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Leiden
The Netherlands

Crossed product algebras associated with topological dynamical systems

Svensson, P.C.

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Chapter 1

Introduction

Consider a pair $\Sigma = (X, \sigma)$ consisting of a compact Hausdorff space X and a homeomorphism σ of X . We shall understand Σ as a topological dynamical system by letting the integers act on X via iterations of σ . Denote by $C(X)$ the algebra of continuous complex-valued functions on X endowed with the supremum norm and the natural pointwise operations. The map $\alpha : C(X) \rightarrow C(X)$ defined, for $f \in C(X)$, by $\alpha(f) = f \circ \sigma^{-1}$ is then easily seen to be an automorphism of $C(X)$. Conversely, given a pair $(C(X), \alpha)$, where X is compact Hausdorff and α is an automorphism of $C(X)$, there exists a unique homeomorphism σ of X such that for all $f \in C(X)$ we have $\alpha(f) = f \circ \sigma^{-1}$. To realize this, denote by $\Delta(C(X))$ the set of all characters, i.e. all non-zero multiplicative linear functionals, of $C(X)$ and note firstly that α permutes this set by composition. Namely, denoting the permutation by $\hat{\alpha}$, we have, for $\zeta \in \Delta(C(X))$, that $\zeta \xrightarrow{\hat{\alpha}} \zeta \circ \alpha$. Secondly, recall that $\Delta(C(X))$ can be shown to coincide with the set of all point evaluations of $C(X)$. Denoting, for $x \in X$, such an evaluation by μ_x , the permutation $\hat{\alpha}$ above induces a bijection, σ , of X by $\mu_x \xrightarrow{\hat{\alpha}} \mu_x \circ \alpha = \mu_{\sigma^{-1}(x)}$. Thus, for $f \in C(X)$, we have indeed that $\alpha(f)(x) = f \circ \sigma^{-1}(x)$ for all $x \in X$ and furthermore one can show that σ is a homeomorphism and unique, as desired. Hence, studying the dynamical system Σ is equivalent to studying the pair $(C(X), \alpha)$, where the integers act on $C(X)$ via iterations of α . Given another compact Hausdorff space, Y , one can use an argument similar to the above to conclude that there exists a homeomorphism between X and Y if and only if $C(X)$ is isomorphic, as an algebra, to $C(Y)$, whence to study the space X is equivalent to study the algebra $C(X)$. Having this appealing correspondence in mind, it is quite natural to try to transplant the pair $(C(X), \alpha)$ above into some algebraic object such that its structure reflects topological dynamical properties of the system $\Sigma = (X, \sigma)$. Since $C(X)$ is a typical commutative unital C^* -algebra, one natural choice of category for the object associated with $(C(X), \alpha)$ would be that of unital C^* -algebras. Given such a pair, one can indeed construct a certain C^* -algebra, a so called C^* -crossed product, which is generated by a copy of $C(X)$ and a unitary element, δ , that implements the action of the integers on $C(X)$ via α . We denote this C^* -algebra by $C^*(\Sigma)$ to indicate that it is associated with the dynamical system Σ . In the literature it is also commonly denoted by $C(X) \rtimes_{\alpha} \mathbb{Z}$. One way of obtaining $C^*(\Sigma)$ is as the completion of a $*$ -algebra, $k(\Sigma)$, in a certain norm.

We shall go through the construction of $C^*(\Sigma)$ in detail in the following section, but we now introduce $k(\Sigma)$ to give the reader a basic idea of how the pair $(C(X), \alpha)$ can be transplanted into an algebraic structure.

Hence, we shall endow the set

$$k(\Sigma) = \{a : \mathbb{Z} \rightarrow C(X) : \text{only finitely many } a(n) \text{ are non-zero}\}$$

with the structure a $*$ -algebra. We define scalar multiplication and addition on $k(\Sigma)$ as the natural pointwise operations. Multiplication is defined by convolution twisted by the automorphism α as follows:

$$(ab)(n) = \sum_{k \in \mathbb{Z}} a(k) \cdot \alpha^k(b(n-k)),$$

for $a, b \in k(\Sigma)$ and $n \in \mathbb{Z}$. The involution, $*$, is defined by

