$. This is a contradiction, and thus $I \cap C(X)' \neq \{0\}$ as asserted. $\square$

6.4. Consequences of the intersection property of $C(X)'$

Theorem 6.3.7 allows us to prove a number of analogues of theorems on the interplay between Σ and $C^*(\Sigma)$ appearing e.g. in [13] and [14]. We begin with the following analogue of [14, Theorem 5.4].

Theorem 6.4.1. *For a topological dynamical system Σ , the following statements are equivalent.*

- (i) Σ is topologically free;
- (ii) $I \cap C(X) \neq 0$ for every non-zero closed ideal I of $\ell^1(\Sigma)$;

(iii) $C(X)$ is a maximal abelian Banach $*$ -subalgebra of $\ell^1(\Sigma)$.

Proof. Equivalence of (i) and (iii) is an immediate consequence of Proposition 6.3.2 together with Proposition 6.3.3. To see that (i) implies (ii) note that, by Proposition 6.3.2, (i) implies that $C(X) = C(X)'$ and thus (ii) follows by Theorem 6.3.7. To show that (ii) implies (i), we shall use the same technique as in the proof of [14, Theorem 5.4]. Suppose that Σ is not topologically free. Then by Lemma 6.3.1 there is a positive integer n such that $\text{Per}^n(\sigma)$ has non-empty interior. Let $f \in C(X)$ be non-zero and such that $\text{supp}(f) \subseteq \text{Per}^n(\sigma)$ and consider the closed ideal I of $\ell^1(\Sigma)$ generated by $f - f\delta^n$. Note that all representations in $\{\tilde{\pi}_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\tilde{\pi}_{y,1} : y \in \text{Per}(\sigma)\}$ vanish on $f - f\delta^n$ and hence on I . To see this, first note that for $x \in \text{Per}^\infty(\sigma)$ and $i \in \mathbb{Z}$ we have $\tilde{\pi}_x(f - f\delta^n)e_i = f \circ \sigma^i(x)e_i - f \circ \sigma^{i+n}(x)e_{i+n} = 0$ since f is zero outside $\text{Per}(\sigma)$. Similarly it follows that $\tilde{\pi}_{y,1}(f - f\delta^n) = 0$ when y does not have period dividing n . If y has period k where $n = r \cdot k$ then for $i \in \{0, 1, \dots, k-1\}$ we have $\tilde{\pi}_{y,1}(f - f\delta^n)e_i = f \circ \sigma^i(y)e_i - 1^r \cdot f \circ \sigma^i(y)e_i = 0$. This clearly implies that the family $\Phi'' = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,1} : y \in \text{Per}(\sigma)\}$ vanishes on I . Suppose now that $g \in C(X) \cap I$. For every point $z \in X$ there is a pure state extension of μ_z in Φ'' . Hence $\mu_z(g) = g(z) = 0$ and we conclude that $g \equiv 0$ and hence that $I \cap C(X) = \{0\}$. $\square$

The following result is analogous to [14, Theorem 5.3], [1, Theorem VIII 3.9] and the main result in [6].

Theorem 6.4.2. *Suppose that X consists of infinitely many points. Then $\ell^1(\Sigma)$ is simple if and only if $\Sigma = (X, \sigma)$ is minimal.*

Proof. Suppose that Σ is not minimal. Then there is a point $x \in X$ such that $\overline{\mathcal{O}_\sigma(x)} \neq X$. Note that $\overline{\mathcal{O}_\sigma(x)}$ is invariant under σ and its inverse. Define

$$I = \{a \in \ell^1(\Sigma) : a_k \in \ker(\overline{\mathcal{O}_\sigma(x)}) \text{ for all integers } k\}$$

