Applications of AdS/CFT in Quark Gluon Plasma
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CHAPTER 4

DRAG FORCE IN 4D KERR-ADS BLACK HOLE

4.1 Introduction

An important feature of RHIC is the spatial anisotropy of the data correlated with non-zero impact parameter of the colliding ions. The spatial anisotropy translates into a non-zero component of the second Fourier harmonic of the particle distribution in the plane transverse to the collision.[1] This coefficient is known as elliptic flow and has been calculated using hydrodynamic evolution of the sQGP in a stunning agreement with data [104, 105][2]. Up to now most AdS computations of sQGP have ignored spatial anisotropy: the black hole background is always the standard static isotropic AdS black hole. There is good reason to do so. Transport coefficients such as the viscosity are defined with respect to the isotropic perfect fluid and for other quantities the experimental indication that the system thermalizes rapidly to an almost perfect, i.e. isotropic, fluid means that anisotropy corrections are small. The exceptions are "local temperature/pressure" approximations as in Bhattacharyya et al [57] and Chesler et al [58]. Yet for a number of them, e.g. photon production or jet-quenching, it is the anisotropic component that is experimentally the most accessible.

[1]The first Fourier component vanishes by symmetry.

[2]A more detailed discussion on this phenomena and others can be found in [106] and references therein.

In this chapter we make a first step towards the study anisotropic effects on jet-quenching from the string theory point of view. Jet-quenching is a characteristic feature of the sQGP phase in RHIC. It signals the strong energy loss of a highly massive quark moving in hot charged plasma. In the frame work of AdS/CFT, a quark is represented by a string suspended from the boundary of asymptotically AdS space into the interior [65, 107, 108]. This set-up was proposed in the context of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at finite temperature [35,39] and has been explored in detail in [59,60] with a beautiful extension to trailing wake of the heavy quark in the sQGP dual to the back-reaction of string on the black-hole geometry. The way we shall introduce anisotropy in the system is to consider non-zero angular momentum. The advantage is that the dual description of this system is straightforward: one considers rotating black holes. The drawback is that the anisotropy primarily responsible for elliptic flow is due to the asymmetric almond-shape overlap of the two non-central colliding nuclei rather than angular momentum. As all non-central collisions the total system carries a significant amount of angular momentum, but most of that is carried away by spectator-nuclei not involved in the formation of the sQGP. At RHIC the angular momentum fraction of the total elliptic flow is thought to be less than 10%, although it is expected to increase to 30% at LHC [61,62] and the references therein. Clearly experimentally more relevant would be a AdS/CFT study of elliptic flow due to non-rotational anisotropy. The problem is that the gravity set-up in this case is unclear. Non-rotational anisotropy dissipates fast as the system equilibrates and isotropizes, and this points to a time-dependent gravity dual, see e.g. Chesler presentation in Amsterdam String Theory Workshop 2010.[3] For that reason we start here with studying anisotropic jet-quenching in a rotating plasma.

4.2 Drag force on a string in a global 4D AdS black hole

In global coordinates, the metric of four dimensional AdS-Schwarzschild is given by

$$\begin{aligned} ds^2 &= -r^2 h(r) dt^2 + \frac{1}{r^2 h(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \qquad (4.1) \\ h(r) &= l^2 + \frac{1}{r^2} - \frac{2M}{r^3}, \end{aligned}$$

where M is proportional to the mass of the black hole and l is the radius of curvature. The Hawking temperature of four dimensional AdS-Schwarzschild can be obtained in a simple way by demanding the periodicity of Euclidean

[3] Alternatively one could consider 4+1 dimensional hairy black holes to break the anisotropy; we thank H. Ooguri for pointing this out.

time to avoid a conical singularity at $r = r_H$. This gives $T_H = \frac{1}{4\pi}\left(\frac{1}{r_H} + 3r_H l^2\right)$, where r_H is the radius of horizon defined as the zero locus $h(r_H) = 0$ [109] and can be written explicitly in terms of parameters l and M:

$$r_H(l, M) = \frac{\left(9Ml^4 + \sqrt{3l^6 + 81M^2l^8}\right)^{1/3}}{3^{2/3}l^2} - \frac{1}{3^{1/3}\left(9Ml^4 + \sqrt{3l^6 + 81M^2l^8}\right)^{1/3}}. \tag{4.2}$$

A string in this background can be described by the following Nambu-Goto action:

$$\begin{aligned} S =& -\frac{1}{2\pi\alpha'}\int d\sigma^2\sqrt{-\text{det}g_{\alpha\beta}} = \int d\sigma^2\mathcal{L}, \\ g_{\alpha\beta} \equiv& G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \end{aligned} \tag{4.3}$$

with σ^α are coordinates of string worldsheet, $X^\mu = X^\mu(\sigma)$ are the embedding of string worldsheet in spacetime, and $G_{\mu\nu}$ is the spacetime metric (4.1). The equation of motions derived from (4.3) are,

$$\nabla_\alpha P^\alpha_\mu = 0, \quad P^\alpha_\mu \equiv -\frac{1}{2\pi\alpha'}G_{\mu\nu}\partial^\alpha X^\nu = -\frac{1}{2\pi\alpha'}\pi^\alpha_\mu \tag{4.4}$$

with P^α_μ is the worldsheet current of spacetime momentum carried by the string, proportional to the canonical worldsheet momentum