$$a^*(n) = \overline{\alpha^n(a(-n))},$$

for $a \in k(\Sigma)$ and $n \in \mathbb{Z}$. The bar denotes the usual pointwise complex conjugation. One can then view $C(X)$ as a $*$ -subalgebra of $k(\Sigma)$, namely as

$$\{a : \mathbb{Z} \rightarrow C(X) : a(n) = 0 \text{ if } n \neq 0\}.$$

A useful way of working with $k(\Sigma)$ is to write an element $a \in k(\Sigma)$ in the form $a = \sum_{k \in \mathbb{Z}} a_k \delta^k$, for $a_k = a(k)$ and $\delta = \chi_{\{1\}}$ where, for $n, m \in \mathbb{Z}$,

$$\chi_{\{n\}}(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

It is then readily checked that $\delta^* = \delta^{-1}$ and that $\delta^n = \chi_{\{n\}}$, for $n \in \mathbb{Z}$. As promised, the unitary element δ implements the action of the integers on $C(X)$ via α . Namely, for $f \in C(X)$, we have

$$\delta f \delta^* = \alpha(f) = f \circ \sigma^{-1}$$

and this clearly implies that, for $n \in \mathbb{Z}$, the relation

$$\delta^n f \delta^{*n} = \alpha^n(f) = f \circ \sigma^{-n}$$

holds. Note that $k(\Sigma)$, and hence $C^*(\Sigma)$, is commutative precisely when the system Σ is trivial in the sense that σ is the identity map of X .

The type of construction that yields $C^*(\Sigma)$ was first used in a systematic way in [2]. Since then, the connections between topological dynamical properties of Σ and the structure of $C^*(\Sigma)$ have been intensively studied. To give the reader an idea of what the nature of such connections can be like, we shall now state three known theorems on this so-called interplay between Σ and $C^*(\Sigma)$ which will play a central role in this thesis.

For $\Sigma = (X, \sigma)$, a point $x \in X$ is called *aperiodic* if for every non-zero $n \in \mathbb{Z}$ we have $\sigma^n(x) \neq x$. The system Σ is called *topologically free* if the set of its aperiodic points is dense in X . An equivalent statement of the following theorem appeared for the first time as [21, Theorem 4.3.5]. It is also to be found as [22, Theorem 5.4].

Theorem 1. *The following three properties are equivalent.*

- Σ is topologically free;
- Every non-zero closed ideal I of $C^*(\Sigma)$ is such that $I \cap C(X) \neq \{0\}$;
- $C(X)$ is a maximal abelian C^* -subalgebra of $C^*(\Sigma)$.

A system $\Sigma = (X, \sigma)$ is said to be *minimal* if there are no non-empty proper closed subsets C of X such that $\sigma(C) \subseteq C$. Equivalently, for every $x \in X$ the orbit of x under σ , $\mathcal{O}_\sigma(x) = \{\sigma^n(x) : n \in \mathbb{Z}\}$, is dense in X . A C^* -algebra is called *simple* if it lacks non-zero proper closed ideals (in $C^*(\Sigma)$, which is unital, this is equivalent to lacking arbitrary non-zero proper ideals). The following classical result follows from the main result of [12]. It is also proved in [22, Theorem 5.3].

Theorem 2. *If X consists of infinitely many points, then $\Sigma = (X, \sigma)$ is minimal if and only if $C^*(\Sigma)$ is simple.*

A system $\Sigma = (X, \sigma)$ is called *topologically transitive* if for every pair U, V of non-empty open subsets of X , there exists an integer n such that $\sigma^n(U) \cap V \neq \emptyset$. A C^* -algebra is *prime* if every pair of non-zero closed ideals have non-zero intersection. For a proof of the following result we refer to [22, Theorem 5.5].

Theorem 3. *If X consists of infinitely many points, then $\Sigma = (X, \sigma)$ is topologically transitive if and only if $C^*(\Sigma)$ is prime.*

Note that all these theorems are concerned with ideals of $C^*(\Sigma)$. This is not a coincidence. Understanding the ideal structure of an algebra of crossed product type is crucial to several major directions of investigation of it, e.g. its representation theory and examinations of the relations between its ideals and its “building block” algebra (or “coefficient algebra”), which for $C^*(\Sigma)$ is $C(X)$. Ideals will play a prominent role in large parts of this work as well.