where $\ker(\overline{\mathcal{O}_\sigma(x)}) = \{f \in C(X) : f \text{ vanishes on } \overline{\mathcal{O}_\sigma(x)}\}$. It is easy to see that I is a proper non-zero closed ideal of $\ell^1(\Sigma)$, which is thus not simple. Conversely, suppose that $\ell^1(\Sigma)$ is not simple and let I be a proper non-zero closed ideal of it. If Σ is minimal the fact that X is infinite clearly implies that Σ is topologically free, since $X = \text{Per}^\infty(\sigma)$. By Theorem 6.4.1 it follows that $I \cap C(X) \neq \{0\}$. It is not difficult to see that $I \cap C(X)$ is a closed ideal of $C(X)$ that is invariant under α and its inverse. It is clearly proper since $I \cap C(X) = C(X)$ would imply that $I = \ell^1(\Sigma)$. As the closed ideals of $C(X)$ are precisely the kernels of closed subsets of X we may write $I \cap C(X) = \ker(C)$, where C is some proper non-empty closed subset of X . It also follows that C is invariant under σ and its inverse, since $I \cap C(X)$ is invariant under α and its inverse. This contradicts the minimality of Σ . $\square$

The assertion does not hold if we drop the condition that X be infinite. Consider for example the case when $\Sigma = (\{x\}, \text{id})$. Then $\ell^1(\Sigma)$ is easily seen to be isometrically $*$ -isomorphic to ℓ^1 , which is not simple since its character space is non-empty. The system Σ , however, is trivially minimal.

We conclude this section by proving the analogue of [14, Theorem 5.5]. To do this, we need the following two easy topological lemmas.

Lemma 6.4.3. *If $\Sigma = (X, \sigma)$ is not topologically transitive, then there exist two disjoint non-empty open sets O_1 and O_2 , both invariant under σ and its inverse, such that $\overline{O_1} \cup \overline{O_2} = X$.*

Proof. As the system is not topologically transitive, there exist non-empty open sets $U, V \subseteq X$ such that for any integer n we have $\sigma^n(U) \cap V = \emptyset$. Now clearly the set $O_1 = \bigcup_{n \in \mathbb{Z}} \sigma^n(U)$ is a non-empty open set invariant under σ and σ^{-1} . Then $\overline{O_1}$ is a closed set invariant under σ and σ^{-1} . It follows that $O_2 = X \setminus \overline{O_1}$ is an open set, invariant under σ and σ^{-1} , containing V . Thus we even have that $\overline{O_1} \cup O_2 = X$, and the result follows. $\square$

Lemma 6.4.4. *If $\Sigma = (X, \sigma)$ is topologically transitive and there is an $n > 0$ such that $X = \text{Per}^n(\sigma)$, then X consists of a single orbit and is thus finite.*

Proof. Assume two points $x, y \in X$ are not in the same orbit. As X is Hausdorff we may separate the points $x, \sigma(x), \dots, \sigma^{n-1}(x), y$ by pairwise disjoint open sets $V_0, V_1, \dots, V_{n-1}, V_y$. Now consider the set

$$U_x := V_0 \cap \sigma^{-1}(V_1) \cap \sigma^{-2}(V_2) \cap \dots \cap \sigma^{-n+1}(V_{n-1}).$$

Clearly the sets $A_x = \bigcup_{i=0}^{n-1} \sigma^i(U_x)$ and $A_y = \bigcup_{i=0}^{n-1} \sigma^i(V_y)$ are disjoint non-empty open sets, both invariant under σ and σ^{-1} , which leads us to a contradiction. Hence X consists of one single orbit under σ . $\square$

Theorem 6.4.5. *Suppose that X consists of infinitely many points. Then $\ell^1(\Sigma)$ is prime if and only if $\Sigma = (X, \sigma)$ is topologically transitive.*