$$\pi^\alpha_\mu = -\frac{(2\pi\alpha')}{\sqrt{-g}}\frac{\delta S}{\delta\partial_\alpha X^\mu}, \tag{4.5}$$

with $g = \text{det}g_{\alpha\beta}$. The total momentum charge in the direction μ carried by the string equals

$$p_\mu = \int d\Sigma_\alpha\sqrt{-g}P^\alpha_\mu, \tag{4.6}$$

where Σ_α is a cross-sectional surface (a line) on the string worldsheet. The proper-force on the string then equals

$$\frac{\partial p_\mu}{\partial\sigma^0} = \sqrt{-g}P^{\sigma^1}_\mu \tag{4.7}$$

which in turn is equal to the canonical worldsheet-momentum

$$\frac{\partial p_\mu}{\partial\sigma^0} = -\frac{1}{2\pi\alpha'}\pi^{\sigma^1}_\mu \tag{4.8}$$

If the configuration is constant in time, the proper force $\frac{\partial p_\mu}{\partial\sigma^0}$ does not depend on the location σ^1 along the worldsheet, thanks to the equation of motion.

$$\frac{\partial}{\partial\sigma^1}\frac{\partial p_\mu}{\partial\sigma^0} = \frac{\partial}{\partial\sigma^1}\sqrt{-g}P^{\sigma^1}_\mu = -\frac{\partial}{\partial\sigma^0}\sqrt{-g}P^{\sigma^0}_\mu \overset{static}{=} 0 \tag{4.9}$$

Using the physical gauge, $\sigma^\alpha = (t, r)$, we can write the action (4.3) as follows

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma^2\sqrt{-g},$$
$$-g = 1 + r^4 h(r)\left(\theta'^2 + \phi'^2\sin^2\theta\right) - \frac{1}{h(r)}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) - r^4\sin^2\theta\left(\dot\theta\phi' - \theta'\dot\phi\right)^2, \tag{4.10}$$

with $\dot{} \equiv \frac{d}{dt}$ and $' \equiv \frac{d}{dr}$. The equations of motion are:

$$\frac{\partial}{\partial t}\left(\frac{r^4}{\sqrt{-g}}\sin^2\theta\left(\dot\theta\phi' - \theta'\dot\phi\right)\theta' - \frac{\sin^2\theta}{h(r)\sqrt{-g}}\dot\phi\right) + \frac{\partial}{\partial r}\left(\frac{r^4 h(r)\sin^2\theta}{\sqrt{-g}}\phi' - \frac{r^4}{\sqrt{-g}}\sin^2\theta\left(\dot\theta\phi' - \theta'\dot\phi\right)\dot\theta\right) = 0, \tag{4.11}$$

and

$$\frac{1}{\sqrt{-g}}\left(r^4 h(r)\phi'^2 - \frac{1}{h(r)}\dot\phi^2 - r^4\left(\dot\theta\phi' - \theta'\dot\phi\right)^2\right)\sin\theta\cos\theta + \frac{\partial}{\partial t}\left(\frac{r^4\sin^2\theta}{\sqrt{-g}}\left(\dot\theta\phi' - \theta'\dot\phi\right)\phi' + \frac{1}{h(r)\sqrt{-g}}\dot\theta\right) - \frac{\partial}{\partial r}\left(\frac{r^4\sin^2\theta}{\sqrt{-g}}\left(\dot\theta\phi' - \theta'\dot\phi\right)\dot\phi + \frac{r^4 h(r)}{\sqrt{-g}}\theta'\right) = 0. \tag{4.12}$$

4.2.1 Great circle at $\theta = \pi/2$

It easy to see that the metric (4.1) has $SO(3)$ symmetry. Specially the boundary is the S^2, and the free motion of a quark on this sphere is a great circle. The motion of string giving arise to a great circle trajectory should therefore reduce to the geodesic equations on the boundary. The simplest solution of the great circles is when we consider the motions in the equatorial coordinates $\phi = \phi(t, r)$ only, while the other angular coordinate we set $\theta = \pi/2$. We take an ansatz for $\phi(t, r)$ as below[4]

$$\phi(t, r) = \omega t + \eta(r), \tag{4.13}$$

where ω is a non-zero constant. Substituting the ansatz (4.13) into the action (4.3), we obtain equation of motion for $\eta(r)$ which is given by a constant of its

[4]With this ansatz, the dynamical field is effectively given by field $\eta(r)$.

momentum conjugate since the Lagrangian is only a functional of $\eta'(r)$

$$\begin{aligned}\pi_\phi^r &= -2\pi\alpha'\frac{\partial\mathcal{L}}{\partial\eta'(r)} = \frac{r^4h(r)\eta'(r)}{\sqrt{1+r^4h(r)\eta'(r)^2-\frac{\omega^2}{h(r)}}},\\ \eta'(r) &= \frac{\pi_\phi^r}{r^4h(r)}\sqrt{\frac{h(r)-\omega^2}{h(r)-\frac{(\pi_\phi^r)^2}{r^4}}},\end{aligned} \tag{4.14}$$

thus we find π_ϕ^r is a constant, by solving the equation of motion, proportional to the momentum conjugate in radial direction. We have chosen positive sign to describe a string that trails out from boundary down to the horizon. In order to make sense of the solution, we have to require that (4.14) must be real everywhere. This requirement gives us a condition

$$\frac{h(r)-\omega^2}{h(r)r^4-(\pi_\phi^r)^2}\geq 0 \tag{4.15}$$

At the boundary $r\to\infty$, the requirement tells us that there is a bound on the velocity of the particle that $l^2\geq\omega^2$. Because (4.14) ought to be real everywhere for $r_H\leq r<\infty$, the only possible choice we can take is for the constant $(\pi_\phi^r)^2=\omega^2r_{Sch}^4$, with $r_{Sch}=r_H(\sqrt{l^2-\omega^2},M)$ defined to be the point where $h(r_{Sch})=\omega^2$. Then the numerator and denominator in (4.15) change their sign at the same point at $r=r_{Sch}$.