Although it is beyond the scope of this thesis, the aforementioned equivalence of homeomorphism of two compact Hausdorff spaces, X_1 and X_2 , and isomorphism of $C(X_1)$ and $C(X_2)$ naturally raises the question whether there is a known relation between two arbitrary topological dynamical systems necessary and sufficient for the existence of an isomorphism between their associated C^* -crossed products. The answer is in the negative. A natural candidate would, for example, be that of topological conjugacy or flip conjugacy since there is an obvious isomorphism between the C^* -crossed products of two systems that are topologically conjugate or flip conjugate. However, one can show that there exist systems $\Sigma_1 = (X_1, \sigma_1)$ and $\Sigma_2 = (X_2, \sigma_2)$ such that the spaces X_1 and X_2 are not even homeomorphic, but where $C^*(\Sigma_1)$ is isomorphic to $C^*(\Sigma_2)$, as is mentioned in [23]. Nevertheless, there are some results available in this direction. For example, let θ_1, θ_2 be two irrational real numbers and denote, for $i = 1, 2$, by Σ_i the dynamical system defined by rotation of the unit circle by the angle $2\pi\theta_i$. Then it follows from the work in [11] and [14] that $C^*(\Sigma_1)$ is isomorphic to $C^*(\Sigma_2)$ if and only if $\theta_1 \equiv \pm\theta_2 \pmod{\mathbb{Z}}$. Furthermore, a well-known result being proved in [3] states that so-called strong orbit equivalence of minimal systems on the Cantor set is equivalent to isomorphism of their associated C^* -crossed products. In

relation to this it is interesting to mention the main result of [13], from which it follows that topological conjugacy of two systems, Σ_1 and Σ_2 , is equivalent to isomorphism of their associated so-called analytic crossed products. The analytic crossed product associated with a system Σ is a certain natural closed non self-adjoint subalgebra of $C^*(\Sigma)$; it is generated, as a Banach algebra, by $C(X)$ and the unitary element δ as introduced above.

In this thesis we focus not only on the C^* -algebra $C^*(\Sigma)$ associated with an arbitrary topological dynamical system $\Sigma = (X, \sigma)$, but also on a Banach $*$ -algebra, $\ell^1(\Sigma)$, and the non-complete $*$ -algebra $k(\Sigma)$ as introduced above, of both of which $C^*(\Sigma)$ contains a dense $*$ -isomorphic copy; $C^*(\Sigma)$ is the so-called enveloping C^* -algebra of $\ell^1(\Sigma)$. The algebras $\ell^1(\Sigma)$ and $C^*(\Sigma)$ can be obtained as completions of $k(\Sigma)$ in two different norms, which we define in the following section. We investigate the interplay between Σ and, respectively, $k(\Sigma)$ and $\ell^1(\Sigma)$, none of which is a C^* -algebra. While studies of connections between Σ and $C^*(\Sigma)$ have an extensive history, considerations of $k(\Sigma)$ and $\ell^1(\Sigma)$ are new. The algebras $C^*(\Sigma)$, $\ell^1(\Sigma)$ and $k(\Sigma)$ all contain a copy of $C(X)$, whose commutant, $C(X)'$, is being investigated in detail in all three algebras. In particular, we are concerned with its intersection properties for ideals of these algebras. We also consider the interplay between algebras generalizing $k(\Sigma)$ and corresponding dynamical systems.

It is worth mentioning that there has been a long-standing strong link between ergodic theory and the theory of von Neumann algebras dating back to the seminal work of Murray and von Neumann (cf. [6], [7], [24]), which appeared before the counterpart for topological dynamical systems and C^* -algebras that serves as departure point for the work in this thesis. There, one associates a crossed product von Neumann algebra with the action of a countable group of non-singular transformations on a standard Borel space equipped with a σ -finite measure. One of the most famous results on this interplay states, under the condition that the action is free, that the associated crossed product is a factor if and only if the action is ergodic, and furthermore gives precise conditions on the measure-theoretic side under which it is a factor of certain types. Another well-known result is the theorem of Krieger ([4],[5]) saying that two such ergodic group actions are orbit equivalent if and only if their associated crossed product von Neumann algebras are isomorphic. This strong ergodic interplay has stimulated studies of the topological case as introduced above.