Proof. Suppose first that the system Σ is not topologically transitive. Then there exists, by Lemma 6.4.3, two disjoint non-empty open sets O_1 and O_2 , both invariant under σ and σ^{-1} , such that $\overline{O_1} \cup \overline{O_2} = X$. Let I_1 and I_2 be the closed ideals generated in $\ell^1(\Sigma)$ by $\ker(\overline{O_1})$ and $\ker(\overline{O_2})$ respectively. It is then not difficult to see that, for $i = 1, 2$, we have

$$I_i = \left\{ \sum_n f_n \delta^n \in \ell^1(\Sigma) : f_n \in \ker(\overline{O_i}) \text{ for all } n \right\}$$

and hence that $E(I_i) = \ker(\overline{O_i})$. Hence

$$E(I_1 \cap I_2) \subseteq E(I_1) \cap E(I_2) = \ker(\overline{O_1}) \cap \ker(\overline{O_2}) = \ker(\overline{O_1} \cup \overline{O_2}) = \ker(X) = \{0\}.$$

Now note that if I is an ideal and $E(I) = \{0\}$, then $I = \{0\}$. Namely, suppose that $a \in I$. Then for an arbitrary integer n we have that $a_n = E(a \delta^{*n}) = 0$ and hence $a = 0$. Applying this to $I_1 \cap I_2$, we see that $I_1 \cap I_2 = \{0\}$, hence $\ell^1(\Sigma)$ is not prime. Next suppose that Σ is topologically transitive. We claim that Σ is topologically free. If not, then by Lemma 6.3.1 there is an integer $n > 0$ such that $\text{Per}^n(\sigma)$ has non-empty interior. As $\text{Per}^n(\sigma)$ is invariant under σ and σ^{-1} and closed, topological transitivity implies that $X = \text{Per}^n(\sigma)$. This, however, is impossible since by Lemma 6.4.4 it would force X to consist of a single orbit and hence be finite. Thus Σ is topologically free after all. Now let I and J be two non-zero proper closed ideals in $\ell^1(\Sigma)$ and assume by contradiction that $I \cap J = \{0\}$. Then

$I \cap C(X)$ and $J \cap C(X)$ are proper closed ideals of $C(X)$ with zero intersection that are invariant under α and its inverse, and topological freeness of Σ assures us that they are non-zero, by Theorem 6.4.1. This implies that there are proper non-empty closed subsets C_1, C_2 of X that are invariant under σ and its inverse and such that $I \cap C(X) = \ker(C_1)$ and $J \cap C(X) = \ker(C_2)$. Now $\{0\} = I \cap J \cap C(X) = \ker(C_1) \cap \ker(C_2) = \ker(C_1 \cup C_2)$ whence $C_1 \cup C_2 = X$. Since C_2 is proper and closed, $(C_1)^0 \supset X \setminus C_2 \neq \emptyset$. Since C_1 is proper and closed, $X \setminus C_1$ is open and non-empty. Invariance of C_1 under σ and its inverse implies that $\sigma^n((C_1)^0) \cap (X \setminus C_1) = \emptyset$ for all integers n . This contradicts topological transitivity of Σ and we conclude that $I \cap J \neq \{0\}$. It follows that $\ell^1(\Sigma)$ is prime. $\square$

This theorem is also false if the condition that X be infinite is dropped. Again, consider the case when we have $\Sigma = (\{x\}, \text{id})$. As mentioned after the proof of Theorem 6.4.2, $\ell^1(\Sigma)$ is isometrically $*$ -isomorphic to ℓ^1 . It is well known that the character space of ℓ^1 can be identified with $\mathbb{T}$ with its standard topology. Consider two proper non-empty closed subsets C_1, C_2 of $\mathbb{T}$ such that $C_1 \cup C_2 = \mathbb{T}$. Then $\ker(C_i) = \{a \in \ell^1 : a \text{ is annihilated by every character in } C_i\}$, for $i = 1, 2$, are two proper closed ideals of ℓ^1 that are non-zero by regularity of ℓ^1 . Semi-simplicity of ℓ^1 implies that $\ker(C_1) \cap \ker(C_2) = \ker(\mathbb{T}) = \{0\}$ and we conclude that ℓ^1 is not prime. Trivially, however, Σ is topologically transitive.