The exact solution for equation (4.14) is quite difficult to find but to compute the drag force it is enough just to use the equation (4.14). To compute the flow of momentum dp_ϕ down the string, we use

$$\Delta P_\phi = \int d\Sigma_\alpha P_\phi^\alpha. \tag{4.16}$$

In this static configuration all momentum flow is radial. Thus the total momentum reduces to

$$\Delta P_\phi = \int_{\mathcal{I}} dt\sqrt{-g}P_\phi^r = \frac{dp_\phi}{dt}\Delta t, \tag{4.17}$$

with $\mathcal{I}$ is some time interval of length Δt. Thus the drag force in ϕ direction is given by

$$\frac{dp_\phi}{dt} = \sqrt{-g}P_\phi^r = -\frac{\pi_\phi^r}{2\pi\alpha'}, \tag{4.18}$$

where the negative value implies that it is the drag force. Explicitly $\pi_\phi^r=\omega r_{Sch}^2$, with $\omega\geq 0$.

4.2.2 General solution of the great circle

In general, great circles are not only the motion in $\theta = \pi/2$ plane. The great circle can be in an arbitrary plane. We can describe this as follows using coordinates transformation:

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta. \tag{4.19}$$

We consider a constrained Nambu-Goto action[5]

$$\begin{aligned} S_{cNG} &= -\frac{1}{2\pi\alpha'}\int d\sigma^2\sqrt{-g}\left[1+\frac{\lambda^2}{2}(x^i x^i - 1)\right], \\ -g &= r^4\left((\dot{x}^i x'^i)^2 - (\dot{x}^i\dot{x}^i - h(r))\left(x'^i x'^i + \frac{1}{r^4 h(r)}\right)\right), \end{aligned} \tag{4.20}$$

where $x^i \equiv (x, y, z)$ and λ is a Lagrange multiplier. The equations of motion for this action are given by a constraint equation $x^i x^i = 1$ and

$$\begin{aligned} \lambda^2 x^i\sqrt{-g} &- \frac{\partial}{\partial t}\left(x'^i\frac{r^4\dot{x}^j x'^j}{\sqrt{-g}} - \dot{x}^i\frac{\left(r^4 x'^j x'^j + \frac{1}{h(r)}\right)}{\sqrt{-g}}\right) \\ &- \frac{\partial}{\partial r}\left(\dot{x}^i\frac{r^4\dot{x}^j x'^j}{\sqrt{-g}} - x'^i\frac{r^4(\dot{x}^j\dot{x}^j - h(r))}{\sqrt{-g}}\right) = 0. \end{aligned} \tag{4.21}$$

Notice that if we substitute the constraint equation back to the action (4.20) then we get back the action (4.3).

radial independent ansatz

A simple solution can be described by radial r independent ansatz, $x^i = x^i(t)$, where the equations of motion become

$$\lambda^2 x^i\sqrt{-g} + \frac{\partial}{\partial t}\left(\frac{\dot{x}^i}{h(r)\sqrt{-g}}\right) = 0, \tag{4.22}$$

with $-g = \frac{1}{h}(h - \dot{x}^i\dot{x}^i)$. Multiplying with x^i and using constraint $x^i\dot{x}^i = 0$, we obtain

$$\dot{x}^i\dot{x}^i = \frac{\lambda^2 h(r)}{1+\lambda^2}. \tag{4.23}$$

[5] Recall that in order to make a world sheet volume invariant under world sheet general coordinate transformation, we have to multiply with $\sqrt{-g}$. A repeated index denotes Einstein summation index.

Assume λ is constant, the equations of motion are now simplified to

$$\ddot{x}^i = -\frac{\lambda^2 h(r)}{1+\lambda^2} x^i. \tag{4.24}$$

The general solution is given by

$$x(t)^i = a^i \sin\left(\lambda\sqrt{\frac{h(r)}{1+\lambda^2}}t\right) + b^i \cos\left(\lambda\sqrt{\frac{h(r)}{1+\lambda^2}}t\right), \tag{4.25}$$

where a^i and b^i are constants. The constraint requires that these constants obey

$$a^i a^i = b^i b^i = 1, \qquad a^i b^i = 0. \tag{4.26}$$

These solutions are the general great circle solutions in the plane spanned by $\vec{a}$ and $\vec{b}$. However, these solutions have an angular velocity which depends on r and therefore they are not consistent with the ansatz.

