Fermions, criticality and superconductivity

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Chapter 7

Conclusions

We have been battling with the signful fermions in this thesis. We have not really gotten very far with the microscopic approach. The reformulation of the Anderson-Higgs mechanism for bosonic systems and the understanding of its absence in fermionic systems from the worldline perspective are interesting stories by themselves [Chapter 2]. The Mott-insulator picture of the free fermion nodal structure deepens our understanding of fermionic statistics [Chapter 3]. However, we are still far from a satisfying general understanding of the fermion sign problem. Probably we theorists should be more modest. It can be that there is actually no way to solve the sign problem once and for all. The NP hardness of this problem already tells us that a mathematically rigorous approach is impossible, which is not necessarily bad news: it hints strongly at surprising new discoveries. The task of theorists is thus to find the prototype, be it in the form of a wavefunction, an effective field theory, or even using the language of string theory, to model emergent phenomena.

From this point of view, the natural question to ask is whether the different worldline theories and different nodal structures can serve as prototypes of new forms of matter. A scalar field theory on the worldline represents a bosonic particle. Inclusion of Grassmann fields changes the statistics to become fermionic. Different statistics are encodes in the different field content of the 0+1-dimensional worldline field theory. Along this line of thinking, it would be interesting to see whether it is possible to construct a more complex worldline field theory that can represent fermions without double occupancy, as appears in the t-J model. Such particles have essentially different statistics as compared to conventional fermions. When the extra worldline fields are gapped, they can be integrate out, leaving us with a normal fermionic particle. In this way the second-order phase transition in the worldline field theory may explain the continuous nature of the transition from the usual fermions to the projected fermions in the t-J model.

In order to understand Mott physics from the worldline formalism, we also need a better way of dealing with the effect of the lattice. The essence of the existence of the lattice is that the number of available states is greatly reduced. This is a necessary condition for the electrons to experience jamming and produce a Mott insulating state. Pictorially the effect of the lattice can be modeled by replacing the point particles by finite size hard core spheres. And the Mott transition corresponds to the jamming transition of such spheres.

A generalization of the 0+1-dimensional worldline field theory to 1+1-dimensional worldsheet would lead to string theory. Having in mind the equivalence of the Anderson-Higgs mechanism and infinite winding, an obvious question to ask is what happens when the strings have long windings. Here one considers a finite density of strings. And to calculate the free energy of this system, one needs to sum over all possible permutations of the initial or finial string configurations. Such a string condensation will backreact to the space-time geometry, and will have important cosmological implications.

We do learn important lessons from wrestling with the fermion nodal structure. The traditional way to understand the different phases of matter is to use symmetry properties of the wave functions. The nodal structure provides a more topological perspective. Wen and collaborators have used the pattern of zeros of the wave functions to characterize the topological order in fractional quantum Hall states [296–298]. There the nodes are at discrete points where two particles coincide. The pattern of zeros describes how fast the wavefunction approaches to zero near the nodes.

The number of nodal cells is an adiabatic invariant for quantum-mechanical systems [299]. We have seen in Chapter 3 that for the Fermi gas, there are two nodal cells. The immediate consequence is that for the Fermi liquid, which can be adiabatically continued to the Fermi gas, there should also be just two nodal cells. This number can be used to characterize the different phases of interacting fermions. In the Mott insulating state, the particles localize and the number of nodal cells is expected to be $N!$, with N the total number of particles. So as the repulsion between the particles increases, and the metallic Fermi liquid ground state is driven to become a Mott insulator, there must be a phase transition in-between to open up more nodal cells. One can imagine that the initial nodal hypersurface becomes more and more wrinkled, as we approach the critical point, and eventually topology changes and more nodal cells are created. At the critical point, the nodal hypersurface has to be scale invariant, thus a fractal structure [229]. This process can happen smoothly, and a second-order phase transition is plausible. It's still not clear to us what is the universality class of this transition. The problem is how to make such intuitive pictures into concrete mathematical equations, and calculate the normal state properties of the system at finite temperatures.

A less ambitious project is to simply do perturbation theory around the Fermi gas. We already have some preliminary results on this problem. The Feynman rules in such first-quantized formalism have been constructed and in principle everything that can be done in the conventional second-quantized formalism can also be done using the worldline language. The partition function and all the correlation functions can be expressed in terms of the N-particle density matrix. The question is what can we learn by doing this. What we can hope for is to write down a sign-free perturbation theory for fermions. From the perturbation corrected density matrix, we can read off the perturbed nodal surface. And then we can carry out the constrained path integral using the new nodal constraints, which will produce a new density matrix. Such a process can be repeated untill the result converges to some fixed point. We can see clearly from this process the difference of the fermionic RG from the usual bosonic RG. The geometric nodal hypersurface provides new dimensions for the parameter space, arising from the constraints on the paths. A natural question to ask at this point is what are the possible fixed-point nodal surfaces. One possible geometry is the one corresponding to free fermions, a $dN-1$-dimensional hypersurface that divides the whole dN-dimensional configuration space into 2 parts. Another possible geometry is the one corresponding to Mott insulators with infinitely large Hubbard U. This fixed-point geometry is simply the whole dN-dimensional space with several discrete points deleted. A more interesting fixed-point geometry is the fractal nodal surface [229]. With such a fractal boundary, one may expect to have "multifractal" behavior in the physical observables, i.e. scaling behavior with a continuous spectrum of exponents.