There is a general theory of C^* -crossed products of which $C^*(\Sigma)$ is a special case. There, one starts with a triple (G, A, β) consisting of a locally compact group G , a C^* -algebra A and a homomorphism $\beta : G \rightarrow \text{Aut}(A)$ such that, for every $a \in A$, the map $g \mapsto (\beta(g))(a)$, from G to A , is norm continuous. With such a triple a C^* -crossed product, $C^*(G, A, \beta)$, is then associated. In our case, G is the group of integers, with the discrete topology, $A = C(X)$ and β is the homomorphism mapping an integer n to α^n . By the Gelfand-Naimark theorem for commutative C^* -algebras, the commutative unital C^* -algebras are precisely the algebras $C(X)$, with X compact Hausdorff, whence the studies of $C^*(\Sigma)$ amount precisely to the special case when A is commutative and unital, and the integers act on A via iterations of a single automorphism, α , of A . Some results holding for the aforementioned general C^* -crossed products can be obtained by simpler means in this particular situation. There are also results on $C^*(\Sigma)$ that have no known analogues in the general context. When A is commutative, the C^* -crossed product is sometimes referred to as a transformation group C^* -algebra in the literature. For the interested reader, we mention [10] and [25] as standard references for the theory of general C^* -crossed products. We

also refer to the work in [8] and [9] for results in the same vein as ours, e.g. on intersection properties for ideals of certain maximal abelian subalgebras, in the context of various purely algebraic crossed product structures, of some of which the algebra $k(\Sigma)$ mentioned above is a special case.

We shall devote Section 1.1 to going through the remaining details of the constructions of the aforementioned algebras to be able to state, in Section 1.2, the questions investigated in this thesis in a clear fashion. We then conclude this chapter by summarizing, in Section 1.3, the contents and some of the main results of the papers corresponding to the remaining chapters.

1.1. Three crossed product algebras associated with a dynamical system

Consider a topological dynamical system $\Sigma = (X, \sigma)$. As usual, X is a compact Hausdorff space, σ a homeomorphism of X and the integers act on X via iterations of σ . Again, we denote by α the automorphism of $C(X)$ defined, for $f \in C(X)$, by $\alpha(f) = f \circ \sigma^{-1}$. Above we introduced the $*$ -algebra $k(\Sigma)$, which has its multiplication defined in terms of α . Although the algebras $\ell^1(\Sigma)$ and $C^*(\Sigma)$ can both be obtained directly as completions of $k(\Sigma)$ in different norms, we shall, to be consistent with the presentation in the following chapters, first define $\ell^1(\Sigma)$ as a completion of $k(\Sigma)$ and then regard $C^*(\Sigma)$ as the enveloping C^* -algebra of $\ell^1(\Sigma)$.

Recall that an arbitrary element a of $k(\Sigma)$ can be written, in a unique way, as a finite sum,

$$a = \sum_k f_k \delta^k,$$

where the f_k are in $C(X)$ and δ is unitary, meaning that $\delta^* = \delta^{-1}$. We endow $k(\Sigma)$ with a norm as follows:

$$\|a\| = \sum_k \|f_k\|_\infty.$$

Completing $k(\Sigma)$ in this norm then yields the Banach $*$ -algebra $\ell^1(\Sigma)$. As in [20], we understand a Banach $*$ -algebra (or involutive Banach algebra) to be a complex Banach algebra with an *isometric* involution. The algebra $\ell^1(\Sigma)$ can be concretely realized as

$$\{a = \sum_k f_k \delta^k : \sum_k \|f_k\|_\infty < \infty\},$$

with the operations of $k(\Sigma)$ extended by continuity. Note that the representation of an element of $\ell^1(\Sigma)$ as such an infinite sum is unique, and that the closed $*$ -subalgebra $\{a : a = f_0 \delta^0 \text{ for some } f_0 \in C(X)\} \subseteq \ell^1(\Sigma)$ constitutes an isometrically $*$ -isomorphic copy of $C(X)$. The C^* -crossed product, $C^*(\Sigma)$, associated with Σ is the enveloping C^* -algebra of $\ell^1(\Sigma)$. This is defined as the completion of $\ell^1(\Sigma)$ in a different norm. This new norm is defined, for $a \in \ell^1(\Sigma)$, by

$$\|a\|_{C^*} = \sup \{\|\tilde{\pi}(a)\| : \tilde{\pi} \text{ is a Hilbert space representation of } \ell^1(\Sigma)\}.$$