There is another solution which is similar to (4.25) where angular velocity v is a constant but $\lambda = \lambda(r)$ depends on radial coordinate,

$$x(t)^i = a^i \sin(vt) + b^i \cos(vt). \tag{4.27}$$

From equation (4.23), we find that

$$\lambda(r)^2 = \frac{v^2}{h(r) - v^2}. \tag{4.28}$$

Unfortunately for this solution $-g$ is not positive definite. This is similar to the non-global case discussed in Herzog et al [35] and we will take as our starting point.

curved equatorial ansatz

Motivated by equatorial solution (4.13) and general great circle solutions (4.27) previously, we take the ansatz depends on time and radial coordinate as follows:

$$x(t,r)^i = a^i \sin(vt + c(r)) + b^i \cos(vt + c(r)), \tag{4.29}$$

with v is a constant and $c(r)$ is a function will be determined later. Using the constraint equation, as we did in radial independent ansatz, the equations of motion now become

$$\sqrt{-g}\frac{\partial}{\partial r}\left(\frac{r^4 h\, x^i}{\sqrt{-g}}\right) = \left(\frac{\lambda^2}{1+\lambda^2} - \frac{v^2}{h}\right) x^i, \tag{4.30}$$

where $-g = \frac{1}{1+\lambda^2}$ and we have $\lambda = \lambda(r)$. The $-g$ can also be written, by substituting the ansatz to (4.30), as

$$-g = 1 + r^4 h(r)\, c'(r)^2 - \frac{v^2}{h(r)}. \tag{4.31}$$

Comparing this with the equatorial solution we can identify $c(r) = \eta(r)$ and $v = \omega$. One can check, using these two expressions of $-g$, that at equatorial we get back the equatorial solution (4.13). Now, the equations of motion reduce to

$$\frac{\partial}{\partial r}\left(\frac{r^4 h\, c'(r)}{\sqrt{-g}}\right) = 0 \tag{4.32}$$

which is the same equation for $\eta(r)$ as in the equatorial case. Furthermore one can also find the expression for λ which is given by

$$\lambda(r)^2 = \frac{v^2}{r^4}\frac{r^4 - r_{Sch}^4}{h(r) - v^2}, \tag{4.33}$$

where r_{Sch} is defined as $h(r_{Sch}) = v^2$. This guarantees that $\lambda(r)$ is a positive definite function.

Adding the constraint into the original Nambu-Goto action, we have manifest rotation symmetry $SO(3)$ in x, y, z coordinates. In this case the drag force is related to angular momentum currents in r direction or torques of the world sheet

$$J_r^i = \frac{dL^i}{dt} = -\frac{r^4}{2\pi\alpha'\sqrt{-g}}\left(\dot{x}^j x'^j \varepsilon_{imn} x^m \dot{x}^n - (\dot{x}^j\dot{x}^j - h(r))\varepsilon_{imn} x^m x'^n\right), \tag{4.34}$$

where ε_{imn} is a totally antisymmetric tensor, with $\varepsilon_{123} = 1$. The angular momentum currents or torques, after substituting the ansatz and the constraint, are

$$J_r^i = \frac{dL^i}{dt} = -\frac{r^4 h(r)}{2\pi\alpha'}\frac{\varepsilon_{ijk}x^j x'^k}{\sqrt{-g}} = -\frac{r^4 h(r) c'(r)}{2\pi\alpha'\sqrt{-g}}\varepsilon_{ijk} b^j a^k = -\frac{r^4 h(r) c'(r)}{2\pi\alpha'\sqrt{-g}} n^i \tag{4.35}$$

where $n^i \equiv (n_x, n_y, n_z)$ is the normal vector of great circle with the norm unity. The equations of motion imply that these angular momentum currents of torques are constants. The norm of total angular momentum current or total torque equals to the norm of drag force (4.18), $J_r^2 \equiv J_r^i J_r^i = \frac{dL^i}{dt}\frac{dL^i}{dt} = \left(\frac{dp_\phi}{dt}\right)^2$. This shows that the total drag force is the same as before.

The drag force of string moving in the background of a four dimensions AdS-Schwarzschild is thus a constant related to momentum of a particle represented by the end of a string at the boundary. We can derive the full motion of the particle, e.g. the friction as it moves in the plasma with a constant angular velocity ω, with the boundary metric

$$ds_B^2 = -dt^2 + \frac{1}{l^2}\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.36}$$

To illustrate the relativistic angular velocity of this particle at the boundary is $u^\mu = \gamma(1,0,\omega)$,in terms of coordinates (t,θ,ϕ), with $\gamma = \left(1-\omega^2/l^2\right)^{-1/2}$ and $\theta = \pi/2$. Taking non-relativistic limit, $\omega \ll l$, we obtain $P^\mu \equiv m(1,0,\omega)\left(1+\frac{1}{2}\frac{\omega^2}{l^2}+\cdots\right)$ with m is the mass of particle. In this limit, the drag force becomes

$$\frac{dp_\phi}{dt} = -\frac{1}{2\pi\alpha'}\frac{p_\phi}{m}r_H^2 + O(\omega) \tag{4.37}$$

and thus the friction coefficient is

$$\mu_\phi = \frac{r_H^2}{2m\pi\alpha'} + O(\omega). \tag{4.38}$$

Recall that $r_H = r_{Sch}(\omega = 0)$.