6.5. Closed ideals of $\ell^1(\Sigma)$ which are not self-adjoint

We will determine when all closed ideals of $\ell^1(\Sigma)$ are self-adjoint. Our approach is based on the following special case of the rather deep general result [7, Theorem 7.7.1].

Theorem 6.5.1. *ℓ^1 contains a closed ideal which is not self-adjoint.*

Now we can establish the following.

Theorem 6.5.2. *Every closed ideal of $\ell^1(\Sigma)$ is self-adjoint if and only if Σ is free.*

Proof. Suppose Σ is free, hence in particular topologically free, and let $I \subseteq \ell^1(\Sigma)$ be a non-zero closed ideal. Denote by $\pi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma)/I$ the natural quotient map. Then $I \cap C(X)$ is easily seen to be a closed ideal of $C(X)$ that is invariant under α and its inverse and that is non-zero by Theorem 6.4.1. Hence we may write $I \cap C(X) = \ker(X_\pi)$ for some closed subset X_π of X that is invariant under σ and its inverse. Denote by σ_π the restriction of σ to X_π and write $\Sigma_\pi = (X_\pi, \sigma_\pi)$. We shall show that π can be factored in a certain way. Denote by $\phi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma_\pi)$ the $*$ -homomorphism defined by $\sum_k f_k \delta^k \mapsto \sum_k f_k|_{X_\pi} \delta^k$. By Tietze's extension theorem every function in $C(X_\pi)$ can be extended to a function in $C(X)$, and an easy application of Urysohn's lemma shows that one can choose an extension whose norm is arbitrarily close to the norm of the function one extends. Using this, it is not difficult to show that the map $\Psi : \ell^1(\Sigma_\pi) \rightarrow \ell^1(\Sigma)/I$ defined by $\sum_k \tilde{f}_k \delta^k \mapsto \sum_k \tilde{f}_k \delta^k + I$, where the $\tilde{f}_k$ are such that $\sum_k \tilde{f}_k \delta^k \in \ell^1(\Sigma)$ and $\tilde{f}_k|_{X_\pi} = f_k$, is a well-defined contractive homomorphism. We note that $\pi = \Psi \circ \phi$. Since $\ker(\Psi)$ is a closed ideal of $\ell^1(\Sigma_\pi)$, $\ker(\Psi) \cap C(X_\pi) = \{0\}$ and Σ_π is free, hence topologically free, it

follows from Theorem 6.4.1 that Ψ is injective. Thus $I = \ker(\pi) = \ker(\phi)$ and the latter is self-adjoint since ϕ is a $*$ -homomorphism.

Conversely, suppose that some $x \in X$ has period $p > 0$. We will use the p -dimensional GNS-representations $\tilde{\pi}_{x,z}$, where $z \in \mathbb{T}$, of $\ell^1(\Sigma)$, which are associated with the periodic point p as described in Section 6.2, to construct a continuous surjective $*$ -homomorphism $\Psi : \ell^1(\Sigma) \rightarrow M_p(\ell^1)$. Here $M_p(\ell^1)$ has its natural structure as a $*$ -algebra and is a Banach space under the norm $\|A\| = \max_{1 \leq i, j \leq p} \|A_{i,j}\|$, where $A \in M_p(\ell^1)$. Then, if I is a closed non-self-adjoint ideal of ℓ^1 as in Theorem 6.5.1, $M_p(I)$ is a closed non-self-adjoint ideal of $M_p(\ell^1)$ and hence, since Ψ is surjective, $\Psi^{-1}(M_p(I))$ is a closed non-self-adjoint ideal of $\ell^1(\Sigma)$. To construct Ψ we first note that ℓ^1 is isomorphic, as a $*$ -algebra, to the algebra $\text{AC}(\mathbb{T})$ of continuous functions on $\mathbb{T}$ with an absolutely convergent Fourier series. The isomorphism is the Fourier transform, given by $\mathcal{F}((\dots, a_{-1}, a_0, a_1, \dots))(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. It yields a natural $*$ -isomorphism $j : M_p(\ell^1) \rightarrow M_p(\text{AC}(\mathbb{T}))$. We will construct a surjective $*$ -homomorphism $\theta : \ell^1(\Sigma) \rightarrow M_p(\text{AC}(\mathbb{T}))$ and then $j^{-1} \circ \theta$ will be the desired continuous $*$ -homomorphism Ψ . To define θ , we recall that the GNS-representations $\tilde{\pi}_{x,z} : \ell^1(\Sigma) \rightarrow M_p(\mathbb{C})$ are such that