While it is readily checked that $\|\cdot\|_{C^*}$ is a C^* -seminorm, it is not obvious that it is actually a norm. One can show, however, that $\ell^1(\Sigma)$ has sufficiently many Hilbert space representations, meaning that for every $a \in \ell^1(\Sigma)$ there is such a representation $\tilde{\pi}$ such that $\tilde{\pi}(a) \neq 0$. This can be done as follows. Consider a Hilbert space representation on $\mathcal{H}$, say, of $C(X)$. We shall construct a representation, $\tilde{\pi}$, of $\ell^1(\Sigma)$ on the Hilbert space $\ell^2(\mathbb{Z}, \mathcal{H})$ of all square summable functions x of $\mathbb{Z}$ into $\mathcal{H}$ endowed with the norm

$$\|x\|_2^2 = \sum_{k \in \mathbb{Z}} \|x(k)\|^2.$$

We first define $\tilde{\pi}$ on the generating set $C(X) \cup \{\delta\} \subseteq k(\Sigma)$ by

$$\begin{aligned} (\tilde{\pi}(f)x)(n) &= \pi(\alpha^{-n}(f))(x(n)), \\ (\tilde{\pi}(\delta)x)(n) &= x(n-1), \end{aligned}$$

for all $f \in C(X)$, $x \in \ell^2(\mathbb{Z}, \mathcal{H})$ and $n \in \mathbb{Z}$. One can then check that setting, for $a = \sum_k f_k \delta^k \in k(\Sigma)$, $\tilde{\pi}(a) = \sum_k \tilde{\pi}(f_k) \tilde{\pi}(\delta)^k$ yields a well-defined representation of $k(\Sigma)$ on $\ell^2(\mathbb{Z}, \mathcal{H})$, that extends by continuity to $\ell^1(\Sigma)$. By the Gelfand-Naimark theorem there exist faithful, hence isometric, Hilbert space representations of $C(X)$. Knowing this, it is not difficult to choose, for a given arbitrary non-zero element $a \in \ell^1(\Sigma)$, a suitable Hilbert space representation π of $C(X)$ and an element $x \in \ell^2(\mathbb{Z}, \mathcal{H})$ such that $\tilde{\pi}(a)x \neq 0$. Similarly one shows that the embedded copy of $C(X)$ in $C^*(\Sigma)$ is isometric with $C(X)$.

To sum up we have that, up to $*$ -isomorphisms,

$$C(X) \subsetneq k(\Sigma) \subsetneq \ell^1(\Sigma) \subsetneq C^*(\Sigma),$$

where the last two inclusions are dense.

1.2. Overview of the main directions of investigation

The research carried out in this thesis can be roughly divided into three parts, all intimately related, the questions of which we now outline briefly.

1.2.1. The algebras $k(\Sigma)$ and $\ell^1(\Sigma)$

Let $\Sigma = (X, \sigma)$ be a topological dynamical system. As stated in Section 1.1 its associated C^* -algebra, $C^*(\Sigma)$, contains a dense $*$ -isomorphic copy of the $*$ -algebra $k(\Sigma)$ and of the Banach $*$ -algebra $\ell^1(\Sigma)$. One of our main directions of investigation is the study of the interplay between Σ and the algebras $k(\Sigma)$ and $\ell^1(\Sigma)$, respectively. We consider analogues of results from the C^* -algebra context for these structures, e.g. Theorems 1-3 above, and also investigate links between their structure and Σ lacking counterparts for $C^*(\Sigma)$. While C^* -algebras have several attractive properties that fail for general Banach $*$ -algebras, $\ell^1(\Sigma)$ has an obvious advantage to $C^*(\Sigma)$ in that its norm is defined such that each of its elements can be written, in a unique way, as an infinite sum of the form $\sum_n f_n \delta^n$. Hence this allows one to approximate it by elements of $k(\Sigma)$ in an obvious manner. This should be compared

to [23, Proposition 1] which provides a more complicated formula for approximating an arbitrary element $a \in C^*(\Sigma)$ by a sequence of elements in $k(\Sigma)$, defined in terms of the so-called generalized Fourier coefficients of a . Naturally, the fact that $\ell^1(\Sigma)$ has $C^*(\Sigma)$ as its enveloping C^* -algebra furthermore allows us to make use of some of the known facts concerning the latter to derive new results on the former. In the case of $k(\Sigma)$ it turns out that, although it lacks the obvious advantage of being a complete normed algebra, the fact that its elements can be written in a unique way as *finite* sums of the form $\sum_n f_n \delta^n$ makes it a very computable object, which enables us to give considerably simpler proofs of theorems on this algebra than of their counterparts for $\ell^1(\Sigma)$ and $C^*(\Sigma)$.