4.3 Anisotropic drag on a string in 4D Kerr-AdS black hole

As has been explained before in the introduction, we are looking for a background metric as a solution to Einstein equation with negative cosmological constant that naturally has anisotropy at the boundary. One such solution is Kerr-AdS black holes. We shall use Kerr-AdS in Boyer-Lindquist coordinates which has less mixing terms than the other coordinates representation and it manifestly reduces to the non-rotating solution of previous section when the rotation parameter a vanishes. A disadvantage is that this coordinates representation does not have manifest $SO(3)$ symmetry at the boundary $r \to \infty$ even though it is restored there, as can be seen by a transformations to another coordinates. The metric of four dimensions Kerr-AdS black hole in Boyer-Lindquist coordinates is explicitly written as [110]

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{\rho^2}\left(dt - \frac{a}{\Xi}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2, \end{aligned} \tag{4.39}$$

where

$$
\begin{aligned}
\rho^2 &= r^2 + a^2\cos^2\theta \\
\Delta_r &= (r^2+a^2)(1+l^2r^2) - 2Mr \\
\Delta_\theta &= 1 - l^2a^2\cos^2\theta \\
\Xi &= 1 - l^2a^2,
\end{aligned} \tag{4.40}
$$

with a is the rotation parameter constrained to $1 > a^2l^2$ in order to have a finite positive value of the horizon area. The event horizon or outer horizon is located at $r = r_{KH}$ which is the largest root of Δ_r. The Hawking temperature is given by

$$
T_H = r_{KH}\frac{3l^2r_{KH}^2 + 1 + a^2l^2 - a^2/r_{KH}}{4\pi(r_{KH}^2 + a^2)}. \tag{4.41}
$$

The Nambu-Goto action for the metric above is

$$
\begin{aligned}
S_{NG} &= -\frac{1}{2\pi\alpha'}\int d\sigma^2\sqrt{-g}, \\
-g &= \Bigg(\left(a\Delta_r - a(r^2+a^2)\Delta_\theta\right)\frac{\sin^2\theta}{\Xi\rho^2}\phi' + \frac{\rho^2}{\Delta_\theta}\dot{\theta}\theta' \\
&\quad + \left(\Delta_\theta(r^2+a^2)^2 - a^2\Delta_r\sin^2\theta\right)\frac{\sin^2\theta}{\Xi^2\rho^2}\dot{\phi}\phi'\Bigg)^2 \\
&\quad - \left(\frac{\rho^2}{\Delta_r} + \frac{\rho^2}{\Delta_\theta}\theta'^2 + \left(\Delta_\theta(r^2+a^2)^2 - a^2\Delta_r\sin^2\theta\right)\frac{\sin^2\theta}{\Xi^2\rho^2}\phi'^2\right)\times \\
&\quad \times\Bigg(\left(a^2\Delta_\theta\sin^2\theta - \Delta_r\right)\frac{1}{\rho^2} + \left(a\Delta_r - a(r^2+a^2)\Delta_\theta\right)\frac{2\sin^2\theta}{\Xi\rho^2}\dot{\phi} \\
&\quad \left(\Delta_\theta(r^2+a^2)^2 - a^2\Delta_r\sin^2\theta\right)\frac{\sin^2\theta}{\Xi^2\rho^2}\dot{\phi}^2 + \frac{\rho^2}{\Delta_\theta}\dot{\theta}^2\Bigg).
\end{aligned} \tag{4.42}
$$

Let us consider the equatorial solution for $\theta = \pi/2$ and take the ansatz (4.13) such that the equation of motion now becomes

$$
\begin{aligned}
\eta'(r) &= \frac{\pi_\phi^r(1-a^2l^2)}{\Delta_r}\sqrt{\frac{(1-a^2l^2-a\omega)^2\Delta_r - f(r)}{\Delta_r - (1-a^2l^2)^2(\pi_\phi^r)^2}}, \\
f(r) &= (a - a^3l^2 - a^2\omega - \omega r^2)^2,
\end{aligned} \tag{4.43}
$$

requiring real solution everywhere demands

$$
\frac{(1-a^2l^2)^2(1-a^2l^2-a\omega)^2\pi_\phi^2 - f(r)}{(1-a^2l^2-a\omega)^2\Delta_r - f(r)} \leq 1. \tag{4.44}
$$

Again, lets first look at $r \to \infty$, then we obtain[6]

$$l^2 \geq \frac{\omega^2}{(1 - a^2 l^2 - a\omega)^2}. \tag{4.45}$$

Limiting the calculation for real positive r, in order to satisfy the inequality (4.44) we need to know the profile of both numerator and denominator on the left hand side of the inequality (4.44) at least in the region $r_{KH} \leq r < \infty$. Unfortunately this is not an easy task unlike in the case of AdS-Schwarzschild. Here we assume that for some values of parameter l, ω, M, and a the numerator and denominator behave like in the case of AdS-Schwarzschild where at some radius $r = r_K$, which is the largest positive root of $(1 - a^2 l^2 - a\omega)^2 \Delta_r - f(r)$, the numerator and denominator change their sign as we move from the boundary at $r \to \infty$ down to the horizon $r = r_{KH}$ with

$$(\pi_\phi^r)^2 = \frac{(a - a^3 l^2 - a^2\omega - r_K^2 \omega)^2}{(1 - a^2 l^2)^2 (1 - a^2 l^2 - a\omega)^2}. \tag{4.46}$$

Then we can compute the drag force as follows

$$\begin{aligned} \frac{dp_\phi}{dt} &= -\frac{\sqrt{f(r_K)}}{2\pi\alpha'(1 - a^2 l^2 - a\omega)(1 - a^2 l^2)}. \\ &= -\frac{\pi_\phi^r}{2\pi\alpha'}. \end{aligned} \tag{4.47}$$

The drag force actually can be easily computed using the fact that the equation of motion for ϕ is the conservation current in radial r direction, i.e. $\partial_r J_\phi^r = 0$. This is true since we are considering the dynamical field in ϕ to depend on the radial direction r. So, the conserved current in radial direction r for ϕ is actually the conjugate momentum of ϕ in radial direction r such that $J_\phi^r = \frac{\partial \mathcal{L}}{\partial \phi'}$ which is just the constant π_ϕ^r. What we have to do then is to determine the values of π_ϕ^r that give physical solution.