$$\tilde{\pi}_{x,z}(f) = \begin{pmatrix} f(x) & 0 & \dots & 0 \\ 0 & f \circ \sigma(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f \circ \sigma^{p-1}(x) \end{pmatrix}$$

for $f \in C(X)$, and

$$\tilde{\pi}_{x,z}(\delta) = \begin{pmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

One sees that

$$\tilde{\pi}_{x,z}(\delta^2) = \begin{pmatrix} 0 & 0 & \dots & 0 & z & 0 \\ 0 & 0 & \dots & 0 & 0 & z \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{\pi}_{x,z}(\delta^3) = \begin{pmatrix} 0 & 0 & \dots & z & 0 & 0 \\ 0 & 0 & \dots & 0 & z & 0 \\ 0 & 0 & \dots & 0 & 0 & z \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 \end{pmatrix},$$

etc., and that

$$\tilde{\pi}_{x,z}(\delta^p) = \begin{pmatrix} z & 0 & \dots & 0 & 0 \\ 0 & z & \dots & 0 & 0 \\ 0 & 0 & z & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & z \end{pmatrix}.$$

Calculating, it is not difficult to see that

$$\tilde{\pi}_{x,z}(\sum_k f_k \delta^k) = \begin{pmatrix} \sum_k f_{kp}(x)z^k & \cdots & \sum_k f_{(k-1)p+1}(x)z^k \\ \sum_k f_{kp+1}(\sigma(x))z^k & \cdots & \sum_k f_{(k-1)p+2}(\sigma(x))z^k \\ \sum_k f_{kp+2}(\sigma^2(x))z^k & \cdots & \sum_k f_{(k-1)p+3}(\sigma^2(x))z^k \\ \vdots & \vdots & \vdots \\ \sum_k f_{(k+1)p-1}(\sigma^{p-1}(x))z^k & \cdots & \sum_k f_{kp}(\sigma^{p-1}(x))z^k \end{pmatrix}.$$

This makes it obvious that the representations $\tilde{\pi}_{x,z}$ combine naturally to a $*$ -homomorphism $\theta : \ell^1(\Sigma) \rightarrow M_p(\text{AC}(\mathbb{T}))$. Moreover, θ is surjective. Indeed, since each complex number $f_k(\sigma^i(x))$ occurs only once (somewhere in row $i+1$) as a Fourier coefficient in the matrix of $\theta(f_k \delta^k)$, one sees that prescribing the image of $\theta(\sum_k f_k \delta^k)$ amounts to prescribing the numbers $f_k(\sigma^i(x))$ in an unambiguous way. The Urysohn lemma therefore implies that θ is surjective. Finally, defining $s(i, j) \in \mathbb{Z}$ such that for all $z \in \mathbb{T}$ the (i, j) entry of $\theta(\sum_k f_k \delta^k)(z)$ is $\sum_{n=-\infty}^{\infty} f_{np+s(i,j)}(\sigma^{i-1}(x))z^n$, we see that the (i, j) entry of $j^{-1} \circ \theta(\sum_k f_k \delta^k)$ is the sequence $\{f_{np+s(i,j)}(\sigma^{i-1}(x))\}_{n=-\infty}^{\infty}$. This makes it obvious that $j^{-1} \circ \theta : \ell^1(\Sigma) \rightarrow M_p(\ell^1)$ is not only a surjective $*$ -homomorphism but also continuous as desired. $\square$

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