1.2.2. Varying the “coefficient algebra” in $k(\Sigma)$

Denote, as above, by α the automorphism of $C(X)$ induced by σ and suppose that $A \subseteq C(X)$ is an arbitrary subalgebra that is invariant under α and its inverse. To the then naturally defined action of the integers on A one can associate a crossed product type algebra constructed in the same way as $k(\Sigma)$, in which A plays the role of “coefficient algebra” as $C(X)$ does in $k(\Sigma)$. As we shall see later, it turns out that many results on the interplay between Σ and $C^*(\Sigma)$ survive if we replace the latter by $k(\Sigma)$, while if one chooses the A above to be the complex numbers, the associated crossed product will be canonically isomorphic to the Laurent polynomial algebra in one variable regardless of the homeomorphism σ . Hence in the latter case the choice of A yields a crossed product independent of the nature of the dynamical system $\Sigma = (X, \sigma)$: all dynamical information is lost. Inspired by these facts, we investigate which choices of α - and α^{-1} -invariant subalgebras A of $C(X)$ that yield interesting connections between Σ and the associated analogue of $k(\Sigma)$. Furthermore, given a pair (B, Ψ) , where B is a commutative Banach algebra and Ψ an automorphism of B , we consider again an associated crossed product type algebra whose construction is analogous to that of $k(\Sigma)$. The automorphism naturally induces a dynamical system on the character space of B , and we investigate connections between the crossed product associated with (B, Ψ) and this system. When B is a commutative unital C^* -algebra this crossed product is precisely $k(\Sigma)$, where Σ is the induced system on the character space of B . Hence this generalizes the situation where the interplay between Σ and $k(\Sigma)$ is considered.

1.2.3. The commutant of $C(X)$

Recall from Section 1.1 that for a topological dynamical system $\Sigma = (X, \sigma)$, $C(X)$ can be naturally embedded into the algebras $k(\Sigma)$, $\ell^1(\Sigma)$ and $C^*(\Sigma)$, respectively, by a $*$ -isomorphism which for the last two algebras is also an isometry (although $k(\Sigma)$ is $*$ -isomorphically embedded in both $\ell^1(\Sigma)$ and $C^*(\Sigma)$, we always regard it as a mere algebra and hence do not make any norm considerations when working with it). An object being analyzed in detail in this thesis is the commutant of $C(X)$ in these algebras, and of the “coefficient algebras” in the crossed product type algebras generalizing $k(\Sigma)$ as discussed in the previous subsection. While Theorem 1 gives statements equivalent to maximal commutativity of $C(X)$ in $C^*(\Sigma)$, we show that the commutant of $C(X)$, which we denote by $C(X)'$, is always commutative in all crossed product structures under consideration here.

Hence $C(X)'$ is always the unique maximal commutative subalgebra that contains $C(X)$. Inspired by this fact together with Theorem 1, from which it follows that $C(X)$ has a certain intersection property for closed ideals of $C^*(\Sigma)$ precisely when it coincides with $C(X)'$, we investigate, for arbitrary Σ , intersection properties of $C(X)'$ for ideals of the various crossed products. Our conclusions turn out to serve as key results when proving e.g. analogues of Theorems 1-3 for other crossed products than $C^*(\Sigma)$. Furthermore, we investigate ideal intersection properties of algebras B such that $C(X) \subseteq B \subseteq C(X)'$ and, for $C^*(\Sigma)$, of $\tilde{\pi}(C(X))'$, where $\tilde{\pi}$ is a Hilbert space representation of the former. Many of our results related to $C(X)'$ rely on the crucial fact that we can describe it explicitly, as well as its character space in the cases of $\ell^1(\Sigma)$ and $C^*(\Sigma)$.

1.3. Brief summary of the included papers

Chapter 2 to Chapter 6 of this thesis correspond to, respectively, [15], [16], [17], [18] and [19]. We shall now summarize their contents briefly.