The exact expression for r_K, as solution to $f(r_K) = (1 - a^2 l^2 - a\omega)^2 \Delta_r(r_K)$, in this case is very long and complicated. In the small limit $a \ll 1$, we can write $r_K = r_0 + a\, r_1 + O(a^2)$, with $r_0 = r_{Sch}$ and

$$r_1 = \frac{\omega l^2 r_0^4 - 2\omega M r_0}{2(l^2 - \omega^2) r_0^3 + r_0 - M}. \tag{4.48}$$

Then we can write π_ϕ^r as expansion of a up to first order as follows

$$\pi_\phi^r \approx \omega r_0^2 - (1 - \omega r_0 (\omega r_0 + 2 r_1))\, a + O(a^2). \tag{4.49}$$

[6] This condition is related to light and time-like region of the spacetime at the boundary where the light-like is given by $\frac{\omega^2}{l^2} = (1 - a^2 l^2 - aw)^2$.

Thus the drag force can be written as

$$\frac{dp_\phi}{dt} = -\frac{1}{2\pi\alpha'}\left[\omega r_0^2 - (1-\omega r_0(\omega r_0 + 2r_1))\, a + O(a^2)\right]. \tag{4.50}$$

The fist term is the drag force of the 4D non-rotating black hole (4.18) and the second term can be considered as correction in the present of small angular momentum a. Explicitly for small a, the drag force is written as

$$\begin{aligned} \frac{dp_\phi}{dt} &= -\frac{1}{2\pi\alpha'}\left[\pi_\phi^0 - a + C\,\pi_\phi^0 a + O(a^2)\right], \\ C &= \omega\left[\frac{2(2l^2-\omega^2)r_0^3 + r_0 - 5M}{2(l^2-\omega^2)r_0^3 + r_0 - M}\right], \end{aligned} \tag{4.51}$$

with π_ϕ^0 is the drag force of non-rotating AdS-Schwarzschild black hole. The first linear term in a is simply due to a change of frame, but the term with coefficient C exhibits a nonlinear-enhancement of the drag force in the presence of angular momentum.

4.3.1 Static solution

Generalization of the arbitrary great circle solutions (4.29) to Kerr-AdS black holes is very difficult because of the complexity of the equations of motion. However, we can still solve these equations for a particular solution where we consider the string to be static. We will define shortly what we mean by static. This kind of solution can already capture the drag force effect of a rotating plasma. In the equatorial plane, the effect should be the same as considering a moving string in a non-rotating black hole by switching observers. For motion that is not equatorial, we will find a new force due to the anisotropy breaking by the angular momentum. We expect it to be centripetal force-like and drive the motion back to equatorial orbits. Specifically this means that this force will not depend on the direction of the angular momentum, but only on its magnitude. To lowest order in a therefore, this contribution goes as a^2. Here we will compute this lowest order consequence of anisotropy by expanding and solving the static equations of motion to order a^2. This will establish the lowest order contribution of anisotropic effects to the drag force. In the rotating black hole case, the static solution is not the time-independent solution with $\theta = \theta(r)$ and $\phi = \phi(r)$.

There is a subtle aspect of the Kerr-AdS metric in Boyer-Lindquist coordinates. It is actually not manifest asymptotically AdS metric at the boundary. We need to do the following coordinate transformation to obtain this.

$$\begin{aligned} T &= t, \qquad \Phi = \phi - al^2 t, \\ y\cos\Theta &= r\cos\theta, \qquad y^2 = \frac{1}{\Xi}\left(r^2\Delta_\theta + a^2\sin^2\theta\right). \end{aligned} \tag{4.52}$$

The full expression for the Kerr-AdS metric in coordinates T, y, Θ, Φ is very complicated. For $M = 0$, however it is the regular AdS metric

$$ds^2 = -(1 + l^2y^2)dT^2 + \frac{1}{1 + l^2y^2}dy^2 + y^2 \left(d\Theta^2 + \sin^2 \Theta d\Phi^2\right). \tag{4.53}$$

In this metric the static straight string, with $\Theta = C_\Theta$ and $\Phi = C_\Phi$ constants, is a solution of the equation of motion. This then corresponds to a "time-dependent" solution in Boyer-Lindquist coordinates

$$\phi = C_\Phi + al^2t, \tag{4.54}$$

$$\theta = \arccos\left(\frac{y}{r}\cos C_\Theta\right), \tag{4.55}$$

with

$$y^2 = \frac{r^2(r^2 + a^2)}{(1 - a^2l^2 \sin^2 C_\Theta)r^2 + a^2 \cos^2 C_\Theta}. \tag{4.56}$$