The macroscopic approach has been more rewarding up to now. The simple scaling theory of superconductivity we proposed for quantum critical metals [Chapter 5] turns out to have surprising connections with other pursuits of strongly correlated electron systems. One such connection is with the numerical work on the two dimensional Hubbard model. The Hubbard model is now accepted as the de facto model for cuprates. Numerical calculations of the Hubbard model strongly support the idea of a finite-doping QCP separating the low-doping region, found to be a non-Fermi liquid, from a higher doping Fermi liquid region. In the vicinity of the QCP, calculations also show that for a wide range of temperatures, the doping and temperature dependence of the single-particle properties are consistent with marginal Fermi liquid behavior. The critical doping seems to be in close proximity to the optimal doping for superconductivity as found both in the context of the Hubbard and t-J models. This proximity already indicates that a QCP enhances pairing, though the detailed mechanism is largely unknown.

We have been working together with Jarrell's group, using the dynamical cluster approximation to understand the proximity of the superconducting dome to the QCP in the Hubbard model [300]. The full pairing susceptibility is decomposed into the bare susceptibility, which is constructed from the dressed one-particle Green's function, and the vertex function. The d-wave channel is found to be the most divergent one. The bare pair susceptibility and the vertex function are then projected to the d-wave channel. At critical doping, as one lowers the temperature, the magnetic susceptibility saturates, the charge susceptibility is strongly enhanced, and the d-wave pairing susceptibility diverges at a certain temperature. An interesting result is that the d-wave pairing vertex falls monotonically with increasing doping. And it changes smoothly across the critical doping. In contrast, the bare d-wave pairing susceptibility exhibits significantly different features close to and away from the QCP. In the underdoped region, it saturates at low temperatures. In the overdoped region, it displays the normal Fermi liquid type log divergence. However at the critical doping, it diverges more quickly with decreasing temperature, roughly following the power-law behavior $1/\sqrt{T}$. Decomposing the pairing vertex into different channels, it is also found that as the QCP is approached, the pairing originates predominantly from the spin channel. The basic observation of the Hubbard model is that pairing in the critical region is due to an algebraic temperature-dependence of the bare pair susceptibility rather than an enhanced d-wave pairing vertex, supporting our QCBCS picture of superconductivity in such systems.

Another interesting connection is with the newly developed string theoretical approach to condensed matter systems. A class of superconductors have been constructed theoretically, which have a mathematical description in terms of charged black holes with nontrivial 'hair' [285]. This approach is based on the idea of the AdS/CFT correspondence, which states that the strong coupling limit of a gauge theory can be described by a weak coupling gravitational theory in one dimension higher and with negative cosmological constant [123–125]. In the gravitational description, black holes will play the role of temperature. Superconductivity follows from the spontaneous breaking of the U(1) symmetry, i.e. the formation of a non-zero condensate. On the gravity side, such condensates correspond to static non-zero fields outside the black holes, usually called black holes 'hair'. In AdS space, the negative cosmological constant plays the role of a confining box, and the charged particles pair-created from the vacuum via the Schwinger effect will be confined to the region near the horizon, producing the black hole 'hair'. A coupled system of gravity, Maxwell field and a charged scalar is enough to produce a superfluid/superconducting condensate. Due to the asymptotic AdS background, the pairing susceptibility in such models will automatically have a power-law behavior at high frequency. Near zero frequency, it also displays the usual hydrodynamic behavior. So this class of models can be looked upon as a more sophisticated way of incorporating scaling in the presence of superconductivity, along the same line as our QCBCS approach. In Chapter 5, we have used as a toy model the scaling function from 1+1 dimensional CFT. The AdS/CFT approach will provide us with the truly high dimensional scaling functions.

The big theme that emerges from this thesis is the instability of QCPs. It is by now an empirical fact based on experiments that the quantum critical metals are more susceptible than normal metals and superconductors. States of matter that can not be constructed from stable states like normal metals or superconductors can be built near the QCPs. We still do not have a good theoretical understanding of this fact. The general reasoning is that since the QCP is a highly degenerate state, a tiny perturbation may make it unstable. Chapter 5 of this thesis is based on such logic, where we incorporate this idea in a scaling theory of superconductivity. For the Fermi liquid, the superconducting instability is driven by a marginally relevant four-point interaction between excitations about the Fermi surface. The exponential form of the gap equation follows from the marginal nature of the interaction. One may speculate that in quantum critical metals, the superconducting instability becomes truly relevant, resulting in the algebraic gap equation of Chapter 5.

A more difficult question is, for particular materials which perturbation will finally dominate and determine the fate of the QCP. In many materials, superconductivity seems to be a plausible end of QCPs. We have seen another possibility in Chapter 4 that, in the presence of competing orders, the second-order phase transitions may become first order at low temperatures, with the immediate consequence that spatially modulated inhomogeneous phases are expected to be present near the QCPs. Other examples include the nematic phase around the metamagnetic QCP in the bilayer ruthenate $Sr_3Ru_2O_7$ [93–96]. A better understanding of this problem will provide theoretical guidance for the search for exotic materials in systems involving QCPs.