Chapter 2: Dynamical systems and commutants in crossed products

To a pair (A, Ψ) of an arbitrary associative commutative complex algebra A and an automorphism Ψ of A , we associate a purely algebraic crossed product containing an isomorphic copy of A . Namely, we endow the set

$$\{f : \mathbb{Z} \rightarrow A : f(n) = 0 \text{ for all but finitely many } n \in \mathbb{Z}\}$$

with operations making it an associative, in general non-commutative, complex algebra, which we denote by $A \rtimes_{\Psi} \mathbb{Z}$. Its multiplication is defined in terms of Ψ in a way analogous to the case of the algebra $k(\Sigma)$, as introduced above, which is the special case here obtained as the crossed product associated with the pair $(C(X), \alpha)$. We show that the commutant, A' , of A is commutative and describe it explicitly in the case when A is a function algebra and Ψ a composition automorphism defined via a bijection of the domain of A . Commutativity of A' implies that it is the unique maximal abelian subalgebra of $A \rtimes_{\Psi} \mathbb{Z}$ that contains A . Given various classes of pairs (X, σ) of a topological space X and a homeomorphism σ of X , which we consider as dynamical systems by letting the integers act on X via iterations of σ as above, we prove that suitable subalgebras A of $C(X)$, invariant under the automorphism $\alpha : C(X) \rightarrow C(X)$ induced by σ and under α^{-1} , constitute maximal abelian subalgebras of $A \rtimes_{\alpha} \mathbb{Z}$ if and only if the set of aperiodic points of (X, σ) is dense in X . We show that a specific class of such A are maximal abelian in $A \rtimes_{\alpha} \mathbb{Z}$ precisely when σ is not of finite order. An example of this is the case when A is the algebra of all holomorphic functions on a connected complex manifold, M , and σ is a biholomorphic function on M .

Furthermore, for pairs (A, Ψ) , where A is a semi-simple commutative Banach algebra and Ψ is an automorphism of A , we introduce a topological dynamical system on the character space, $\Delta(A)$, of A naturally induced by Ψ and prove e.g. that when A is also completely regular we have equivalence between maximal commutativity of A in $A \rtimes_{\Psi} \mathbb{Z}$ and density of the set of aperiodic points of the associated system on $\Delta(A)$. When A is the algebra $L_1(G)$ of integrable functions on a locally compact abelian group G with connected dual

group, we use the above results to conclude that $L_1(G)$ is maximal abelian in the crossed product precisely when the automorphism Ψ of $L_1(G)$ is not of finite order.

All these results should be compared to the first and third statement of Theorem 1.

Chapter 3: Connections between dynamical systems and crossed products of Banach algebras by $\mathbb{Z}$

Here we start the investigation of the ideal structure of algebraic crossed products as defined in Chapter 2. We mainly focus on crossed products associated with pairs (A, Ψ) where A is a commutative semi-simple completely regular Banach algebra and Ψ is an automorphism of A . We prove that A has non-zero intersection with every non-zero ideal precisely when A is a maximal abelian subalgebra of $A \rtimes_{\Psi} \mathbb{Z}$, which is in turn equivalent to density of the aperiodic points of the associated dynamical system on $\Delta(A)$ as introduced in Chapter 2. Thus we obtain a result analogous to Theorem 1. When A is the algebra $L_1(G)$ of integrable functions on a locally compact abelian group G with connected dual group, the set of aperiodic points of $\Delta(A)$ is dense precisely when the automorphism Ψ is not of finite order. We prove equivalence between simplicity of $A \rtimes_{\Psi} \mathbb{Z}$ and minimality of the system on $\Delta(A)$, provided that A is unital and $\Delta(A)$ is infinite. This is analogous to Theorem 2 in the C^* -algebra context. We also show that every non-zero ideal of $A \rtimes_{\Psi} \mathbb{Z}$ always has non-zero intersection with A' . Finally, we show that for unital A such that $\Delta(A)$ is infinite, topological transitivity of the system on $\Delta(A)$ is equivalent to primeness of $A \rtimes_{\Psi} \mathbb{Z}$ and hence find the analogue of Theorem 3.