That is nevertheless describing a static string. For finite M, we expect these constants to be function of r in Boyer-Lindquist coordinates. Thus the "static" ansatz as an expansion in small a is taken to be

$$\theta(r) = \theta_0(r) + a^2\theta_1(r) + O(a^4), \tag{4.57}$$

$$\phi(t, r) = \phi_0(r) + al^2t\ \phi_t(r) + a\phi_1(r) + O(a^2). \tag{4.58}$$

Let's consider $\phi_0(r) = P_0$, $\theta_0(r) = \Theta_0$, and $\phi_t(r) = P_t$ constants. From the mapping of coordinates above, immediately P_t can be set to 1. Solving the equations of motion order by order in power of a, we obtain

$$\phi_1(r) = -\int dr \frac{P_1}{r^4h(r)}, \tag{4.59}$$

with P_1 a constant. The solution for θ_1 is quite complicated, but for our purpose we will just need the near boundary solution which can be computed using Mathematica. Substituting the solutions, we find that the world sheet conjugate momenta in radial direction as an expansion of small a near the boundary are given by

$$\pi^r_\theta \approx \left(-3l^2r + \frac{2T_2}{\sin(2\Theta_0)} + (1 - P_1^2)\frac{\log(r)}{M} - \frac{3}{r} + \cdots\right)\frac{a^2}{2}\sin(2\Theta_0) + O(a^4), \tag{4.60}$$

$$\pi^r_\phi \approx P_1 \sin(\Theta_0)^2 a + O(a^2), \tag{4.61}$$

with T_2 a constant. P_1 can be fixed by comparing the result at $P_t = 0$ with the equatorial solution (4.49) at zero angular velocity which gives us $P_1 = -1$.

Interestingly, this also fixes $T_2 = 0$. The conjugate momentum π^r_θ has a singularity at $r \to \infty$ which goes linearly in r. This singularity at $r \to \infty$ corresponds to infinite mass of the quark [35]. In order to have a more realistic picture, we consider a finite large mass of quark by introducing a cut-off in the geometry near the boundary at $r = r_c$ which in the bulk can be interpreted as the location of a probe D-brane where the string can end. Following [35], the static thermal rest mass of quark can be computed at leading order in a by setting $P_t = 0$,

$$m_{rest} = \frac{1}{2\pi\alpha'}\left(r_c - \frac{1}{3l^2}\left(2\pi T + \sqrt{4\pi^2T^2 - 3l^2}\right)\right), \tag{4.62}$$

with T is the temperature of plasma written as (4.41). Then by evaluating conjugate momenta above at $r = r_c$ we obtain the leading contribution of the conjugate momenta

$$\pi^r_\theta \approx -\left(6\pi\alpha' l^2 m_{rest} + 2\pi T + \sqrt{4\pi^2T^2 - 3l^2}\right)\frac{a^2}{2}\sin(2\Theta_0), \tag{4.63}$$

$$\pi^r_\phi \approx -\sin(\Theta_0)^2 a, \tag{4.64}$$

4.3.2 Drag force

Having the solution in the a expansion, now we are ready to compute the drag forces to leading order in double expansions of small a and ω:

$$\frac{dp_\theta}{dt} \approx \pi^r_\theta(a \neq 0 \ll 1, \omega = 0)_{r\to\infty} + \pi^r_\theta(a = 0, \omega \neq 0 \ll 1)_{r\to\infty}, \tag{4.65}$$

$$\frac{dp_\phi}{dt} \approx \pi^r_\phi(a \neq 0 \ll 1, \omega = 0)_{r\to\infty} + \pi^r_\phi(a = 0, \omega \neq 0 \ll 1)_{r\to\infty}. \tag{4.66}$$

The $(a = 0, \omega = 0)$ term is just a constant which can be set to zero. We have included in this expression the lowest order $(a = 0, \omega \neq 0)$ solution, which is valid as long as both a and ω are small. The leading contribution to $\pi^r_\theta(a = 0)$ and $\pi^r_\theta(a = 0)$ is simply $\pi^r_\theta(a = 0) \approx \omega_\theta$ and $\pi^r_\phi(a = 0) \approx \omega_\phi$, with ω_θ and ω_ϕ are small. So, we obtain the leading contribution to the drag forces

$$\frac{dp_\theta}{dt} \approx \left(3l^2 m_{rest} + \frac{1}{2\pi\alpha'}\left(2\pi T + \sqrt{4\pi^2T^2 - 3l^2}\right)\right)\frac{a^2}{2}\sin(2\Theta_0) - \frac{\omega_\theta}{2\pi\alpha'}, \tag{4.67}$$

$$\frac{dp_\phi}{dt} \approx -\frac{1}{2\pi\alpha'}\left(\omega_\phi - \sin(\Theta_0)^2 a\right). \tag{4.68}$$

Here we can see that at the poles, defined at $\Theta_0 = 0$ and $\Theta_0 = \pi$, there is no additional drag forces to the static quark, with $\omega_\theta = \omega_\phi = 0$. These are unstable points for a generic value of Θ_0 that the drag force in θ direction tends to drag the static quark to the equator, $\Theta_0 = \pi/2$. The general situation at instantaneous time is illustrated in figures 4.1, 4.2, 4.3, and 4.4. The figures describe motion of quarks at different positions on the boundary with an uniform velocity (blue arrow), in Cartesian coordinates, or with

$$\omega_\theta = \frac{1}{10}(\cos\phi - \sin\phi),$$
$$\omega_\phi = -\frac{1}{10\theta}(\cos\phi + \sin\phi), \tag{4.69}$$

in Polar coordinates, being seen from the north pole of S^2 projected into a plane. The circle, with bold line, denotes the equator of S^2. The circular brown arrow is the direction of angular momentum of the black hole. The red arrows show direction of the drag force effect of the plasma with its length proportional to the strength how much the drag force needed to drive the quarks to the equatorial. We have drawn the figures for different values of a and $M_T = (3l^2 m_{rest} + 2\pi T + \sqrt{4\pi^2 T^2 - 3l^2})/2$, with $1/\alpha' = 2\pi$.