Chapter 4: Dynamical systems associated with crossed products

We give simplified proofs of generalizations of some results from Chapter 3, which we thereby show to hold in a broader context than when A is a certain kind of Banach algebra. For example, we give an elementary proof of the fact that if (A, Ψ) is a pair consisting of an arbitrary commutative associative complex algebra A and an automorphism Ψ of A , the associated crossed product $A \rtimes_{\Psi} \mathbb{Z}$ is such that A' has non-zero intersection with every non-zero ideal. This allows us to prove the analogue of Theorem 1 in a greater generality than in Chapter 3. We also investigate subalgebras properly between A and its commutant A' and show that for suitable (A, Ψ) , one may find two such subalgebras, B_1 and B_2 , of $A \rtimes_{\Psi} \mathbb{Z}$ where B_1 has non-zero intersection with every non-zero ideal and where B_2 does not have this property. Finally, we start the investigation of the Banach algebra crossed product, $I_1^{\sigma}(\mathbb{Z}, C_0(X))$, associated with a pair (X, σ) of a locally compact Hausdorff space X and a homeomorphism σ of X where, as usual, the integers act on X via iterations of σ . When X is compact, this is precisely the algebra $\ell^1(\Sigma)$ as introduced in Section 1.1. We determine the closed commutator ideal of $I_1^{\sigma}(\mathbb{Z}, C_0(X))$ in terms of the set of fixed points, $\text{Per}^1(X)$, of (X, σ) and furthermore find a bijection between the characters of $I_1^{\sigma}(\mathbb{Z}, C_0(X))$ and $\text{Per}^1(X) \times \mathbb{T}$.

Chapter 5: On the commutant of $C(X)$ in C^* -crossed products by $\mathbb{Z}$ and their representations

In this chapter, which is focused entirely on $C^*(\Sigma)$ for a topological dynamical system $\Sigma = (X, \sigma)$, we analyze the commutant of $C(X)$, denoted as usual by $C(X)'$, in detail. We describe its elements explicitly in terms of their generalized Fourier coefficients and conclude that it is commutative. More generally, we analyze $\tilde{\pi}(C(X))'$, where $\tilde{\pi}$ is an arbitrary Hilbert space representation of $C^*(\Sigma)$. We describe the spectrum of $\tilde{\pi}(C^*(\Sigma))'$ and consider a topological dynamical system on it which is derived from the action of the integers on $C(X)$ via iterations of α , the automorphism induced by σ . Inspired by the results on $C(X)'$ in $k(\Sigma)$ and its generalizations as obtained in Chapter 3 and 4, we prove that $C(X)'$ always has non-zero intersection with every non-zero (not necessarily closed or self-adjoint) ideal of $C^*(\Sigma)$, and that $\tilde{\pi}(C(X))'$ has the corresponding property if a certain condition on $\tilde{\pi}(C^*(\Sigma))$ holds. This enables us to provide a sharpened version of Theorem 1 above that holds for arbitrary ideals of $C^*(\Sigma)$ rather than just for closed ones. Furthermore, we consider C^* -subalgebras properly between $C(X)$ and $C(X)'$ (cf. Chapter 4) and conclude that, as soon as $C(X) \neq C(X)'$, one may find two such subalgebras, B_1 and B_2 , of $C^*(\Sigma)$ where B_1 has non-zero intersection with every non-zero ideal and where B_2 does not have this property. Finally, we discuss existence of norm one projections of $C^*(\Sigma)$ onto $C(X)'$.

Chapter 6: On the Banach $*$ -algebra crossed product associated with a topological dynamical system

Having focused mainly on $C^*(\Sigma)$ and $k(\Sigma)$, and its generalizations, in the previous chapters, we now consider the Banach $*$ -algebra $\ell^1(\Sigma)$ associated with a topological dynamical system $\Sigma = (X, \sigma)$. As in the other algebras, we describe $C(X)'$ explicitly and conclude that it is commutative. By invoking the theory of ordered linear spaces, we also determine its character space and furthermore prove that it has non-zero intersection with every non-zero closed ideal of $\ell^1(\Sigma)$. Using this result, we deduce analogues of Theorems 1-3 for $\ell^1(\Sigma)$. We conclude by proving a result whose analogue in the C^* -algebra context is clearly false: $\ell^1(\Sigma)$ has non self-adjoint closed ideals if and only if Σ has periodic points.

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