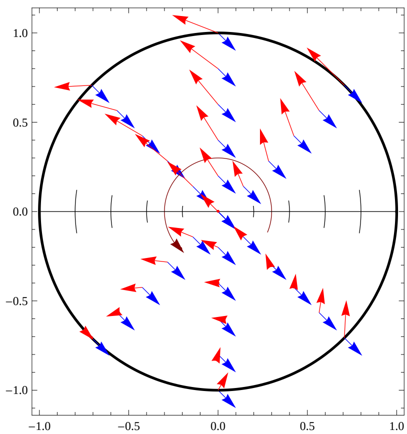

Figure 4.1: $a = 0.1, M_T = 10$

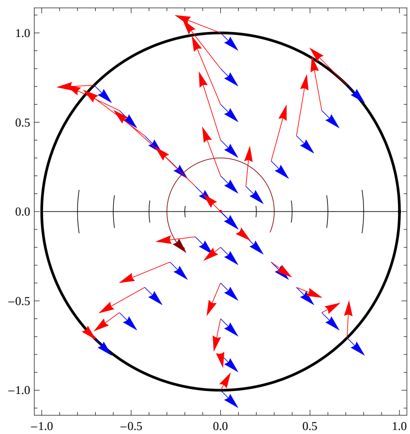

Figure 4.2: $a = 0.1, M_T = 30$

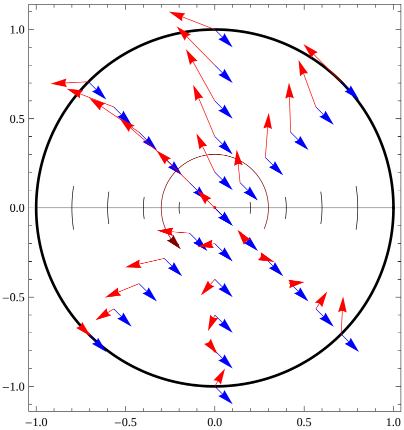

Figure 4.3: $a = 0.1$, $M_T = 20$

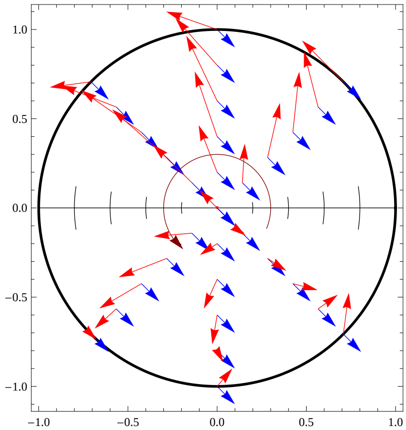

Figure 4.4: $a = 0.118$, $M_T = 20$

4.4 Discussion and conclusion

In a non-rotating 4D global AdS-Schwarzschild black hole, the moving curved string solution (4.13), or in general (4.29), corresponds to a heavy quark moving in relativistic plasma living on a sphere S^2. The motion of this quark follows the geodesic of S^2 namely a great circle. A novel physical consistency condition of this solution is that the string velocity ω must be smaller than curvature radius of the black hole l. In this setting, we obtained the drag force as a function of the velocity ω times the square of critical radius r_{Sch} defined by the largest positive root of $h(r) = \omega^2$. In the non-relativistic limit, $\omega \to 0$, this friction coefficient is simply the square of horizon r_H as in the flat case [35, 39]. Unlike the flat case, the friction coefficient has non-linear dependence on the temperature. Furthermore, the temperature of plasma in this background is bounded from below limited by the Hawking-Page transition to Euclidean AdS [9].

Our main result is to take a first step towards the study of anisotropic effects on drag force. For this, we considered the background metric of the rotating 4D Kerr-AdS black hole. For equatorial motion, the drag force is simplfy solved in the same way as the non-rotating case. The velocity ω is now bounded non-trivially by the angular momentum a and curvature radius l (4.45). Also unlike the previous non-rotating case, at zero velocity, $\omega = 0$, the drag force does not vanish but is proportional to the angular momentum of the black hole a . This can be understood at the boundary as the drag force effect of dropping a static quark into a rotating plasma where the rotating plasma forces the quark to move accordingly with the plasma. In order to get a vanishing drag force, the quark must move at a critical velocity

$$\omega_C = a\left(\frac{1-a^2l^2}{a^2+r_{KH}^2}\right). \tag{4.70}$$

The drag force changes its direction when the quark's velocity crosses this critical velocity.

The generalization to arbitrary great circle motion is rather difficult. Taking a "static" solution in Boyer-Lindquist coordinates, corresponding to a static quark in a rotating plasma, we were able to compute the leading contribution to the drag force. Based on parity this drag force in the θ-direction should be an even function of the angular momentum a whereas the drag force in the ϕ-direction to linear order in a should be a generalization of the equatorial motion. As illustrated in figures 4.1 to 4.4, we conclude that the resulting drag force in a rotating strongly coupled plasma tends to drive the quark back to the equatorial plane and the amount of force is proportional to the static thermal rest mass of the quark m_{rest} and temperature of the plasma T in an analytic expression (4.67).